Uniqueness of Gibbs measures relative to Brownian motion

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Abstract

We consider the set of Gibbs measures relative to Brownian motion for given potential V. V is assumed to be Kato-decomposable but general otherwise. A Gibbs measure for such a potential is in many cases given by a reversible Itô diffusion μ. We show that if V is growing at infinity faster than quadratically and in a sufficiently regular way, then μ is the only Gibbs measure that exists. For general V we specify a subset of the configuration space Ω such that μ is the only Gibbs measure for V supported on this subset. We illustrate our results by some examples.

1 Introduction

In the classical theory of Gibbs measures [9] we are given a (countable) lattice, say $\mathbb{Z}^n$, a measurable state space E, and a measure μ_0 on E. A Gibbs measure μ is any measure on the product σ-field over $\Omega = E^{\mathbb{Z}^n}$ which is locally absolutely continuous with respect to the reference measure $\mu_0^{\mathbb{Z}^n}$ and satisfies the DLR-equations, i.e. for which the regular conditional expectations given the configuration in the outside of a finite subset of $\mathbb{Z}^n$ are of a prescribed form. This specific form is determined by the potential, which is regarded as the central parameter of the theory.

The setup in which we study Gibbs measures is somewhat different from the usual one. The main difference is that $\mathbb{Z}^n$ is replaced by $\mathbb{R}$, and $E^{\mathbb{Z}^n}$ is replaced by $\Omega = C(\mathbb{R}, \mathbb{R}^d)$. Ω can not be written as a countable product of measurable spaces. As a reference measure we take Brownian bridge, and our potential V is assumed to depend only on single points $\omega(t), t \in \mathbb{R}$, of a configuration $\omega \in \Omega$. In the lattice theory, this would be a one-dimensional system of unbounded spins with single site potential and would result in a trivial model with no interaction between different sites, but in our case, dependence equivalent to an interaction is present in the Brownian bridge.

As in the classical theory, a Gibbs measure will be any probability measure fulfilling the DLR equations, cf. Definition 2.1 below. We will refer to any Gibbs measure arising from the above setup as a Gibbs measure relative to Brownian motion. In this work we investigate uniqueness of such Gibbs measures. For many potentials, it is easy to see that a Gibbs measure exists and is given by a reversible diffusion process, but it is not clear whether this is the only one. In fact, the well studied example $V(x) = |x|^2$ shows that infinitely many Gibbs measures can exist for the same potential. However, in that case all non-stationary Gibbs measures share the feature that they are supported on paths that are growing exponentially at infinity. Thus by restricting the configuration space to a subset $\Omega^* \subset \Omega = C(\mathbb{R}, \mathbb{R}^d)$ which is characterized by a condition on the growth of paths at infinity, one can expect to retain uniqueness. We show this for a wide class of potentials V. Moreover, we find that in the case where there exists $b > 2$ such that $V(x)$ grows faster than $|x|^b$ but not faster than $|x|^{2b-2}$ at infinity, no restriction of Ω is necessary in order to obtain uniqueness. In some sense, this shows that the potential $V(x) = |x|^2$ mentioned above is a borderline case.

Gibbs measures relative to Brownian motion have been first treated in the 1970s and have seen a renaissance in recent years. The best understood case is the one described above with a potential depending only on single points $\omega(t)$ of a configuration ω. It is also the case we will study in this work. In [6] and [19], such Gibbs measures are characterized by an invariance property, and the structure of the set of all Gibbs measures is explored under the assumption that V is a polynomial. [18] investigates uniqueness of such Gibbs measures under the same assumption. As we will explain in Example 4.2, we are able to improve considerably on the cited results.

A different point of view is taken in [11], where Gibbs measures are constructed as reversible measures for some stochastic process with infinite dimensional state space. In yet another approach [17] consider Gibbs measures with the torus as state space. In this case, many of the usual difficulties arising from a non-compact state space can be avoided.

The recent developments on Gibbs measures relative to Brownian motion start with [15], where for the first time the more difficult case of an explicit interaction potential is treated mathematically. Various results on such Gibbs measures under different conditions have been obtained in [2, 10, 12]. [3, 13, 14] use them to investigate a model of quantum field theory. In order to gain better control over these Gibbs measures, we have to achieve as much understanding as possible for the 'easy' case involving no pair interaction. The aim of the present paper is to add to this understanding.

2 Preliminaries and basic definitions

Let $I \subset \mathbb{R}$ be a finite union of (bounded or unbounded) intervals. We denote by $C(I, \mathbb{R}^d)$ the space of all continuous functions $I \to \mathbb{R}^d$, and endow it with the σ-field $\mathcal{F}_I$ generated by the point evaluations. The same symbol $\mathcal{F}_I$ will be used to denote the σ-field on $\Omega = C(\mathbb{R}, \mathbb{R}^d)$ generated by the point evaluations at time-points inside I. For the point evaluations $C(I, \mathbb{R}^d) \to \mathbb{R}^d$ we will use both of the notations $t \mapsto \omega_t$ and $t \mapsto \omega(t)$, choosing whichever of the two makes a nicer notation. We write $\mathcal{F}$ instead of $\mathcal{F}_{\mathbb{R}}$, $\mathcal{F}_T$ instead of $\mathcal{F}_{[-T,T]}$, and $\mathcal{T}_T$ instead of $\mathcal{F}_{[-T,T]^c}$ for $T > 0$, where $[-T, T]^c$ denotes the complement of $[-T, T]$. For $s, t \in \mathbb{R}$ with $s < t$ and $x, y \in \mathbb{R}^d$ we denote by $\mathcal{W}^{x,y}_{[s,t]}$ the measure of the

Brownian bridge starting in x at time s and ending in y at time t. $\mathcal{W}_{[s,t]}^{x,y}$ is a measure on $C([s,t],\mathbb{R}^d)$. For $T>0$, we write $\mathcal{W}_T^{x,y}$ instead of $\mathcal{W}_{[-T,T]}^{x,y}$. For $\bar{\omega}\in\Omega$, let $\delta_T^{\bar{\omega}}$ be the Dirac measure on $C([-T,T]^c,\mathbb{R}^d)$ concentrated in $\bar{\omega}$. Note that $\delta_T^{\bar{\omega}}$ does not depend on the part of $\bar{\omega}$ inside $[-T,T]$. Finally, we define

$$\mathcal{W}_T^{\bar{\omega}} := \mathcal{W}_T^{\bar{\omega}(T_1),\bar{\omega}(T_2)} \otimes \delta_T^{\bar{\omega}}. \tag{2.1}$$

We can (and will) regard $\mathcal{W}_T^{\bar{\omega}}$ as a measure on $C(\mathbb{R},\mathbb{R}^d)$.

We now proceed to the definition of Gibbs measure relative to Brownian motion. Let $V:\mathbb{R}^d\to\mathbb{R}$ be a measurable function such that

$$Z_T(x,y) := \int \exp\left(-\int_{-T}^{T} V(\omega_s)\,ds\right)\,d\mathcal{W}_T^{x,y}(\omega) < \infty \tag{2.2}$$

for all $T>0$ and all $x,y\in\mathbb{R}^d$. For $T>0$ we define a probability kernel μ_T from $(\Omega,\mathcal{T}_T)$ to $(\Omega,\mathcal{F})$ by

$$\mu_T(A,\bar{\omega}) := \frac{1}{Z_T(\bar{\omega}_{-T},\bar{\omega}_T)}\int 1_A(\omega)e^{-\int_{-T}^{T}V(\omega_s)\,ds}\,d\mathcal{W}_T^{\bar{\omega}}(\omega) \quad (A\in\mathcal{F},\bar{\omega}\in\Omega). \tag{2.3}$$

Note that $Z_T(\bar{\omega}_{-T},\bar{\omega}_T)$ is indeed the correct normalization. This follows from the Markov property of Brownian motion.

Definition 2.1 *A probability measure μ over Ω is a* **Gibbs measure relative to Brownian motion** *for the potential V if for each $A\in\mathcal{F}$ and $T>0$,*

$$\mu_T(A,\cdot) = \mu(A|\mathcal{T}_T) \qquad \mu\text{-almost surely}, \tag{2.4}$$

where $\mu(A|\mathcal{T}_T)$ denotes conditional expectation given $\mathcal{T}_T$.

Equation (2.4) is the continuum analog to the DLR equations in the lattice context [9]. While the integrability condition (2.2) is enough to define Gibbs measures, for the remainder of this work we put some further restrictions on V.

Definition 2.2 *A measurable function $V:\mathbb{R}^d\to\mathbb{R}$ is said to be in the* **Kato class** *[21] $\mathcal{K}(\mathbb{R}^d)$, if*

$$\sup_{x\in\mathbb{R}}\int_{\{|x-y|\le 1\}} |V(y)|\,dy < \infty \qquad \text{in case } d=1,$$

and

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{\{|x-y|\le r\}} g(x-y)|V(y)|\,dy = 0 \qquad \text{in case } d\ge 2.$$

Here,

$$g(x) = \begin{cases} -\ln|x| & \text{if } d=2 \\ |x|^{2-d} & \text{if } d\ge 3. \end{cases}$$

V is locally in Kato class, i.e. in $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$, if $V1_K\in\mathcal{K}(\mathbb{R}^d)$ for each compact set $K\subset\mathbb{R}^d$. V is **Kato decomposable** *[4] if*

$$V = V^+ - V^- \quad \text{with} \quad V^-\in\mathcal{K}(\mathbb{R}^d), \quad V^+\in\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d),$$

where V^+ is the positive part and V^- is the negative part of V.

It is possible to characterize $\mathcal{K}(\mathbb{R}^d)$ in terms of Wiener integrals [21]. The main feature of Kato-decomposable functions V is that the Schrödinger operator

$$H = -\tfrac{1}{2}\Delta + V \tag{2.5}$$

acting in $L^2(\lambda^d)$ and the corresponding Schrödinger semigroup $(e^{-tH})_{t\geq 0}$ have many nice properties. In (2.5), λ^d is the d-dimensional Lebesgue measure, Δ is the Laplace operator and V acts as multiplication operator. We give a list of those properties that we will need in this work, see [21] for many more.

Remark 2.3 Let H be a Schrödinger operator with Kato-class potential.

1) H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and bounded from below.

2) Consider the semigroup $\{e^{-tH} : t \geq 0\}$ of bounded operators on $L^2(\mathbb{R}^d)$ defined via functional calculus. The operator e^{-tH} is bounded from L^p to L^q for every $1 \leq p \leq q \leq \infty$ and every $t > 0$. In addition, $e^{-tH}f$ is a continuous function for every $f \in L^p, p \in [1, \infty]$ and every $t > 0$.

3) For every $t > 0$, e^{-tH} is an integral operator with continuous, bounded kernel K_t. Moreover, the map $(t, x, y) \mapsto K_t(x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, and the Feynman-Kac formula

$$K_{t-s}(x, y) = \int e^{-\int_s^t V(\omega_r)\, dr}\, d\mathcal{W}_{[s,t]}^{x,y}(\omega) \quad \forall s < t \in \mathbb{R},\ \forall x, y \in \mathbb{R}^d \tag{2.6}$$

holds.

4) If the bottom of the spectrum of H is an eigenvalue, then it necessarily has multiplicity one. In this case, the corresponding eigenfunction ψ_0 can be chosen to be continuous and strictly positive.

For the remainder of the paper, we will make the standing assumption that V is Kato-decomposable and that $H = -\frac{1}{2}\Delta + V$ has an eigenvalue at the bottom of its spectrum. By adding a constant to V if necessary, we may (and do) assume that this eigenvalue is 0, and choose the corresponding unique eigenfunction ψ_0 to be strictly positive and of $L^2(\lambda^d)$-norm one. ψ_0 will be referred to as the ground state of H.

The question whether a given Schrödinger operator has a ground state is studied in detail in [16]. In particular, whenever V is Kato-decomposable and $\liminf_{|x|\to\infty} V(x) = \infty$, the bottom of the spectrum is an eigenvalue separated from the rest of the spectrum by a gap γ, and in particular a ground state ψ_0 exists.

Given a Schrödinger operator with Kato decomposable potential V and ground state ψ_0, we define a probability measure μ on $(\Omega, \mathcal{F})$ (i.e., a stochastic process) by putting

$$\mu(A) = \int dx\, \psi_0(x) \int dy\, \psi_0(y) \int 1_A(\omega) e^{-\int_{-T}^T V(\omega_s)\, ds}\, d\mathcal{W}_T^{x,y}(\omega) \tag{2.7}$$

for $A \in \mathcal{F}_T$ and extending the above to a measure on $\mathcal{F}$. This extension is possible: $e^{-tH}\psi_0 = \psi_0$ and $\left\|\psi_0^2\right\|_2 = 1$ together with the Feynman-Kac-formula and the Markov property of Brownian motion imply that measures defined on $\mathcal{F}_T$, $T > 0$ in (2.7) define a consistent family of probability measures. By the same reasons, μ also fulfills (2.4) and thus is a Gibbs measure relative to Brownian motion for the potential V.

In fact, μ is the measure of a reversible diffusion process with invariant measure $d\nu = \psi_0^2\, d\lambda^d$ and stochastic generator H_ν acting in $L^2(\nu)$ as

$$H_\nu f = \frac{1}{\psi_0} H(\psi_0 f) = -\frac{1}{2}\Delta f - \left\langle \frac{\nabla \psi_0}{\psi_0}, \nabla f \right\rangle_{\mathbb{R}^d}.$$

Such processes are called $P(\phi)_1$-processes in [21], although in probability theory they are better known as Itô-diffusions. The transition probabilities for μ are given by

$$\mu(f(\omega_{t+s})|\omega_s = x) = \int Q_t(x,y) f(y)\, d\nu(y), \tag{2.8}$$

where

$$Q_t(x,y) = \frac{K_t(x,y)}{\psi_0(x)\psi_0(y)} \tag{2.9}$$

is the transition density of μ with respect to its invariant measure.

3 Uniqueness

We have seen in the last section that the existence of a Gibbs measure for the potential V follows from the existence of a ground state for the corresponding Schrödinger operator. The question we ask in this section is whether this Gibbs measure is unique, i.e. we want to know if there are any other probability measures on $(\Omega, \mathcal{F})$ fulfilling the DLR equations (2.4) for the same potential.

We begin by giving a motivating example which demonstrates that uniqueness need not hold in general. This example is in fact well known and has been treated several times in various forms in the literature [19, 7, 1].

Example 3.1 [Ornstein-Uhlenbeck process] Take $V(x) = \frac{1}{2}(x^2 - 1), H := -\frac{1}{2}\frac{d^2}{dx^2} + V$. Then the ground state of H is $\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$, and the diffusion process μ corresponding to H is a one-dimensional Ornstein-Uhlenbeck process. Moreover, Mehler's formula gives explicitly the integral kernel $K_t(x,y)$ of e^{-tH}, i.e.

$$K_t(x,y) = \frac{1}{\sqrt{\pi(1-e^{-2t})}} \exp\left(\frac{4xye^{-t} - (x^2+y^2)(1+e^{-2t})}{2(1-e^{-2t})} \right). \tag{3.1}$$

Now fix $\alpha, \beta \in \mathbb{R}$ and define for $s, x \in \mathbb{R}$

$$\psi_s^l(x) := \pi^{-1/4} \exp\left(-\frac{1}{2}(x + \alpha e^{-s})^2 \right) \exp\left(\frac{\alpha e^{-s}}{2} \right)^2,$$
$$\psi_s^r(x) := \pi^{-1/4} \exp\left(-\frac{1}{2}(x + \beta e^{+s})^2 \right) \exp\left(\frac{\beta e^{+s}}{2} \right)^2.$$

An explicit calculation using (3.1) shows that

$$e^{-tH}\psi^l_s = \psi^l_{s+t}, \quad e^{-tH}\psi^r_s = \psi^r_{s-t}, \quad \text{and} \quad \left\langle \psi^l_s, \psi^r_s \right\rangle = e^{\alpha\beta/2}.$$

Therefore we can define measures $\mu_{\alpha,\beta}$ by

$$\mu_{\alpha,\beta}(A) := e^{-\alpha\beta/2} \int dx\, \psi^l_T(x) \int dy\, \psi^r_T(y) \int 1_A(\omega) e^{-\int_{-T}^{T} V(\omega_s)\,ds}\, d\mathcal{W}^{x,y}_T(\omega) \tag{3.2}$$

for all $A \in \mathcal{F}_T$ in the same way as we did in (2.7). $\mu_{\alpha,\beta}$ is the measure of a Gaussian Markov process, which is stationary if and only if $\alpha = \beta = 0$. By checking (2.4) directly we see that for every $\alpha, \beta \in \mathbb{R}$, $\mu_{\alpha,\beta}$ is a Gibbs measure for $V = \frac{1}{2}(x^2 - 1)$. For this potential we thus have uncountably many Gibbs measures.

We now give a simple criterion allowing to check if a Gibbs measure is the only one supported on a given set. Recall that a probability measure ν is said to be supported on a set A if $\nu(A) = 1$.

Lemma 3.2 *Let $\Omega^* \subset \Omega$ be measurable and let ν be a Gibbs measure for the potential V supported on Ω^*. For $N \in \mathbb{N}, \bar{\omega} \in \Omega$, define $\nu_N(A, \bar{\omega})$ as in (2.3). Suppose that for each $T > 0$, each $A \in \mathcal{F}_T$ and each $\bar{\omega} \in \Omega^*$, $\nu_N(A, \bar{\omega}) \to \nu(A)$ as $\mathbb{N} \ni N \to \infty$. Then ν is the only Gibbs measure for V supported on Ω^*.*

Proof: Let $\tilde{\nu}$ be any Gibbs measure supported by Ω^*. For each $T < N$ and $A \in \mathcal{F}_T$, $\bar{\omega} \mapsto \tilde{\nu}(A|\mathcal{T}_N)(\bar{\omega})$ is a backward martingale in N, thus convergent almost everywhere to $\tilde{\nu}(A|\mathcal{T})(\bar{\omega})$. By the DLR equations (2.4) , $\tilde{\nu}(A|\mathcal{T}_N)(\bar{\omega}) = \nu_N(A, \bar{\omega})$ $\tilde{\nu}$-almost surely, and thus for $\tilde{\nu}$-almost every $\bar{\omega} \in \Omega^*$, we find

$$\tilde{\nu}(A|\mathcal{T})(\bar{\omega}) = \lim_{N\to\infty} \tilde{\nu}(A|\mathcal{T}_N)(\bar{\omega}) = \lim_{N\to\infty} \nu_N(A, \bar{\omega}) = \nu(A).$$

Here we put $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n$. Taking $\tilde{\nu}$-expectations on both sides of the above equality shows $\tilde{\nu}(A) = \nu(A)$. Since this is true for each $A \in \mathcal{F}_T$ and each $T > 0$, $\tilde{\nu} = \nu$. $\square$

In order to apply the above lemma we need to know for which $\bar{\omega} \in \Omega$ the convergence $\mu_N(A, \bar{\omega}) \to \mu(A)$ holds. The next lemma gives a sufficient condition for this in terms of the transition densities Q_T of μ, cf. (2.9).

Lemma 3.3 *Let V be a potential from the Kato-class, $H = -\frac{1}{2}\Delta + V$ the corresponding Schrödinger operator, and assume that H has a ground state ψ_0. Let $Q_t(x, y)$ be as in (2.9) and suppose that for some $\bar{\omega} \in \Omega$ we have*

$$\sup_{x,y\in\mathbb{R}^d} \left(\left| \frac{Q_{N-T}(\bar{\omega}_{-N}, x) Q_{N-T}(y, \bar{\omega}_N)}{Q_N(\bar{\omega}_{-N}, \bar{\omega}_N)} - 1 \right| \psi_0(x)\psi_0(y) \right) \to 0 \quad \text{as } \mathbb{N} \ni N \to \infty \tag{3.3}$$

for all $T > 0$. Then for each $T > 0$ and every $A \in \mathcal{F}_T$, $\mu_N(A, \bar{\omega}) \to \mu(A)$ as $N \to \infty$.

Proof: Let $A \in \mathcal{F}_T$, and let us assume for the start that $A \subset \{\omega \in \Omega : |\omega_{\pm T}| < M\}$ for some $M > 1$. For $N > T$ and $\bar{\omega} \in \Omega$, the Markov property of Brownian motion and the Feynman-Kac formula give

$$\begin{aligned}\mu_N(A, \bar{\omega}) &= \frac{1}{Z_N(\bar{\omega})} \int dx \int dy \left(\int e^{-\int_{-N}^{-T} V(\omega_s)\, ds}\, d\mathcal{W}_{[-N,-T]}^{\bar{\omega}(-N),x}(\omega) \right) \times \\ &\times \left(\int e^{-\int_{-T}^{T} V(\omega_s)\, ds} 1_A(\omega)\, d\mathcal{W}_{[-T,T]}^{x,y}(\omega) \right) \left(\int e^{-\int_{T}^{N} V(\omega_s)\, ds}\, d\mathcal{W}_{[T,N]}^{y,\bar{\omega}(N)}(\omega) \right) = \\ &= \int dx \int dy\, \frac{K_{N-T}(\bar{\omega}_{-N}, x) K_{N-T}(y, \bar{\omega}_N)}{K_{2N}(\bar{\omega}_{-N}, \bar{\omega}_N)} \left(\int e^{-\int_{-T}^{T} V(\omega_s)\, ds} 1_A(\omega)\, d\mathcal{W}_{[-T,T]}^{x,y}(\omega) \right).\end{aligned}$$

By our restriction on A and the boundedness of $K_{2T}(x, y)$, the last factor in the above formula is a bounded function of x and y with compact support, thus integrable over $\mathbb{R}^{2d}$. As a consequence, the claim will be proven for A once we can show that

$$\frac{K_{N-T}(\bar{\omega}_{-N}, x) K_{N-T}(y, \bar{\omega}_N)}{K_{2N}(\bar{\omega}_{-N}, \bar{\omega}_N)} \xrightarrow{N \to \infty} \psi_0(x) \psi_0(y) \tag{3.4}$$

uniformly in $x, y \in \mathbb{R}^d$. By the definition of Q_T, (3.4) is equivalent to (3.3).

For general $A \in \mathcal{F}_T$, consider

$$B_M = \{\omega \in \Omega : \max\{|\omega_T|, |\omega_{-T}|\} < M\}$$

with $M \in \mathbb{N}$, and $A_M = A \cap B_M$. Since $B_M \nearrow \Omega$ as $M \to \infty$, for given $\varepsilon > 0$ we may pick $M \in \mathbb{N}$ with $\mu(B_M^c) < \varepsilon$. Moreover, since both A_M and B_M fulfill the assumptions of the above paragraph, we find $N_0 \in \mathbb{N}$ such that for all $N > N_0$, both

$$|\mu_N(A_M) - \mu(A_M)| < \varepsilon \qquad \text{and} \qquad |\mu_N(B_M) - \mu(B_M)| < \varepsilon.$$

It follows that $\mu_N(B_M^c) < 2\varepsilon$ for all $N > N_0$, and thus

$$\begin{aligned}|\mu_N(A) - \mu(A)| &= |\mu_N(A_M) + \mu_N(A \setminus B_M) - \mu(A_M) - \mu(A \setminus B_M)| \leq \\ &\leq |\mu_N(A_M) - \mu(A_M)| + \mu_N(B_M^c) + \mu(B_M^c) \leq 4\varepsilon.\end{aligned}$$

This shows $\mu_N(A) \to \mu(A)$. □

Equipped with the two above results, we can now tackle the uniqueness question. Our conclusions are most complete when a restricted class of potentials is used.

Theorem 3.4 *Let V^+ denote as before the positive part of V. Suppose there exist constants $C_1, C_3 > 0$, $C_2, C_4 \in \mathbb{R}$ and a, b with $2 < a < b < 2a - 2$ such that*

$$C_1 |x|^a + C_2 \leq V^+(x) \leq C_3 |x|^b + C_4. \tag{3.5}$$

Then μ is the unique Gibbs measure for V supported on Ω.

Proof: It is shown in [8] that Schrödinger operators with potentials as given in (3.5) are intrinsically ultracontractive. In our context this means that for each $N > 0$, $C_N = \|Q_N\|_{L^\infty(\mathbb{R}^{2d})} < \infty$. By the semigroup property of Q_N and the fact that $\int Q_N(x,y)d\nu(y) = 1$ for each x, for $N > 2$ we have

$$\begin{aligned} |Q_N(x,y) - 1| &= \left| \int d\xi \int d\eta\, Q_1(x,\xi)\psi_0^2(\xi)(Q_{N-2}(\xi,\eta) - 1)\psi_0^2(\eta)Q_1(\eta,y) \right| \\ &\leq C_1^2 \int d\xi \int d\eta\, \psi_0(\xi) \Big| K_{N-2}(\xi,\eta) - \psi_0(\xi)\psi_0(\eta) \Big| \psi_0(\eta) \\ &\leq C_1^2 \left(\int d\xi \int d\eta (K_{N-2}(\xi,\eta) - \psi_0(\xi)\psi_0(\eta))^2 \right)^{1/2} = (*) \end{aligned} \tag{3.6}$$

From $\|Q_N\|_{L^\infty(\mathbb{R}^{2d})} < \infty$ it follows that $K_N \in L^2(\lambda^{2d})$ for each $N > 0$. Thus e^{-NH} is a Hilbert-Schmidt operator for each $N > 0$ and in particular H has a purely discrete spectrum with eigenvalues $0 = E_0 < E_1 \leq E_2 \leq \ldots E_n \leq \ldots$. Writing $P_{\psi_0} : L^2(\lambda^d) \to L^2(\lambda^d)$ for the projection onto the subspace of $L^2(\lambda^d)$ spanned by ψ_0, the second factor on the right hand side of (3.6) is just the Hilbert-Schmidt norm of $e^{-(N-2)H} - P_{\psi_0}$, which implies

$$(*) = C_1^2 e^{-(N-2)E_1} \left(\sum_{k=1}^{\infty} e^{-2(N-2)(E_k - E_1)} \right)^{1/2}. \tag{3.7}$$

By dominated convergence, the sum on the right hand side of (3.7) converges to the multiplicity of the second eigenvalue E_1 of H as $N \to \infty$, and by uniqueness of ψ_0 we have $E_1 > 0$. Thus $Q_N(x,y) \to 1$ uniformly in x and y, which implies (3.3). □

As indicated by Example 3.1, a result like Theorem 3.4 can not hold for general Kato-decomposable potentials. In the general case we obtain uniqueness only by restricting the configuration space Ω.

Theorem 3.5 *In the context of Lemma 3.3, assume that the ground state ψ_0 of the Schrödinger operator H is not only in $L^2(\lambda^d)$ but also in $L^1(\lambda^d)$. Denote again by $\gamma = \inf(\mathrm{spec}(H) \setminus 0)$ the spectral gap of H, and put*

$$\Omega^* = \left\{ \omega \in \Omega : \lim_{N \to \pm\infty} \frac{e^{-\gamma|N|}}{\psi_0(\omega_N)} = 0 \right\}. \tag{3.8}$$

Then μ is the unique Gibbs measure for V supported on Ω^*.

Proof: Let P_{ψ_0} be again the projection onto the one-dimensional subspace spanned by ψ_0, and put

$$L_t := e^{-tH} - P_{\psi_0}.$$

L_t is an integral operator with kernel $\tilde{K}_t(x,y) = K_t(x,y) - \psi_0(x)\psi_0(y)$. By the assumption $\gamma > 0$ we have

$$\|L_t\|_{2,2} = e^{-\gamma t}. \tag{3.9}$$

Here and below $\|.\|_{p,q}$ denotes the norm of an operator from L^p to L^q. For estimating $\tilde{K}_t$ note that

$$\sup_{x,y\in\mathbb{R}^d}\left|\tilde{K}_t(x,y)\right| = \sup_{q\in L^1,\|q\|_1=1}\left\|\int \tilde{K}_t(x,y)q(y)\,dy\right\|_\infty = \|L_t\|_{1,\infty}\,,$$

and since $e^{-tH}P_{\psi_0} = P_{\psi_0}e^{-tH} = P_{\psi_0}$ for all $t>0$, we have

$$\|L_t\|_{1,\infty} = \left\|e^{-H}(e^{-(t-2)H} - P_{\psi_0})e^{-H}\right\|_{1,\infty} \le \left\|e^{-H}\right\|_{2,\infty}\|L_{t-2}\|_{2,2}\left\|e^{-H}\right\|_{1,2}.$$

By Remark 2.3 (2), both $\left\|e^{-H}\right\|_{2,\infty}$ and $\left\|e^{-H}\right\|_{1,2}$ are finite. It thus follows that for every $t\le N$,

$$|K_{N-t}(x,y) - \psi_0(x)\psi_0(y)| \le C_t e^{-\gamma N}, \tag{3.10}$$

where $C_t = \left\|e^{-H}\right\|_{2,\infty}\left\|e^{-H}\right\|_{1,2} e^{\gamma(2+t)}$ is independent of x,y and N.

In terms of Q_N, using (3.10) we find for all $\bar\omega\in\Omega^*$

$$\begin{aligned}
|Q_{N-T}(\bar\omega_{-N},x)-1|\psi_0(x) &\le C_T e^{-\gamma N}/\psi_0(\bar\omega_{-N}) \to 0,\\
|Q_{N-T}(x,\bar\omega_N)-1|\psi_0(y) &\le C_T e^{-\gamma N}/\psi_0(\bar\omega_N) \to 0, \qquad \text{and}\\
|Q_{2N}(\bar\omega_{-N},\bar\omega_N)-1| &\le C_0 e^{-2\gamma N}/(\psi_0(\bar\omega_{-N})\psi_0(\bar\omega_N)) \to 0
\end{aligned}$$

as $N\to\infty$, proving (3.4).

It remains to show that μ is in fact supported on Ω^*. By time reversibility, it will be enough to show that

$$\mu\left(\limsup_{N\to\infty}\frac{e^{-\gamma N}}{\psi_0(\omega_N)} > q\right) = 0 \tag{3.11}$$

for each $q>0$. To prove (3.11), note that by stationarity of μ,

$$\begin{aligned}
\mu\left(\frac{e^{-\gamma N}}{\psi_0(\omega_N)} > q\right) &= \mu\left(\psi_0(\omega_0) < \frac{\exp(-\gamma N)}{q}\right) = \\
&= \int 1_{\{\psi_0<\exp(-\gamma N)/q\}}\psi_0^2\,d\lambda^d \le \\
&\le \frac{\exp(-\gamma N)}{q}\|\psi_0\|_{L^1}\,.
\end{aligned}$$

The right hand side of the last expression is summable in N for each q, and so the Borel-Cantelli Lemma implies (3.11), finishing the proof. □

The additional assumption that $\psi_0\in L^1$ is very weak. In fact, for many potentials V it is known that ψ_0 decays exponentially at infinity. A large class of examples with $\psi_0\in L^1$ will be given in Proposition 4.1.

4 Examples and discussion

In Theorem 3.5, Ω^* in (3.8) depends directly on the decay of ψ_0 at infinity. Therefore results which connect the potential V with the ground state ψ_0 of the Schrödinger operator are of interest for us. One of the strongest results in this direction has been obtained by R. Carmona in [5]. A consequence of it is

Proposition 4.1 *Let $V = V_1 - V_2$ with V_1 bounded from below and in $L^{d/2+\varepsilon}_{\text{loc}}$ for some $\varepsilon > 0$, and $0 \leq V_2 \in L^p$ with $p > \max\{1, d/2\}$. Suppose that $H = -\frac{1}{2}\Delta + V$ has a ground state ψ_0.*

a) If there exists $\alpha \geq 0$ such that $V_1(x) \leq C|x|^{2\alpha}$ outside a bounded set, then there exist $D_1 > 0, b_1 > 0$ such that

$$D_1 \exp(-b_1|x|^{\alpha+1}) \leq \psi_0(x) \tag{4.1}$$

for each $x \in \mathbb{R}^d$.

b) If there exists $\alpha \geq 0$ such that $V(x) \geq C|x|^{2\alpha}$ outside a bounded set, then there exist $D_2 > 0, b_2 > 0$ such that

$$\psi_0(x) \leq D_2 \exp(-b_2|x|^{\alpha+1}) \tag{4.2}$$

for each $x \in \mathbb{R}^d$.

A direct check reveals that all potentials considered in Proposition 4.1 are Kato-decomposable.

We start our examples with the case that has been treated most in the literature.

Example 4.2 Suppose that V is a non-constant polynomial which is bounded below. This in particular implies that the degree of V is even. Then, according to the second paragraph after Remark 2.3, H has a unique ground state and a spectral gap. From Theorem 3.4 and Example 3.1 it follows that the Gibbs measure for V is unique if and only if the degree of V is greater than 2. This improves a result of G. Royer [18] who showed that there exists at most one *euclidean* Gibbs measure, i.e. one Gibbs measure which is in addition a reversible process. Our results also shed new light on the results in [19]: Except for the case of quadratic P, the cone C of measures investigated there is in fact just the ray generated by the reversible process μ, i.e. $C = \{r\mu : r > 0\}$.

The purpose of our next example is to study more closely the set Ω^* introduced in Theorem 3.5.

Example 4.3 Let $V(x) = |x|^{2\alpha}$ with $\alpha > 0$. Again, H has a unique ground state ψ_0 and a spectral gap γ. In case $\alpha \geq 1$, Theorem 3.4 and Example 3.1 completely solve the question of uniqueness. In case $0 < \alpha < 1$, (4.2) implies that Theorem 3.5 is applicable, and thus we get uniqueness on a set $\Omega^*(\alpha)$. By (4.1) and (4.2), we obtain fairly sharp estimates on $\Omega^*(\alpha)$: let b_1 and b_2 be the constants from (4.1) and (4.2), respectively, and γ the spectral gap of H. Then

$$\left\{\omega \in \Omega : \limsup_{T\to\pm\infty} \frac{|\omega_{\pm T}|^{\alpha+1}}{T} < \frac{\gamma}{b_1}\right\} \subset \Omega^*(\alpha) \subset \left\{\omega \in \Omega : \limsup_{T\to\pm\infty} \frac{|\omega_{\pm T}|^{\alpha+1}}{T} < \frac{\gamma}{b_2}\right\}. \tag{4.3}$$

Note that $\Omega^*(\alpha)$ becomes smaller as α increases towards 1, in contrast with the intuition that a fast growing potential should bring the path back to stationarity more quickly.

This intuition is in fact confirmed eloquently in the case $\alpha > 1$, and the phenomenon in case $\alpha < 1$ is certainly an artifact of our proof of Theorem 3.5. There we relied on the (rather crude) estimate (3.10), and thus a fast decay of ψ_0 had to be compensated by a slow growth of $\bar{\omega}$.

The case $\alpha = 1$ suggests that when it comes to determining a maximal subset of configuration space on which μ is the unique Gibbs measure, a set consisting of exponentially growing paths is much closer to truth than our Ω^*, but to obtain it is beyond the method of our proof. K. Iwata obtains such sets of exponentially growing paths as sets of uniqueness for the Gibbs measure in [11], but he has to assume convexity of the function $V(x) - \kappa|x|^2$ for some $\kappa > 0$, so the case $\alpha < 1$ is not covered by his results.

The advantage of our method is that we are not limited to polynomials and not even to continuous or semibounded functions in terms of V. In particular, we may add local singularities and other perturbations to V in the above examples without altering the conclusions. This corresponds to the intuition that, since Brownian motion is so strongly mixing, only the behavior of V at infinity should determine whether uniqueness of the Gibbs measure holds on a given set.

As a final example, we shall treat some special potentials of interest, all of them containing local singularities.

Example 4.4 A physically very important example is the Coulomb potential in three dimensions, i.e. $d = 3$ and $V(x) = 1/|x|$. With this choice, H has a ground state as well as a spectral gap [16], and using Theorem 3.5, we find that Ω^* consists of functions ω that are growing at most linearly at infinity. We do not know whether non-uniqueness of the Gibbs measure for the Coulomb potential can occur.

In dimension one, results of J. T. Cox on entrance laws [7] yield some interesting examples of non-uniqueness. When written in our notation, Example 3) of [7] says that there exist non-stationary Gibbs measures for the potential $V = \delta|x|^{\delta-1} + |x|^{2\delta}$ if $1/3 < \delta \leq 1$, the case $\delta = 1$ being again the Ornstein-Uhlenbeck process. While [7] is a mere existence result, from Example 4.3 and the remark following it we can see that none of these non-stationary Gibbs measures can have support on the set of paths growing slower than $|t|^{1/(\delta+1)}$ as $t \to \pm\infty$.

The second part of the last example shows that the phenomenon of non-uniqueness of the Gibbs measure is not restricted to the Ornstein-Uhlenbeck process. Two interesting open questions arise: Is it possible to prove or disprove the existence of more than one Gibbs measure in case of potentials growing slower than $|x|^{2/3}$ at infinity or in more than one dimension, e.g. for the Coulomb potential? Is it possible to obtain better bounds on the subset Ω^* on which the stationary diffusion is the unique Gibbs measure?

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