The Normal Form of a Planar Curve and Its Application to Curve Design

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Abstract. Analogously to the Hessian normal form of a line, the normal form $h(\boldsymbol{x}) = 0$ of a planar curve is defined. Thus any planar curve for which the normal form exists can be considered as an implicit curve and all such curves can be treated in a uniform way. Hence the simple implicit blending methods are applicable. Offsets are just level curves of the oriented distance function h. Bisectors of two curves have simple implicit representations.

§1. Introduction

In CAGD a planar curve is usually defined as a parametric curve $\boldsymbol{x} = \boldsymbol{\gamma}(t), t \in [a, b]$ or as an implicit curve $f(\boldsymbol{x}) = 0$. But there are important curves which have no such standard representations, for example (except for special cases) the offset curves of an implicit curve, the bisectors of two curves, and blend curves of two parametric curves. The idea of the normal form (introduced below) allows us to consider any curve as an implicit one, and all the powerful methods defined for implicit curves are applicable. Furthermore, the simple geometric meaning of the oriented distance function h yields a *uniform treatment of all curves independent of their definitions.*

In Section 2 we define the normal form for a planar parametric curve and generalize the idea to arbitrary (planar) curves. Special attention is paid to implicit curves. Section 3 shows that the normal form is a powerful tool for solving basic problems in CAGD. In [3] the idea of the normal form is extended to surfaces.

§2. The Normal Form of a Planar Curve

We introduce first the normal form for a parametric curve, generalize it to arbitrary curves, and show how to handle implicit curves.

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2.1 The Normal Form of a Parametric Curve

Let

- $\Gamma : \boldsymbol{x} = \boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t)),\ t \in [a, b]$ be a smooth plane curve,
- $\boldsymbol{n}(t) := (\gamma_2(t), -\gamma_1(t))/\|...\|$ the unit normal,
- $D \subset \mathbb{R}^2$ such that for any $\boldsymbol{x} \in D$ the function $d_{\boldsymbol{x}}(t) := \|\boldsymbol{x} - \boldsymbol{\gamma}(t)\|$ has exactly one minimum,
- $t_m(\boldsymbol{x})$ the parameter of the minimum, $\boldsymbol{\gamma}(t_m(\boldsymbol{x}))$ the footpoint (of point $\boldsymbol{x}$).
- $h(\boldsymbol{x}) := sign\big((\boldsymbol{x} - \boldsymbol{\gamma}(t_m(\boldsymbol{x}))) \cdot \boldsymbol{n}(t_m(\boldsymbol{x}))\big)\ d_{\boldsymbol{x}}(t_m(\boldsymbol{x}))$ the oriented distance of $\boldsymbol{x}$ to curve Γ.

The equation $h(\boldsymbol{x}) = 0$ is an implicit representation of curve Γ and is called the normal form of Γ.

Remark: The normal form of a line is the well known Hessian normal form: $h(\boldsymbol{x}) := \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_1) = 0$ where $\boldsymbol{n}$ is a unit normal and $\boldsymbol{x}_1$ a point of the line.

Essential Properties of the Oriented Distance Function:

(O) Level curves of the oriented distance function h are offset curves of curve Γ.

(D) h is differentiable: $\nabla h(\boldsymbol{x}) = \boldsymbol{n}(t_m(\boldsymbol{x}))$ (unit normal at the footpoint belonging to $\boldsymbol{x}$).

The second property can be derived by considering the offset curves $\Gamma_h : \boldsymbol{x} = \boldsymbol{\gamma}(t) + h\boldsymbol{n}(t)$ of Γ. Applying the implicit function theorem to the equation $\boldsymbol{x} - \boldsymbol{\gamma}(t) + h\boldsymbol{n}(t) = 0$ for given $\boldsymbol{x}$ yields the existence of the real function $h(\boldsymbol{x})$ and statement (D).

Evaluation of the Distance Function:

The usage of the normal form depends on the possibility of evaluating the oriented distance function h. This means solving for a given $\boldsymbol{x}$ the equation

$$(\boldsymbol{x} - \boldsymbol{\gamma}(t)) \cdot \dot{\boldsymbol{\gamma}}(t) = 0.$$

This can be done either by

- Newton iteration (the second derivative of $\boldsymbol{\gamma}$ is necessary) or
- repeatedly determining footpoints on tangents (only the first derivative of $\boldsymbol{\gamma}$ is used): $t_{i+1} = t_i + \frac{(\boldsymbol{x} - \boldsymbol{\gamma}(t_i)) \cdot \dot{\boldsymbol{\gamma}}(t_i)}{\dot{\boldsymbol{\gamma}}(t_i)^2}$.

The necessary starting parameter can be determined from the minimum of $\{(\boldsymbol{x} - \boldsymbol{\gamma}(t_1))^2, ..., (\boldsymbol{x} - \boldsymbol{\gamma}(t_n))^2\}$ for an equally spaced polygon $\{\boldsymbol{\gamma}(t_1), ..., \boldsymbol{\gamma}(t_n)\}$ on Γ or from the footpoint of a point $\boldsymbol{x}'$ in the vicinity of $\boldsymbol{x}$.

2.2 Generalization

It is not necessary that the curve Γ from the previous subsection be given by a parametric representation. The only necessary properties are

(NF1) Γ has to be *smooth* and

(NF2) there is a region $D \subset \mathbb{R}^2$ in the vicinity of Γ such that for any point $\boldsymbol{x} \in D$ it is possible to determine a (unique) *footpoint* (point on Γ with minimal distance to $\boldsymbol{x}$).

Conclusion: Any planar curve Γ with properties (NF1) and (NF2) can be considered as an implicit curve $h(\boldsymbol{x}) = 0$, where $h(\boldsymbol{x})$ is the oriented distance of point $\boldsymbol{x} \in D$ to the curve Γ and $\nabla h(\boldsymbol{x})$ is the unit normal at the footpoint belonging to $\boldsymbol{x}$. The equation $h(\boldsymbol{x}) = 0$ is called the normal form of Γ.

So any algorithm for implicit curves which only needs $h(\boldsymbol{x})$ and $\nabla h(\boldsymbol{x})$ for a point $\boldsymbol{x}$ of consideration are applicable to Γ. For example, the tracing algorithm or Newton iteration for determining intersection points (see Section 3).

Remarks: a) The determination of the region D, for which a unique footpoint exists, is usually a difficult task and subject to further investigation.

b) The convergence of the footpoint algorithm becomes critical for points near the boundary of D. Statements on the quality of convergence are difficult.

2.3 The Normal Form of an Implicit Curve

Let $\Gamma_0 : \ f = 0$ be a smooth implicit curve. Usually the function value $f(\boldsymbol{x})$ has no geometric meaning. The advantage of the normal form $h = 0$ of a curve is the simple geometric meaning of the function value $h(\boldsymbol{x})$ (oriented distance) and the level curves $h(\boldsymbol{x}) = c$ (offset curves). So, whenever the distance of a point or an offset curve of an implicit curve is needed, one should apply the normal form of the curve.

Evaluation of the Distance Function:

Let $\Gamma_0 : \ f = 0$ be an implicit curve and $\boldsymbol{x}$ a point within the vicinity of Γ_0. A footpoint $\boldsymbol{x}_0$ of $\boldsymbol{x}$ fulfills the following two conditions:

1) $f(\boldsymbol{x}_0) = 0$ and 2) $\nabla f(\boldsymbol{x}_0)$ is parallel to $\boldsymbol{x} - \boldsymbol{x}_0$.

For the determination of $\boldsymbol{x}_0$ we use the following procedure curvepoint which calculates a *curve point* $\boldsymbol{c}$ along the steepest way for a given point $\boldsymbol{p}$:

(CP0) $\boldsymbol{q}_0 = \boldsymbol{p}$

(CP1) repeat $\boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \frac{f(\boldsymbol{q}_k)}{\nabla f(\boldsymbol{q}_k)^2} \nabla f(\boldsymbol{q}_k)$ (Newton step)

until $\|\boldsymbol{q}_{k+1} - \boldsymbol{q}_k\|$ is "sufficiently" small.

$\boldsymbol{c} = \boldsymbol{q}_{k+1}$.

Determination of *footpoint* $\boldsymbol{x}_0$ (belonging to $\boldsymbol{x}$):

(FP0) $\boldsymbol{c}_0 = \texttt{curvepoint}(\boldsymbol{x})$

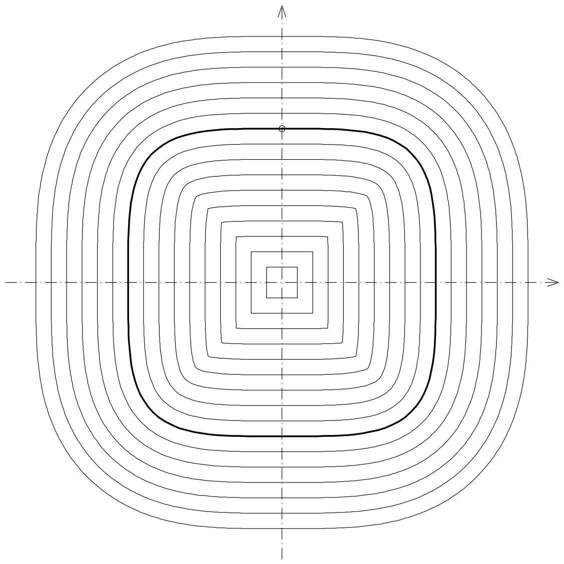

Fig. 1. Offset curves of the implicit curve $f(\boldsymbol{x}) = x^4 + y^4 - 1 = 0$ (bold curve).

(FP1) repeat $\boldsymbol{t}_{i+1} = \boldsymbol{x} - \frac{(\boldsymbol{x}-\boldsymbol{c}_i)\cdot\nabla f(\boldsymbol{c}_i)}{\nabla f(\boldsymbol{c}_i)^2}\nabla f(\boldsymbol{c}_i)$ (near point on tangent line),

$\boldsymbol{c}_{i+1} = \texttt{curvepoint}(\boldsymbol{t}_{i+1})$.

until $\|\boldsymbol{c}_{i+1} - \boldsymbol{c}_i\|$ is "sufficiently" small.

$\boldsymbol{x}_0 = \boldsymbol{c}_{i+1}$.

Displaying an Implicit Curve:

All curves in Section 3 are either implicit curves or represented implicitly by their normal forms and *traced* by the following algorithm for regular implicit curves:

Let $\Gamma_0 : \; f = 0$ be an implicit curve.

(IC0) Choose a starting point $\boldsymbol{q}_1$ in the vicinity of Γ

(IC1) $\boldsymbol{p}_i := \texttt{curvepoint}(\boldsymbol{q}_i)$ (see algorithm `curvepoint` above)

$\boldsymbol{t}_i := (-f_y(\boldsymbol{p}_i), f_x(\boldsymbol{p}_i))/\|...\|$ (unit tangent)

$\boldsymbol{q}_{i+1} := \boldsymbol{p}_i + \delta \boldsymbol{t}_i$ (δ: steplength)

The tracing algorithm stops if $\boldsymbol{p}_i$ is "near" a prescribed endpoint (or another termination).

§3. Applications of the Normal Form

3.1 Offset Curves of Implicit Curves

Offset curves of an implicit curve $f = 0$ are level curves of the oriented distance function: $h = c$.

Example: Figure 1 shows the curve $f(\boldsymbol{x}) = x^4 + y^4 - 1 = 0$ (bold curve) and offset curves. One recognizes that some of the inner level curves possess singularities at lines of symmetry.

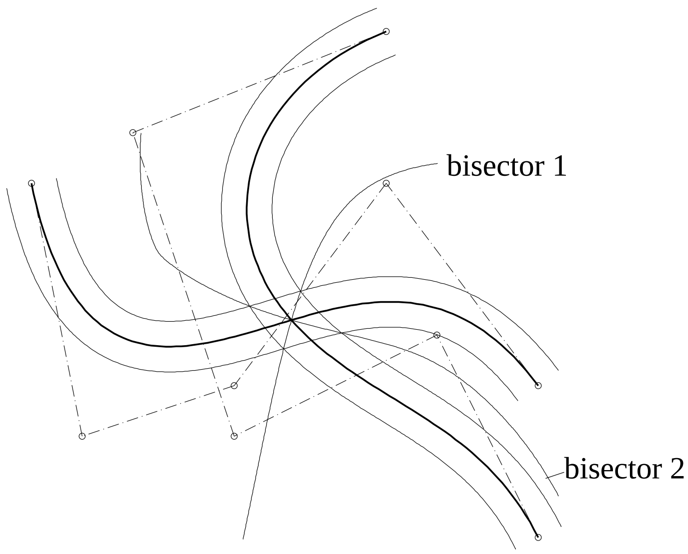

Fig. 2. Offset curves and bisectors of two Bezier curves (bold curves).

3.2 Bisectors of Two Curves

Let Γ_1, Γ_2 be two smooth parametric or implicit curves, and $h_1 = 0, h_2 = 0$ their normal forms. Hence the equations

$$h_1(\boldsymbol{x}) - h_2(\boldsymbol{x}) = 0, \quad h_1(\boldsymbol{x}) + h_2(\boldsymbol{x}) = 0$$

are implicit representations of the bisector curves (points, which have equal distance to Γ_1 and Γ_2).

The bisectors can be traced by any marching algorithm that uses only the first derivative (see Section 2.3).

Example 1: Figure 2 shows two Bezier curves (bold curves) together with two offset curves and their bisectors.

Example 2: Figure 3 shows the curves

$x^2 + y^2 - r = 0$ (circle), $\quad (x - 1.5)^4 + (y - 0.5)^4 - 1 = 0$ (hypercircle)

and one bisector for the cases: $r = 1,\ r = 0.5,\ r = 0.2$, and $r = 0.002$.

Example 3: Figure 4 shows the bisector curve of

a) the ellipse $x^2 + \frac{y^2}{4} = 1$ and the point $(-0.5, 1)$
b) the hypercircle $x^4 + y^4 = 1$ and the point $(-0.4, 0.6)$.

While tracing the bisectors one should pay attention to singularities on the lines of symmetry of the ellipse and the hypercircle respectively. Further information on bisectors is contained in [1,2].

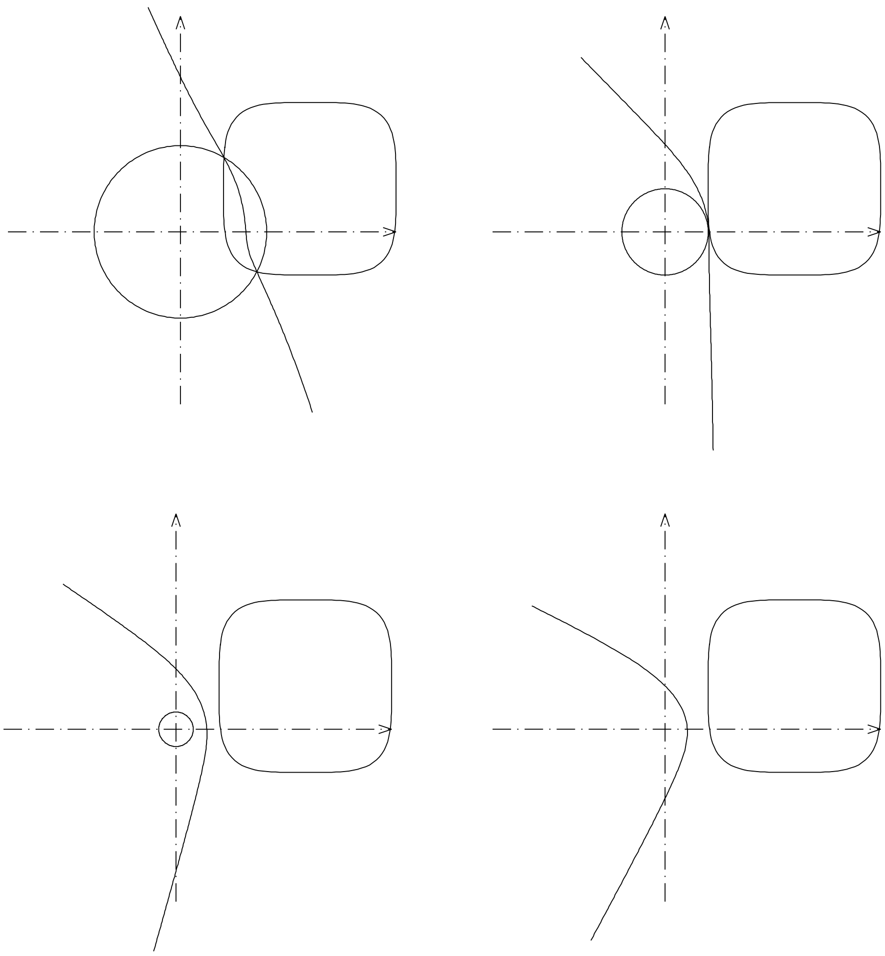

Fig. 3. One bisector of a circle with various radii and a hypercircle .

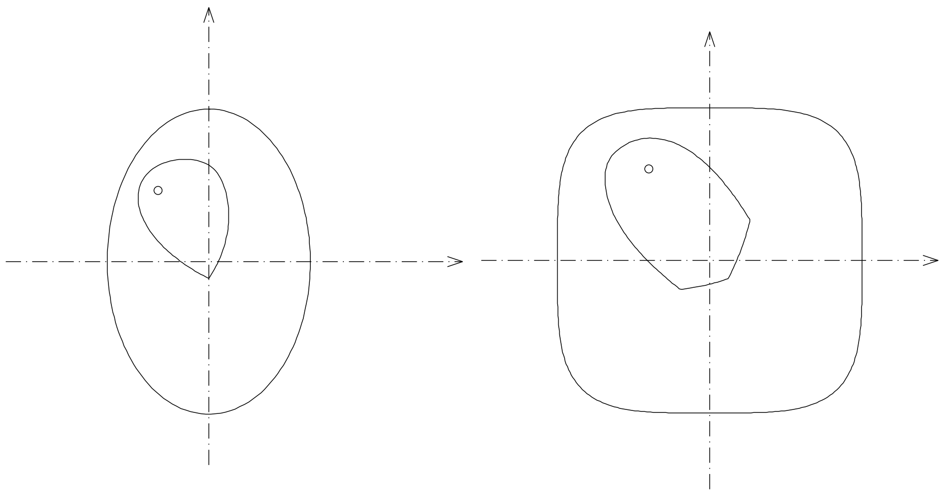

Fig. 4. The bisector of an ellipse/hypercircle and a point.

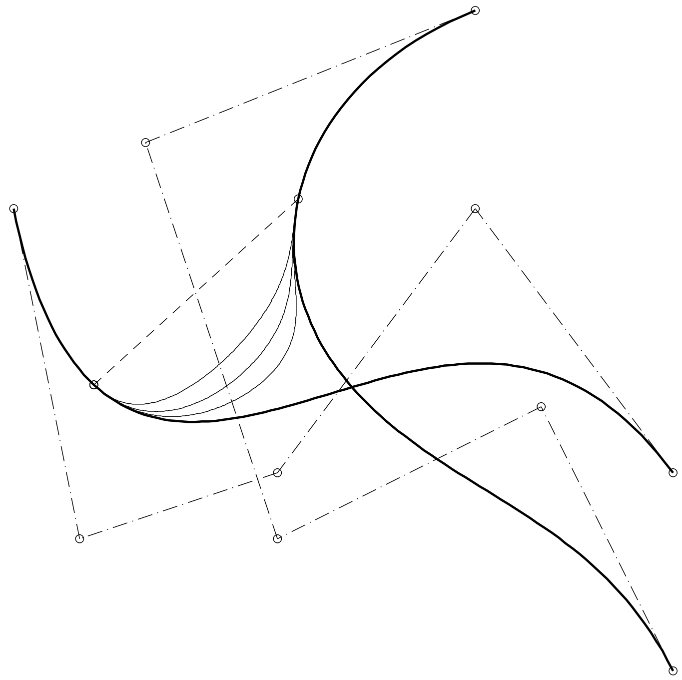

Fig. 5. G^2–Blending curves of two Bezier curves.

3.3 G^n–Blending of Curves

Let $\Gamma_1 : f_1(\boldsymbol{x}) = 0,\ \Gamma_2 : f_2(\boldsymbol{x}) = 0$ be two implicit curves and $\Gamma_0 : f_0(\boldsymbol{x}) = 0$ a line which intersects Γ_1 and Γ_2 (f_i differentiable enough). Then

$$\Phi_\mu : (1-\mu) f_1 f_2 - \mu f_0^{n+1} = 0,\ \ 0 < \mu < 1,$$

is for any μ a curve with G^n-contact to the curves Γ_1 and Γ_2 at the intersection points $\Gamma_1 \cap \Gamma_0,\ \Gamma_2 \cap \Gamma_0$. The blending curves Φ_μ are called parabolic functional splines (see [4]).

Example: Using the normal form this (and any other) implicit blending method is applicable to almost arbitrary curves, especially to parametric curves. Figure 5 shows G^2–blending curves of two Bezier curves for $\mu = 0.01, 0.03$ and 0.1, respectively.

3.4 Intersection Points of Curves

Using normal forms of curves we are able to determine intersection points by an algorithm for implicit curves (Newton iteration).

Example: The intersection points of a blending curve (normal form: $h(\boldsymbol{x}) = 0$) of the previous subsection with a bisector curve (normal form: $b(\boldsymbol{x}) = 0$) of the given Bezier curves can be determined by solving the nonlinear system

$$h(\boldsymbol{x}) = 0, \quad b(\boldsymbol{x}) = 0$$

with a Newton iteration.

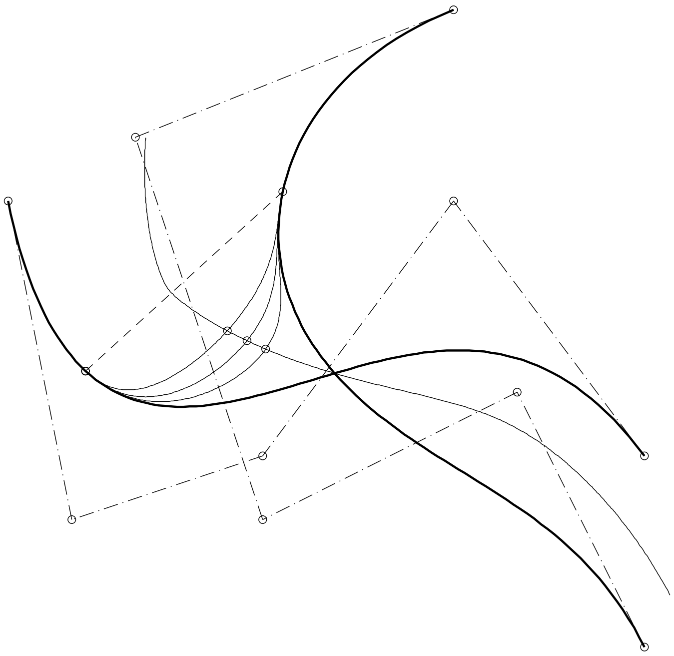

Fig. 6. Intersection points of blending curves and a bisector.

References

1. Farouki, R. T. and J. K. Johnstone, The bisector of a point and a plane parametric curve, Comput. Aided Geom. Design **11** (1994), 117–151.
2. Farouki, R. T. and J. K. Johnstone, Computing point/curve and curve/curve bisectors, in *Design and Application of Curves and Surfaces: Mathematics of Surfaces V*, R. B. Fisher (ed.), Oxford University Press, 1994, 327–354.
3. Hartmann, E., Numerical Implicitization for Intersection and G^n-continuous Blending of Surfaces, Comput. Aided Geom. Design, to appear.
4. Li, J., J. Hoschek, and E. Hartmann, G^{n-1}-functional splines for interpolation and approximation of curves, surfaces and solids, Comput. Aided Geom. Des. **7** (1990), 209–220.

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