On the lower tail probabilities of some random sequences in l_p

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Abstract

We investigate the behaviour of the logarithmic small deviation probability of a sequence $(\sigma_n\theta_n)$ in l_p, $0 < p \leq \infty$, where (θ_n) are i.i.d. random variables and (σ_n) is a decreasing sequence of positive numbers. In particular, the example $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$ is studied thoroughly. Contrary to the existing results in the literature, the rate function and the small deviation constant are expressed explicitly in the present treatment. The restrictions on the distribution of θ_1 are kept to an absolute minimum. In particular, the usual variance assumption is removed. As an example, the results are applied to stable and Gamma-distributed random variables.

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1 INTRODUCTION

Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, that are not concentrated at a single point. Furthermore, let (σ_n) be a sequence of positive numbers that is strictly decreasing and tends to zero. We consider vectors of the form $(\sigma_n\theta_n) = (\sigma_1\theta_1, \sigma_2\theta_2, \ldots)$ in l_p, i.e.

$$\|(\sigma_n\theta_n)\|_p := \begin{cases} \sup_{n\geq 1} |\sigma_n\theta_n| & p = \infty, \\ \left(\sum_{n\geq 1} |\sigma_n\theta_n|^p\right)^{1/p} & 0 < p < \infty. \end{cases}$$

The question that is addressed in this article is the so-called small deviation probability – or lower tail probability – of the vector $(\sigma_n\theta_n) \in l_p$, i.e. the behaviour of the quantity

$$\log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right), \qquad \text{as } \varepsilon \to 0+.$$

We tacitly assume that $\|(\sigma_n\theta_n)\|_p < \infty$ almost surely, since otherwise the task is trivial. The question of the small deviation probability was studied by many authors. In particular, for $0 < p < \infty$, the problem of sums of independent random variables was studied thoroughly, cf. Lifshits(9), Dunker *et al.*(3), and Rozovsky(10). These works are based on the treatment in Lifshits(9). The results are more precise than those in this paper. However, their approach has the great disadvantage that only random variables possessing variance are covered. We shall see that this is not necessary. In fact, the moment condition for θ depends on the speed of decrease of the sequence (σ_n). This is a rather natural relation, contrary to a general moment assumption.

Another advantage of the present treatment is the rather explicit nature of the rate function and the small deviation constant contrasting the implicit results of the above-mentioned papers. Additionally, we obtain yet greater generality since we only require to know the order (in the sense of strong asymptotics) of the sequence (σ_n) not the particular form. Furthermore, the considerations in this article bring to light many similarities between the small deviation of sums of independent random variables and the somewhat simpler case of the supremum.

This work started in the framework of stable distributions and was motivated by the fact that the above-mentioned results cannot be applied to stable random variables owing to the lack of variance for the non-Gaussian stable distributions. No results seem to be known in the stable case as well as the case of general Gamma-distributed random variables. The results of this paper fill these gaps. The basis for this work is the author's Ph.D. thesis (cf. Aurzada(1)).

Certainly, in the case of Gaussian (i.e. 2-stable) random variables, the results mentioned above apply and lead to very precise bounds. However, the question was investigated much earlier by other methods, cf. Hoffmann-Jørgensen *et al.*(6) and Li(7), where the first-mentioned also consider the case $p = \infty$.

The organization of this article is as follows. In Section 2, the main idea is presented. We prove general results in the cases $p = \infty$ and $p < \infty$. These results lead to general upper bounds in the case that an additional restriction is imposed on the sequence (σ_n). This is investigated in Section 3.

In Section 4, we concentrate on the example of polynomical behaviour with logarithmic correction, i.e. the example

$$\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}, \qquad n=1,2,\ldots,$$

for some $\mu>0$ and $\nu\in\mathbb{R}$, where as usual $\sigma_n\sim\tilde{\sigma}_n$ means that $\sigma_n/\tilde{\sigma}_n\to 1$.

It seems intuitively clear that the rate of the small deviation function and the finiteness of the small deviation constant only depend on the behaviour of the distribution at the origin and the tail behaviour. The main result of this paper expresses exactly this fact. On the one hand, condition (O) (cf. Definition 4.1 below) represents the behaviour of the distribution of θ at the origin. This is a very mild condition; it is surely satisfied if, for example, θ has a continuous, non-vanishing density in a neighbourhood of the origin. On the other hand, we have to make an assumption for the tail behaviour of θ. The result is as follows.

Theorem 1.1. *Let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu>0$ and $\nu\in\mathbb{R}$, and let $\theta,\theta_1,\theta_2,\ldots$ be a sequence of i.i.d. random variables satisfying condition (O) with $r>\mu$. Then*

$$\lim_{\varepsilon\to 0+} \varepsilon^{1/(\mu-1/p)}\left(-\log\varepsilon\right)^{\nu/(\mu-1/p)}\log\mathbb{P}\left(\|(\sigma_n\theta_n)\|_p\le\varepsilon\right)=-C_p,$$

with the constant $C_p\in]0,\infty]$ given by

$$C_p:=\begin{cases}\mu^{\nu/\mu}\left(-\int_0^\infty \log\mathbb{P}\left(|\theta|\le z^\mu\right)\,dz\right) & p=\infty,\\ \left[\frac{(\mu-1/p)^{\mu-1/p+\nu}}{\mu^\mu p^{1/p}}\left(-\int_0^\infty\log\mathbb{E}e^{-|z^{-\mu}\theta|^p}\,dz\right)^{\mu}\right]^{1/(\mu-1/p)} & p<\infty.\end{cases}\tag{1.1}$$

The constant C_p is finite if and only if $\mathbb{E}|\theta|^{1/\mu}<\infty$ and $\mu>1/p$.

The proof of this theorem will be given at the end of Section 4.5. As usual, we use the notion $1/p=0$, for $p=\infty$. Note that $C_p\to C_\infty$, as $p\to\infty$. Finally, in Section 4.6 and Section 4.7, the cases of stable and, respectively, Gamma-distributed random variables are considered.

It seems worth to remark two rather surprising aspects of Theorem 1.1. On the one hand, the condition for the small deviation constant C_p to be finite depends on the (upper) tail of the distribution of θ. One might not expect this from the beginning, since we are considering a lower tail problem.

On the other hand, the behaviour of the lower tail of the distribution of θ near to the orgin does not matter at all as long as the (mild) condition (O) is satisfied. Again, this must seem a surprise, since we assert a property of the lower tail.

To illustrate both facts, note that the rate of the small deviation probability is the same for symmetric α-stable distributions (having a continous, non-vanishing density everywhere) and for random variables with mass 1/2 concentrated at zero and one. However, the small deviation constant is infinite for those α-stable distributions with too heavy *upper* tails ($\alpha\le 1/\mu$).

2 MAIN ARGUMENT

Let us look at the sequence (σ_n). The main idea is to interpolate this sequence by a smooth function. Since (σ_n) is strictly decreasing and tends to zero, we can find a function $S : [1,\infty[\to \mathbb{R}_{>0}$ which interpolates the σ_n, i.e. $S(n) = \sigma_n$, for all $n = 1, 2, \ldots$, is twice continuously differentiable and strictly decreasing. From the fact that (σ_n) tends to zero and the continuity of S it follows that $S(x) \to 0$, as $x \to \infty$.

First, let us consider the case $p = \infty$.

Theorem 2.1. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is strictly decreasing to zero, and S a function as constructed above. Then, for all $\varepsilon > 0$,*

$$\log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{\sigma_1}\right) - \int_0^{\sigma_1/\varepsilon} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{d}{dy}\left[S^{-1}(\varepsilon y)\right] dy$$
$$\leq \log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \leq \varepsilon\right) \leq - \int_0^{\sigma_1/\varepsilon} \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) \frac{d}{dy}\left[S^{-1}(\varepsilon y)\right] dy.$$

Proof: Because of the independence of the (θ_n), we can write

$$\log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \leq \varepsilon\right) = \log \prod_{n=1}^{\infty} \mathbb{P}\left(|\sigma_n \theta_n| \leq \varepsilon\right) = \sum_{n=1}^{\infty} \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{\sigma_n}\right).$$

Keeping in mind the last expression, let us define the function

$$F_\varepsilon(x) := \log \mathbb{P}\left(|\theta| \leq \frac{\varepsilon}{S(x)}\right), \qquad x \geq 1.$$

Using the properties of S, note that the function $F_\varepsilon : [1,\infty[\to \mathbb{R}_{\leq 0}$ is non-positive, increasing, and tends to zero as $x \to \infty$. A simple comparison of sum and integral shows that, for all $\varepsilon > 0$,

$$F_\varepsilon(1) + \int_1^\infty F_\varepsilon(x)\, dx \leq \sum_{n=1}^{\infty} F_\varepsilon(n) \leq \int_1^\infty F_\varepsilon(x)\, dx.$$

We transform the integral setting $S(x) = \varepsilon y$ in order to separate the distribution of θ from ε. This exactly leads to the asserted inequalities. ■

This is a rather general result. For a given sequence (σ_n) one can construct an appropriate function S with the properties mentioned above and calculate both – the integral and the remaining probability term on the left-hand side. Since ε does not appear in connection with the distribution in the integrand, one can – at least asymptotically – separate ε from the integral. This will be demonstrated later (Section 3) in the case that S can be chosen to be a regularly varying function at infinity.

Now let us deal with the case $p < \infty$. Let us consider a smooth function S that interpolates the sequence (σ_n), as above. Again, the decisive idea is to express the desired term as a sum, replace that sum by an integral, and use an integral transformation that separates the distribution of θ from ε. However, in the case $p < \infty$, another approach is needed.

Recall that we are interested in the behaviour of the quantity

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty} |\sigma_n \theta_n|^p \leq \varepsilon^p\right).$$

In order to study this, we are going to pass over to the logarithmic Laplace transform of the random variable $\sum_{n=1}^{\infty} |\sigma_n \theta_n|^p$. This is suggested by the well-known fact that question of small deviation probabilities can be formulated in terms of the Laplace transform. Later on, we are going to use the so-called Exponential Tauberian Theorem of de Bruijn (cf. Theorem 4.12.9 in Bingham *et al.*(2) or Theorem 3.5 in Li and Shao(8)).

The result is as follows.

Theorem 2.2. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is strictly decreasing to zero, and S a function as constructed above. Then, for all $\lambda > 0$,*

$$\log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p} - \int_0^{\sigma_1 \lambda^{1/p}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy}\left[S^{-1}(y\lambda^{-1/p})\right]\, dy$$
$$\leq \log \mathbb{E} e^{-\lambda \sum_n |\sigma_n\theta_n|^p} \leq - \int_0^{\sigma_1 \lambda^{1/p}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy}\left[S^{-1}(y\lambda^{-1/p})\right]\, dy.$$

Proof: Let us consider

$$\log \mathbb{E} e^{-\lambda \sum_n |\sigma_n\theta_n|^p} = \log \prod_{n=1}^{\infty} \mathbb{E} e^{-\lambda|\sigma_n\theta_n|^p} = \sum_{n=1}^{\infty} \log \mathbb{E} e^{-\lambda|\sigma_n\theta|^p},$$

as $\lambda \to \infty$. Keeping this in mind, we define the function

$$G_\lambda(x) := \log \mathbb{E} e^{-\lambda|S(x)\theta|^p}, \qquad x \geq 1.$$

Note that, using the properties of S, we see that the function $G_\lambda : [1, \infty[\to \mathbb{R}_{\leq 0}$ is non-positive, increasing, and tends to zero, as x tends to infinity. The same argument as above – a comparison between integral and sum – shows that, for all $\lambda > 0$,

$$G_\lambda(1) + \int_1^\infty G_\lambda(x)\,dx \le \sum_{n=1}^\infty G_\lambda(n) \le \int_1^\infty G_\lambda(x)\,dx.$$

Finally, we substitute $\lambda S(x)^p = y^p$, which gives us the assertion. ■

Again this is a very general result, which enables us to obtain bounds for $\log \mathbb{E} e^{-\lambda \sum_n |\sigma_n\theta_n|^p}$ for a given sequence (σ_n). By virtue of the above-mentioned Tauberian theorem, this implies bounds for the small deviation probability of $\sum_n |\sigma_n\theta_n|^p$ and so finally of $(\sum_n |\sigma_n\theta_n|^p)^{1/p}$. This will be demonstrated in the following section.

Note that Theorem 2.1 and Theorem 2.2 are identical if one replaces $\mathbb{E}e^{-|.|^p}$ by $\mathbb{P}(|.| \le 1)$ and $\lambda^{-1/p}$ by ε.

3 UPPER BOUNDS FOR REGULARLY VARYING FUNCTIONS

In this section, we are going to investigate the question of the lower tail probabilities in the case that the function S interpolating the sequence (σ_n) can be chosen to be a regularly varying function. We refer to Bingham *et al.*[(2)] for a detailed study of properties of these functions.

We shall see that the behaviour of the integrals in Theorems 2.1 and 2.2 can be quantified in a very precise way if certain assumptions for S are made.

For the remaining part of Section 3, let us assume that S can be chosen such that it is a regularly varying function with exponent $-\gamma < 0$ at infinity and its derivative S' is continuous and increasing (i.e. $S'' > 0$) for large enough arguments.

It follows from the theory of regularly varying functions that $S^{-1} :]0, \sigma_1] \to \mathbb{R}_{>0}$ is regularly varying at zero with exponent $-1/\gamma$ (cf. Theorem 1.5.12 in Bingham *et al.*[(2)]). Furthermore, note that S^{-1} is strictly decreasing.

Let us start with the case $p = \infty$.

Theorem 3.1. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is decreasing to zero, and S a function as constructed above. Then*

$$\overline{\lim_{\varepsilon \to 0+}} \frac{\log \mathbb{P}(\sup_n |\sigma_n\theta_n| \le \varepsilon)}{S^{-1}(\varepsilon)} \le \int_0^\infty \log \mathbb{P}(|\theta| \le z^\gamma)\,dz.$$

Proof: Theorem 2.1 suggests to investigate the quantity

$$I_\infty(\varepsilon) := \int_0^{\sigma_1/\varepsilon} \log \mathbb{P}\left(|\theta| \le \frac{1}{y}\right) \frac{d}{dy}\left[S^{-1}(\varepsilon y)\right] dy, \qquad \text{as } \varepsilon \to 0+.$$

Note that I_∞ is positive, since S^{-1} is decreasing. Let us consider

$$\varliminf_{\varepsilon\to 0+} \frac{I_\infty(\varepsilon)}{S^{-1}(\varepsilon)} = \varliminf_{\varepsilon\to 0+} \int_0^{\sigma_1/\varepsilon} \log \mathbb{P}\left(|\theta| \le \frac{1}{y}\right) \frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] dy. \tag{3.1}$$

As $\varepsilon \to 0+$, we have

$$\frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] = \frac{(S^{-1})'(\varepsilon y)\varepsilon}{S^{-1}(\varepsilon)} = \frac{1}{y}\frac{(S^{-1})'(\varepsilon y)(\varepsilon y)}{S^{-1}(\varepsilon y)}\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)} \to \frac{1}{y}\left(-\frac{1}{\gamma}\right) y^{-1/\gamma}, \tag{3.2}$$

by the fact that S^{-1} is regularly varying and a result on regularly varying functions (cf. problem 13 on page 59 in Bingham *et al.*[2] or reference therein), where we need the assumption on S' for large enough arguments.

Therefore, by Fatou's Lemma,

$$\varliminf_{\varepsilon\to 0+} \frac{I_\infty(\varepsilon)}{S^{-1}(\varepsilon)} \ge \int_0^{\infty} \log \mathbb{P}\left(|\theta| \le \frac{1}{y}\right) \frac{1}{y}\left(-\frac{1}{\gamma}\right) y^{-1/\gamma}\, dy.$$

Using Theorem 2.1 and transforming the integral gives the assertion. ■

Note that we have not put *any* restriction on the distribution of θ yet. This is represented by the finiteness of expression on the right-hand side in the theorem. We are going to investigate conditions for the finiteness of this integral in Section 4.5.

The case of the sum of independent random variables is entirely similar.

Theorem 3.2. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables, (σ_n) a sequence of positive numbers that is decreasing to zero, and S a function as constructed above. Then*

$$\varlimsup_{\lambda\to\infty} \frac{\log \mathbb{E}\exp\left(-\lambda \sum_n |\sigma_n \theta_n|^p\right)}{S^{-1}(\lambda^{-1/p})} \le \int_0^\infty \log \mathbb{E} e^{-|z^{-\gamma}\theta|^p}\, dz.$$

Proof: Theorem 2.2 suggests the investigation of the quantity

$$I_p(\lambda) := \int_0^{\sigma_1\lambda^{1/p}} \log \mathbb{E} e^{-|y\theta|^p} \frac{d}{dy}\left[S^{-1}(\lambda^{-1/p} y)\right] dy, \qquad \text{as } \lambda \to \infty.$$

Note the correspondence to the case $p = \infty$, for $\varepsilon = \lambda^{-1/p}$. Thus, as in (3.2),

$$\frac{d}{dy}\left[\frac{S^{-1}(\lambda^{-1/p} y)}{S^{-1}(\lambda^{-1/p})}\right] \to \frac{1}{y}\left(-\frac{1}{\gamma}\right) y^{-1/\gamma} = \left(-\frac{1}{\gamma}\right) y^{-1/\gamma - 1},$$

as $\lambda \to \infty$. Therefore, by Fatou's Lemma,

$$\begin{aligned}\varliminf_{\lambda\to\infty} \frac{I_p(\lambda)}{S^{-1}(\lambda^{-1/p})} &\geq \int_0^\infty \log \mathbb{E}e^{-|y\theta|^p} \left(-\frac{y^{-1/\gamma-1}}{\gamma}\right) dy \\ &= -\int_0^\infty \log \mathbb{E}e^{-|z^{-\gamma}\theta|^p}\, dz. \end{aligned} \tag{3.3}$$

Using Theorem 2.2 finishes the proof. ■

4 EXAMPLES

4.1 Assumption for the Distribution in a Neighbourhood of the Origin

At this point, we make an assumption for the distribution of θ in the neighbourhood of the origin. This condition limits the generality slightly. However, we shall do so in order to avoid technical complications. Since the focus of this paper is on proving small deviation results also for heavy-tailed distributions, we do not seek full generality with respect to the distribution near the origin.

Roughly speaking, we assume that the distribution of θ does have some mass near the origin. More precisely, we assume the following.

Definition 4.1. *We are going to say that the distribution of θ satisfies condition (O) with $r > 0$ if there exists a constant $C_1 > 0$ such that*

$$\mathbb{P}\left(|\theta| \leq t\right) \geq e^{-C_1 t^{-1/r}}, \qquad \textit{for all } 0 < t \leq 1. \tag{O}$$

With the help of the independence of the (θ_n) and the inequality $\|.\|_{l_p^N} \leq N^{1/p}\|.\|_{l_\infty^N}$, for $N \geq 1$, it is easy to see that if the distribution of θ satisfies condition (O) with $r > 0$ we have, for all $0 < p \leq \infty$ and for all decreasing sequences (σ_n),

$$\mathbb{P}\left(\left\|(\sigma_n\theta_n)_{n=1}^N\right\|_p \leq t\right) \geq e^{-C_N t^{-1/r}}, \qquad \text{for all } 0 < t \leq 1, \tag{4.1}$$

with some constant $C_N > 0$ depending on N and σ_1 only.

4.2 Modification of the Sequence (σ_n)

Before we come to concrete examples of sequences, we have to look at what happens if we modify the sequence (σ_n). On the one hand, this is necessary since the inverse of a function S interpolating the sequence (σ_n) cannot be computed explicitly in many cases. On the other hand, it gives greater generality to our results. Namely, we shall see that only the behaviour of the sequence at infinity really matters if we calculate the probability on the log-level. Thus, we have to know only the order of (σ_n) as n tends to infinity.

Lemma 4.1. *Let $\theta, \theta_1, \theta_2, \dots$ be a sequence of i.i.d. random variables, (σ_n) and $(\tilde{\sigma}_n)$ be two decreasing sequences of positive numbers that tend to zero and that satisfy $\sigma_n \sim \tilde{\sigma}_n$. Let $C > 0$ and $T :]0,1] \to \mathbb{R}_{>0}$ be a regularly varying function with exponent $\gamma > 0$; and let us furthermore assume that the distribution of θ satisfies condition (O) with $r > 1/\gamma$.*

Then

$$\lim_{\varepsilon \to 0+} T(\varepsilon) \log \mathbb{P}\left(\|(\sigma_n \theta_n)\|_p \leq \varepsilon \right) = -C$$

if and only if

$$\lim_{\varepsilon \to 0+} T(\varepsilon) \log \mathbb{P}\left(\|(\tilde{\sigma}_n \theta_n)\|_p \leq \varepsilon \right) = -C.$$

Proof: The proof uses the standard arguments for log-level comparisions of small deviations as they can be found e.g. in Gao *et al.*[4] and Gao and Li[5]. However, these arguments have to be modified slightly because we have no regularly varying assumption for the lower tail of the distribution.

Let us consider the case $p = \infty$. Note that T can be written as $T(x) = x^\gamma R(x)$, with a slowly varying function R. Let $0 < \delta < 1$. Then there is an n_0 such that, for all $n > n_0$, $1 - \delta \leq \sigma_n / \tilde{\sigma}_n \leq 1 + \delta$. This implies that

$$\|(\sigma_n \theta_n)\|_\infty \leq \max\left(\sup_{1 \leq n \leq n_0} |\sigma_n \theta_n|, (1+\delta) \sup_{n > n_0} |\tilde{\sigma}_n \theta_n| \right).$$

Thus, by the independence of the (θ_n),

$$\mathbb{P}\left(\|(\sigma_n \theta_n)\|_\infty \leq \varepsilon\right) \geq \mathbb{P}\left(\|(\sigma_n \theta_n)_{n=1}^{n_0}\|_\infty \leq \varepsilon\right) \mathbb{P}\left(\|(\tilde{\sigma}_n \theta_n)\|_\infty \leq \varepsilon/(1+\delta)\right).$$

Taking the logarithm of both sides, using (4.1), multiplying by $\varepsilon^\gamma R(\varepsilon)$, and letting first ε and then δ tend to zero gives us one side of the assertion. Exchanging the role of (σ_n) and $(\tilde{\sigma}_n)$ gives the other side.

The case $p < \infty$ contains exactly the same argument. We therefore omit this part of the proof. ■

4.3 Purely Polynomical Decrease

Let us consider the case when $\sigma_n \sim n^{-\mu}$, with $\mu > 0$.

Theorem 4.1. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables and let us assume $\sigma_n \sim n^{-\mu}$, with $\mu > 0$. Then, if the distribution of θ satisfies condition (O) with $r > \mu$, we have*

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/\mu} \log \mathbb{P}\left(\sup_n |\sigma_n \theta_n| \le \varepsilon\right) = \int_0^\infty \log \mathbb{P}\left(|\theta| \le z^\mu\right) dz. \tag{4.2}$$

Proof: By Lemma 4.1, it is sufficient to consider the case $\sigma_n = n^{-\mu}$. As an interpolating function, we can choose $S(x) = x^{-\mu}$, $x \ge 1$. If the distribution of θ satisfies condition (O) with $r > \mu$ we can estimate the remaining probability term in Theorem 2.1 as follows:

$$\log \mathbb{P}\left(|\theta| \le \varepsilon/\sigma_1\right) \ge -C_1(\varepsilon/\sigma_1)^{-1/r} = -C_1' \varepsilon^{-1/r}.$$

Thus, since $r > \mu$,

$$\varliminf_{\varepsilon \to 0+} \frac{\log \mathbb{P}\left(|\theta| \le \varepsilon/\sigma_1\right)}{S^{-1}(\varepsilon)} \ge -C_1' \varlimsup_{\varepsilon \to 0+} \varepsilon^{1/\mu - 1/r} = 0.$$

On the other hand, for $I_\infty(\varepsilon)$, we obtain precisely

$$I_\infty(\varepsilon) = \left(\int_0^{1/\varepsilon} \log \mathbb{P}\left(|\theta| \le \frac{1}{y}\right)\left(-\frac{1}{\mu}\right) y^{-1/\mu - 1}\, dy\right) \varepsilon^{-1/\mu}.$$

This shows the assertion, by Theorem 2.1. ■

Fully analogous, we obtain the result for $p < \infty$.

Theorem 4.2. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables let $\sigma_n \sim n^{-\mu}$, with $\mu > 1/p$. Then, if the distribution of θ satisfies condition (O) with $r > \mu$, we have*

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/(\mu - 1/p)} \log \mathbb{P}\left(\sum_n |\sigma_n \theta_n|^p \le \varepsilon^p\right) = -(\mu - 1/p)\left[\frac{K^\mu}{p^{1/p}\mu^\mu}\right]^{1/(\mu - 1/p)}, \tag{4.3}$$

where

$$K := -\int_0^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz. \tag{4.4}$$

Proof: Again, by Lemma 4.1, it is sufficient to consider $\sigma_n = n^{-\mu}$; so we can use $S(x) = x^{-\mu}$, for $x \ge 1$. Condition (O) helps us to take care of the remaining term in Theorem 2.2, since

$$\log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p} \ge \log \mathbb{E} e^{-\lambda|\sigma_1\theta|^p} \mathbb{I}_{\{-\lambda|\sigma_1\theta|^p \ge -\sigma_1^p\}} \ge \log\left(e^{-\sigma_1^p} \mathbb{P}\left(|\theta| \le \lambda^{-1/p}\right)\right), \tag{4.5}$$

which is handled the same way as in the proof of Theorem 4.1. By Theorem 2.2, this shows that the correct order of $\log \mathbb{E}\exp(-\lambda \sum_n |\sigma_n\theta_n|^p)$ is $S^{-1}(\lambda^{-1/p}) = \lambda^{1/(\mu p)}$; and the constant the quotient tends to is $-K$. Using Theorem 4.12.9 in Bingham *et al.*(2), one obtains the small deviation rate and the small deviation constant. ■

If we apply Theorem 4.1 and Theorem 4.2 we have to check the finiteness of the small deviation constants in (4.2) and (4.4). This question is investigated in Section 4.5. However, the equality in (4.2) and (4.3) is to be understood in the sense that either both sides are finite and equal or both sides are $-\infty$.

4.4 Polynomical Decrease with Logarithmic Correction

Let us finally consider the case $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}$, $\nu \neq 0$. In the case $p = \infty$, we obtain the following result.

Theorem 4.3. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables and let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}$, $\nu \neq 0$. Let us assume that the distribution of θ satisfies condition (O) with $r > \mu$. Furthermore, if $\nu < 0$ let*

$$\int_0^1 -\log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) y^{-1/\mu-1}(1-\log y)^{-\nu/\mu}\, dy < \infty. \tag{4.6}$$

Then we have

$$\lim_{\varepsilon\to 0+} \varepsilon^{1/\mu}\left(-\log\varepsilon\right)^{\nu/\mu} \log \mathbb{P}\left(\sup_n |\sigma_n\theta_n| \leq \varepsilon\right) = \mu^{\nu/\mu}\int_0^\infty \log \mathbb{P}\left(|\theta| \leq z^\mu\right) dz. \tag{4.7}$$

Again, the formulation of the result for $p < \infty$ is slightly more complicated. However, the proof is essentially the same.

Theorem 4.4. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. random variables and let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}$, $\nu \neq 0$. Let us assume that the distribution of θ satisfies condition (O) with $r > \mu$. Furthermore, if $\nu < 0$ let*

$$\int_0^1 -\log \mathbb{E}e^{-|y\theta|^p} y^{-1/\mu-1}(1-\log y)^{-\nu/\mu}\, dy < \infty. \tag{4.8}$$

Then we have

$$\begin{aligned}\lim_{\varepsilon\to 0+} & \varepsilon^{1/(\mu-1/p)}\left(-\log\varepsilon\right)^{\nu/(\mu-1/p)} \log \mathbb{P}\left(\sum_n |\sigma_n\theta_n|^p \leq \varepsilon^p\right) \\ &= -\left[\frac{(\mu-1/p)^{\mu-1/p+\nu}}{\mu^\mu p^{1/p}}\left(-\int_0^\infty \log \mathbb{E}e^{-|z^{-\mu}\theta|^p}\, dz\right)^\mu\right]^{1/(\mu-1/p)},\end{aligned} \tag{4.9}$$

Proof of Theorem 4.3 and Theorem 4.4: By Lemma 4.1, it is sufficient to deal with the sequence $\tilde{\sigma}_n = n^{-\mu}(1+\log n)^{-\nu}$, with $\mu > 0$ and $\nu \in \mathbb{R}$, $\nu \neq 0$. The logical choice for the interpolation function is

$$\tilde{S}(y) = y^{-\mu}(1+\log y)^{-\nu}, \qquad y \geq 1.$$

However, since we cannot compute the inverse of $\tilde{S}$ explicitly, we pass over to the sequence (σ_n) given by $\sigma_n := S(n)$, where S is the inverse of

$$S^{-1}(x) := \mu^{\nu/\mu} x^{-1/\mu} (1 - \log x)^{-\nu/\mu}, \qquad x \leq 1.$$

According to Lemma 4.1, we have to check whether $\lim_{n\to\infty} \sigma_n/\tilde{\sigma}_n \to 1$, which can be verified easily in our concrete example.

Theorem 3.1 and Theorem 3.2 already give the correct upper bounds for the desired small deviation probabilities. However, since we would like to have lower bounds as well, we have to prove that the upper limits in (3.1) and (3.3) tend to the same constants.

For simplicity, let us define

$$P(y) := \begin{cases} \left(-\frac{1}{\mu}\, y^{-1/\mu-1}\right) \log \mathbb{P}\left(|\theta| \leq \frac{1}{y}\right) & p = \infty, \\ \left(-\frac{1}{\mu}\, y^{-1/\mu-1}\right) \log \mathbb{E} e^{-|y\theta|^p} & p < \infty. \end{cases}$$

Note that the function P is non-negative. It is easy to calculate that in our concrete example

$$(S^{-1})'(x) = \mu^{\nu/\mu} \left(-\frac{1}{\mu}\right) x^{-1/\mu-1} (-\log(x/e))^{-\nu/\mu} \left(1 + \frac{\nu}{\log(x/e)}\right).$$

Therefore,

$$\frac{d}{dy}\left[\frac{S^{-1}(\varepsilon y)}{S^{-1}(\varepsilon)}\right] = \frac{(S^{-1})'(\varepsilon y)\varepsilon}{S^{-1}(\varepsilon)} = \left(\frac{y^{-1/\mu-1}}{-\mu}\right) \left(1 + \frac{\log y}{\log(\varepsilon/e)}\right)^{-\nu/\mu} \left(1 + \frac{\nu}{\log(y\varepsilon/e)}\right).$$

Thus, we have to prove that

$$\overline{\lim_{\varepsilon\to 0+}} \int_0^{1/\varepsilon} P(y) \left(1 + \frac{\log y}{\log(\varepsilon/e)}\right)^{-\nu/\mu} \left(1 + \frac{\nu}{\log(y\varepsilon/e)}\right) dy \leq \int_0^\infty P(y)\, dy, \tag{4.10}$$

provided the right-hand side is finite. If it is not finite, already Theorem 3.1 and Theorem 3.2 give the assertion. We prove (4.10) for the integrals from 0 to 1, and from 1 to ∞ separately.

Integral from 0 *to* 1, *case* $\nu > 0$. Note that, if $0 \leq y \leq 1$ and $\nu > 0$, we have, for $\varepsilon < e$, $1 + \log y/(\log(\varepsilon/e)) \geq 1$ and $1 + \nu/(\log(\varepsilon y/e)) \leq 1$. Thus,

$$\overline{\lim_{\varepsilon\to 0+}} \int_0^{1} P(y) \left(1 + \frac{\log y}{\log(\varepsilon/e)}\right)^{-\nu/\mu} \left(1 + \frac{\nu}{\log(y\varepsilon/e)}\right) dy \leq \int_0^1 P(y)\, dy$$

which implies the respective part of (4.10).

Integral from 0 *to* 1, *case* $\nu < 0$. In this case, for $\varepsilon \leq \min(1, e^{1-2\nu})$, we have

$$\left(1 + \frac{\log y}{\log(\varepsilon/e)}\right)^{-\nu/\mu} \leq \left(1 + \frac{\log y}{\log(1/e)}\right)^{-\nu/\mu} = (1 - \log y)^{-\nu/\mu}$$

and $1 + \nu/\log(y\varepsilon/e) \leq 1 + \nu/\log e^{-2\nu} = 1/2$. Since we assume that $P(y)(1 - \log y)^{-\nu/\mu}$ is integrable over $[0, 1]$, by Lebesgue's Theorem, we have proved the existence of the respective part of (4.10).

The estimates for the integrals from 1 to infinity are similar. We therefore omit them.

In order to apply Theorems 2.1 and 2.2, we have to take care of the remaining terms on the left-hand side in those theorems. This is handled the same way as in the case $\nu = 0$ with the help of condition (O).

Making the assumption (O) not only disposes of the remaining probability terms, it also allows us to return from the sequence σ_n generated by the inverse of S^{-1} to the original sequence $\tilde{\sigma}_n = n^{-\mu}(1 + \log n)^{-\nu}$ using Lemma 4.1.

For $p = \infty$, we are finished, whereas, for $p < \infty$, we have to use the mentioned Tauberian theorem (cf. Theorem 4.12.9 in Bingham *et al.*[(2)]) in order to return from the logarithmic Laplace transform to the small deviation rate. ■

Similarly to the last section, it is necessary to check the finiteness of the small deviation constants in (4.7) and (4.9). This question is investigated in the following section. However, independent of their finiteness, the equalities in (4.7) and (4.9) are to be understood in the sense that either both sides are finite and equal or both sides are $-\infty$.

4.5 The Small Deviation Constant

In many cases (Theorems 4.1, 4.2, 3.1, 3.2, 4.3, and 4.4) we have to check the finiteness of the small deviation constant. The crucial part of this is represented by

$$K := \begin{cases} -\int_0^\infty \log \mathbb{P}\left(|\theta| \leq z^\mu\right) dz & p = \infty, \\ -\int_0^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p} dz & p < \infty. \end{cases}$$

Of course, this only depends on the distribution of θ. However, it is not transparent when this expression is finite. Nevertheless, it is intuitively clear that only the behaviour of the distribution near zero and infinity matters. First of all, we shall see that the finiteness of the integral from zero to one is always ensured under condition (O).

Lemma 4.2. *Let $\mu > 0$ and assume that the distribution of θ satisfies condition (O) with $r > \mu$. Then $K_1 < \infty$, where*

$$K_1 := \begin{cases} -\int_0^1 \log \mathbb{P}\left(|\theta| \leq z^\mu\right) dz & p = \infty, \\ -\int_0^1 \log \mathbb{E} e^{-|z^{-\mu}\theta|^p} dz & p < \infty. \end{cases}$$

Proof: This is trivial if $p = \infty$. In the case $p < \infty$ it follows from the argumentation (4.5). ■

Thus, as long as condition (O) is satisfied with sufficiently large r, no additional restriction to ensure the convergence of the integral defining K at zero is needed. For the remaining integral, we have to distinguish the cases $p = \infty$ and $p < \infty$. For $p < \infty$, the following simple necessary and sufficient condition holds.

Lemma 4.3. *We have*

$$-\int_1^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz < \infty \tag{4.11}$$

if and only if $\mathbb{E}|\theta|^{1/\mu} < \infty$ *and* $\mu > 1/p$.

Proof: For $z \geq 1$, $\mathbb{E}e^{-|z^{-\mu}\theta|^p} \geq \mathbb{E}e^{-|\theta|^p} > 0$. Therefore, for some $C > 0$,

$$\int_1^\infty \log \mathbb{E} e^{-|z^{-\mu}\theta|^p}\, dz \geq \int_1^\infty C\left(\mathbb{E}e^{-|z^{-\mu}\theta|^p} - 1\right) dz.$$

On the other hand, the reverse inequality is true with $C = 1$. Thus, (4.11) holds if and only if

$$\int_1^\infty \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right) dz < \infty. \tag{4.12}$$

It is clear that $0 \leq \int_0^1 \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right) dz \leq 1$. Thus, (4.12) is true if and only if

$$\int_0^\infty \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right) dz < \infty.$$

To finish the proof we observe that

$$\int_0^\infty \mathbb{E}\left(1 - e^{-|z^{-\mu}\theta|^p}\right) dz = \mathbb{E}\int_0^\infty 1 - e^{-|z^{-\mu}\theta|^p}\, dz = \mathbb{E}|\theta|^{1/\mu}\int_0^\infty 1 - e^{-y^{-p\mu}}\, dy.$$

■

The last result shows that the range of eligible $\mu > 0$ is also strongly determined by the *upper* tail behaviour of the distribution of θ, a fact one might not expect when dealing with *lower tails.* The same is true in the case $p = \infty$.

Lemma 4.4. *We have*

$$-\int_1^\infty \log \mathbb{P}\left(|\theta| \leq z^\mu\right) dz < \infty$$

if and only if $\mathbb{E}|\theta|^{1/\mu} < \infty$.

The proof is similar to the one of Lemma 4.3; and we omit it. Now we are in the position to proof the main result mentioned in Section 1.

Proof of Theorem 1.1: The same considerations as in Lemma 4.4 and Lemma 4.2 can be applied to the conditions (4.6) and (4.8). We combine Lemma 4.3, Lemma 4.4, and Lemma 4.2 with Theorem 4.3 and Theorem 4.4, respectively. This leads to the main result, Theorem 1.1. ■

4.6 Stable Random Variables

First, let us consider the example of i.i.d. **non-Gaussian**, α-stable random variables $\theta, \theta_1, \theta_2, \ldots$ with parameters $0 < \alpha < 2$, $\sigma > 0$, $\beta \in [-1, 1]$, and $\mu \in \mathbb{R}$ (cf. Samorodnitsky and Taqqu[11], Chapter 1, for the description of the parameters and the properties). If $|\beta| = 1$ and $0 < \alpha < 1$, we say that θ is totally skewed.

Corollary 4.1. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. non-trivial α-stable random variables that are not totally skewed and let $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$ with $\mu > \max(1/\alpha, 1/p)$ and $\nu \in \mathbb{R}$. Then*

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/(\mu-1/p)} \left(-\log \varepsilon\right)^{\nu/(\mu-1/p)} \log \mathbb{P}\left(\|(\sigma_n \theta_n)\|_p \leq \varepsilon\right) = -C_p,$$

where C_p is the finite, positive constant given in (1.1). For $\mu \leq \max(1/p, 1/\alpha)$, we have $C_p = \infty$.

Proof: It is well-known that θ has a continuous non-vanishing density on the whole $\mathbb{R}$ if θ is not totally skewed. Therefore, the distribution satisfies condition (O) for all $r > 0$.

Furthermore, it is well-known (cf. Chapter 1 in Samorodnitsky and Taqqu[11]) that $\mathbb{E}|\theta|^{1/\mu} < \infty$ if and only if $1/\mu < \alpha$ (i.e. $\mu > 1/\alpha$). Applying Theorem 1.1 gives the assertion. ■

Open cases: The last statement clarifies the small deviation rate for most of the cases where the sequence $(\sigma_n \theta_n)$ is in l_p a.s. Indeed, by Theorem 11.3.2 in Samorodnitsky and Taqqu[11] (in the case $\alpha = 1$, we have to assume that θ is symmetric), $(\sigma_n \theta_n) \in l_p$ a.s. if and only if

$$\begin{cases} \sum_n \sigma_n^p < \infty & p < \alpha \\ \sum_n \sigma_n^\alpha \log \sigma_n^{-1} < \infty & p = \alpha \\ \sum_n \sigma_n^\alpha < \infty & p > \alpha \text{ or } p = \infty. \end{cases}$$

In our case, $\sigma_n \sim n^{-\mu}(1 + \log n)^{-\nu}$, Corollary 4.1 solves most of the cases. However, the following cases remain open:

(a) $\mu = \max(1/p, 1/\alpha)$ and $\nu > \max(1/p, 1/\alpha)$ if $p \neq \alpha$,

(b) $\mu = 1/\alpha$ and $\nu > 2/\alpha$ for $p = \alpha$,

(c) the case of totally skewed stable random variables: $|\beta| = 1$ and $0 < \alpha < 1$.

In the first two cases, the order from Corollary 4.1 is not the correct one apparently, as the second part of that corollary shows. Let us illustrate this with the help of an example.

Example: If $p = \infty$, $\mu = 1/\alpha$, $\nu = b/\alpha$, with some $b > 1$, then $(\sigma_n\theta_n) \in l_\infty$ almost surely, by the result above. Using Theorem 2.1, it can be seen that

$$-C_1 \le \varepsilon^\alpha(-\log\varepsilon)^{b-1} \log \mathbb{P}\left(\sup_n |n^{-1/\alpha}(1+\log n)^{-b/\alpha}\theta_n| \le \varepsilon\right) \le -C_2,$$

for some positive constants C_1, C_2 and all $0 < \varepsilon < 1$. Thus, in this case, an additional logarithmic term appears in the order. Presumably, a similar behaviour will show in the other remaining cases of (a) and (b).

A more detailed investigation was carried out in the author's Ph.D. thesis (cf. Aurzada[(1)]).

On the other hand, for **Gaussian** (i.e. 2-stable) random variables the result is much easier.

Corollary 4.2. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. Gaussian random variables and let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$ with $\mu > 1/p$ and $\nu \in \mathbb{R}$. Then*

$$\lim_{\varepsilon\to 0+} \varepsilon^{1/(\mu-1/p)} \left(-\log\varepsilon\right)^{\nu/(\mu-1/p)} \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \le \varepsilon\right) = -C_p,$$

where C_p is the finite, positive constant in (1.1). For $\mu \le 1/p$, we have $C_p = \infty$.

Proof: It is well-known that $\mathbb{E}|\theta|^p < \infty$, for all $p > 0$, and that θ satisfies condition (O) for all $r > 0$. Theorem 1.1 shows the assertion. ■

For $p = 2$ and any $\mu > 1/2$, the constant can be calculated explicitly. The result is

$$C_2 = \left(\frac{2\mu-1}{2}\right)^{1+2\nu/(2\mu-1)} \left(\frac{\pi}{2\mu\sin\frac{\pi}{2\mu}}\right)^{2\mu/(2\mu-1)}.$$

Note that – on the logarithmic level – Corollary 4.3 in Dunker *et al.*[(3)] is a special case of the last statement if one sets $\nu = 0$ and $p = 2$ and substitutes $A = 2\mu$ and $a = 2\mu/(2\mu-1)$.

4.7 Gamma Distributions

Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. Gamma-distributed random variables, i.e.

$$\mathbb{P}\left(\theta \le t\right) = \int_0^t \frac{x^{b-1}e^{-x/a}}{\Gamma(b)a^b}\, dx,$$

for some fixed parameters $a, b > 0$. This particularly includes exponential distributions, so-called Erlang distributions, and χ^2-distributions.

Corollary 4.3. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of i.i.d. Gamma-distributed random variables and let $\sigma_n \sim n^{-\mu}(1+\log n)^{-\nu}$ with $\mu > 1/p$ and $\nu \in \mathbb{R}$. Then*

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/(\mu-1/p)} \left(-\log \varepsilon\right)^{\nu/(\mu-1/p)} \log \mathbb{P}\left(\|(\sigma_n\theta_n)\|_p \leq \varepsilon\right) = -C_p,$$

where C_p is the finite, positive constant in (1.1). For $\mu \leq 1/p$, we have $C_p = \infty$.

Proof: It is elementary to check that $\mathbb{E}|\theta|^p < \infty$, for all $p > 0$, and that θ satisfies condition (O) for all $r > 0$. Theorem 1.1 shows the assertion. ■

If $p = 1$ it is possible to calculate the constant explicitly. The result is

$$C_1 = (\mu-1)^{1+\nu/(\mu-1)} a^{1/(\mu-1)} b^{\mu/(\mu-1)} \left(\frac{\pi/\mu}{\sin(\pi/\mu)}\right)^{\mu/(\mu-1)}.$$

Note that also this recovers the logarithmic level of the result in Corollary 4.3 in Dunker *et al.*[3] if one sets $a = 2$, $b = 1/2$, $\nu = 0$, and $\mu = A$, $a = A/(A-1)$.

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