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Dear Alan and Jonathan

It appears that this approach requires a close look on description of isomorphisms onto the duals – which I could not manage, yet (cf Hypothesis 1.3)

Best regards

Christian

PS The (lecture) notes I have sent, recently, may be ignored – they dont deal with real Jordan form, anyway

1. Case I over the reals

1.1. **Linear theory.** Endomorphisms of a real vector space V, $\dim V = d$, admit a basis giving a description in real Jordan form: block diagonal compositions of indecomposable blocks A_i; the latter are Jordan blocks with real eigenvalue λ_i or matrices of 2×2 blocks, the diagonal blocks

$$\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$$

of which encode the same pair $\lambda_i, \bar{\lambda}_i$ of complex numbers while the first off-diagonal consist of unit blocks.

Indecomposable endomorphisms correspond to matrices with a single block A_1. The characteristic (and minimal) polynomial of such is $p(x)^k$ where $p(x) = x - \lambda$ and $k = d$ in the first case, $2k = d$ and the irreducible $p(x) = (x - \lambda)(x - \bar{\lambda})$ in $\mathbb{R}[x]$ in the second, the minimal invariant (and elementary) divisor.

The quadruples associated with endomorphisms of d-dimesnional real vector spaces are isomorphic iff the endomorphisms admit the same real Jordan form iff the endomorphisms have the same minimal invariant divisors.

Over the reals, there is a bijective correspondence between isomorphism types of case I sextuplets in dimension d (the dimension of the vector space underlying the basic endomorphism φ) and $d \times d$ matrices in real Jordan form (referring to an order on complex numbers which controls the order of blocks). This correspondence matches indecomposable sextuplets with single blocks (so that in this case no order is needed). This correspondence relies on a real Jordan basis for φ and its canonical extension to the sextuplet (via the frame).

1.2. **Self-duality.** We know that, over any field of characteristic $\neq 2$, an indecomposable case I sextuplet, based on an endomorphism having an eigenvalue λ (in the base field F, and thus admitting associated Jordan normal form) has its dual also case I with eigenvalue $1-\lambda$.

We know that an indecomposable case I sextuplet is isomorphic to its dual iff it admits a symmetric or symplectic form; and that the latter is then unique up to scaling. And that in case of eigenvalue $\frac{1}{2}$ the form is symplectic iff d is even.

Claim 1.1. *Over the reals, an indecomposable case I sextuplet, based on the endomorphism φ, is isomorphic to its dual iff φ has real Jordan normal form with unique block given by $\lambda = \frac{1}{2}$ or, with $b > 0$, by*

$$\begin{pmatrix} \frac{1}{2} & b \\ -b & \frac{1}{2} \end{pmatrix}$$

1.3. **Diagonal subdirect product and direct sum.** Consider a case I sextuplet (V, U_i, W_i) based on the endomorphism φ of A_2. The diagonal direct sum $2V$ is given as $V \times V,\ U_i \times U_i,\ W_i \times W_i)$

Let η the embedding

$$\eta : V \to V \times V,\ \eta(x) = (x, x)$$

Then $(\eta(V), \eta(U_i), \eta(W_i))$ is again a case I sextuplet isomorphic to (V, U_i, W_i).

Then involutive sublattices generated by the sextuplets $2V$ and $\eta(V)$, resp. are isomorphic. If V does not admit symmetric resp. symplectic forms then so do $\eta(v)$ and $2V$.

1.4. **Limits of representations.**

Claim 1.2. *In fixed d and k, consider for each n some $b_n > 0$ and system B_n of matrices describing an indecomposable case I sextuplet V_{b_n} based on*

$$\begin{pmatrix} \frac{1}{2} & b_n \\ -b_n & \frac{1}{2} \end{pmatrix}$$

Let $b_n \to b$. Then $B_n \to B$ where B describes such sextuplet base on $b > 0$ or, if $b = 0$ an $2B_0$, where B_0 is an indecomposable case I with eigenvalue $\frac{1}{2}$, given k and dimension $\frac{d}{2}$.

Hypothesis 1.3. *Assume that to each B_b we can associate a matrix A_b describing an isomorphism onto the dual and such that the entries of both the A_b and A_b^{-1} are bounded, uniformly for all $b > 0$.*

Then, for a free ultrafilter, the ultralimit A of the A_n exists and is invertible – thus describing an isomorphism of B onto its dual. Moreover, if the $A_n + A_n^t$ are nilpotent then so is $A + A^t$. Similarly for $A - A^t$.

1.5. **Symmetric versus symplectic.** Let k be even. We want to show (under the hypothesis) that the indecomposables V_b based on real Jordan form with $b > 0$ all admit symplectic forms.

Claim 1.4. *There is $c > 0$ such that no V_b with $0 < b < c$ admits a symmetric form*

Assume the contrary. Then we have $b_n \to 0$ such that V_{b_n} has $A_n - A_n^t$ nilpotent. Then so does the limit $2B_0$. Contradiction.

Consider

$$X_- = \{b \geq \frac{c}{2} \mid A_b + A_b \text{ nilpotent } \}$$
$$X_+ = \{b \geq \frac{c}{2} \mid A_b - A_b \text{ nilpotent } \}$$

Claim 1.5. $[\frac{c}{2}, \infty)$ *is the disjoint union of its subsets X_+ and X_-, both are closed.*

Now, X_- is nonempty, whence X_+ must be empty and $X_- = (\frac{c}{2}, \infty)$, that is, all V_b admit a symplectic form.

1.6. **Remark.** I do not know how daring the hypothesis is. Establishing this, if it is true, could be a bit easier than establishing the symplectic forms, directly – say for the case $k = 2$ (that is $p(\varphi)^2 0 = 0$) to which we might reduce as in the case of eigenvalue $\frac{1}{2}$.

Generalizing this approach to subfields seems to require a parametrization of irreducible polynomials $\prod_i (x - \lambda_i)(x - 1 + \lambda_i)$ over a connected set.