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# ON MODULAR LATTICES GENERATED BY TWO COMPLEMENTED PAIRS

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In [1; Problem 43] G. Birkhoff called for a description of the modular lattice FM(B) freely generated by four elements a, b, c, and d satisfying a+d = b+c = 1 and ad = bc = 0. Examples of subdirectly irreducible factors (as it turns out all of them) are provided in the paper [2] of A. Day, R. Wille et al. . There the modular lattice $FM(J_1^4)$ freely generated by elements $\bar{a},\bar{b},\bar{c},\bar{d}$ subject to the relations $\bar{a}\bar{b} = \bar{a}\bar{c} = \bar{a}\bar{d} = \bar{b}\bar{c} = \bar{b}\bar{d} = \bar{c}\bar{d} = 0$ and $\bar{a}+\bar{b} = \bar{a}+\bar{c} = \bar{a}+\bar{d} = \bar{b}+\bar{d} = \bar{c}+\bar{d} = 1$ has been determined and shown to be subdirectly irreducible. Moreover, defining recursively $e_o = 1$, $e_{n+1} = \bar{b}e_n + \bar{c}e_n$ if n+1 is odd, and $e_{n+1} = \bar{a}e_n + \bar{d}e_n$ if n+1 is even, they have finite subdirectly irreducible sections $S(n,4) = [e_n,1]$ generated by $\bar{a}+e_n$, $\bar{b}+e_n$, $\bar{c}+e_n$, and $\bar{d}+e_n$.

Subsequently, Sauer, Seibert, and Wille showed in [4] that the only subdirectly irreducible homomorphic images of FM(B) which, in addition, satisfy ab = ac = bd = cd = 0 are $FM(J_1^4)$ and the length two lattice $M_4$ with four atoms (an alternative proof shall be outlined in §4). From this and the lemma below our main result follows immediately.

THEOREM. *$M_4$, $FM(J_1^4)$ and its dual, and for $n < \infty$ the $S(n,4)$ are exactly the subdirectly irreducible homomorphic images of FM(B).*

Since in [2] the word problem for $FM(J_1^4)$ has been solved and it has been shown that $FM(J_1^4)$ is a homomorphic image of any 4-generated subdirect product of infinitely many S(n,4)'s, this establishes a solution of the word problem for any finitely presented homomorphic image of FM(B). An explicit description of FM(B) shall be given in §5.

To formulate the quoted Lemma (which is an essential tool in the analysis of the free modular lattice with four generators, too) we introduce polynomials $g_0(w,x,y,z) = w+x+y+z$ and $g_{n+1}(w,x,y,z) = (wg_n+xg_n)(yg_n+zg_n)$.

LEMMA. *Let M be a subdirectly irreducible modular lattice with generators* *a,b,c,d which is not isomorphic to $M_4$ nor to any of the S(n,4). Then either*

$$ab = ac = ad = bc = bd = cd = 0 = \prod_{n<\infty} g_n(a,b,c,d)g_n(a,c,b,d)g_n(a,d,b,c)$$

*or the dual statement holds.*

The proof of the lemma is based on the following two propositions the first of which is a variant of Proposition 7 in Wille [5].

PROPOSITION 1. *Let L and M be modular lattices, L finite and M subdirectly irreducible. Let $\gamma$ ($\sigma$) be a meet (join) homomorphism of L in M. Suppose that $\sigma x \leq \gamma x$ holds for all x in L and that M is generated by the union of the intervals $[\sigma x, \gamma x]$, x in L. Then either M is a homomorphic image of L or $\sigma p \leq \gamma q$ holds for all prime quotients p/q of L.*

PROPOSITION 2. *Let M be a modular lattice, u an element of M, S a lattice and $\alpha$ an order preserving map of S into M such that $u\alpha x + u\alpha y = u\alpha(x+y)$ and $(u+\alpha x)(u+\alpha y) = u + \alpha(xy)$ hold for all x,y in S. Moreover, let M be generated by the union of the sets $E_x = [u\alpha x, \alpha x] \cup [u\alpha x, u]$ $(x \in S)$. Then u is a neutral element of M.*

Before we prove the Propositions we shall outline the proof of the Lemma. Let $A_\infty$ be the lattice of fig. 1 a precise description of which is given in Section 1. Obviously, its interval $A_n = [m_n, 1]$ has S(n,4) as its only subdirectly irreducible image. We are going to apply Proposition 1 to the $A_n$. Given a modular lattice M and a map $\epsilon$ of $\{a_o, b_o, c_o, d_o\}$ into M we construct inductively a map $\gamma_M^\epsilon$ of $A_\infty$ into M such that $\gamma_M^\epsilon m_n = g_n(\epsilon a_o, \epsilon b_o, \epsilon c_o, \epsilon d_o)$. For any lattice L we denote the dual by L* and write x* = x for x in L. Let $\sigma_M^\epsilon$ be the map of $A_\infty^*$ in M defined dually to $\gamma_M^\epsilon$, i.e. $\sigma_M^\epsilon = \gamma_{M^*}^\epsilon$ holds with respect to the underlying sets. By induction we prove:

CLAIM 2. If M is subdirectly irreducible, generated by the image of $\epsilon$, and not a homomorphic image of any $A_k$ with $k \leq n$, then $\gamma_M^\epsilon | A_n$ ($\sigma_M^\epsilon | A_n^*$) is a meet (join) homomorphism of $A_n$ ($A_n^*$) into M and, for all $i,j \leq n$ with $i + j \leq n + 1$, it holds $\sigma_M^\epsilon m_i \leq \gamma_M^\epsilon m_j$.

To check, in the inductive step, $\sigma_M^\epsilon m_i \leq \gamma_M^\epsilon m_j$ for all $i,j \leq n + 1$ with $i + j \leq n + 2$ we apply Proposition 1 with $L = A_{n+1}$, $\gamma = \gamma_M^\epsilon | A_{n+1}$, and $\sigma = \sigma_M^\epsilon \omega_{n+1}$ where $\omega_{n+1}$ is the nontrivial dual automorphism of $A_{n+1}$ which maps $m_i$ onto $m_{n+1-i} = m_{n+1-i}^*$ and leaves the generators fixed. One should be aware of the fact that $\sigma_M^\epsilon m_i^* \leq \gamma_M^\epsilon m_j$ is not true in $A_{n+2}$ if $i + j > n + 2$.

Then, coming to the proof of the Lemma, we consider a subdirectly irreducible modular lattice M with generators a,b,c,d which is not a homomorphic image of any $A_n$. We define

$$u_n = g_n(a,b,c,d)g_n(a,c,b,d)g_n(a,d,b,c)$$

and take u to be the filter of M generated by the $u_n$. Then, we verify the hypotheses of Proposition 2 for $L = M_4$, u, and the sublattice M′ generated by a,b,c,d and u in the filter lattice of M. To do so, we make use of the downward continuity and the fact that the $\gamma_M^\epsilon | A_n$ are meet homomorphisms according to Claim 2. We conclude that u must be neutral in M′ and therefore equal $0_M$ or $1_M$. By duality, we get the same statement for the ideal v which is defined, dually. But, the validity of $u = 1_M$ and $v = 0_M$ would imply that M is isomorphic to $M_4$. Thus, we have $u = 0_M$ or $v = 1_M$ and this immediately yields the claim of the Lemma.

PROOF OF PROPOSITION 1. Suppose that M is not a homomorphic image of L. Let $\psi$ be the largest congruence on L which does not contain p/q and $\pi$ the canonical projection onto $L/\psi$. Then $L/\psi$ is subdirectly irreducible and not isomorphic to M. We define $\sigma',\gamma'$: $L/\psi \rightarrow L$ by mapping any congruence class on its smallest respectively greatest element. Then $\sigma\sigma'$ and $\gamma\gamma'$ are join and meet homomorphism of $L/\psi$ into M respectively, and because of $\sigma'\pi x \leqslant x \leqslant \gamma'\pi x$ it holds $\sigma\sigma' y \leqslant \gamma\gamma' y$ for all $y \in L/\psi$ and M is generated by the union of the $[\sigma\sigma' y, \gamma\gamma' y]$, y in L. Since $\pi p/\pi q$ is a prime quotient in $L/\psi$ we can apply Proposition 7 in [5] to get $\sigma\sigma'\pi p \leqslant \gamma\gamma'\pi q$. But by the definition of $\pi$ we have $\sigma'\pi p \nleqslant q$ and $\gamma'\pi q \ngeqslant p$, hence $p = q + \sigma'\pi p$ and $q = p\gamma'\pi q$. Now, $\sigma p = \sigma(q+\sigma'\pi p) = \sigma\sigma'\pi p + \sigma q \leqslant \gamma\gamma'\pi q$ and, finally, $\sigma p \leqslant \gamma p \gamma\gamma'\pi q = \gamma(p\gamma'\pi q) = \gamma q$ follow.

PROOF OF PROPOSITION 2. Let $M_x$ be the sublattice generated by $E_x$, i.e. the set of all a in M with $a = a\alpha x + au \geqslant u\alpha x$. Then for a in $M_x$ and $z \geqslant x$ it follows $a + u\alpha z \in M_z$. Therefore, we conclude for b in $M_y$, $a + b = a + u\alpha x + b + u\alpha y = a + b + u\alpha(x+y) \in M_{x+y}$. Hence the union of the $M_x$ ($x \in S$) is closed under joins and, dually, meets, i.e. a sublattice of M and equal to M. Thus, for any a and b in M there are x and y in S such that $a \in M_x$ and $b \in M_y$ and one calculates $u(a+b) = au + bu + u(a\alpha x + b\alpha y) \leqslant au + bu + u\alpha(x+y) = au + u\alpha x + bu + u\alpha y = au + bu$. This shows that u is neutral in M. Moreover $M_x$ is the interval $[u\alpha x, u+\alpha x]$ in M, since $u\alpha x \leqslant a \leqslant u + \alpha x$

implies $a\alpha x + au = a(\alpha x+u) = a$.

1. **The Pitzer-lattice.** Let $A_\infty$ be the sublattice of $FM(J_1^4) \times FM(J_1^4)$ which is generated by $a_o = (\bar{a},\bar{b})$, $b_o = (\bar{d},\bar{c})$, $c_o = (\bar{b},\bar{a})$ and $d_o = (\bar{c},\bar{d})$. Define recursively $m_o = l_o = r_o = 1$, $l_{n+1} = a_n + b_n$, $r_{n+1} = c_n + d_n$, $m_{n+1} = l_{n+1}r_{n+1}$, and $x_{n+1} = x_o m_{n+1}$ for $x = a,b,c,d$. Define $m_\infty = 0$.

As an easy consequence of the description of $FM(J_1^4)$ given in [2] we get the following claim.

CLAIM 1. $A_\infty$ equals the set of all elements $m_n (0 \leqslant n \leqslant \infty)$, $l_n$ and $r_n$ $(1 \leqslant n < \infty)$, and $a_i + m_n$, $b_i + m_n$, $c_i + m_n$, $d_i + m_n$ with $2 \leqslant n \leqslant \infty$ and $0 \leqslant i \leqslant n-2$. Moreover, this representation is unique and it holds for $x,y = a,b,c,d$

$$x_i + m_n \leqslant x_j + m_k \text{ if and only if } i \geqslant j \text{ and } n \geqslant k$$

in the case that $x = y$ or $\{x,y\} \in \{\{a,b\}, \{c,d\}\}$ takes place. In the remaining cases it holds

$$x_i + m_n \leqslant x_j + m_k \text{ if and only if } k \geqslant i.$$

All this results in the diagram of $A_\infty$ which is given by figure 1 (cf. [4]).

In fact, this diagram is crucial for the proof of our result and overcoming our pedantry we could have defined $A_\infty$ by this diagram, as well.

Now, let $A_n$ denote the section $[m_n,1]$ of $A_\infty$. We observe that $A_n$ is generated by $a_o + m_n$, $b_o + m_n$, $c_o + m_n$ and $d_o + m_n$ and thus a subdirect product of two copies of $S(n,4)$. Let $\omega_n$: $A_n \to A_n$ be defined by $\omega_n m_i = m_{n-i}$, $\omega_n l_i = l_{n+1-i}$, $\omega_n r_i = r_{n+1-i}$, and $\omega_n(x_j+m_i) = (x_o+m_{n-j})m_{n-i}$. By evidence, $\omega_n$ is a dual automorphism of $A_n$.

Finally, if M is a modular lattice generated by a,b,c,d and $\epsilon$ a map of $\{a_o,b_o,c_o,d_o\}$ onto $\{a,b,c,d\}$, then we define a map $\gamma = \gamma^\epsilon$ of $A_\infty$ into M recursively:

$$\gamma m_o = 1 = g_o(\epsilon a_o,\epsilon b_o,\epsilon c_o,\epsilon d_o),$$

$\gamma l_{n+1} = \epsilon a_o \gamma m_n + \epsilon b_o \gamma m_n$, $\qquad$ $\gamma r_{n+1} = \epsilon c_o \gamma m_n + \epsilon d_o \gamma m_n$,

for $x = a,b$, $\qquad$ for $x = c,d$,

$\gamma(m_{n+1} + x_o) = \epsilon x_o + \gamma l_{n+1}$, $\qquad$ $\gamma(m_{n+1} + x_o) = \epsilon x_o + \gamma r_{n+1}$,

for $1 \leqslant i \leqslant n-1$, $\gamma(m_{n+1} + x_i) = \gamma(m_{n+1} + x_o)\gamma(m_n + x_i)$

$$\gamma m_{n+1} = \gamma l_{n+1}\gamma r_{n+1} = g_{n+1}(\epsilon a_o,\epsilon b_o,\epsilon c_o,\epsilon d_o),$$

$\gamma x_k = \epsilon x_o \gamma m_n$ $\qquad$ for $x = a,b,c,d$,

$\gamma m_\infty = 0$

Let $\sigma^\epsilon : A^*_\infty \to M$ be defined dually.

**2. Exclusion of S(n,4) - Proof of Claim 2.** We proceed by induction on n. For $n = 1$ the hypotheses of Proposition 1 are satisfied trivially and we may conclude, in particular, $\sigma^\epsilon m_1^* = \epsilon a_o \epsilon b_o + \epsilon c_o \epsilon d_o \leq (\epsilon a_o + \epsilon b_o)(\epsilon c_o + \epsilon d_o) = \gamma^\epsilon m_1$. Thus in the inductive step $n \to n+1$ we may assume $\epsilon a_o \epsilon b_o + \epsilon c_o \epsilon d_o = \sigma^\epsilon m_1 \leq \sigma^\epsilon m_n \leq \gamma^\epsilon m_n$. First, we show that $\gamma = \gamma^\epsilon | A_{n+1}$ is a meet homomorphism. Evidently, $\gamma$ is order preserving. Moreover, we have $\epsilon a_o \gamma(b_o + m_{n+1}) = \epsilon a_o(\epsilon b_o + \gamma l_{n+1}) = \epsilon a_o \epsilon b_o + \epsilon a_o \gamma m_n \leq \gamma l_{n+1}$ and $\gamma l_{n+1} \leq \gamma(m_{n+1} + a_i)\gamma(m_{n+1} + b_i) \leq \gamma(m_{n+1} + a_o)\gamma(m_{n+1} + b_o) \leq \gamma l_{n+1}$. Now, for $x = a,b$, $y \geq m_n$ and $y(m_n + x_i) = m_n + x_j$ we conclude $\gamma(m_{n+1} + x_i)\gamma y = \gamma(m_{n+1} + x_o)\gamma(m_n + x_i)\gamma y = \gamma(m_{n+1} + x_o)\gamma(m_n + x_i)y = \gamma(m_{n+1} + x_o)\gamma(m_n + x_j) = \gamma(m_{n+1} + x_j) = \gamma y(m_{n+1} + x_i)$. This proves that $\gamma | [l_{n+1}, 1]$ and, symmetrically, $\gamma | [r_{n+1}, 1]$ preserve meets. Finally, for $x = a,b$, $y = c,d$ one gets $\gamma(m_{n+1} + x_i)\gamma(m_{n+1} + y_j) \leq \gamma(m_n + x_i)\gamma(m_n + y_j) = \gamma(m_n + x_i)(m_n + y_j) = \gamma m_n$ and, thus,

$$\gamma(m_{n+1} + x_i)\gamma(m_{n+1} + y_j) = \gamma(m_{n+1} + x_i)\gamma m_n \gamma(m_{n+1} + y_j) = \gamma l_{n+1} \gamma r_{n+1} = \gamma m_{n+1}.$$

This shows that $\gamma$ is a meet homomorphism. By duality, $\sigma = \sigma^\epsilon \circ \omega_{n+1} : A_{n+1} \to M$ is a join homomorphism. We observe that $\sigma x \leq \gamma x$ holds for all x in $A_{n+1}$. Namely, by definition we have $\gamma m_o = 1$, $\sigma m_{n+1} = 0$ and $\sigma(x_o + m_{n+1}) \leq \epsilon x_o \leq \gamma(x_o + m_{n+1})$ for $x = a,b,c,d$. Moreover, the inductive hypothesis states that $\sigma m_i = \sigma^\epsilon m^*_{n+1-i} \leq \gamma m_i$ for $1 \leq i \leq n$.

This implies

$$\sigma(x_o + m_{n+1})m_i \leq \sigma(x_o + m_{n+1})\sigma m_i \leq \gamma(x_o + m_{n+1})\gamma m_i = \gamma(x_o + m_{n+1})m_i$$

for $x = a,b,c,d$. Since any element u of $A_{n+1}$ can be written as join $u = v + w$ of two elements of this kind $\sigma u = \sigma v + \sigma w \leq \gamma v + \gamma w \leq \gamma u$ follows for any $u \in A_{n+1}$.

This and $\sigma(x_o + m_{n+1}) \leq \epsilon x_o \leq \gamma(x_o + m_{n+1})$ show that the hypotheses of Proposition 1 are satisfied.

To check the inductive claim we consider $i,j \leq n+1$ such that $i + j \leq n+2$. Then with $k = n+1-i$ and $h = n+2-i$ we have $k = h-1$, $h \geq j$, $m_k = l_h + r_h$ and the fact that $l_h/m_h$ and $r_h/m_h$ are prime quotients in $A_{n+1}$.

Thus, Proposition 1 yields

$$\sigma^\epsilon m_i^* = \sigma m_k = \sigma l_h + \sigma r_h \leq \gamma m_h \leq \gamma^\epsilon m_j.$$

3. **Exclusion of $M_4$ - Proof of the Lemma.** Let M be a subdirectly irreducible modular lattice with generators a,b,c,d which is not a homomorphic image of any $A_n$. Let $\Delta$ consist of the (three) maps $\epsilon$ of $\{a_o,b_o,c_o,d_o\}$ onto $\{a,b,c,d\}$ which map $a_o$ onto a and $b_o$ onto b,c, and d, respectively, and $\epsilon c_o$ lexicographically before $\epsilon d_o$. Define $u_n = \prod_{\epsilon\in\Delta} \gamma^\epsilon m_n = g_n(a,b,c,d)g_n(a,c,b,d)g_n(a,d,b,c)$. Let M be embedded into its filter lattice $F$(M) and $u = \prod_{n<\infty} u_n$. Let M′ be the lattice generated by a,b,c,d, and u. We observe that, for all $x \neq y$ in $\{a,b,c,d\}$,

(1) $x(y + u) \leqslant u$.

Namely, by Claim 2 we have

$$x(y + u_{n+1}) \leqslant (x + u_{n+1})(y + u_{n+1}) \leqslant \prod_{\epsilon\in\Delta} (x + \gamma^\epsilon m_{n+1})(y + \gamma^\epsilon m_{n+1})$$
$$\leqslant \prod_{\epsilon\in\Delta} \gamma^\epsilon(\epsilon^{-1}x + m_{n+1})\gamma^\epsilon(\epsilon^{-1}y + m_{n+1}) \leqslant \prod_{\epsilon\in\Delta} \gamma^\epsilon m_n = u_n.$$

By the (downward) continuity of $F$(M) we conclude

$$x(y + u) = \prod_{n<\infty} x(y + u_n) = \prod_{n<\infty} x(y + u_{n+1}) \leqslant \prod_{n<\infty} u_n = u.$$

By modularity it follows

(2) $(x + u)(y + u) = x(y + u) + u = u$,

i.e. that the sublattice generated by x,y, and u is distributive. Then, since $x + y \geqslant u_1 \geqslant u$ by definition, we get

(3) $xu + yu = (x + y)u = u$.

An application of Proposition 2 to M′ with $S = M_4$ (and atoms a,b,c,d), $\alpha 0 = 0$, $\alpha 1 = 1$, and $\alpha x = x$ for x = a,b,c,d yields that u is a neutral element of M′ and $x \mapsto (xu, x + u)$ an embedding of M′ into $[0_{M'},u] \times [u,1_{M'}]$. Since M was assumed to be subdirectly irreducible we have either $M \subset [u,1_{M'}]$ or $M \subset [0_{M'},u]$. In the first case by (2) it follows $u = xy$ for all $x \neq y$ in a,b,c,d and, in particular, $u = 0_M$.

In the second case we get $u = x + y = 1$ for all $x \neq y$ in $\{a,b,c,d\}$. Now, let v be the element corresponding to u in the dual construction. By the principle of duality we have either $v = x + y = 1$ for all $x \neq y$ in $\{a,b,c,d\}$ or $v = xy = 0$ for all $x \neq y$ in $\{a,b,c,d\}$. But the latter would imply $M \cong M_4$, contradicting the hypothesis.

4. **Marginalia.** A short proof of the quoted result of Sauer, Seibert, and Wille is contained in the following two claims.

CLAIM 3. Let M be a subdirectly irreducible modular lattice generated by elements a,b,c,d such that $ab = ac = ad = bc = bd = cd = 0$ and $a + d = b + c = 1$. Then either $a + b = c + d = 1$ or $a + c = b + d = 1$.

PROOF. With $\alpha,\beta \in \Delta$ such that $\alpha b_o = b$ and $\beta b_o = c$, an easy induction yields

(1) $a\gamma^\alpha m_k\gamma^\beta m_n + d\gamma^\alpha m_k\gamma^\beta m_n = b\gamma^\alpha m_k\gamma^\beta m_n + c\gamma^\alpha m_k\gamma^\beta m_n = \gamma^\alpha m_k\gamma^\beta m_n$

(2) $\gamma^\alpha l_k + \gamma^\beta l_n = \gamma^\alpha r_k + \gamma^\beta r_n = \gamma^\alpha m_{k-1}\gamma^\beta m_{n-1}$

(3) $\gamma^\alpha l_k\gamma^\beta l_n + \gamma^\alpha r_k\gamma^\beta r_n \geqslant \gamma^\alpha m_{k-1}\gamma^\beta m_{n-1}$

(4) $\gamma^\alpha m_k + \gamma^\beta m_n = 1$

(5) $(x + \gamma^\alpha m_k)\gamma^\beta m_n \leqslant x + \gamma^\alpha m_{k-1}\gamma^\beta m_{n-1}$ for $x = a,b,c,d$.

Now, let M be embedded into its filter lattice $F$(M) and $u = \prod_{n<\infty}\gamma^\alpha m_n$, $v = \prod_{n<\infty}\gamma^\beta m_n$, and M′ the sublattice generated by a,b,c,d,u, and v. By (4) it follows that $u + v = 1$, by the Lemma, $uv = 0_{M'}$ and by (5), $x = (x + u)(x + v)$ for $x = a,b,c,d$. Thus, M is embedded into $[u,1]_{M'} \times [v,1]_{M'}$ and either $u = 1$ or $v = 1$.

CLAIM 4. (Pitzer, unpublished) If the hypotheses of Claim 3 and, in addition, $a + c = b + d = 1$ are satisfied then $\gamma^\alpha$ is a homomorphism of $A_\infty$ onto M.

The proof proceeds straightforwardly first showing by induction that the $\gamma^\alpha|A_n$ are join-homomorphisms, too. The details of both proofs are left to the reader.

We see that we did not use the fact that $FM(J_1^4)$ is freely generated. We only need to know that it is a subdirectly irreducible modular lattice and besides $M_4$ the only homomorphic image of $A_\infty$. But this is obvious by the diagrams of both lattices.

Finally, we remark that the lattices listed in the Theorem are exactly the subdirectly irreducible modular 2-distributive (i.e. satisfying $w(x + y + z) = w(x + y) + w(x + z) + w(y + z)$) lattices - M - with four generators - - a,b,c,d.

The basic tool in the easy proof is the observation that an element of a modular 2-distributive lattice is neutral if it is neutral with respect to a set of generators. If M is not isomorphic to $M_4$ or any of the S(n,4) then, by the Lemma, we may assume that $ab = ac = ad = bc = bd = cd = 0 = \prod_{n<\infty} u_n$. By arguments similar to those used in the proof of the Lemma we get $\prod_{n<\infty}\gamma^\epsilon m_n \in \{0_M, 1_M\}$ for all $\epsilon \in \Delta$. Because of the Theorem we are left to consider the case that $\prod_{n<\infty}\gamma^\epsilon m_n = 0$ for all $\epsilon \in \Delta$. Similarily, we get that for any triple $(n_\epsilon|\epsilon \in \Delta)$, $\sum_{\epsilon\in\Delta}\gamma^\epsilon m_{n_\epsilon}$ equals 1 or that there is $(n_\epsilon|\epsilon \in \Delta)$ with $\sum_{\epsilon\in\Delta}\gamma^\epsilon m_{n_\epsilon} = 0$. In the first case we conclude $0 = 1$ by the continuity of $F$(M). To treat the second case, one easily shows by induction applied to the sublattice generated by a(c + d), b(c + d), c(a + b) and d(a + b), that any modular 2-distributive lattice satisfying $ab = ac = ad = bc = bd = cd = 0 = \sum_{\epsilon\in\Delta} \gamma^{\epsilon} m_{n_\epsilon}$ is a subdirect product of S(1,4) and S(2,4) -- using the $D_2$- and $M_3$-Lemma of Wille [5], evidently.

This generalizes the corresponding result of R. Freese [3] on breadth two modular lattices.

**5. The structure of FM(B).** According to the Theorem and the remarks following it, FM(B) is a subdirect product of the S(n,4). Now, we consider the lattices $A_n$ and their generators $a'_n = a_n + m_n$, $b'_n = b_n + m_n$, $c'_n = c_n + m_n$, $d'_n = d_n + m_n$. In $A_n^2$ we define $\bar{a}_n = (a'_n,a'_n)$, $\bar{b}_n = (b'_n,c'_n)$, $\bar{c}_n = (c'_n,b'_n)$, and $\bar{d}_n = (d'_n,d'_n)$. Obviously, $\bar{a}_n + \bar{d}_n = \bar{b}_n + \bar{c}_n = 1$ and $\bar{a}_n\bar{d}_n = \bar{b}_n\bar{c}_n = (m_n,m_n) = 0_{A_n^2}$. Because of $g_n(\bar{a}_n,\bar{b}_n,\bar{c}_n,\bar{d}_n) = (0_{A_n^2},1)$ and $g_n(\bar{a}_n,\bar{c}_n,\bar{b}_n,\bar{d}_n) = (1,0_{A_n^2})$ these elements generate $A_n^2$. Thus there is a homomorphism $\varphi$ of FM(B) onto $A_n^2$ mapping x onto $\bar{x}_n$ for x = a,b,c,d. One easily checks that any homomorphism of FM(B) onto S(n,4) (there are four of them) factorizes through $\varphi$. Therefore, FM(B) is the subdirect product of the $A_n^2$ with generators $a = (\bar{a}_n | n < \infty)$, $b = (\bar{b}_n | n < \infty)$, $c = (\bar{c}_n | n < \infty)$, and $d = (\bar{d}_n | n < \infty)$.

To give a more precise description of FM(B) we observe that for any n and k with $n \leqslant k$ there are two embeddings $\gamma_{nk}$ and $\pi_{nk}$ of $A_n$ into $A_k$ with image $[m_k,m_{k-n}]$ and $[m_n,1]$ respectively, and $\gamma_{nk}x \leqslant \pi_{nk}x$ for all $x \in A_n$. Let $\gamma_{nk}^2$ and $\pi_{nk}^2$ denote the square maps of $A_n^2$ in $A_k^2$ and, for $x \in \prod_{n<\infty} A_n^2$, $x_n$ the n-th component of x. Then

$$L: =\{x | x \in \prod_{n<\infty} A_n^2 \text{ and, for all } n \leqslant k,\ \gamma_{nk}^2 x_n \leqslant x_k \leqslant \pi_{nk}^2 x_n\}$$

is a sublattice of $\prod_{n\lesssim\infty} A_n^2$ containing E = { a,b,c,d,0,1} and, therefore, FM(B). We write $\tilde{a} = d$, $\tilde{d} = a$, $\tilde{b} = c$, $\tilde{c} = b$, $\tilde{0} = 1$, and $\tilde{1} = 0$.

CLAIM 5. FM(B) consists exactly of those elements w of L for which there is $y \in E$, and even n and $x_n$ and $z_n$ in $A_n^2$ with $x_n \leqslant (m_{n/2},m_{n/2}) \leqslant z_n$, $x_n \leqslant \tilde{y}_n \leqslant z_n$, such that $w_k = \gamma_{nk}^2 x_n + y_k \pi_{nk}^2 z_n$ holds for all $k \geqslant n$. Choosing n minimal, this representation becomes unique.

Replacing "square" by "cube" everywhere, the subdirect product over all S(n,4) with four generators which split arbitrarily into two complemented pairs can be determined in the same way. According to §4 this is just the 2-distributive lattice

$FD_2(B')$ freely generated by four elements subject to certain relations which exclude those factors S(1,4) and S(2,4) the generators of which do not partition into two complemented pairs.

To prove the claim, we show firstly, that the set of all elements of L which have a representation of the required type form a sublattice of L. Since the representation is equivalent to its dual, we have to consider joins only. Thus, let $w'$ and $w''$ be given. Obviously, w has a representation with y = 1 if and only if there is an r such that $w_k \geq (m_r,m_r)$ holds for all k. For $w = w' + w''$ this is the case if $y' = 1$ (or $y'' = 1$) or if $y',y'',0$, and 1 are all different. Namely $w_k \geq z'_k \geq (m_{n'},m_{n'})$ and - - with $r = \max(n',n'')$ - $w_k \geq y'(m_r,m_r) + y''(m_r,m_r) \geq (m_{r+1},m_{r+1})$ hold for all k in the first and second case respectively. If $y' = y''$ is different from 0 and 1 then $y = y'$, $n = \max(n',n'')$, $x_n = x'_n + x''_n$, and $z_n = z'_n + y_n z''_n$ yield a representation for w. The case $y' = y'' = 0$ is trivial. Thus, finally, we consider $y'' = 0$ and $y' \neq 0,1$. We choose $y = y'$, $n = 2\max(n',n'')$, $z_n = z''_n$, and $x_n \leq \tilde{y}_n$ such that there is $u \leq y_n(m_{n/2},m_{n/2})$ with $x_n + u = x'_n + x''_n$. A look at Figure 1 shows that this can be done. This proves that all elements of FM(B) have a representation of the required kind since the generators do so, trivially.

To prove the uniqueness we have (after a second look at the diagram of $A_n$) to check the uniqueness of y, only. But y has to be 0 (1) if there is an n such that $w_k = \pi^2_{nk} w_n$ $(\gamma^2_{nk} w_n)$ holds for all $k \geq n$ and the unique element in $E - \{0,1\}$ for which there is an n such that the length of $[w_k y_k, y_k]$ is less than n for all k, otherwise.

We are left to show that any element of L which has a representation belongs to FM(B). This is easy once it is done for all elements w with $y \in \{0,1\}$. Namely, given w with $y \neq 0,1$ we observe that $w_n y_n + \tilde{y}_n = y_n z_n + \tilde{y}_n = z_n$ and define $z'_k = w_k y_k + \tilde{y}_k$ for $k \leq n$, $z'_k = \pi_{nk} z_n$ for $k \geq n$, $x'_k = w_k$ for $k \leq n$, and $x'_k = \gamma_{nk} w_n$ for $k \geq n$. Then $z'$ and $x'$ are in L, obviously, and belong to FM(B) by our assumption. Thus, $w = x' + yz'$ belongs to FM(B), too.

As announced, we are going to show that for given N the sublattice of L which consists of all w with $w_k \geq (m_N,m_N)$ for all k is contained in FM(B). Using the polynomials $g_n$ as above, we see that this is just the interval $[g_N(a,b,c,d)g_N(a,c,b,d),1]$ and the direct product of $[g_N(a,b,c,d),1]$ and $[g_N(a,c,b,d),1]$. Therefore, we will be done after having showed that e.g. the first factor is generated by the $x^N = x + g_N(a,b,c,d)$ with $x \in E$. This is done by induction on N. Let $N \geqslant 1$ and $w \geqslant g_N(a,b,c,d)$ be given. We observe that the second component of any component of w equals 1, always. Let $g_n^*$ denote the polynomial dual to $g_n$. Since any element of $A_n$ is the meet of some y and z with $y \geqslant x_o + m_n$ and $z \geqslant \tilde{x}_o + m_n$ for suitable $x \in \{a,b,c,d\}$, there are x and $i,j \leqslant N$ such that, with $u = x^N + g_i^*(a^N,b^N,c^N,d^N)$ and $v = \tilde{x}^N + g_j^*(a^N,b^N,c^N,d^N)$, $w_N = u_N v_N$. Realizing that, for any $k \leqslant N$, $u_{k-1}$ is the join of all elements. covering $u_k$ (there is at most one!) and (by the definition of L) $w_{k-1}$ is a join of elements covering $w_k$, we get $u_k \geqslant w_k$ and $v_k \geqslant w_k$ for all $k \leqslant N$. Define z by $z_k = w_k$ for $k \leqslant N - 1$ and $z_k = w_{N-1}$ for $k \geqslant N - 1$. Then $z \geqslant g_{N-1}(a,b,c,d)$ holds and z belongs to the sublattice generated by the $x^N$ because of the inductive hypothesis and the fact that $x^{N-1} = x^N + g_{N-1}(a^N,b^N,c^N,d^N)$. Then so does $w = uvz$.

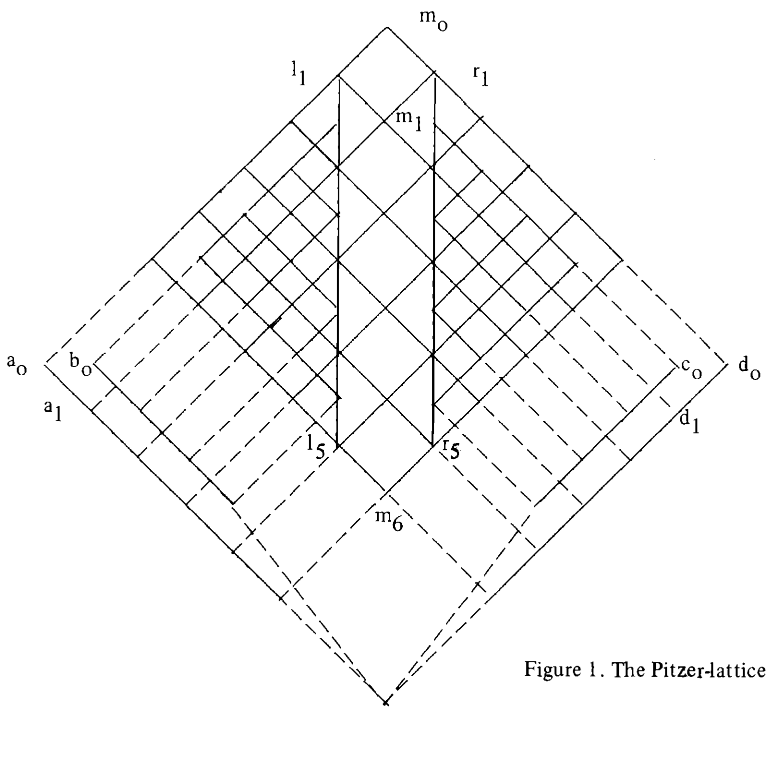


Figure 1. The Pitzer-lattice $A_\infty$

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