December 31, 2016
This combines and corrects com6 and com7

0.1. **From D-F to D-R for homogeneous.** Following Jonathan, we consider the continuous reps of D-F excluding $\Delta_1(k;1)$ and $\Delta_2(K_1)$ since for these $\sum F_i = V$ is not equivalent to $\bigcap X_i = 0$. For the others, at most one of the $\alpha_i$, $\beta_j$ is not an isomorphism. Up to symmetry (permutation of indices) we may assume that all but $\beta_3$ are isomorphisms. Since the maps $\alpha_j \times \beta_j$ are injective, we may identify $Y_j$ with its image. Now, put

$$A_1 = Y_1,\ A_2 = X_2,\ A_3 = Y_2,\ A_{12} = X_1,\ A_{23} = Y_3$$

$$C = (Y_2 + Y_3) \cap (X_1 + X_2)$$

and observe

$$C \oplus A_1 = A_1 + A_2$$

Form the sextuplet

$$U_1 = A_1,\ W_1 = A_1 + A_2,\ U_2 = A_3,\ W_2 = A_2 + A_3$$

$$C \oplus A_1 = A_1 + A_2,\ U_3 = (C \oplus U_2) \cap (U_1 \oplus A_{23}),\ W_3 = A_{12} + U_3$$

and observe that

$$U_i = Y_i,\ W_1 = X_i + X_{i+1}$$

is a case I sextuplet

0.2. **Correction: Duals of homogeneous ones.** When turning (via tensoring) an indecomposable representation into a representation over the algebraic closure, the latter will, in general, become decomposable. We relax the definition of case I allowing decomposable endomorphism $\varphi$. In the sequel, any sextuplet will be case I, any field of characteristic 0. A sextuplet will be addressed by its ambient space $V$.

1

**Claim 0.1.** *For an endomorphism $\varphi$ of an $F$-vector space and $\lambda \in F$ the following are equivalent*

(1) *$\lambda$ is the unique zero of the minimal (characteristic) polynomial of $\varphi$*
(2) *$\varphi$ has minimal elementary divisors $(x - \lambda)$*
(3) *there is a basis such that $\varphi$ is in Jordan normal form w.r.t. $\lambda$*

*Moreover, the following are equivalent*

(4) *$\varphi$ is indecomposable and $\lambda$ is the unique zero of the minimal (characteristic) polynomial of $\varphi$*
(5) *$\varphi$ is indecomposable and has minimal invariant divisor $(x - \lambda)$*
(6) *there is a basis such that $\varphi$ is in Jordan normal form w.r.t. $\lambda$ with unique block*

(7) *$\varphi$ is indecomposable and has an eigenvector w.r.t. $\lambda$*

*In particular, for an indecomposable sextuplet over algebraically closed $F$ there is unique $\lambda \in F$, the* eigenvalue *of the sextuplet, such that some/all of (4)-(7) apply to $\varphi$.*

1b

**Claim 0.2.** *Let $\varphi$ be an (indecomposable) endomorphism over $F$ and $\bar{F}$ the algebraic closure. $\lambda \in \bar{F}$ is a zero of the minimal polynomial of $\varphi$ over $F$ iff $\varphi$ has over $\bar{F}$ a Jordan block (i.e. an indecomposable direct summand) with eigenvalue $\lambda$*

1c

**Claim 0.3.** *An endomorphism over $F$ is indecomposable iff its minimal invariant divisor over $F$ is irreducible.*

2

**Claim 0.4.** *If a sextuplet is indecomposable with eigenvalue $\lambda$ then its dual is indecomposable with eigenvalue $1 - \lambda$.*

The proof is that of Lemma 4.4.

3

**Claim 0.5.** *If a sextuplet $V$ is a direct sum of indecomposables $V_i$ with eigenvalues $\lambda_i$ then its dual is the direct sum of indecomposables $V_i^*$ with eigenvalues $1 - \lambda_i$.*

4

**Claim 0.6.** *If a sextuplet $V$ is a direct sum of indecomposables $V_i$ with eigenvalues $\lambda_i$, $i \in I$, and isomorphic to its dual, then then there is a permutation $\sigma$ of $I$ such that $\sigma^2 = \mathsf{id}$ and $\lambda_{\sigma(i)} = 1 - \lambda_i$ and* $\dim V_{\sigma(i)} = \dim V_i$

By Krull-Remak-Schmidt and claim 0.5

Now consider an indecomposable sextuplet $V$ which is isomorphic to its dual. Let $\bar{F}$ the algebraic closure of $F$ and consider $V$ as $\bar{V}$ over $\bar{F}$. Decompose into indecomposables $\bar{V}_i$. These have eigenvalues $\lambda_i \in \bar{F}$. By claim 0.3 the minimal invariant divisor $p(x)$ of $\varphi$ over $F$ is irreducible and so has no multiple zeros. If $\lambda = \frac{1}{2}$ is a zero then it is the only one and we are done. Otherwise, by claims 0.6 and 0.2 its zeros in $\bar{F}$ are $\lambda_i, 1 - \lambda_i$, $i = 1, \ldots, k$, all pairwise distinct.

Such can occur, e.g. if $\varphi$ is given by the matrix

$$\begin{array}{cc} 0 & 5 \\ 1 & 1 \end{array}$$

Of course, adjoining the zeros of the minimal polynomial, we get only an isomorphism onto the dual over the extension field. So we should stay with the base field and use the rational canonical form for the endomorphism.

0.3. **Continuing case I in characteristic 0.** Given indecomposable case I sextuplet $V$ based on the indecomposable endomorphism $\varphi$ of the subspace $A_2$ we have

(1) there is an irreducible polynomial $p(x)$ (the minimal invariant divisor) such that the characteristic (and also minimal) polynomial of $\varphi$ is $p(x)^k$, where $k \cdot \deg p = \dim A_2 =: d$
(2) Let $\bar{V}$ denote $V$ tensored by the algebraic closure of $\bar{F}$ of $F$. $\bar{=} \bigoplus_i V_i$ where the $V_i$ are indecomposable case I with eigenvalue $\lambda_i \in \bar{F}$, all pairwise distinct
(3) $\lambda_i = \frac{1}{2}$ for some $i$ then $p(x) = x - \frac{1}{2}$ and we are done. Exclude this case in the sequel.
(4) Let $V^*$ the dual. It is indecomopsable case I based on an indecomposable endomorphism $\psi$ with minimal invariant divisor $q(x)$. Also, $\bar{V}^* = \bigoplus_i V_i^*$ where the $V_i^*$ are indecomposable case I with eigenvalue $1 - \lambda_i$
(5) The following are equivalent
    (a) $V \cong V^*$
    (b) $\bar{V} \cong \bar{V}^*$
    (c) The zeros of $p(x)$ come in a pairs $\lambda,\ 1-\lambda$
    (d) $p(x) = q(x)$
    (e) The quadruples given by $\varphi$ and $\psi$, resp., are isomorphic

Thus, (c) is equivalent to $V$ having an admisible symplectic or symmetric bilinear form (unique up to scaling). It remains to find out which is case. Equivalently, we may consider the form extended to $\bar{V}$. Observe that $\dim V$ is even (as eigenvalues over $\bar{F}$ come in pairs). The reduction method might apply with the action of $p(x)$ on $V$ or $\bar{V}$ to reduce the question to the case $k = 1$ (i.e. $\varphi$ diagonizable over $\bar{F}$) resp. $k = 2$.

The interesting case is $F$ the reals; here, $p(x)$ is quadratic and $\lambda = \frac{1}{2} \pm bi$ with $b \neq 0$. To see what happens for $b \to 0$ one can use the real Jordan normal form: in the limit one gets a direct sum of 2 isomorphic indecomposables with eigenvalue $\frac{1}{2}$. Can we say something about possible admissible form for such?

PS

Dear Alan and Jomnathan

sorry that my mails are quite short and that questions remain unanswered – mostly, what I can send is a second or third version. So I agree with your suggestions on the next steps

Best regards Christian