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Definable relations in finite-dimensional subspace lattices with involution

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In memory of Bjarni Jónsson.

Abstract. For a large class of finite dimensional inner product spaces V, over division $*$-rings F, we consider definable relations on the subspace lattice $\mathsf{L}(V)$ of V, endowed with the operation of taking orthogonals. In particular, we establish translations between the relevant first order languages, in order to associate these relations with definable and invariant relations on F—focussing on the quantification type of defining formulas. As an intermediate structure we consider the $*$-ring $\mathsf{R}(V)$ of endomorphisms of V, thereby identifying $\mathsf{L}(V)$ with the lattice of right ideals of $\mathsf{R}(V)$, with the induced involution. As an application, model completeness of F is shown to imply that of $\mathsf{R}(V)$ and $\mathsf{L}(V)$.

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1. Introduction

Translating geometric concepts into algebraic ones is a well established method since Descartes. For Projective Geometry, Whiteley [11,12] has discussed this in the context of first order logic. In particular, "geometric configurations" appear as sets of systems of subspaces, definable in terms of the subspace lattice. Well known examples are the centrally and the axially perspective

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configurations related to the Arguesian law of Bjarni Jónsson (cf. [3]) as well as harmonic quadruples and the more powerful quadrangular sextuples of Jónsson [8].

The Logic point of view also governed the study [6] of complexity of satisfiability problems in ortholattices of subspaces of finite dimensional real and complex Hilbert spaces (the structures of finite dimensional Geometric Quantum Logic): via interpretation of polynomial feasibility problems in one direction, by non-deterministic Linear Algebra algorithms in the other.

The present note takes up ideas from both approaches considering definable relations on lattices $\mathsf{L}(V)$ (with the induced involution) of subspaces of (inner product) spaces V over division ($*$-) rings F and their coordinate-wise description. Notice that F is in the signature of rings with involution but no operation symbol to capture multiplicative inverse. Of course, translating from F, inversion can be dealt with as a defined operation.

A structure well suited to relate $\mathsf{L}(V)$ with the coordinate domain F is the ($*$-) ring $\mathsf{R}(V)$ of endomorphisms of V: its lattice (with involution) of right ideals is canonically isomorphic to $\mathsf{L}(V)$. While the translation from $\mathsf{L}(V)$ to $\mathsf{R}(V)$ is obvious, the converse relies on von Neumann's [10] Coordinatization Theory (requiring dimension ≥ 3), based on the concept of frames, refined to capture relations. Passing from $\mathsf{R}(V)$ to F, sets Φ of n-tuples of endomorphisms are associated with sets K of n-tuples of matrices over F via a fixed basis, but depending only on the isometry type (up to scaling) of that basis. The corresponding invariance property follows from definability on the side of $\mathsf{R}(V)$, it has to be required on the side of F (that is, on the side of matrices). The reverse translation from F into $\mathsf{R}(V)$ proceeds via existential or universal quantification over systems of matrix units: the ring analogues of frames. Accordingly, translations come in pairs, one intended to preserve quantification types Σ_k, the other Π_k.

The crucial observation is that, for any orthogonal basis, one can express (in terms of the associated frame or matrix units) the quotients of the diagonal entries of the matrix describing the form – this generalizes the concept of "orthonormal frame" from [6]. Definability requires that for some basis these quotients are defined in F via first order formulas. Moreover to grant, for any such system of quotients, automorphisms of $\mathsf{L}(V)$ matching the associated frames, we require that F is commutative or that vectors can be normalized (e. g. in quaternionic Hilbert spaces). The intended preservation of quantification type becomes possible if the formulas defining the form are in Σ_1 respectively Π_1. Our main Theorems 6.2 and 10.4 translate thus definable n-ary relations in one structure to relations of appropriate arity in another while preserving their descriptive complexity.

The translations allow to deduce 'model completeness' of $\mathsf{L}(V)$ and $\mathsf{R}(V)$ (i. e. quantifier elimination up to one existential block) from that of F; cf. Proposition 12.2. They also allow to relate decidability and axiomatizability of $\mathsf{L}(V)$ and $\mathsf{R}(V)$ with that of F; see Corollaries 10.5 and 10.7.

A particular feature of the anisotropic case is that any quantifier-free formula is, within $\mathsf{L}(V)$, equivalent to a positive primitive one (Corollary 12.1);

this extends to $\mathsf{R}(V)$ with the additional operation of Moore-Penrose-Rickart pseudo-inversion (Theorem 12.3).

All translations in this note are effective; it is not hard to see that, in the purely relational setting, these translations can be carried out in polynomial time – but a detailed discussion shall be postponed to subsequent work.

2. Algebraic preliminaries

Statements presented as "Fact" are well known or obvious; proofs will be omitted or sketched. In the sequel, let F be a division ring with involution $r \mapsto r^*$ and V a right F-vector space of dimension $\dim V = d < \infty$ and endowed with a non-alternate non-degenerate $*$-hermitean form $\langle .|. \rangle$, that is: additive in both arguments and

$$\langle vr|ws\rangle = r^*\langle v|w\rangle s, \quad \langle w|v\rangle = \langle v|w\rangle^*$$

as well as $\langle v|v\rangle \neq 0$ for some v, and $\langle w|v\rangle = 0$ for all $w \in V$ only if $v = 0$ cf. [5, Chapter I]. We write $|v| = \langle v|v\rangle$ (without taking square roots which may not exist in F). A basis $\bar{v} = (v_1, \dots, v_d)$ is *orthogonal* if $\langle v_i|v_j\rangle = 0$ for $i \neq j$; we will speak of a $\perp$-*basis*. Recall that such always exist [5, II §2 Corollary 1]: any $v_1 \neq 0$ can be completed to a $\perp$-basis; and one has for such

$$\Big\langle \sum_i v_i r_i \Big| \sum_j v_j s_j \Big\rangle = \sum_k r_k^* |v_k| s_k$$

with $|v_k| = |v_k|^* \neq 0$; we write $|\bar{v}| = (1, |v_1|^{-1}|v_2|, \dots, |v_1|^{-1}|v_d|)$. Observe that $|\bar{v}|$ determines the isometry type of $\bar{v}$ up to a scaling.

Up to (isometric) isomorphism the spaces V are the F^d with the form

$$\langle \bar{r}|\bar{s}\rangle = \sum_k r_k^* \delta_k s_k$$

where $\delta_i = \delta_i^* \neq 0$. Moreover, for any endomorphism f of V there is a unique endomorphism f^*, the *adjoint* of f, such that $\langle fv|w\rangle = \langle v|f^*w\rangle$ for all v, w. Indeed, if $fv_j = \sum_k v_k a_{kj}$ then $b_{\ell i}$ with $f^*v_i = \sum_\ell v_\ell b_{\ell i}$ are determined from

$$a_{ij}^*|v_i| = \langle fv_j|v_i\rangle = \langle v_j|f^*v_i\rangle = |v_j|b_{ji}.$$

We say that f is *orthogonal* if $f^* = f^{-1}$, equivalently, if f is bijective and $\langle fv|fw\rangle = \langle v|w\rangle$ for all $v, w \in V$.

A $*$-*ring* is a ring with unit 1 and with an *involution* $r \mapsto r^*$ that is an anti-automorphism of order 2; 1 and 0 are considered as constants, $+, \cdot, -, {}^*$ as fundamental operations.

The endomorphism ring of V is also a $*$-ring $\mathsf{R} = \mathsf{R}(V)$ with involution $f \mapsto f^*$. It is *von Neumann regular*: for any a there is b such that $aba = a$ (cf. [10]). Any right ideal I of R is principal and, if $I = a\mathsf{R}$, then $I = ab\mathsf{R}$ with idempotent ab for any b such that $aba = a$. Observe that for any idempotent $e \in \mathsf{R}$ one has

$$a\mathsf{R} \subseteq e\mathsf{R} \Leftrightarrow ea = a.$$

Associating with I the subspace $U = \bigcup\{\mathsf{im}\, f \mid f \in I\}$ of V, one can choose e as the projection onto U associated with any direct decomposition $V = U \oplus W$; thus $U = \mathsf{im}\, e$.

We will consider lattices L with bounds $\mathbf{0}$ and $\mathbf{1}$, joins being written as $a+b$ and meets as $a \cap b$, with meets having preference over joins in notation. We write $a \oplus b = c$ if $a+b = c$ and $a \cap b = 0$; and $a \oplus^{\perp} b = c$ if, in addition, $a \leq b^{\perp}$. An *involution* on L is a map $a \mapsto a^{\perp}$ such that $a^{\perp\perp} = a$ and $a \leq b$ if and only if $b^{\perp} \leq a^{\perp}$. Considering such as an additional operation, L becomes an *involutive lattice*, IL for short. It is an *ortholattice* if $a \cap a^{\perp} = 0$. The lattice of linear subspaces of V with involution $U \mapsto U^{\perp} = \{v \in V \mid \forall u \in U.\ \langle v|u\rangle = 0\}$ forms an IL $\mathsf{L} = \mathsf{L}(V)$ which is an ortholattice if the form on V is anisotropic. Moreover, $\mathsf{L}(V) = \mathsf{L}(V')$ if V' arises from V by scaling.

The right ideals of R form an IL with join $I + J$, meet $I \cap J$, bounds $\mathbf{0} = \{0\} \subset \mathsf{R}$ and $\mathbf{1} := \mathsf{R}$, and involution

$$(e\mathsf{R})^{\perp} = (1 - e^{*})\mathsf{R} \text{ for idempotent } e.$$

This lattice is modular and a complement of $e\mathsf{R}$ is given by $(1-e)\mathsf{R}$. Moreover, an isomorphism onto $\mathsf{L}(V)$ is given by $\Omega(f\mathsf{R}) = \mathsf{im}\, f$. We will identify these two structures under this isomorphism.

3. Preliminaries from logic

We consider first order languages with countably many variables, equality, and finitely many operation symbols, but no relation symbols. Finite strings of variables or elements are written e. g. as $\bar{x}$ and $\bar{a}$, the length being given by context. We also use matrices $X = (x_{ij})_{ij}$ of variables in an obvious way. Given a structure A and a formula $\varphi(\bar{x}, \bar{z})$ in the first order language Λ built on the signature of A and constants $\bar{a}$ from A let $\mathrm{Mod}_A(\varphi(\bar{x}, \bar{a})) = \{\bar{u} \in A^n \mid A \models \varphi(\bar{u}, \bar{a})\}$. $\mathrm{Th}(A)$ denotes the first order theory of A. A subset M of A^n is *definable* if $M = \mathrm{Mod}_A(\varphi(\bar{x}))$ for some formula $\varphi(\bar{x}) \in \Lambda$. The following tool will later be applied to pass from definitions involving coordinates to defining formulas in lattices Λ_R and Λ_L. The formula ρ is meant to capture coordinate systems.

Fact 3.1. *Let G a subgroup of the automorphism group of A and $\rho(\bar{x})$ and $\varphi(\bar{x}, \bar{z})$ formulas in Λ such that*

(i) *There is $\bar{a}$ in A such that $A \models \rho(\bar{a})$.*
(ii) *For all $\bar{a}$ in A with $A \models \rho(\bar{a})$ one has $\mathrm{Mod}_A(\varphi(\bar{x}, \bar{a}))$ closed under the component-wise action of G on A^n.*
(ii) *For all $\bar{a}, \bar{b}$ in A, if $A \models \rho(\bar{a})$ and $A \models \rho(\bar{b})$ then there is $\omega \in G$ such that $\omega\bar{a} = \bar{b}$.*

Then for all $\bar{a}, \bar{b}$ in A such that $A \models \rho(\bar{a})$ and $A \models \rho(\bar{b})$

$$\begin{aligned}\mathrm{Mod}_A(\varphi(\bar{x}, \bar{a})) = \mathrm{Mod}_A(\varphi(\bar{x}, \bar{b})) &= \mathrm{Mod}_A(\forall \bar{z}.\rho(\bar{z}) \Rightarrow \varphi(\bar{x}, \bar{z})) \\ &= \mathrm{Mod}_A(\exists \bar{z}.\rho(\bar{z}) \wedge \varphi(\bar{x}, \bar{z})).\end{aligned}$$

Proof. Assume that $A \models \rho(\bar{a}), \rho(\bar{b})$ and consider any $\bar{u} \in \mathrm{Mod}_A(\varphi(\bar{x}, \bar{a}))$. By (iii) there is $\omega \in G$ such that $\omega\bar{a} = \bar{b}$. By (ii), $A \models \varphi(\omega^{-1}\bar{u}, \bar{a})$. Applying ω, it follows $A \models \varphi(\bar{u}, \bar{b})$. In view of (i) this proves that there is $M \subseteq A^n$ such that $M = \mathrm{Mod}_A(\varphi(\bar{x}, \bar{a}))$ for all $\bar{a}$ with $A \models \rho(\bar{a})$. Also, $\subseteq$ in the second equality follows while $\subseteq$ in the third is obvious. Finally, to show $\bar{u} \in M$ for $\bar{u} \in \mathrm{Mod}_A(\exists\bar{z}.\rho(\bar{z}) \wedge \varphi(\bar{x}, \bar{z}))$ choose witnessing $\bar{a}$. □

The *quantification type* Σ_k or Π_k of a formula $\varphi(\bar{x})$ is considered up to logical equivalence and cumulative; in particular, $\exists\bar{y}\varphi$ is in Σ_1 if φ is quantifier-free; and *positive primitive* if, in addition, φ is a conjunction of equations. A *positive* Σ_1-formula is a disjunction of positive primitives. A *basic equation* or *basic atomic formula* is of the form $y = x$ or $y = f(\bar{x})$ where f is an operation symbol.

Fact 3.2. *To every quantifier-free formula $\varphi(\bar{x})$ there is a conjunction of basic equations, $\varphi'(\bar{x}, \bar{z}, \bar{y})$, with new variables $\bar{z}, \bar{y}$ and a boolean combination $\varphi''(\bar{y})$ of equations between variables from $\bar{y}$ such that $\varphi(\bar{x})$ is logically equivalent to both*

$$\varphi^{\exists}(\bar{x}, \bar{z}, \bar{y}) \equiv \exists\bar{z}\bar{y}.\, \varphi'(\bar{x}, \bar{z}, \bar{y}) \wedge \varphi''(\bar{y})$$
$$\varphi^{\forall}(\bar{x}, \bar{z}, \bar{y}) \equiv \forall\bar{z}\bar{y}.\, \varphi'(\bar{x}, \bar{z}, \bar{y}) \Rightarrow \varphi''(\bar{y}).$$

In particular, any positive Σ_1-formula is equivalent to one where the atomic formulas are basic equations.

Moreover, φ' is such that satisfying values for $\bar{z}, \bar{y}$ are given and uniquely determined by those for $\bar{x}$.

Proof. With each term $t(\bar{x})$ one can associate new variables y_t and $\bar{z}_t$ and a conjunction $\varphi_t(\bar{x}, \bar{z}_t, y_t)$ of basic equations such that $t(\bar{x}) = y_t$ is equivalent to $\exists\bar{z}_t.\, \varphi_t(\bar{x}, \bar{z}_t, y_t)$ as well as to $\forall\bar{z}_t.\, \varphi_t(\bar{x}, \bar{z}_t, y_t)$ (the $\bar{z}_t$ capture the intermediate values when evaluating the term $t(\bar{x})$). Thus, $t_1(\bar{x}) = t_2(\bar{x})$ is equivalent to both

$$\exists\bar{z}_{t_1}\bar{z}_{t_2}.\, \varphi_{t_1}(\bar{x}, \bar{z}_{t_1}, y_{t_1}) \wedge \varphi_{t_2}(\bar{x}, \bar{z}_{t_2}, y_{t_2}) \wedge y_{t_1} = y_{t_2}$$
$$\forall\bar{z}_{t_1}\bar{z}_{t_2}.\, \varphi_{t_1}(\bar{x}, \bar{z}_{t_1}, y_{t_1}) \wedge \varphi_{t_2}(\bar{x}, \bar{z}_{t_2}, y_{t_2}) \Rightarrow y_{t_1} = y_{t_2}$$

Now, given quantifier-free $\varphi(\bar{x})$ replace any equation $t_1(\bar{x}) = t_2(\bar{x})$, occurring in $\varphi(\bar{x})$ by $y_{t_1} = y_{t_2}$ to obtain φ'' in variables y_t. Also, let $\varphi'(\bar{x}, \bar{z}, \bar{y})$ the conjunction of all $\varphi_t(\bar{x}, \bar{z}_t, y_t)$ where $t(\bar{x})$ occurs in $\varphi(\bar{x})$ (as side of an equation) and where $\bar{z}$ comprises all $\bar{z}_t$ and $\bar{y}$ all y_t. □

In the following definition k ranges over all integers $k > 0$.

- We say that a map τ from one language to another *preserves type* if $\tau(\varphi)$ is (i) in Σ_k whenever φ is; (ii) in Π_k whenever φ is; and (iii) positive primitive whenever φ is.
- We say that τ *shifts type* (modulo a formula α) if the following hold: (i) If φ is in Σ_k (and $\alpha \in \Sigma_1$), then so is $\tau(\varphi)$ in case k is odd; otherwise in Σ_{k+1}. (ii) If φ is in Π_k (and $\alpha \in \Pi_1$), then so is $\tau(\varphi)$ in case k is even; otherwise in Π_{k+1}. (iii) If φ is positive primitive, then so is $\tau(\varphi)$.

- A pair $\tau^\exists, \tau^\forall$ *preserves type* (modulo a formula α) if $\tau^\exists(\varphi)$ is (i) of type Σ_k whenever φ is (and $\alpha \in \Sigma_1$); (ii) of type Π_k whenever φ is (and $\alpha \in \Sigma_1$); and (iii) positive primitive whenever φ is.

We consider also bounded quantifications

$$\exists \bar{y}.\ \rho(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{y}) \quad \text{and} \quad \forall \bar{y}.\ \rho(\bar{x}, \bar{y}) \Rightarrow \psi(\bar{x}, \bar{y})$$

where $\rho(\bar{x}, \bar{y})$ is a conjunction of equations and no variable from $\bar{y}$ occurs bounded in ψ. The *bounded quantification type* Σ_k^b resp. Π_k^b of a formula with prenex bounded quantifiers is defined in the obvious manner.

Fact 3.3. *Any formula of type Σ_k^b (Π_k^b) is logically equivalent to a formula of type Σ_k (Π_k).*

Qx will denote a quantification where Q stands for $\exists$ or $\forall$; also, a block of quantifications is denoted in the form $\bar{Q}\,\bar{x}$. Such will give rise to bounded quantifications $\bar{Q}^\tau \bar{x}$ where for each x_k a binding formula has to be specified.

We say that a structure A is *model complete* (cf. Exercise 3.4.12d in [9]) if for any formula $\varphi(\bar{x})$ there is $\varphi'(\bar{x})$ in Σ_1 equivalent to $\varphi(\bar{x})$ within A. By the Tarski–Seidenberg Theorem, this applies to any real-closed field (in the signature of rings resp. $*$-rings with identity involution), e. g. the field $\mathbb{R}$ of real numbers. It also applies to $*$-fields $F = F^*(i)$ where $F^* = \{r \in F \mid r = r^*\}$ is real closed, $i \notin F^*$, $i^2 = -1$ and $(a + bi)^* = a - bi$ for $a, b \in F^*$; e. g. the $*$-field $\mathbb{C}$ of complex numbers with conjugation as involution. This provides axiomatizations of $\mathrm{Th}(\mathbb{R})$ and $\mathrm{Th}(\mathbb{C})$, given by finitely many schemes. However, neither theory is finitely axiomatizable.

Observation 3.4. If F is a $*$-subfield of $\mathbb{C}$ then any real algebraic number contained in F is definable in F.

4. Overview

In the sequel, F will be a division $*$-ring, V a vector space of $\dim V = d$ over F and endowed with a $*$-hermitean form as above, R its $*$-ring of endomorphisms, and L the involutive lattice of (principal) right ideals of R. From Section 8 to 10 we require that one of the following holds.

(A) F is commutative.

(B) V allows *normalization*: for any $v \neq 0$ there is $r \in F$ such that $|vr| = 1$.

In case (B), speaking about a *basis* we require also $|v_1| = 1$. The Condition (B) applies, for example, to spaces $\mathbb{H}^d$ with canonical scalar products, $\mathbb{H}$ the quaternions. In Sections 8 and 10 we require

(C) V admits a $\perp$-basis $\bar{v}$ such that each component α_i from $\alpha = |\bar{v}|$ is first order definable within F by some formula $\alpha_i^\#(u_i)$. Thus, α is defined by $\bigwedge_i \alpha_i^\#(u_i)$.

Also, when translating from ring or field language into lattice language (with or without involution), in order to make use of coordinatization methods, we have to require

(D) $\dim V \geq 3$.

In the sequel, both F and V (vector space plus form) will be arbitrary under the above restrictions.

We consider the languages Λ_F and Λ_R in the signature of $*$-rings; Λ_L in the signature of ILs. Any space V has associated R and L with a natural correspondence between R^n and L^n, for fixed n, given by $\bar{u} = \bar{f}\mathsf{R}$, meaning that, component-wise, $u_k = f_k\mathsf{R}$. That is

$$\theta_{RL} = \{(\bar{f}, \bar{u}) \in \mathsf{R}^n \times \mathsf{L}. \mid \bar{u} = \bar{f}\mathsf{R}\}, \quad \theta_{LR}\{(\bar{u}, \bar{f}) \in \mathsf{L}^n \times \mathsf{R}^n \mid \bar{u} = \bar{f}\mathsf{R}\}$$

and thus inducing the maps

$$\theta_{RL}(\Phi) = \{\bar{f}\mathsf{R} \mid \bar{f} \in \Phi\} \subseteq \mathsf{L}^n, \quad \theta_{LR}(M) = \{\bar{f} \mid \bar{f}\mathsf{R} \in M\} \subseteq \mathsf{R}^n$$

between the power sets of R^n and L^n. For an isometry type α of orthogonal bases, one can also associate with such Φ and M relations $\theta_{R\alpha F}(\Phi)$ and $\theta_{L\alpha F}(M)$ over F, more precisely subsets of $(F^{d\times d})^n$.

Our main objective will be to establish translations (with a focus on quantification types) between the languages of L, R, and F (denoted by τ with suitable subscripts) which match defining formulas for relations matched by one of the above correspondences. These translations depend on $\dim V$ and, if F is involved, have refer to the isometry type (supposed to be definable, due to condition (C)) of an orthogonal basis. In order to deal with quantification type Σ_k resp. Π_k, these translations come in pairs $\tau^{\exists}, \tau^{\forall}$, also denoted by τ^Q, $Q \in \{\exists, \forall\}$ (though, translating from Λ_L there is no $\tau^{\forall}$, cf. Problem 13.1). They can be viewed as interpretations (cf. [7]), with the exception of the τ^Q_{RL} which are to capture the "many-to-one" relation θ_{RL}.

The concept of frame, the lattice version of a coordinate system, is introduced in Section 5 and used to reduce Σ_1-formulas to positive primitive ones. Section 6 gives the translation from Λ_L to Λ_R. The converse translation in Section 7, from Λ_R to Λ_L relies on Coordinatization Theory. Translation to Λ_F means considering R as a matrix ring and requires reference to a basis representing its isometry type (Section 8). In order to get a one-one correspondence (in Section 10), one needs to consider invariance properties of definable sets of matrices (Section 9). Counterexamples in Section 11 show that quantification is required when translating from F to L. The special case of anisotropic forms is dealt with in Section 12.

The results in Sections 6 to 10 remain valid – in simplified versions – if F is just a division ring, V a vector space, $\bar{v}$ any basis, R the endomorphism ring, L the lattice of all linear subspaces of V, isomorphic to the lattice of right ideals of R (we will indicate where presence or absence of involution does matter). Here, one uses any systems of matrix units and frames, proofs are simplified versions of the ones given.

5. Frames

Frames are the lattice analogues of coordinate systems in projective geometry. Fix $d \geq 1$. A *frame* in a bounded lattice L is a system $\bar{a} = (a_{ij} \mid 1 \leq i, j \leq d)$

of elements such that for pairwise distinct i, j, k (where $a_i = a_{ii}$)

$$\mathbf{1} = \bigoplus_{\ell} a_\ell, \;\; a_{ij} = a_{ji}, \;\; a_i + a_j = a_i \oplus a_{ij}, \;\; a_{ik} = (a_i + a_k) \cap (a_{ij} + a_{jk}).$$

Let $\rho(\bar{z})$ the obvious conjunction of bounded lattice equations such that $\bar{a}$ is a frame in L if and only if $L \models \rho(\bar{a})$. We always require that the *order* d of $\bar{a}$ is the height of L, e. g. $d = \dim V$ for $L = \mathsf{L}(V)$. Thus, the a_i and a_{ij} are atoms and the $a_i' = \sum_{j \neq i} a_j$ are coatoms such that $\bigcap_i a_i' = \mathbf{0}$. The frame $\bar{a}$ of $\mathsf{L}(V)$ is *associated* with the basis $\bar{v}$ of V if

$$a_i = v_i F, \; a_{ij} = (v_i - v_j)F \;\text{ for } i \neq j.$$

Clearly, any basis gives rise to an associated frame; and for any frame $\bar{a}$ of $\mathsf{L}(V)$ and $0 \neq v_1 \in a_1$ there is a unique basis $\bar{v}$ extending v_1 and having $\bar{a}$ associated. Moreover, completing a basis of U to a basis of V one gets the following fact.

Fact 5.1. *For $U \in \mathsf{L}(V)$ one has $\dim U = k$ if and only if there is a frame $\bar{a}$ such that $U = \sum_{i=1}^{k} a_i$.*

A frame $\bar{a}$ in a lattice with involution is *orthogonal* or a $\perp$*-frame* if $a_i \leq a_j^{\perp}$ for all $i \neq j$. The above remarks on "association" apply to $\perp$-frames and $\perp$-bases, analogously. In the context of lattices with involution and fixed dimension, $\rho(\bar{z})$ will denote the conjunction of equations defining orthogonal frames of order d.

Frames allow to derive equivalences of formulas in Λ_L, assuming fixed dimension. With $z_i' = \sum_{j \neq i} z_j$ and $\bar{x} = (x_1, \ldots, x_k)$ define the lattice term

$$D_k(\bar{x}, \bar{z}) = \bigcap_{h=1}^{k} \sum_{i=1}^{d} \Big((x_h + z_i') \cap z_i + \sum_{j \neq i} ((x_h + z_i') \cap z_i + z_{ij}) \cap z_j \Big)$$

Fact 5.2. *For any frame $\bar{a}$ in $\mathsf{L}(V)$ and any $\bar{b} \in \mathsf{L}(V)^k$,*

$$D_k(\bar{b}, \bar{a}) = \begin{cases} \mathbf{1} & \text{if } b_h \neq \mathbf{0} \text{ for all } h \leq k, \\ \mathbf{0} & \text{if } b_h = \mathbf{0} \text{ for some } h \leq k. \end{cases}$$

The claim follows from the fact that, due to the frame relations, a_i is perspective to a_j via a_{ij} if $i \neq j$.

Proposition 5.3. (i) *Within the classes of all $\mathsf{L}(V)$ (with or without involution), $\dim V = d$ fixed, any Σ_1-formula $\varphi(x)$ is equivalent to a positive Σ_1-formula.*

(ii) *In the presence of involution, given d, for any quantifier-free formula $\varphi(\bar{x}) \in \Lambda_L$ there is a term $t(\bar{x}, \bar{y}) \in \Lambda_L$ such that for any V of $\dim V = d$ one has $\varphi(\bar{x})$ equivalent to $\exists \bar{y}.\, t(\bar{x}, \bar{y}) = \mathbf{0}$ in $\mathsf{L}(V)$.*

Proof. Of course, we may assume $\varphi(\bar{x})$ quantifier-free. First observe that for any $u, w \in L = \mathsf{L}(V)$ one has (due to modularity and existence of complements and frames)

$$u = w \Leftrightarrow L \models \exists y.\, (u + w) \cap y = \mathbf{0} \;\wedge\; u \cap w + y = \mathbf{1},$$

$$u \neq w \Leftrightarrow L \models \exists \bar{z}.\ \rho_L(\bar{z}) \wedge\ z_1 = z_1 \cap (u + w)\ \wedge\ z_1 \cap u \cap w = \mathbf{0}.$$

Thus, any conjunction $\varphi_i(\bar{x})$, $1 \leq i \leq k$, of atomic and negated atomic formulas is equivalent to a formula $\exists \bar{y}_i.\ \bigwedge_j t_{ij}(\bar{x}, \bar{y}_i) = \mathbf{0} \wedge s_{ij}(\bar{x}, \bar{y}_i) = \mathbf{1}$ and the latter to $\exists \bar{y}_i.\ t_i(\bar{x}, \bar{y}_i) = \mathbf{0} \wedge s_i(\bar{x}, \bar{y}_i) = \mathbf{1}$ where $t_i = \sum_j t_{ij}$ and $s_i = \bigcap_j s_{ij}$. This proves the first claim.

Assuming the presence of involution, $s_i(\bar{x}, \bar{y}_i) = \mathbf{1}$ is equivalent to $s_i(\bar{x}, \bar{y}_i)^\perp = \mathbf{0}$ and $\varphi_i(\bar{x})$ to $\exists \bar{y}_i.\ r_i(\bar{x}, \bar{y}_i) = \mathbf{0}$ with $r_i = t_i + s_i^\perp$. Of course, the $\bar{y}_i$ may be assumed to have no variables in common.

Now, the conjunction $\varphi_0(\bar{z})$ of equations defining frames of order d is equivalent to $\exists y_0.\, r_0(\bar{z}, \bar{y}_0) = \mathbf{0}$ for some term r_0. Then, by Fact 5.2, with $\bar{y}$ comprising all the $\bar{y}_i$ (that is, $\bar{y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_k)$, one has that $\bigvee_{i=1}^k \varphi_i(\bar{x})$ is equivalent to

$$\exists \bar{z} \exists \bar{y}.\ r_0(\bar{z}, \bar{y}_0) + D_k(r_1(\bar{x}, \bar{y}_1), \ldots, r_k(\bar{x}, \bar{y}_k), \bar{z}) = \mathbf{0}.$$ □

Concerning the Grassmann-Cayley algebra point of view one has the following where $\mathsf{L}^{GC}(V)$ denotes the set of all linear subspaces of V endowed with the ternary relations

$$j(a,b,c) \Leftrightarrow a = b \oplus c, \quad m(a,b,c) \Leftrightarrow\ a = b \cap c\ \wedge\ b + c = \mathbf{1}$$

and Λ_{GC} the associated language.

Corollary 5.4. *For any fixed d and any Σ_1 (positive primitive) formula $\varphi(\bar{x})$ in the language of bounded lattices there is a Σ_1 (positive primitive) formula $\psi(\bar{x})$ in Λ_{GC} such that, for any V of $\dim V = d$, $\mathsf{L}(V) \models \varphi(\bar{u})$ if and only if $\mathsf{L}^{GC}(V) \models \psi(\bar{u})$.*

Proof. In view of Proposition 5.3 and Fact 3.2 we may assume that $\varphi(\bar{x})$ is positive quantifier-free with all atomic subformulas being basic (a conjunction of equations). Thus, it suffices to encode the latter into positive primitive formulas. Since $\mathsf{L}(V)$ is complemented, this is achieved for $x = y + z$ by

$$\exists u \exists v \exists w.\ j(y,u,v) \wedge j(z,v,w) \wedge j(x,y,w)$$

and dually for $x = y \cap z$. □

6. Interpreting lattices in rings

This is the easy part when translating between Λ_L and Λ_R, not requiring coordinate systems, whence uniform independent of dimensions.

The basic relation between L and R is given by $u = f\mathsf{R}$. We write $\bar{u} = \bar{f}\mathsf{R}$ if this holds component-wise and define for $M \subseteq \mathsf{L}^n$

$$\theta_{LR}(M) = \{\bar{f} \in \mathsf{R} \mid \bar{f}\mathsf{R} \in M\}.$$

Translation from L to R relies on the following.

Fact 6.1. *For any idempotents e, f, g in R one has the following.*

$$e\mathsf{R} = f\mathsf{R} \;\Leftrightarrow\; ef = f \wedge fe = e$$

$$g\mathsf{R} = e\mathsf{R} + f\mathsf{R} \;\Leftrightarrow\; ge = e \wedge gf = f \wedge \exists r \exists s.\, g = er + fs,$$

$$g\mathsf{R} = e\mathsf{R} \cap f\mathsf{R} \;\Leftrightarrow\; eg = fg = g \wedge \exists r \exists s.\, (1-g) = r(1-e) + s(1-f).$$

Also, $(e\mathsf{R})^\perp = (1-e^*)\mathsf{R}$, $\mathbf{0} = 0\mathsf{R}$, *and* $\mathbf{1} = 1\mathsf{R}$.

Proof. The first claims are obvious. Considering the opposite ring, the right hand side of the third equivalence amounts to $\mathsf{R}(1-g) = \mathsf{R}(1-e) + \mathsf{R}(1-f)$ which in turn to $g\mathsf{R} = e\mathsf{R} \cap f\mathsf{R}$ by [10, Lemma II.2.3]. □

We have to translate any $\varphi(\bar{x})$ in Λ_L into $\tau^Q_{LR}(\varphi(\bar{x}))$ in Λ_R, $Q \in \{\exists, \forall\}$, such that for any $\bar{f}$ in R

$$\mathsf{L} \models \varphi(\bar{f}\mathsf{R}) \;\Leftrightarrow\; \mathsf{R} \models \tau^Q_{LR}(\varphi(\bar{f})). \tag{$*$}$$

Let the variables x of Λ_R also serve as variables of Λ_L; though, for each x add a specific variable $\hat{x}$. If $\varphi(\bar{x})$ is a basic equation, then by Fact 6.1 there is $\tau^{0\exists}_{LR}(\varphi(\bar{x}))$ in the same free variables $\bar{x}$ which is positive primitive such that $(*)$ holds when substituting $\bar{f}$ with idempotent f_k. Now, given quantifier-free $\varphi(\bar{x}) \in \Lambda_L$, in view of Proposition 5.3 and Fact 3.2 there is a positive Σ_1-formula $\exists \bar{y}.\, \psi(\bar{x}, \bar{y})$ with basic atomic formulas, equivalent to φ within L. Translating these basic equations as above, we obtain a positive Σ_1-formula $\tau^{0\exists}_{LR}(\varphi(\bar{x}))$ such that $(*)$ holds when substituting $\bar{f}$ with idempotent f_k. Finally, given prenex $\varphi(\bar{x}) = \bar{Q}\bar{y}.\, \psi(\bar{x}, \bar{y})$ define $\tau^{\exists}_{LR}\varphi$ as

$$\exists.\, \hat{\bar{x}} \bigwedge_k x_k \hat{x}_k x_k = x_k \wedge \bar{Q}^\tau \bar{y}.\, \tau^{0\exists}_{RL}(\psi'(\bar{x}, \hat{\bar{x}}, \bar{y}))$$

where ψ' arises from ψ substituting $x_k \hat{x}_k$ for x_k and where quantification Qy_ℓ in $\bar{Q}\bar{y}$ gives rise to Qy_ℓ bounded by $y_\ell^2 = y_\ell$. This is comprised in the following main result:

Theorem 6.2. *If M is definable in L^n (by prenex $\varphi(\bar{x})$) then $\theta_{LR}(M)$ is definable in R^n (by $\tau^{\exists}_{LR}(\varphi(\bar{x}))$). Moreover, $\tau^{\exists}_{LR}$ shifts type.*

For example, $x_1 \cap x_2 = \mathbf{0}$ translates into

$$\exists \hat{x}_1 \exists \hat{x}_2.\, x_1 \hat{x}_1 x_1 = x_1 \wedge x_2 \hat{x}_2 x_2 = x_2$$
$$\wedge \exists y_1 \exists y_2.\, y_1^2 = y_1 \wedge y_2^2 = y_2 \wedge 1 = y_1(1 - x_1\hat{x}_1) + y_2(1 - x_2\hat{x}_2).$$

7. Interpreting rings within lattices

A subset Φ of R^n determines a subset of L^n, canonically,

$$\theta_{RL}(\Phi) = \{\bar{f}\mathsf{R} \mid \bar{f} \in \Phi\}.$$

The objective is to show, by means of a translation, that $\theta_{RL}(\Phi)$ is definable in L if so is Φ in R. This requires $d \geq 3$, since for $d \leq 2$, L degenerates to a set with involution and two constants. Thus, fixed $d \geq 3$ is a general assumption for this section. Also, the translation will depend on d, being uniform otherwise. We use concepts from Coordinatization Theory — but proofs are by elementary Linear Algebra.

The "coordinate systems" for R are systems $\bar{e} = (e_{ij} \mid 1 \leq i, j \leq d)$ of $*$-*matrix units* that is elements of R such that (where $e_j = e_{jj}$)

$$\sum\nolimits_{j=1}^{d} e_j = \mathrm{id}_V,\ \ e_j^* = e_j,\ \ e_{k\ell}e_{ij} = \begin{cases} e_{kj} & \text{if } \ell = i, \\ 0 & \text{if } \ell \neq i. \end{cases}$$

(Since we read from right to left when applying maps we read indices the same way.)

Observe that $V = \bigoplus_j^{\perp} \mathsf{im}\, e_j$ with $\dim \mathsf{im}\, e_j = 1$ and $\mathsf{im}\, e_{ij} = \mathsf{im}\, e_{ji}^* = \mathsf{im}\, e_i$ and that the restriction $e_{ij}|\,\mathsf{im}\, e_j$ is an isomorphism of $\mathsf{im}\, e_j$ onto $\mathsf{im}\, e_i$ and that $e_{ij}|\sum_{k\neq j} \mathsf{im}\, e_k = 0$ whence $e_{ij}^*|\,\mathsf{im}\, e_i$ an isomorphism of $\mathsf{im}\, e_i$ onto $\mathsf{im}\, e_j$ and $e_{ij}^*|\sum_{k\neq i} \mathsf{im}\, e_k = 0$. The system $\bar{e}$ of $*$-matrix units is *associated* with the $\perp$-basis $\bar{v}$ if

$$e_{ij}v_k = \begin{cases} v_i & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Of course, for any $\perp$-basis there is an associated system of $*$-matrix units and vice versa: choose $0 \neq v_1 \in \mathsf{im}\, e_1$ and $v_i = \varepsilon_{i1}v_1$. As for frames, for systems of matrix units in R we suppose the *order* d to be $d = \dim V$.

In order to relate frames to matrix units, recall that for subspaces U, W of V such that $U \cap W = 0$ there is a 1-1-correspondence between linear maps $f : U \to W$ and subspaces X such that $X \cap W = 0$ and $X + W = U + W$ given by $G(f) = \{u - f(u) \mid u \in U\}$, the (negative) graph of f (as considered by von Neumann). Under the isomorphism Ω, identifying the right ideal lattice with the subspace lattice, if $U = \mathsf{im}\, e_j$ and $W = \mathsf{im}\, e_i$, where $i \neq j$, then, for $f \in e_i\mathsf{R}e_j$, $\Omega((e_j - f)\mathsf{R})$ is the graph of the induced map. This allows to think of $(e_j - f)\mathsf{R}$ as a graph. We define $\mathsf{R}_{ij} = \{u \in \mathsf{L} \mid u \oplus a_i = a_i + a_j\}$.

Fact 7.1. *There is a 1-1-correspondence between systems $\bar{e}$ of $*$-matrix units of R and $\perp$-frames $\bar{a}$ of L given by $a_i = e_i\mathsf{R}$ and $a_{ij} = (e_j - e_{ij})\mathsf{R}$ for $i \neq j$. Moreover, for $i \neq j$, $u \in \mathsf{R}_{ij}$ if and only if $u = (e_j - f)\mathsf{R}$ for some (unique) $f \in e_i\mathsf{R}e_j$.*

First, we establish a d^2-dimensional interpretation of R in L; that is, we view R as a matrix ring (via a system $\bar{e}$ of $*$-matrix units) and capture the coefficients by lattice elements (via the associated $\perp$-frame $\bar{a}$). Indeed, $f \in \mathsf{R}$ is uniquely determined by its "matrix coefficients" e_ife_j; these in turn by the graphs $(e_j - e_ife_j)\mathsf{R}$ in case $i \neq j$ while in case $i = j$ one has to choose $k \neq j$ and to use the graph $(e_j - e_{kj}fe_j)\mathsf{R}$.

Thus, we choose n_{ij} uniformly such that $n_{ij} = i$ if $i \neq j$ and $n_{jj} \neq j$. Then, under the correspondence of Fact 7.1 one has an injective map τ (depending on $\bar{a}$) associating with each $f \in \mathsf{R}$ a $d \times d$-array τf of elements $\tau_{ij}f$ of L (the "lattice coefficients" of f)

$$\tau_{ij}f = (e_j - e_{n_{ij}i}fe_j)\mathsf{R}.$$

Observe that $\tau_{ij}f = \tau_{ij}e_ife_j = \tau_{ij}e_if = \tau_{ij}fe_j$ and that, for $i \neq j$, $\tau_{ij}f = (e_j - e_ife_j)\mathsf{R}$ while $\tau_{jj}f = \tau_{n_{jjj}}e_{n_{jjj}}f$.

A matrix $X = (u_{ij})_{ij} \in \mathsf{L}^{d\times d}$ is in the image of τ iff $u_{ij} \in \mathsf{R}_{n_{ijj}}$ for all i,j; indeed, under this assumption, we have $u_{ij} = (e_j - f_{ij})\mathsf{R}$ for unique $f_{ij} \in e_{n_{ij}}\mathsf{R}e_j$ and choose $f = \sum_{ij} e_{in_{ij}}f_{ij}$ to obtain $\tau f = X$.

Further, we have to express the fundamental operations of R (we use the binary operation of subtraction in place of addition and its inversion) in terms of the lattice coefficients and the frame. Choose specific variables $\bar{z} = (z_{ij})_{ij}$ (where $z_{ii} = z_i$) for frame elements. With additional variables x, y define for pairwise distinct i,j,k the *perspectivity terms* and further terms tailored to capture subtraction, involution, and multiplication (where h is minimal such that $h \neq i,j$); these terms involve the variables $\bar{z}$ though not listed, explicitly.

$$\begin{aligned}
p^{ij}_{kj}(x) &= (z_k + z_j) \cap (z_{ki} + x)\\
p^{ij}_{ik}(x) &= (z_i + z_k) \cap (z_{kj} + x)\\
y \ominus_{ij} x &= (z_i + z_j) \cap (z_h + (z_i + z_{hj}) \cap (y + p^{ij}_{ih}(x)))\\
x^{\#_{ij}} &= (z_j + z_i) \cap (z_i \ominus_{ij} x)^{\perp}\\
y \otimes_{ij} x &= (z_i + z_j) \cap (y + x).
\end{aligned}$$

Observe that for pairwise distinct i,j,k and $f,g \in \mathsf{R}$

$$p^{ij}_{kj}(\tau_{ij}f) = \tau_{kj}e_{ki}f, \quad p^{ij}_{ik}(\tau_{ij}f) = \tau_{ik}fe_{jk}$$
$$\tau_{ik}g \otimes_{ij} \tau_{kj}f = \tau_{ij}e_ige_kfe_j = \tau_{ij}ge_kf.$$

and for any i,j

$$\tau_{ij}g \ominus_{n_{ijj}} \tau_{ij}f = \tau_{ij}(g-f), \quad (\tau_{ij}f)^{\#_{n_{ijj}}} = \tau_{ji}f^*,$$
$$\tau_{ij}0 = a_j, \quad \tau_{ij}e_{ij} = a_{n_{ijj}}$$

whence $\tau_{ij}\,\mathrm{id} = \tau_{ij}(\sum_k e_k) = a_{n_{jjj}}$ if $i = j$, a_j else. Also observe

$$\tau_{jj}f = \tau_{n_{jjj}}e_{n_{jjj}}f = a_{n_{jj}\ell} \otimes_{n_{jjj}} \tau_{\ell j}e_jf \text{ for any } \ell \neq j, n_{jj}.$$

This leaves us to deal with multiplication, mimicking matrix multiplication. In order to have sums available, define the lattice terms $s^{\ell}_{ij}(x_1,\dots,x_\ell)$, recursively,

$$s^0_{ij} = z_j, \quad s^{\ell+1}_{ij} = x_{\ell+1} \ominus_{ij} (z_j \ominus_{ij} s^{\ell}_{ij}(x_1,\dots,x_\ell))$$

to obtain

$$\tau_{ij}\sum_{h=1}^{\ell} f_h = s^{\ell}_{ij}(\tau_{ij}f_1,\dots,\tau_{ij}f_\ell).$$

Since $\tau_{ij}(gf) = \sum_{h=1}^d \tau_{ij}ge_hf$, once we express the $\tau_{ij}e_ige_hfe_j$ in terms of $\tau_{ih}g$, $\tau_{hj}f$, and the $a_{k\ell}$, it is obvious how to construct the required lattice terms. This requires choices of indices, though, these depend only on equalities between indices and can be done uniformly.

Assume $i \neq j$. If $h \neq i,j$ then $n_{ih} = i$ and $n_{hj} = h$ whence $\tau_{ij}e_ige_hfe_j = \tau_{ih}g \otimes_{ij} \tau_{hj}f$. Otherwise, choose $k \neq i,j$ and $g' = e_ige_{hk}$ and $f' = e_{kh}fe_j$; then $e_ige_hfe_j = g'e_kf'$ whence $\tau_{ij}e_ige_hfe_j = \tau_{ik}g' \otimes_{ij} \tau_{kj}f'$; if $h = j$ then

$\tau_{ik}g' = \tau_{ij}g \otimes_{ik} \tau_{jk}e_{jk} = \tau_{ij}g \otimes_{ik} a_{jk}$ and $\tau_{kj}f' = \tau_{jj}f$ if $k = n_{jj}$, $\tau_{kj}f' = p_{kj}^{n_{jj}j}\tau_{jj}f$ else; if $h = i$ then $\tau_{kj}f' = a_{ki} \otimes_{kj} \tau_{ij}f$ and, for any $\ell \neq i, k$, $\tau_{ik}g' = a_{i\ell} \otimes_{ik} (\tau_{li}e_{li}g \otimes_{lk} a_{ik})$ where $\tau_{li}e_{li}g$ is $\tau_{ii}g$ if $\ell = n_{ii}$, $p_{n_{ii}i}^{\ell i}\tau_{ii}g$, else.

Assume $i = j$ and put $k = n_{jj}$. If $h \neq j, k$ then $\tau_{jj}e_jge_hfe_j = \tau_{kj}e_{kj}e_jge_hfe_j = (a_{kj} \otimes_{kh} \tau_{jh}g) \otimes_{kj} \tau_{hj}f$. If $h = j$ choose $\ell \neq j, k$ and observe that

$$\begin{aligned}\tau_{jj}e_jge_jfe_j &= \tau_{kj}e_{kj}ge_{j\ell}e_{\ell j}fe_j = \tau_{k\ell}e_{kj}ge_{j\ell} \otimes_{kj} \tau_{\ell j}e_{\ell k}e_{kj}fe_j \\ &= (\tau_{kj}e_{kj}ge_j \otimes_{k\ell} a_{j\ell}) \otimes_{kj} (a_{\ell k} \otimes_{\ell j} \tau_{jj}fe_j) \\ &= (\tau_{jj}ge_j \otimes_{k\ell} a_{j\ell}) \otimes_{kj} (a_{\ell k} \otimes_{\ell j} \tau_{jj}fe_j).\end{aligned}$$

If $h = k$ choose $\ell \neq k, j$ to obtain

$$\begin{aligned}\tau_{jj}e_jge_hfe_j &= \tau_{kj}e_{kj}ge_kfe_j = \tau_{kj}e_{kj}ge_kfe_{j\ell}e_{\ell j} = (\tau_{k\ell}e_{kj}ge_kfe_{j\ell}) \otimes_{kj} a_{\ell j} \\ &= (a_{kj} \otimes_{k\ell} \tau_{j\ell}ge_kfe_{j\ell}) \otimes_{kj} a_{\ell j} = (a_{kj} \otimes_{k\ell} (\tau_{jk}g \otimes_{j\ell} \tau_{k\ell}fe_{j\ell})) \otimes_{kj} a_{\ell j}\end{aligned}$$

where $\tau_{k\ell}fe_{j\ell} = \tau_{kj}f \otimes_{k\ell} a_{j\ell}$.

Choose for each variable $x \in \Lambda_R$ a $d \times d$-array of variables $\hat{\tau}_{ij}x$ in Λ_L, all pairwise distinct. In view of the above, for any equation $\varphi(\bar{x})$ in Λ_R there is a conjunction $\sigma(\varphi)(\bar{X}, \bar{z})$ of equations in Λ_L where $X_k = (\hat{\tau}_{ij}x_k)_{ij}$ such that, for any $\perp$-frame $\bar{a}$ of L and $\bar{f}$ in R

$$\mathsf{R} \models \varphi(\bar{f}) \ \Leftrightarrow\ \mathsf{L} \models \sigma(\varphi)((\tau_{ij}f)_{ij}, \bar{a}). \qquad (*)$$

Observe that $\tau_{ij}f$ is obtained substituting f into $*$-ring terms (based on a system of $*$-matrix units and associated $\perp$-frame) and so obtaining a lattice element while $\hat{\tau}_{ij}x$ is a lattice variable to denote such element; in particular, $\hat{\tau}_{ij}x$ is NOT a lattice term which would yield $\tau_{ij}f$ if $f\mathsf{R}$ is substituted for x.

For example, consider dimension $d = 3$ and the formula $\psi(x, y)$ given as $x \cdot y = y \cdot x$. With the ring variables x, y we have the associated 3×3-matrices $X = (x_{ij})_{ij}$ and $Y = (y_{ij})_{ij}$ of of lattice variables where, for convenience, we write $x = x_1$, $y = x_2$, and $x_{ij} = \hat{\tau}_{ij}x$ and $y_{ij} = \hat{\tau}_{ij}y$. Recall the variables $\bar{z}$ for the 3-frame, not mentioned explicitly. Define $n_{11} = 2$, $n_{22} = 3$, $n_{33} = 1$. Now $\sigma(\psi)(X, Y, \bar{z})$ is the formula

$$\bigwedge_{i,j=1}^{3} s_{ij}^3(x_i \times_{i1j} y_j, x_i \times_{i2j} y_j, x_i \times_{i3j} y_j) = s_{ij}^3(y_i \times_{i1j} x_j, y_i \times_{i2j} y_j, y_i \times_{i3j} x_j)$$

where $y \times_{ihj} x$ is the lattice term in variables $x, y, \bar{z}$ given as follows. First, consider $i \neq j$. $y \times_{ihj} x \equiv y \otimes_{ij} x$ if $h \neq i, j$. Otherwise, let $k \neq i, j$; if $h = j$ and $k = n_{jj}$ then $y \times_{ihj} x \equiv (y \otimes_{ik} z_{jk}) \otimes_{ij} x$; if $h = j$ and $k \neq n_{jj}$ then $y \times_{ihj} x \equiv (y \otimes_{ik} z_{jk}) \otimes_{ij} p_{kj}^{n_{jj}j}x$; if $h = i$ and $j = n_{ii}$ then $y \times_{ihj} x \equiv (z_{ij} \otimes_{ik} (y \otimes_{jk} z_{ik})) \otimes_{ij} (z_{ki} \otimes_{kj} x)$; if $h = i$ and $j \neq n_{ii}$ then $y \times_{ihj} x \equiv (z_{ij} \otimes_{ik} (p_{n_{ii}i}^{ji}y \otimes_{jk} z_{ik})) \otimes_{ij} (z_{ki} \otimes_{kj} x)$.

Assume $i = j$ and put $k = n_{jj}$. If $h \neq j, k$ then $y \times_{ihj} x \equiv (z_{kj} \otimes_{kh} g) \otimes_{kj} x$. If $h = j$ and $\ell \neq j, k$ then $y \times_{ihj} x \equiv (y \otimes_{k\ell} z_{j\ell}) \otimes_{kj} (z_{\ell k} \otimes_{\ell j} x)$. If $h = k$ and $\ell \neq k, j$ then $y \times_{ihj} x \equiv (z_{kj} \otimes_{k\ell} (y \otimes_{j\ell} (x \otimes_{k\ell} z_{j\ell})) \otimes_{kj} z_{\ell j}$.

Define $\sigma(\varphi)$ for arbitrary quantifier-free φ replacing any equation occurring in φ by the corresponding conjunction of equations; then $(*)$ holds, too.

Corollary 7.2. *$\perp$-frames provide a d^2-dimensional interpretation of* R *in* L*, uniformly for all V of* $\dim V = d$*, as established, above.*

In order to capture the relation θ_{RL} between rings and lattices we need, in addition, to express $f\mathsf{R}$ in terms of the lattice coefficients of f.

Lemma 7.3. *There is a lattice term $t((x_{ij})_{ij}, \bar{z})$ such for any V of* $\dim V = d$ *and $\perp$-frame $\bar{a}$ of* L *and $f \in \mathsf{R}$*

$$f\mathsf{R} = t((\tau_{ij}f)_{ij}, \bar{a}).$$

Proof. In defining t we use a $\perp$-basis $\bar{v}$, the associated $\perp$-frame $\bar{a}$, and $*$-matrix units $\bar{e}$, but our definition does not depend on these. Recall the isomorphism Ω identifying right ideals with subspaces: $f\mathsf{R} \mapsto \mathsf{im}\, f$; for readability, we write $\bar{a}$ in place of $\Omega(\bar{a})$. For $r \in F$ we define

$$\tau_{ij}r = (v_j - v_{n_{ij}}r)F$$

and observe that

$$\begin{aligned} a_j = v_jF = \tau_{ij}0,\ a_{n_{ij}j} &= \tau_{ij}1 = (v_j - v_{n_{ij}})F, \\ \tau_{ij}r = \tau_{jn_{ij}}r^{-1} &\text{ and } \tau_{ij}r \oplus a_j = a_{n_{ij}} + a_j \quad \text{if } r \neq 0, \\ \Omega(\tau_{ij}f) = \tau_{ij}r_{ij} &\quad \text{where } e_{n_{ij}i}fv_j = v_{n_{ij}}r_{ij}. \end{aligned}$$

Since $f\mathsf{R} = \sum_j fe_j\mathsf{R}$ and $\tau_{ij}f = \tau_{ij}fe_j$ it suffices to consider the case $f = fe_j$, say for $j = 1$. We put $n_{i1} = k_i$, $r_{i1} = r_i$, and $U := \Omega(fe_1\mathsf{R}) = \mathsf{im}\, f = (\sum_i v_ir_i)F$. If $r_\ell \neq 0$ then

$$U = (\sum_i v_ir_ir_\ell^{-1})F = \bigcap_{1\leq i\leq d,\, i\neq\ell} \big(\tau_{i\ell}(-r_ir_\ell^{-1}) + \sum_{k\neq i,\ell} a_k\big).$$

Below, we define lattice terms $t_0(\bar{x}, \bar{z})$ such that

$$t_0(\tau_{k_11}r_1, \ldots, \tau_{k_d1}r_d, \bar{a}) = \begin{cases} \mathbf{1} & \text{if } r_\ell \neq 0 \text{ for some } \ell, \\ \mathbf{0} & \text{else,} \end{cases}$$

and $t_{i\ell}(x, y, \bar{z})$, $i \neq \ell$, such that

$$t_{i\ell}(\tau_{k_i1}r, \tau_{k_\ell1}s, \bar{a}) = \begin{cases} \tau_{i\ell}(-rs^{-1}) & \text{if } s \neq 0, \\ a_i + a_\ell & \text{if } s = 0. \end{cases}$$

Thus

$$U = t_0(\tau_{k_11}r_1, \ldots, \tau_{k_d1}r_d, \bar{a}) \cap \bigcap_{i\neq\ell}\big(t_{i\ell}(\tau_{k_i1}r_i, \tau_{k_\ell1}r_\ell, \bar{a}) + \sum_{k\neq i,\ell} a_k\big).$$

Thus, we have a term t_1 to deal with the case $f = fe_1$; similarly, terms t_j to deal with $f = fe_j$. The term required in the lemma is then $t = \sum_j t_j$. Still, we have to establish t_0 and the $t_{i\ell}$. Let

$$t_0 = \sum_k ((x_1 + z_1) \cap z_{k_1} + z_{k_1k}) \cap z_k + \sum_{i\geq 2}\sum_k ((x_i + z_1) \cap z_i + z_{ik}) \cap z_k$$

– compare the proof of Fact 5.2. Now define for pairwise distinct i, j, k

$$t_{ijk}(x, y, \bar{z}) = \big([(x + y) \cap (a_k + a_i) + a_j] \cap (a_{kj} + a_i) + a_k\big) \cap (a_j + a_i)$$

and observe that

$$t_{ijk}(\tau_{ji}r, \tau_{jk}s, \bar{a}) = \begin{cases} \tau_{ji}(-rs^{-1}) & \text{if } s \neq 0, \\ a_j + a_i & \text{if } s = 0, \end{cases}$$

$$p^{ij}_{kj}((v_j - v_i r)F), \bar{a}) = (v_j - v_k r)F, \;\; p^{ij}_{ik}((v_j - v_i r)F, \bar{a}) = (v_k - v_i r)F.$$

Thus, substituting and applying perspectivity terms, one derives $t_{i\ell}$ for any of the cases $i \neq 1 \neq \ell$, $i = 1$, and $\ell = 1$. □

Define, for the given $d \geq 3$, the translation $\tau^0_{RL} : \Lambda_R \to \Lambda_L$ as follows: for prenex $\varphi(\bar{x}) \equiv \bar{Q}\bar{y}.\, \psi(\bar{x}, \bar{y})$ define $\tau^0_{RL}(\varphi)(\bar{x}, \bar{X}, \bar{z})$ as

$$\bar{Q}^\tau \bar{y}.\, \sigma(\psi)(\bar{x}, \bar{X}, \bar{Y}, \bar{z})$$

where $\sigma(\psi)$ is prenex as defined preceding Corollary 7.2, where $X_k = (\hat{\tau}_{ij} x_k)_{ij}$ and $Y_\ell = (\hat{\tau}_{ij} y_\ell)_{ij}$, and where quantification Qy_ℓ in $\bar{Q}\bar{y}$ gives rise in $\bar{Q}^\tau \bar{y}$ to Qy_ℓ bounded by $\forall Y_\ell.\, y_\ell = t(Y_\ell, \bar{z})$ (if $Q = \forall$) resp. $\exists Y_\ell.\, y_\ell = t(Y_\ell, \bar{z})$ (if $Q = \exists$).

Fact 7.4. *For any $\perp$-frame $\bar{a}$, $\bar{f} \in \mathsf{R}$, and $\bar{B}$ given by the $B_k = (\tau_{ij} f_k)_{ij}$ in $\mathsf{L}^{d\times d}$ one has $\mathsf{R} \models \varphi(\bar{f})$ if and only if $\mathsf{L} \models \tau^0_{RL}(\varphi)(\bar{f}\mathsf{R}, \bar{B}, \bar{a})$.*

Proof. By Lemma 7.3, one has for $k = 1, \dots, n$

$$u_k = t((\tau_{ij} f_k)_{ij}), \bar{a}) \;\Leftrightarrow\; u_k = f_k \mathsf{R}.$$

The claim follows from condition $(*)$ satisfied by σ. □

Let $\rho_0(\bar{z})$ the conjunction of equations such that $\mathsf{L} \models \rho_0(\bar{a})$ if and only if $\bar{a}$ is an $\perp$-frame. Define $\tau^{0\exists}_{RL}(\varphi(\bar{x}))$ as

$$\exists \bar{X}.\, \bigwedge_k x_k = t(X_k, \bar{z}) \wedge \tau^0_{RL}(\varphi)(\bar{x}, \bar{X}, \bar{z})$$

and $\tau^{0\forall}_{RL}(\varphi(\bar{x}))$ as

$$\forall \bar{X}.\, \bigwedge_k x_k = t(X_k, \bar{z}) \Rightarrow \tau^0_{RL}(\varphi)(\bar{x}, \bar{X}, \bar{z}).$$

and then $\tau^\exists(\varphi(\bar{x}))$ and $\tau^\forall(\varphi(\bar{x}))$ as

$$\exists \bar{z}.\, \rho_0(\bar{z}) \wedge \tau^{0\exists}_{RL}(\varphi(\bar{x})) \text{ and } \forall \bar{z}.\, \rho_0(\bar{z}) \Rightarrow \tau^{0\forall}_{RL}(\varphi(\bar{x})).$$

Recall Fact 3.3 to obtain the following theorem:

Theorem 7.5. *Fix $d \geq 3$. If $\Phi \subseteq \mathsf{R}^n$ is Λ_R-definable (by prenex $\varphi(\bar{x})$) then $\theta_{RL}(\Phi)$ is Λ_L-definable in L^n (by $\tau^Q_{RL}(\varphi(\bar{x}))$ for $Q \in \{\exists, \forall\}$). Moreover, the pair $\tau^\exists_{RL}, \tau^\forall_{RL}$ preserves type.*

For example, if $\psi(x, y)$ is the formula $x \cdot y = y \cdot x$, considered in the above example, then $\varphi(x) \equiv \forall y.\, \psi(x, y)$ translates to $\tau^\forall_{RL}(\varphi)(x)$ given by

$$\forall \bar{z}(\rho_0(\bar{z}) \Rightarrow \forall X(x = t(X) \Rightarrow \forall y \forall Y(y = t(Y, \bar{z}) \Rightarrow \sigma(\psi)(X, Y, \bar{z})))).$$

We leave it as an exercise to find a short defining formula for $\theta_{RL}(\Phi)$ where $\Phi = \{r \in \mathsf{R} \mid \mathsf{R} \models \varphi(r)\}$.

8. Interpretations referring to a basis

Recall that we require condition (A) or (B) as well as condition (C); that is, in case (A) F is commutative and in case (B) $\perp$-bases $\bar{v} = (v_1, \ldots, v_d)$ with $|v_1| = 1$ exist and that we consider only such. Thus, in these cases, given a $\perp$-basis $\bar{v}$ and $\alpha = |\bar{v}| = (1, |v_1|^{-1}|v_2|, \ldots, |v_1|^{-1}|v_d|)$ from Section 2, one has the well known description of R. Namely,

$$\Omega_{\bar{v}}(f) = (a_{ij})_{ij} \text{ where } fv_j = \sum_i v_i a_{ij}$$

defines an isomorphism of R onto the $*$-ring $F_\alpha^{d\times d}$: the matrix ring $F^{d\times d}$ endowed with the involution

$$A^{\dagger_\alpha} = D_\alpha^{-1} A^* D_\alpha$$

where D_α is the diagonal matrix with diagonal α and $A^* = (a^*_{ji})_{ij}$ the conjugate transpose of A. Indeed, in case (A), *scaling* the form on V to obtain the space V' with form $\langle x|y\rangle' = |v_1|^{-1}\langle x|y\rangle$, one does not change adjoints, that is $\mathsf{R}(V) = \mathsf{R}(V')$. Thus, there is a conjunction $^\dagger(\bar{u}, X, Y)$ (with $d \times d$-matrices X, Y of variables) of equations such that $B = A^{\dagger_\alpha}$ is equivalent to $F \models {}^\dagger(\bar{r}, A, B)$, provided that $|\bar{r}| = \alpha$.

Also, assuming (C), there are an orthogonal basis $\bar{v}$ with $|\bar{v}| = \alpha = (\alpha_1, \ldots, \alpha_d)$ and first order formulas $\alpha_i^\#(u_i)$ such that for any $r_i \in F$ one has $r_i = \alpha_i$ if and only if $F \models \alpha_i^\#$; then $\alpha^\#(\bar{u}) \equiv \bigwedge_i \alpha_i^\#(u_i)$ defines α.

For $F_\alpha^{d\times d}$ one has the canonical $*$-matrix units where e_{ij} has 1 in position (i, j), 0 else; and the isomorphism $\Omega_{\bar{v}}$ matches the given system of $*$-matrix units with the canonical one.

Now, for any fixed n, component-wise application of the isomorphism $\Omega_{\bar{v}} : \mathsf{R} \to F_\alpha^{d\times d}$ gives rise to $\theta_{R\bar{v}F} \subseteq \mathsf{R}^n \times (F^{d\times d})^n$. In order to see that definable subsets of R^n are related with such of $F^{d^2 n}$, with any variable x in Λ_R we associate a $d\times d$-array $\tau(x) = (\tau_{ij}x)_{ij}$ of specific variables in Λ_F and will translate $\varphi(\bar{x}) \in \Lambda_R$ into $\tau^Q_{R\alpha F}(\bar{X}) \in \Lambda_F$ where $X_k = (\tau_{ij}x_k)_{ij}$ and $Q \in \{\exists, \forall\}$. Via the entry-wise description of operations in $F_\alpha^{d\times d}$ any basic ring equation is translated into a conjunction of equations in Λ_F while $x_2 = x_1^*$ is translated into $^\dagger(\bar{u}, X_1, X_2)$, adding in front an existential resp. universal quantifier for $\bar{u}$ which is bounded by $\alpha^\#(\bar{u})$. Clearly, (for any $\perp$-basis $\bar{v}$ with $|\bar{v}| = \alpha$ and in case (B) also $|v_1| = 1$) one has

$$\mathsf{R} \models \varphi(\bar{f}) \;\Leftrightarrow F \models \tau^Q_{R\alpha F}(\varphi)(\bar{A}) \text{ where } A_k = \Omega_{\bar{v}}(f_k).$$

In view of Fact 3.2, this extends, canonically, to a translation $\tau^Q_{R\alpha F}(\varphi)$ for quantifier-free formulas into Σ_1 resp. Π_1 formulas and then for arbitrary prenex formulas – the bounded Q-quantification for $\bar{u}$ to be put in front of the formula. As mentioned above, in this particular context, basic equations are translated into conjunctions of equations, Thus, one obtains the following fact.

Fact 8.1. *If $\Phi \subseteq \mathsf{R}^n$ is Λ_R-definable (by $\varphi(\bar{x})$) within R then $\theta_{R\bar{v}F}(\Phi) \subseteq (F^{d\times d})^n$ is Λ_F-definable (by $\tau^Q_{R\alpha F}(\varphi(\bar{x}))$ within F. Moreover, the pair $\tau^\exists_{R\alpha F}, \tau^\forall_{R\alpha F}$ preserves types modulo $\alpha^\#$.*

For example, let $d = 3$, $F = \mathbb{R}$ (or any real closed field) with identity involution, and $\alpha = (1, \sqrt{2}, -\sqrt{3})$. Then α is defined by the formula $\alpha^{\#}(\bar{u})$ given as $u_1 = 1 \wedge u_2^2 = 2 \wedge u_2 \geq 0 \wedge u_3^2 = 3 \wedge -u_3 \geq 0$ where $u \geq 0$ is the formula $\exists v.\ v^2 = u$. This results in ${}^{\dagger}(\bar{u}, X, Y)$ as

$$\bigwedge_{ij} u_i y_{ji} = x_{ij} u_j.$$

Now, the formula $x^2 = x \wedge xx^* = 0$, defining the set of idempotent e such that $e\mathsf{R}$ is co-isotropic, translates to

$$\exists \bar{u}.\ \alpha^{\#}(\bar{u}) \wedge \bigwedge_{ij} x_{ij} = \sum_h x_{ih} x_{hj}$$
$$\wedge \exists (y_{ij})_{ij}.\ \big({}^{\dagger}(\bar{u}, (x_{ij})_{ij}, (y_{ij})_{ij}) \wedge \bigwedge_{ij} \sum_h x_{ih} y_{hj} = 0\big)$$

alternatively to

$$\forall \bar{u}.\ \alpha^{\#}(\bar{u}) \Rightarrow \big(\bigwedge_{ij} x_{ij} = \sum_h x_{ih} x_{hj}$$
$$\wedge \forall (y_{ij})_{ij}.\ \big({}^{\dagger}(\bar{u}, (x_{ij})_{ij}, (y_{ij})_{ij}) \Rightarrow \bigwedge_{ij} \sum_h x_{ih} y_{hj} = 0\big)\big).$$

The translation from L to F has that to R as an intermediate step. Define $\theta_{L\bar{v}F}$ as the relational product $\theta_{R\bar{v}F} \circ \theta_{LR}$ and $\tau^{\exists}_{L\alpha F} = \tau^{\exists}_{R\alpha F} \circ \tau^{\exists}_{LR} : \Lambda_L \to \Lambda_F$. Observe that for $\bar{u} \in \mathsf{L}^n$ one has (with $A_k = (a^k_{ij})_{ij}$)

$$\theta_{L\bar{v}F}(\bar{u}) := \{\bar{A} \in (F^{d\times d})^n \mid \Omega(u_k) = \sum_{j=1}^{d} (\sum_{i=1}^{d} v_i a^k_{ij}) F, k = 1, \ldots n\}.$$

Fact 8.2. *If $M \subseteq \mathsf{L}^n$ is Λ_L-definable (by $\varphi(\bar{x})$) within L then $\theta_{L\bar{v}F}(M) \subseteq (F^{d\times d})^n$ is Λ_F-definable (by $\tau^{\exists}_{L\alpha F}(\varphi(\bar{x}))$ within F. Moreover, $\tau^{\exists}_{L\alpha F}$ shifts types modulo $\alpha^{\#}$.*

In the converse direction, let $\bar{e}$ the system of $*$-matrix units and $\bar{a}$ the $\perp$-frame associated with $\bar{v}$ – and observe that these are also associated with $\bar{v}r$ for any $r \neq 0$; in other words, scaling the form with $r \neq 0$ does not change the relation of "association". Let $\theta_{F\bar{v}R} = \theta^{-1}_{R\bar{v}F}$. The basic interpretation of F in R (giving rise to one in L) is the following fact.

Fact 8.3. *$e_1\mathsf{R}e_1$ is a $*$-subring of R, but with unit e_1 and there is an isomorphism $\omega_{\bar{v}}$ of F onto $e_1\mathsf{R}e_1$ such that $\omega_{\bar{v}}(r)v_1 = v_1 r$.*

We give a translation of formulas not depending on $\bar{v}$ so that it can be used later with varying coordinate systems. Let the $X_k = (X_{kij})_{ij}$ be $d \times d$-arrays of variables, all pairwise distinct, and $\psi(\bar{X}) \equiv \bar{Q}\bar{y}.\psi'(\bar{X}, \bar{y})$ a formula in Λ_F with quantifier-free ψ', with free variables $\bar{X}$, and quantified y_ℓ according to the string $\bar{Q}$ of quantifiers. Consider the y_ℓ as variables in Λ_R and choose pairwise distinct new variables x_k and z_{ij} in Λ_R. Translate $\psi(\bar{X})$ into $\tau^0_{FR}(\psi)(\bar{x}, \bar{z}) \in \Lambda_R$ given as $\bar{Q}\bar{y}\,\hat{\psi}$ where $\hat{\psi}$ arises from ψ' replacing any occurrence of x_{kij} by $z_{1i} x_k z_{j1}$, any occurrence of y_ℓ by $z_{11} y_\ell z_{11}$. Observe that, in view of Fact 8.3, one has $\mathsf{R} \models \hat{\psi}(\bar{f}, \bar{e})$ if and only if $F \models \psi(\bar{A})$ where $A_k = \Omega_{\bar{v}}(f_k)$.

Fact 8.4. *If $K \subseteq (F^{d\times d})^n$ is Λ_F-definable within F (by prenex $\psi(\bar{X})$) then $\theta_{F\bar{v}R}(K)$ is Λ_R-definable within R (by $\tau^0_{FR}(\psi)(\bar{x}, \bar{e})$) where $\bar{e}$ is associated with $\bar{v}$. Moreover, τ^0_{FR} preserves types.*

For example, consider $\psi(X_1, X_2)$ given as $\exists y_1.\ \bigwedge_{ij} y_1 x_{1ij} = x_{2ij}$. Then this translates to $\exists y_1.\ \bigwedge_{ij} z_{11} y_1 z_{11} z_{1i} x_{1ij} z_{j1} = z_{1i} x_{2ij} z_{j1}$.

Again, the translation from F to L has that to R as intermediate step: Define $\theta_{F\bar{v}L} = \theta_{RL} \circ \theta_{F\bar{v}R}$ and $\tau^0_{FL} = \tau^0_{RL} \circ \tau^0_{FR}$ where τ^0_{RL} is the restriction to $e_1\mathsf{R}e_1$ and R_{ij} of the interpretation of R in L (w.r.t. associated $\bar{e}$ and $\bar{a}$) of Corollary 7.2, that is the well known interpretation of F in L. By Fact 7.4 we conclude the follow consequence:

Fact 8.5. *If $K \subseteq (F^{d\times d})^n$ is Λ_F-definable within F (by prenex $\psi(\bar{X})$) then $\theta_{F\bar{v}L}(K)$ is Λ_L-definable within L (by $\tau^0_{FL}(\psi)(\bar{x}, \bar{a})$) where $\bar{a}$ is associated with $\bar{v}$. τ^0_{FL} preserves types.*

9. Invariance

Clearly, definable subsets of R^n or L^n are invariant under automorphisms. In order to apply Fact 3.1, we have to choose suitable groups of automorphisms. In case (B) let $F^+ = \{1\}$ and $\mathrm{O}^+(V)$ consist of all orthogonal maps. In case (A) let F^+ the multiplicative subgroup $\{r \mid 0 \neq r = r^* \in F\}$ of F and $\mathrm{O}^+(V)$ consist of all *scaled orthogonal maps* g: for some $r \in F^+$ and orthogonal h

$$gv = h(vr) \text{ for all } v \in V.$$

In view of commutativity, $g = rh := (r\,\mathrm{id}) \circ h$; equivalently

$$\langle gv|gw\rangle = r\langle v|w\rangle \text{ for all } v, w \in V.$$

For $\perp$-bases $\bar{v}, \bar{w}$ we write $\bar{v} \sim \bar{w}$ if $|\bar{w}| = r|\bar{v}|$ for some $r \in F^+$. Observe that this relation and the group $Q^+(V)$ are not changed if the form on V is scaled by an element of F^+.

Fact 9.1. (i) *$\mathrm{O}^+(V)$ is a subgroup of $\mathrm{GL}(V)$.*

(ii) *For any $g \in \mathrm{O}^+(V)$, the maps $f \mapsto g^R(f) = gfg^{-1}$ and $f\mathsf{R} \mapsto g^L(f\mathsf{R}) = gf\mathsf{R}$ are automorphisms of R and L, respectively. Moreover, $g \mapsto g^R$ and $g \mapsto g^L$ define group homomorphisms.*

(iii) *$\bar{v} \sim \bar{w}$ if and only if $\bar{w} = g\bar{v}$ for some $g \in \mathrm{O}^+(V)$.*

Proof. Let r, h, g be as above. Then $(rh)^{-1} = r^{-1}h^{-1}$, $(rh)^* = rh^* = rh^{-1}$, and $gf\mathsf{R} = gfg^{-1}\mathsf{R}$ which yields (i) and (ii). Now, if $|\bar{w}| = r|\bar{v}|$ and $r \in F^+$, define $g \in \mathrm{GL}(V)$ by $gv_i = w_i$ to obtain, for $v = \sum_i v_i r_i$ and $w = \sum_j v_j s_j$, that $\langle gv|gw\rangle = \big\langle \sum_i w_i r_i \big| \sum_j w_j s_j \big\rangle = \sum_i r_i^* |w_i| s_i = r \sum_i r_i^* |v_i| s_i = r\langle v|w\rangle$. □

We say that $\Phi \subseteq \mathsf{R}^n$ and $M \subseteq \mathsf{L}^n$ are *invariant* if they are invariant under the component-wise action of the g^R resp. g^L, $g \in \mathrm{O}^+(V)$. Clearly, definable Φ

and M are invariant; if Φ is invariant then so is $\theta_{RL}(\Phi)$ and if M is invariant then so is $\theta_{LR}(M)$.

Dealing with the $*$-ring $F_\alpha^{d\times d}$, we define $\mathrm{O}_\alpha^+(F,d) = \mathrm{O}^+(F_\alpha^d)$, consisting of the $T \in \mathrm{GL}(F,d)$ such that $T^{\dagger_\alpha} = rT^{-1}$ for some $0 \neq r \in F^+$, and consider the action $A \mapsto TAT^{-1}$ on $F^{d\times d}$. We call $K \subseteq (F^{d\times d})^n$ *α-invariant* if it is invariant under the component-wise action of $\mathrm{O}_\alpha^+(F,d)$; *right invariant* if $(A_1T_1,\dots,A_nT_n) \in K$ for all $(A_1,\dots,A_n) \in K$ and $T_i \in \mathrm{GL}(F,d)$; and *α-bi-invariant* if both conditions are satisfied, i.e. if $(TA_1T_1,\dots,TA_nT_n) \in K$ for all $(A_1,\dots,A_n) \in K$, $T \in \mathrm{O}_\alpha^+(F,d)$, and $T_i \in \mathrm{GL}(F,d)$.

Of course, given a first order formula, right invariance of the subset of $(F^{d\times d})$ it defines can be stated by a first order sentence; similarly α-invariance if α is definable.

Fact 9.2. *$\theta_{L\bar{v}F}$ and $\theta_{F\bar{v}L}$ induce mutually inverse bijections between the set of all subsets M of L^n and the set of all right invariant subsets K of $(F^{d\times d})^n$.*

Given α, define $\theta_{R\alpha F}(\Phi)$, $\theta_{F\alpha R}(K)$, $\theta_{L\alpha F}(M)$, and $\theta_{F\alpha L}(K)$, respectively, as the union of the $\theta_{R\bar{v}F}(\Phi)$, $\theta_{F\bar{v}R}(K)$, $\theta_{L\bar{v}F}(M)$, and $\theta_{F\bar{v}L}(K)$ where $\bar{v}$ ranges over all $\perp$-bases $\bar{v}$ with $|\bar{v}| = \alpha$.

Proposition 9.3. *Fix a $\perp$-basis $\bar{v}$ and $\alpha = |\bar{v}|$.*

(i) *$\theta_{R\bar{v}F}$ and $\theta_{F\bar{v}R}$ induce mutually inverse bijections between the set of all invariant $\Phi \subseteq \mathsf{R}^n$ and the set of all α-invariant $K \subseteq (F^{d\times d})^n$. Moreover, for such Φ and K, $\theta_{R\bar{v}F}(\Phi) = \theta_{R\alpha F}(\Phi)$ and $\theta_{F\bar{v}R}(K) = \theta_{F\alpha R}(K)$.*

(ii) *$\theta_{L\bar{v}F}$ and $\theta_{F\bar{v}L}$ induce mutually inverse bijections between the set of all invariant $M \subseteq \mathsf{L}^n$ and the set of all α-bi-invariant $K \subseteq (F^{d\times d})^n$; moreover, for such M and K, $\theta_{L\bar{v}F}(M) = \theta_{L\alpha F}(M)$ and $\theta_{F\bar{v}L}(K) = \theta_{F\alpha L}(K)$.*

Returning to the example of Φ consisting of all idempotent e such that $e\mathsf{R}$ is co-isotropic, consider $A \in K = \theta_{R\bar{v}F}(\Phi)$ that is $A^2 = A$ and $A^{\dagger_\alpha}A = 0$. Consider also $T \in O_\alpha^+(F,3)$, that is $T^{\dagger_\alpha} = rT^{-1}$ for some $0 \neq r \in F$. It follows $(TAT^{-1})^2 = TAT^{-1}$ and $(TAT^{-1})^{\dagger_\alpha}TAT^{-1} = (T^{-1})^{\dagger_\alpha}A^{\dagger_\alpha}T^{\dagger_\alpha}TAT^{-1} = r(T^{-1})^{\dagger_\alpha}A^{\dagger_\alpha}AT^{-1} = 0$ showing α-invariance of K. Now, let $\Phi' = \theta_{LR}(M)$ where $M = \{U \in \mathsf{L} \mid U^\perp \leq U\}$ is invariant in L. Thus, $K' = \theta_{L\alpha F}(\Phi')$ is α-bi-invariant. On the other hand, Φ' is the union of all $e\mathsf{R}$, $e \in \Phi$, whence $K' = K\,\mathrm{GL}(F,3)$ which is right invariant, obviously. To see α-invariance, directly, consider AB where $A \in K$ and $B \in \mathrm{GL}(F,3)$ and $T \in O_\alpha^+(F,3)$; then $TABT^{-1} = TAT^{-1}TBT^{-1} \in K'$ since $TAT^{-1} \in K$.

Proof. Observe that, by Fact 9.1(iii), $w_j = \sum_i v_i t_{ij}$ defines a $\perp$-basis $\bar{w} \sim \bar{v}$ if and only if $T = (t_{ij})_{ij} \in \mathrm{O}_\alpha^+(F,d)$ if and only if $\bar{w} = g(\bar{v})$ for some $g \in \mathrm{O}^+(V)$ with $\Omega_{\bar{v}}(g) = T$. To prove (i) it suffices to observe that, for $A = \Omega_{\bar{v}}(f)$, basis transformation yields $T^{-1}AT = \Omega_{\bar{v}}(g^{-1}fg) = \Omega_{\bar{w}}(f)$. Concerning (ii), assume that M is invariant. Thus, with $\bar{w} = g(\bar{v})$, $\bar{A} \in \theta_{L\bar{v}F}(\bar{u})$ implies

$$g(u_k) = \sum_{j=1}^{d}(\sum_{i=1}^{d} w_i a_{ij}^k)F$$

that is, $\bar{A} \in \theta_{L\bar{w}F}(g(\bar{u}))$. This proves that $\theta_{L\bar{v}F}(M) = \theta_{L\alpha F}(M)$. To prove α-bi-invariance of $\theta_{L\bar{v}F}(M)$, assume $T \in \mathrm{O}^+_\alpha(F, d)$ related to $\bar{w}$ and g as above and observe that $T\bar{A} \in \theta_{L\bar{v}F}(\bar{u})$ due to

$$g(u_k) = \sum_{j=1}^{d}(\sum_{h=1}^{d}(\sum_{i=1}^{d} v_i t^k_{ih}) a^k_{hj}) F = \sum_{j=1}^{d}(\sum_{i=1}^{d} v_i (\sum_{h=1}^{d} t^k_{ih} a^k_{hj}))F.$$

The same kind of reasoning applies for the reverse direction, while Fact 9.2 proves that the induced maps are inverses of each other. □

10. Definability

While, according to Section 8, definability in R or L gives rise to definability in F, depending only on $\alpha = |\bar{v}|$ in view of Proposition 9.3, the converse requires to capture α in terms of $*$-matrix units respectively $\perp$-frames. For the latter, we combine Corollary 7.2 and 8.3 into the following fact:

Fact 10.1. *For any basic $*$-ring operation $q(\bar{x})$ there is an ortholattice term $\hat{q}(\bar{x}, \bar{z})$ such that for any $\perp$-basis $\bar{v}$ and associated system $\bar{e}$ of $*$-matrix units and $\perp$-frame $\bar{a}$, and for any $i \geq 2$, the set $R_{i1}(\bar{a}) = \{u \in \mathsf{L} \mid u \oplus a_i = a_1 + a_i\}$ becomes a $*$-ring under the operations $\hat{q}(\bar{x}, \bar{a})$, and $\omega^i_{\bar{v}}(r) = (e_1 - e_{i1} \circ \omega_{\bar{v}}(r))\mathsf{R}$ defines an isomorphism of F onto $R_{i1}(\bar{a})$.*

Recall that in condition (C) we require that there are $\alpha^\#_i(x) \in \Lambda_F$ such that $r = \alpha_i$ if and only if $F \models \alpha^\#_i(r)$. Then there is $\alpha^@_i(x)$ in Λ_F such that $F \models \alpha^@_i(r)$ if and only if $r = -\alpha_i^{-1}$. We say that a system $\bar{e}$ of $*$-matrix units is an *α-system* if

$$e_1 \mathsf{R} e_1 \models \alpha^\#_i(e^*_{i1} e_{i1}) \ \text{ for } i = 2, \dots, d;$$

and a $\perp$-frame $\bar{a}$ is an *α-frame* if

$$R_{i1}(\bar{a}) \models \alpha^@_i((a_1 + a_i) \cap a^\perp_{1i}) \ \text{ for } i = 2, \dots, d.$$

Lemma 10.2. *Let $\bar{v}$ a $\perp$-basis with associated system $\bar{e}$ of $*$-matrix units and $\perp$-frame $\bar{a}$. Then, for any $\alpha \in F^d$, one has $|\bar{v}| = r\alpha$ for some $r \in F^+$ if and only if $\bar{e}$ is an α-system if and only if $\bar{a}$ is an α-frame.*

Proof. In view of scaling we may assume $|v_1| = 1$. For $i > 1$ let $f_i = e^*_{i1} e_{i1}$. Then, in view of Fact 8.3, $f_i = e_1 f_i e_1 = \omega_{\bar{v}}(r_i)$ for some $r_i \in F$ and it follows $r_i = |v_1| r_i = \langle v_1 | v_1 r_i\rangle = \langle v_1 | f_i v_1\rangle = \langle e_{i1} v_1 | e_{i1} v_1\rangle = \langle v_i | v_i \rangle = |v_i|$; in particular, $e_1 \mathsf{R} e_1 \models \alpha^\#_i(f_i)$ for all $i > 1$ if and only if $F \models \alpha^\#_i(r_i)$ for all $i > 1$ if and only if $r_i = \alpha_i$ for all $i > 1$ if and only if $|\bar{v}| = \alpha$.

Concerning the lattice case, observe that the isomorphism identifying right ideal with subspaces takes $\omega^i_{\bar{v}}(r)$ to $(v_1 - v_i r)F$. Again, denote the image frame in the subspace lattice also by $\bar{a}$. It suffices to show that $(a_1 + a_i) \cap a^\perp_{1i} = (v_1 + v_i \alpha_i^{-1})F$: in other words: $\langle v_1 - v_i r | v_1 - v_i\rangle = 0$ if and only if $r = -|v_i|^{-1}|v_1|$. The latter is easily verified. □

Lemma 10.3. *For given α such that $\alpha = r|\bar{v}|$ for some $\perp$-basis $\bar{v}$ and $r \in F^+$, the groups of all g^R and of all g^L, $g \in \mathrm{O}^+(V)$, act transitively on the set of all α-systems respectively all α-frames.*

Proof. Let $\bar{e}$ the α-system associated with $\bar{v}$. It suffices to consider any α-system $\bar{f}$ and to find $g \in \mathrm{O}^+(V)$ with $g^R\bar{e} = \bar{f}$. For that purpose, choose w_1 with $\mathsf{im}\, f_1 = w_1F$ (and $|w_1| = 1$ in case (B)) and complete to a $\perp$-basis $\bar{w}$ associated to $\bar{f}$. By Lemma 10.2 we have $\bar{w} \sim \bar{v}$. Hence, $g\bar{v} = w$ for some $g \in \mathrm{O}^+(V)$ and $g^R\bar{e} = \bar{f}$. The reasoning for frames is analogous. □

For $\psi(\bar{X}) \in \Lambda_F$ and $N \in \{R, L\}$ define $\tau^{\exists}_{F\alpha N}(\psi)$ and $\tau^{\forall}_{F\alpha N}(\psi)$ in Λ_R as

$$\exists \bar{z}.\ \rho_{\alpha N}(\bar{z}) \wedge \tau^0_{FN}(\psi) \text{ and } \forall \bar{z}.\ \rho_{\alpha N}(\bar{z}) \Rightarrow \tau^0_{FN}(\psi)$$

where $\rho_{\alpha R}(\bar{z})$ and $\rho_{\alpha L}(\bar{z})$ are the obvious first order formulas defining the concept of α-system and α-frame, respectively. Observe that the latter formulas are in Σ_k (in Π_k, positive primitive) of so are the $\alpha_i^{\#}$.

Theorem 10.4. *Assume that one of conditions (A), (B) holds and that (C) holds, as stated in Section 4 with α defined by $\alpha^{\#}$. Consider the bijections of Proposition 9.3.*

(i) *$\theta_{R\alpha F}$ and $\theta_{F\alpha R}$ match Λ_R-definable $\Phi \subseteq \mathsf{R}(V)^n$ with Λ_F-definable and α-invariant $K \subseteq (F^{d\times d})^n$.*
(ii) *Let $\dim V \geq 3$. $\theta_{L\alpha F}$ and $\theta_{F\alpha L}$ match Λ_L-definable $M \subseteq \mathsf{L}(V)^n$ with Λ_F-definable and α-bi-invariant $K \subseteq (F^{d\times d})^n$.*

Associated translations in (i) are provided by $\tau^Q_{R\alpha F}$ and $\tau^Q_{F\alpha R}$, $Q \in \{\exists, \forall\}$ so that the $(\exists, \forall)$-pairs of translations preserve type modulo $\alpha^{\#}$. In (ii) the translations are $\tau_{L\alpha F}$ which shifts type modulo $\alpha^{\#}$ and the pair $\tau^{\exists}_{F\alpha L}, \tau^{\forall}_{F\alpha L}$ which preserves types modulo $\alpha^{\#}$. If F is model complete then so is R and, in case $\dim V \geq 3$, L.

Observe that, in general, α-bi-invariant K are not definable in our sense. For example, consider $V = \mathbb{R}^3$ with canonical scalar product and M consisting of all $(\bar{a}, (e^*_j - f)\mathsf{R})$, $f \in e_i\mathsf{R}e_j$, such that $e_{ji}fe_{ji} = ke_j$ for some $k \in \mathbb{N}$ where $\bar{a}$ is a $\perp$-frame, $\bar{e}$ the associated system of $*$-matrix units. Then $\theta_{L\alpha F}(M)$ is bi-invariant but not definable. Similarly, one can choose M definable by the conjunction of infinitely many formulas, requiring $e_{ji}fe_{ji} \neq ke_j$ for all $k \in \mathbb{N}$.

Proof. Translating $\psi(\bar{X})$ in Λ_F to $\tau^Q_{FN}(\psi)(\bar{X})$ in Λ_N, $N \in \{R, L\}$, based on Facts 8.4 and 8.5, we apply Fact 3.1 with $\rho(\bar{z})$ given by $\rho_{\alpha N}(\bar{z})$ and $\varphi(\bar{x}, \bar{z})$ by $\tau^0_{FN}(\psi)(\bar{x}, \bar{z})$. Here, the system of matrix units respectively the frame associated with $\bar{v}$ witnesses condition (i), while (ii) is granted by Proposition 9.3 and (iii) by Lemma 10.3. The converse translations are given by Facts 8.1 and 8.2.

Now, assume F model complete whence the α_i defined by Σ_1-formulas $\alpha_i^{\#}(u_i)$. Translating from Λ_R to Λ_F, one has $\tau^{\exists}_{R\alpha F}(\varphi(\bar{x}))$ in Fact 8.1 of the form $\exists \bar{u}.\ \alpha^{\#}(\bar{u}) \wedge \psi(\bar{X})$ and for any $\perp$-basis $\bar{v}$ with $|\bar{v}| = \alpha$ one has $\mathsf{R} \models \varphi(\bar{f})$ iff $F \models \psi(\Omega_{\bar{v}}(\bar{f}))$. By model completeness, $\psi(\bar{X})$ is equivalent in F to a Σ_1-formula $\psi'(\bar{X})$, that is $\mathsf{R} \models \varphi(\bar{f})$ iff $F \models \psi'(\Omega_{\bar{v}}(\bar{f}))$. Translating back from Λ_F to Λ_R via Fact 8.4 one obtains $\chi \equiv \tau^0_{FR}(\psi'(\bar{X}))(\bar{X},\bar{z})$ in Σ_1 such that $F \models \psi'(\bar{B})$ iff $\mathsf{R} \models \chi((\omega_{\bar{v}}(\bar{b}),\bar{e})$ where $\bar{e}$ is the system of $*$-matrix units associated with $\bar{v}$, $B_k = \Omega_{\bar{v}}(f_k)$, $\bar{b}$ a (systematic) listing of all the entries of the B_K, and $\omega_{\bar{v}}$ the isomorphism of F onto $e_1\mathsf{R}e_1$. The a_{kij} are obtained from f_k as $\omega_{\bar{v}}^{-1}(e_{1i}e_if_ke_je_{j1})$. This allows to rewrite χ to $\chi'(\bar{x},\bar{z})$, also in Σ_1, such that $\mathsf{R} \models \chi'(\bar{f},\bar{e})$ iff $F \models \psi'(\bar{A})$ iff $\mathsf{R} \models \varphi(\bar{f})$ (namely, substitute $z_{1i}z_ix_kz_jz_{j1}$ for the entry x_{kij} of X_k). In view of Fact 3.1 one gets $\varphi(\bar{x})$ equivalent in R to $\exists\bar{z}.\, \rho_{\alpha R}(\bar{z}) \wedge \chi'(\bar{x},\bar{z})$ which is in Σ_1. This proves model completeness of R. The translation $\tau^{\exists}_{RL} \circ \tau^{\exists}_{LR}$ (cf. Theorems 6.2 and 7.5) yields then model completeness of L. □

The translations τ apply also in the case $n = 0$, providing translations between the first order theories of F, R, and L. Recall the assumptions of Theorem 10.4.

Corollary 10.5. Th(F) *is decidable if and only* Th(R) *is decidable if and only if, in case* $d \geq 3$, Th(L) *is decidable. The analogous result applies to* Σ_1*-fragments.*

Corollary 10.6. *Let* V_F *and* $V'_{F'}$ *be spaces admitting* $\perp$*-bases* $\bar{v}$ *and* $\bar{v}'$ *of cardinality* d *such that there is* $\alpha^{\#}(\bar{y}) \in \Lambda_F$ *defining* $|\bar{v}|$ *in* F *and* $|\bar{v}'|$ *in* F'. *Then* Th(F) = Th(F') *if and only* Th$(\mathsf{R}(V))$ = Th$(\mathsf{R}(V'))$ *if and only if, in case* $d \geq 3$, Th$(\mathsf{L}(V))$ = Th$(\mathsf{L}(V'))$.

Proof. The first equivalence follows from Fact 8.3 in one direction, from $\mathsf{R}(V) \cong F^{d\times d}_{|\bar{v}|}$ in the other. The second equivalence follows from Theorem 7.5. □

Corollary 10.7. *Let* Σ *an axiomatization of* Th(F).

(i) *An axiomatization of* Th(R) *is given by* $\tau^{\forall}_{F\alpha R}(\Sigma)$ *along with the finitely many axioms requiring that* R *is a regular* $*$*-ring of module height* d *admitting an* α*-system of matrix units* (*transferring this concept to abstract* $*$*-rings*).

(ii) *In case* $d \geq 3$, *an axiomatization of* Th(L) *is given by* $\tau^{\forall}_{F\alpha L}(\Sigma)$ *together with the finitely many axioms requiring that* L *is an involutive Arguesian lattice of height* d *admitting an* α*-frame.*

Th(F) *is finitely axiomatizable if and only* Th(R) *is finitely axiomatizable if and only if, in case* $d \geq 3$, Th(L) *is finitely axiomatizable.*

Proof. Consider the $\perp$-basis $\bar{v}$ required in Theorem 10.4 and the associated α-system $\bar{e}$ and α-frame $\bar{a}$. Let R' be a model of the axioms in (i). The matrix units $\bar{e}'$ yield a ring isomorphism $\mathsf{R}' \to {F'}^{d\times d}$ for the $*$-ring $F' = e'_1\mathsf{R}'e'_1$ which is a division $*$-ring since e'_1 is a minimal projection. Moreover, since $\bar{e}'$ is an α-system, this isomorphism is easily seen to be an isomorphism $\mathsf{R}' \to {F'}^{d\times d}_{\alpha}$ of $*$-rings. That is, up to isomorphism, $\mathsf{R}' = \mathsf{R}(V'_{F'})$ with $\perp$-basis $\bar{v}'$ satisfying the requirements of Corollary 10.6 and $\bar{e}'$ the α-system associated with $\bar{v}'$. Then, due to the translation of Σ into the language of R, one has Th(F') = Th$(e'_1\mathsf{R}'e'_1)$ = Th$(e_1\mathsf{R}e_1)$ = Th(F) and it follows Th(R) = Th(R') by Corollary 10.6.

Assume that Th(R) is finitely axiomatizable but that Th(F) is not. By the Compactness Theorem one can replace $\tau^{\forall}_{F\alpha R}(\Sigma)$ in the axiomatization of Th(R) by $\tau^{\forall}_{F\alpha R}(\Gamma)$ for some finite subset Γ of Σ. On the other hand, since Th(F) is not finitely axiomatizable, there are F' and $\psi \in \mathrm{Th}(F)$ such that $F' \models \Gamma' \cup \{\neg\psi\}$ where Γ' is Γ together with the finitely many axioms granting that ${F'}^d_\alpha$ satisfies the hypotheses of Corollary 10.6. If follows $\mathrm{Th}(\mathsf{R}') = \mathrm{Th}(\mathsf{R})$ for $\mathsf{R}' = {F'}^{d\times d}_\alpha$ but $\tau^{\forall}_{F\alpha R}(\psi) \notin \mathrm{Th}(\mathsf{R}')$, contradiction.

The lattice case is shown, similarly: Arguesian lattices of height ≥ 3 with a $\perp$-frame are coordinatized by vector spaces (cf. [2, Theorem 13.4]), the involution is then induced by a hermitean form (cf. [4, §14]). □

11. Counterexamples

Example 11.1. *The set* $M = \{u \in \mathsf{L} \mid \dim u = k\}$ *is positive primitive definable in* L*, without using involution, but not quantifier-free in the IL* L *if* $d \geq 3$*;* $\theta_{LF}(M) = \{A \in F^{d\times d} \mid \mathrm{rk}(A) = k\}$ *is quantifier-free definable in* F*.*

Proof. The positive claims are obvious (cf. Fact 5.1). Assume a quantifier-free definition φ_k of $\dim u = k$ in the IL $\mathsf{L}(V)$. The involutive sublattice S_u generated by any u consists of $\mathbf{0}, u \cap u^\perp, u, u^\perp, u + u^\perp, \mathbf{1}$ whence any $\varphi(x)$ is equivalent to a Boolean combination of $x \cap x^\perp = \mathbf{0}$, $x \leq x^\perp$, and $x^\perp \leq x$. Since V admits an orthogonal basis, for any $0 < k < d$ there is u_k with $\dim u_k = k$ and S_{u_k} the 4-element boolean algebra. In particular, $S_{u_1} \cong S_{u_2}$ with $u_1 \mapsto u_2$. Thus, both u_1 and u_2 satisfy φ_1 and φ_2. Contradiction. □

Example 11.2. *For a field* F*, the set* M *of all collinear harmonic quadruples is positive primitive definable in* L *without involution but not quantifier-free in the IL* L *if* $d \geq 3$*. Also,* $\theta_{LF}(M)$ *is quantifier-free definable in* F*.*

Proof. Let $\varphi(\bar{x})$ express that the $\bar{x}$ are a harmonic quadruple of points on a line – referring to points not on the line. Now, the involutive sublattice S generated by a quadruple of points on a line l is isomorphic to the direct product of a height 2 IL if $l \cap l^\perp = \mathbf{0}$; otherwise, S consist, besides $\mathbf{0}, \mathbf{1}$, only of points on l and lines through $l^\perp$. Thus, a quantifier-free formula equivalent in L to $\varphi(\bar{x})$ can state only some boolean combination of equalities between the x_i and their orthogonals on the line. □

12. The anisotropic case

We now derive some stronger results in case that V is *anisotropic*, that is, if $\langle v|v\rangle = 0$ implies $v = 0$. Then L is an ortholattice. Here, F may be any division ring with involution.

Corollary 12.1. *Fix* d*. For any* Σ_1*-formula* $\varphi(\bar{x})$ *there is a formula of the form* $\exists\bar{y}.\ t(\bar{x},\bar{y}) = \mathbf{0}$ *with a term* t *such that this formula is equivalent to* $\varphi(\bar{x})$ *within* L *for any anisotropic* V *of* $\dim V = d$*.*

Proof. This follows from Proposition 5.3. Here, we can simplify the construction observing that any equation $u = w$ is equivalent, within L, to $t = \mathbf{0}$ where $t = (u^\perp \cap (u+w)) + (w^\perp \cap (u+w))$. And, in view of Fact 5.2, given a frame $\bar{a}$, we have $u \neq w$ equivalent to $D_1(t, \bar{a})^\perp = \mathbf{0}$. Thus, referring to $\bar{a}$, any conjunction $\varphi_i(\bar{x})$ of atomic and negated atomic formulas is equivalent to $r_i(\bar{x}, \bar{a}) = \mathbf{0}$ for some terms r_i and $\bigvee_{i=1}^k \varphi_i(\bar{x})$ to $D_k(r_1(\bar{x}, \bar{a}), \ldots, r_k(\bar{x}, \bar{a})) = \mathbf{0}$. Finally, reference to $\bar{a}$ can be replaced by quantification $\exists \bar{z}$, bounded by the equation $r_0(\bar{z}) = \mathbf{0}$ comprising the equations defining a frame. □

Proposition 12.2. *For any formula $\varphi(\bar{x}) \in \Lambda_L$ there is a quantifier-free formula $\hat{\varphi}(\bar{x}) \in \Lambda_L$ equivalent to $\varphi(\bar{x})$ in L for any anisotropic V of $\dim V = 2$ over an infinite field F.*

Proof. We show that the class $\mathcal{C}$ of all infinite ortholattices of height 2 admits quantifier elimination. Observe that here both $z = x \vee y$ and $z = x \wedge y$ are equivalent to quantifier-free formulas in the language with operation symbols $\mathbf{0}, \mathbf{1}$, only; and the ortholattices L reduce to infinite sets with constants $\mathbf{0}, \mathbf{1}$ and a fixedpoint-free involution interchanging these. In particular, for any assignments $\bar{x} \mapsto \bar{a}^i$ in $L_i \in \mathcal{C}$ of equal cardinality there is an isomorphism $\omega : L_1 \to L_2$ with $\omega(\bar{a}^1) = \bar{a}^2$ provided that these assignments satisfy the same quantifier-free formulas – choose $\omega(b^\perp) = (a_i^2)^\perp$ if $b = a_i^1$ and match the remaining pairs of orthocomplements of L_1 with those of L_2. Thus, (c) of [7, Theorem 8.4.1] together with the Löwenheim-Skolem-Theorem apply to prove quantifier elimination. □

In the anisotropic case, R owns also the operation $f \mapsto f^+$ of Moore-Penrose-Rickart *pseudo-inversion*, uniquely determined by well known identities — we write R^+ if that operation is added and Λ_R^+ for the associated first order language.

Theorem 12.3. *Fix $d \geq 3$. For any Σ_1-formula $\psi(\bar{x}) \in \Lambda_R^+$ there is a term $p(\bar{x}, \bar{y}) \in \Lambda_R^+$ such that $\psi(\bar{x})$ is equivalent to $\exists \bar{y}\, p(\bar{x}, \bar{y}) = 0$ within R^+ for any anisotropic V of $\dim V = d$.*

Proof. In view of the Λ_R-equations having the pseudo-inverse as unique solution, we may replace any equation in Λ_R^+ by a positive primitive formula in Λ_R. Thus, we may assume that $\psi(\bar{x})$ is a Σ_1-formula in Λ_R. By Theorem 7.5, $\tau_{RL}(\psi)$ is a Σ_1-formula in Λ_L which, by Corollary 12.1 is equivalent within L to one of the form $\exists \bar{y}\, t(\bar{x}, \bar{y}) = \mathbf{0}$. Now, within R^+, for any $f \in \mathsf{R}$ one has $L(f)\mathsf{R} = f\mathsf{R}$ and $\mathsf{R}R(f) = \mathsf{R}f$ with left projection $L(f) := ff^+$ and right projection $R(f) := f^+f$ and the fundamental operations of L can be expressed by $*$-ring terms with the additional operations of left and right projection [1, Proposition §I.3.7]; in particular, they can be expressed by terms of Λ_R^+. Thus, $t(\bar{x}, \bar{y})$ translates into a term $p(\bar{x}, \bar{y})$ in Λ_R^+ and $\exists \bar{y}\, t(\bar{x}, \bar{y}) = \mathbf{0}$ into $\exists \bar{y}\, p(\bar{x}, \bar{y}) = 0$. □

13. Open problems

Of course, an interpretation of L within R can also be given considering joins and meets as smallest upper and greatest lower bounds. Though, this would not preserve type Π_1.

Problem 13.1. Is there a map $\tau : \Lambda_L \to \Lambda_R$ such that $\tau(\varphi)$ is in Π_1 if so is φ and $\tau(\varphi)$ defines $\theta_{LR}(M)$ if φ defines M?

For anisotropic V, one can replace θ_{LR} by a bijection π of L onto the set P of self-adjoint idempotents of R. If the operation of Moore-Penrose-Rickart inversion is added to R, the following has a positive answer.

Problem 13.2. Is there a map $\tau : \Lambda_L \to \Lambda_R$ such that $\tau(\varphi)$ is in Π_1 if so is φ and $\tau(\varphi)$ defines $\pi(M)$ (within P) if φ defines M?

In the presence of an orthonormal basis $\bar{v}$, for any $*$-ring term $t(\bar{x})$ there are $*$-ring terms $t_{ij}(\bar{X})$ such that $\Omega_{\bar{v}}(t(\bar{f})) = (t_{ij}(\bar{A}))_{ij}$ where $A_k = \Omega_{\bar{v}}(f_k)$. Thus, in this case Fact 8.1 can be improved by a map τ which preserves types and quantifier freeness.

Problem 13.3. In the presence of an orthonormal basis $\bar{v}$, is there a map $\tau : \Lambda_L \to \Lambda_F$ such that $\tau(\varphi)$ defines $\theta_{L\bar{v}F}(M)$ if φ defines M and such that $\tau(\varphi)$ is quantifier-free if so is φ?

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