DIRECT FINITENESS OF REPRESENTABLE REGULAR $*$-RINGS. ERRATUM

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Dedicated to the memory of Susan M. Roddy

ABSTRACT. We show that a von Neumann regular ring with involution is directly finite provided that it admits a representation as a $*$-ring of endomorphisms of a vector space endowed with a non-degenerate orthosymmetric sesquilinear form.

0. ERRATUM

There is no proof of Thm 3 since there is no proof of $et = er$ in Lemma 4. Actually, there is a counterexample given is the new paper: Direct finiteness of representable regular $*$-rings: A counterexample.

1. INTRODUCTION

A $*$-ring, that is ring with involution, is called *finite* if $rr^* = 1$ implies $r^*r = 1$. This is a basic notion in the classification of von Neumann algebras; in particular, as shown by Murray and von Neumann, a finite von Neumann algebra admits a finite $*$-ring of quotients. This ring is also $*$-regular and *directly finite*, that is $rs = 1$ implies $sr = 1$. As Ara and Menal [1] have shown, any $*$-regular ring is at least finite, while direct finiteness remains an open question, as stated by Handelman [5, Problem 48]. The present note gives a positive answer for certain [von Neumann] regular rings with involution.

For $*$-rings, there is a natural and well established concept of [faithful] representation in a vector space V_F endowed with a non-degenerate orthosymmetric sesquilinear form: an embedding into the $*$-ring $\mathsf{End}^*(V_F)$ of those endomorphisms of V_F which admit an adjoint. Famous examples are due to Gel'fand-Naimark-Segal (C^*-algebras in Hilbert space) and Kaplansky (primitive $*$-rings with a minimal right ideal). For

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$*$-regular rings of classical quotients of finite Rickart C^*-algebras existence of representations has been established in [8], jointly with M. Semenova. N. Niemann [11, 6] has shown that a subdirectly irredcucible $*$-regular ring is representable if and only its ortholattice lattice of principal right ideals is representable within the ortholattice of closed subspaces of some V_F.

According to joint work with Susan M. Roddy [7], representability of modular ortholattices is equivalent to membership in a variety generated by finite height members. Using ideas from Tyukavkin [12], the analogue for $*$-regular rings was obtained by F. Micol [10]. Here, we rely on the presentation given in [9]: A regular $*$-ring can be represented within V_F, respectively some ultrapower thereof, if and only if it can be obtained via formation of ultraproducts, regular $*$-subrings, and homomorphic images from the class of the $\mathsf{End}^*(U_F)$, U_F ranging over finite dimensional non-degenerate subspaces of V_F. It will be shown that direct finiteness is inherited under the particular construction of [9] which proves the reduction to finite dimensions.

Thanks are due to Ken Goodearl for the hint to reference [4].

2. Preliminaries

When mentioning rings, we always mean associative rings R with unit 1_R, considered as a constant. In any ring, if a has a *left inverse* x, that is $xa = 1$, then a is *left cancellable*, that is $ay = az$ implies $y = z$. R is *directly finite* if, for all $r, s \in R$, $sr = 1$ implies $rs = 1$. In such ring, a has a left inverse if and only if a is a unit (and $x = a^{-1}$). The endomorphism ring $\mathsf{End}(V_F)$ of a vector space is directly finite if and only if $\dim V_F < \omega$. A *$*$-ring* is a ring endowed with an involution $r \mapsto r^*$; an element e of such a ring is a *projection*, if $e = e^2 = e^*$.

A ring R is [*von Neumann*] *regular* if for any $a \in R$, there is an element $x \in R$ such that $axa = a$; such an element is called a *quasi-inverse* of a. If, for all a, x can be chosen a unit, then R is *unit regular*. Examples of such are the $\mathsf{End}(V_F)$, $\dim V_F < \omega$. A detailed discussion of direct finiteness in regular rings is given in Goodearl [5]. A regular $*$-ring is *$*$-regular* if $xx^* = 0$ only for $x = 0$.

In a regular ring, any left cancellable a has a left inverse; indeed $axa = a$ implies $xa = 1$. If $xa = 1$ and $aua = a$ with a unit u then $a = u^{-1}$ and $x = a^{-1}$. It follows

Fact 1. *In a directly finite regular ring every left cancellable element is a unit – similarly on the right. Every unit regular ring is directly finite.*

We recall some basic concepts and facts from [9] (here, Λ can be taken the $*$-ring of integers). In the sequel, F will be a division ring endowed with an involution and V_F a [right] F-vector space of $\dim V_F > 1$ endowed with a non-degenerate sesquilinear form $\langle . \mid . \rangle$ which is *orthosymmetric*, that is $\langle v \mid w\rangle = 0$ iff $\langle w \mid v\rangle = 0$. Such space will be called *pre-hermitean* and denoted by V_F, too. Within such space, any endomorphism φ has at most one adjoint φ^*; and these φ form a subring of $\mathsf{End}(V_F)$ which is a $*$-ring $\mathsf{End}^*(V_F)$ under the involution $\varphi \mapsto \varphi^*$. For $\dim V_F < \omega$, $\mathsf{End}^*(V_F)$ contains all of $\mathsf{End}(V_F)$. A [faithful] *representation* of a $*$-ring R is an embedding of R into some $\mathsf{End}^*(V_F)$.

Consider a linear subspace U of V_F, $1 < \dim U_F < \omega$. With the induced sesquilinear form, U_F is pre-hermitean if and only if $V = U \oplus U^\perp$; in particular, there is a projection $\pi_U \in \mathsf{End}^*(V_F)$ such that $U = \mathsf{im}\,\pi_U$ and such that the inclusion map $\varepsilon_U : U \to V$ is the adjoint of π_U (here, considerd as a map $V \to U$). We write in this case $U \in \mathbb{O}(V_F)$ and say that U is a *finite-dimensional orthogonal summand*. A crucial fact is that V_F is the directed union of the U_F, $U \in \mathbb{O}(V_F)$. Let $C(F)$ denote the center of F and, for $U \in \mathbb{O}(V_F)$,

$$\begin{aligned} B_U &= \{\varepsilon_U\varphi\pi_U + \lambda\,\mathsf{id}_V \mid \varphi \in \mathsf{End}^*(U_F),\ \lambda \in C(F)\} \\ &= \{\psi \in \mathsf{End}(V_F) \mid \psi(U) \subseteq U\ \&\ \exists\lambda \in C(F)\ \psi|U^\perp = \lambda\,\mathsf{id}_{U^\perp}\} \end{aligned}$$

(φ and ψ are related via $\varphi(v) = \psi(v) + \lambda v$). Thus, B_U is a $*$-subring of $\mathsf{End}^*(V_F)$ and embeds into $\mathsf{End}^*(W_F)$ for any $W \in \mathbb{O}(V_F)$, $U \subset W \neq U$. In particular, B_U is directly finite. Moreover, B_U is unit regular; indeed, $\chi \in B_U$ is a unit quasi-inverse of $\psi \in B_U$ if $\chi|U$ is one of $\psi|U$ and $\chi|U^\perp = (\psi|U^\perp)^{-1}$ (considering these as endomorphisms of U and $U^\perp$, respectively). We put

$$\begin{aligned} J(V_F) &= \{\varphi \in \mathsf{End}^*(V_F) \mid \dim \mathsf{im}\,\varphi < \omega\} \\ \hat{J}(V_F) &= \{\varphi + \lambda\,\mathsf{id}_V \mid \varphi \in J(V_F),\ \lambda \in C(F)\} \end{aligned}$$

According to [9, Proposition 4.4] $\hat{J}(V_F)$ is a $*$-subring of $\mathsf{End}^*(V_F)$ and $J(V_F)$ is an ideal of $\mathsf{End}^*(V_F)$ closed under the involution. Also, the following holds.

($*$) For any finite $\Phi \subseteq J(V_F)$ there is $U \in \mathbb{O}(V_F)$ such that $\varphi = \pi_U\varphi = \varphi\pi_U$ for all $\varphi \in \Phi$.

Thus, $\hat{J}(V_F)$ is the directed union of the B_U, $U \in \mathbb{O}(V_F)$, whence unit-regular.

Lemma 2. *Every regular $*$-subring R of $\mathsf{End}^*(V_F)$ extends to a regular $*$-subring $\hat{R}$ of $\mathsf{End}^*(V_F)$ containing $\hat{J}(V_F)$ and such that $J(V_F)$ is an ideal of $\hat{R}$.*

Proof. $\{\lambda\varphi \mid \varphi \in R,\, \lambda \in C(F)\}$ is a regular $*$-subring R' of $\mathsf{End}^*(V_F)$ and [9, Proposition 4.5] applies to R'. ☐

Recall that a [faithful] representation of a $*$-ring R within a pre-hermitian space V_F is an embedding $\varepsilon\colon R \to \mathsf{End}^*(V_F)$. It is convenient to consider representations as unitary R-F-bimodules ${}_RV_F$ (where the action of R is given as $rv = \varepsilon(r)(v)$) with sesquilinear form on V_F; that is, a 3-sorted structure with sorts V, R, and F. Considering a $*$-subring A of R we may add a fourth sort, A, and the embedding map. to obtain $({}_RV_F; A)$. Any elementary extension $({}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A})$ is again such a structure, that is, a representation of $\tilde{R}$ and a $*$-ring $\tilde{A}$ which may be considered as $*$-subring of $\tilde{R}$. It is a *modestly saturated* extension if, for each set $\Sigma(\bar{x})$ of first order formulas in finitely many [sorted] variables and with parameters from $({}_RV_F; A)$, one has $({}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A}) \models \exists\bar{x}.\Sigma(\bar{x})$, provided that $\Sigma(\bar{x})$ is *finitely realized* in $({}_RV_F; A)$, that is $({}_RV_F; A) \models \exists\bar{x}.\Psi(\bar{x})$ for every finite subset $\Psi(\bar{x})$ of $\Sigma(\bar{x})$. Such extension always exists, cf. [2, Corollary 4.3.1.4].

3. Main result

Theorem 3. *Every representable regular $*$-ring is directly finite.*

Proof. We recall the relevant steps of the proof of [9, Theorem 10.1]. Given a representation ${}_RV_F$ of the regular $*$-ring R, we may assume that $\dim V_F \geqslant \omega$. In view of Proposition 2, we also may assume that R is a $*$-subring of $\mathsf{End}^*(V_F)$ containing $A = \hat{J}(V_F)$ and having ideal $J(V_F)$. Choose $({}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A})$ a modestly saturated elementary extension of $({}_RV_F; A)$.

Let J_0 denote the set of projections in $J(V_F)$. For $a \in \tilde{A}$ and $r \in R$, we put $a \sim r$ if $ae = re$ and $a^*e = r^*e$ for all $e \in J_0$. According to Claims 1–4 in the proof of [9, Theorem 10.1], $S = \{a \in \hat{A} \mid a \sim r \text{ for some } r \in R\}$ is a regular $*$-subring of $\hat{A}$ and there is a surjective homomorphism $g : S \to R$ such that $g(a) = r$ if and only if $a \sim r$.

Being an elementary extension of A, $\tilde{A}$ is directly finite and so is its subring S. Now, assume $sr = 1$ in R. Consider a finite set $E \subseteq J_0$. According to $(*)$, there is $e \in J_0$ such that $ef = f$ and $er^*f = r^*f$ for all $f \in E$. Take $a = re$ and observe that $af = ref = rf$ and $a^*f = er^*f = r^*f$ for all $f \in E$. Thus, the set

$$\Sigma(x) = \big\{[xe = re] \;\&.\; [x^*e = r^*e]\&[\exists y.\, yx = 1] \mid e \in J_0\big\}$$

of formulas with a free variables x, y of type A and R, respectively, is finitely realized in $({}_RV_F; A)$. Indeed, given a finite subset Ψ of $\Sigma(x)$ there is finite $E \subseteq J_0$ containing all $f \in J_0$ which occur in $\Sigma(x)$; choose

e for E as above, g, t, u according to Lemma 4 below, and $x = t+1-g$, $y = u + 1 - g$.

By saturation, there are $a \in \tilde{A}$ and $b \in \tilde{R}$ with $ba = 1$ and $a \sim r$, whence $a \in S$ and $g(a) = r$. Moreover, a is left cancellable in $\tilde{R}$ whence in the subring S and so a unit of S by regularity. Hence, $r = g(a)$ is a unit of R and $s = r^{-1}$ whence $rs = 1$. □

Lemma 4. *Consider a regular ring R with ideal I such that each eRe, $e \in I$, is unit-regular. Then for any $r, s \in R$ with $sr = 1$ and idempotent $e \in I$ there are an idempotent $g \in I$, $e \in gRg$, and $t, u \in gRg$ such that $ut = g$, $te = re$, and $et = er$.*

Proof. Following [3] we consider R the endomorphism ring of a (right) R-module, namely $M_R = R_R$. Observe that r is an injective endomorphism of M_R. Let $U = \mathsf{im}\, e$, $W_1 = U + r^{-1}(U)$, $W_2 = r(W_1)$; in a particular, these are submodules of M_R and $r|W_1$ is an isomorphism of W_1 onto W_2. By (the proof of) [4, Lemma 2] there is an idempotent $g \in I$ such that $e, re, se \in S := gRg$. Put $W = \mathsf{im}\, g$ which is a submodule of M_R, and an S-module under the induced action of S, so that $S = \mathsf{End}(W_S) = \mathsf{End}(W_R)$.

By hypothesis, S is unit-regular whence, in particular, directly finite. Due to regularity of S, for any $h \in S$ and S-linear map $\phi : hS \to W$ there is an extension $\bar{\phi} \in S$, namely $\bar{\phi}|(g - h)S = 0$. Due to direct finiteness, any injective such ϕ has an inverse in S. Also, by regularity, the submodules $W_1 = \mathsf{im}\, e + \mathsf{im}\, se$ and $W_2 = \mathsf{im}\, e + \mathsf{im}\, re$ are of the form $W_i = \mathsf{im}\, g_i$ with idempotents $g_i \in S$.

Let $X_i = \mathsf{im}(g - g_i)$ whence $W = W_i \oplus X_i$. Since $r|W_1 : W_1 \to W_2$ is an S-linear isomorphism, according to [3, Theorem 3] there is an S-linear isomorphism $\varepsilon : X_1 \to X_2$. Put $\delta(v) = \varepsilon(v) + g_2(r(v))$ for $v \in X_1$. If $\delta(v) = w \in W_2$ then $\varepsilon(v) \in W_2 \cap X_2$ whence $\varepsilon(v) = 0$ and $v = 0$; it follows that δ is an S-linear isomorphism of X_1 onto $Y \subseteq W$ where $Y \cap W_2 = 0$. Also, $g_2(\delta(v)) = g_2(r(v))$ since $g_2(X_2) = 0$. Define $t \in S$ as $t(v + w) = r(v) + \delta(w)$ for $v \in W_1$ and $w \in X_1$. t is injective whence it has inverse u in S. □

An example of a simple regular $*$-ring which is not finite is obtained as follows: Let V_F a vector space of countably infinite dimension, and $R = \mathsf{End}(V_F)/J(V_F)$. Of course, R is not directly finite. Define the involution on the direct product $R \times R^{op}$ by exchange: $(r, s)^* = (s, r)$ to obtain the $*$-ring S. Now, if $rs = 1$ but $sr \neq 1$ then $xx^* = 1$ but $x^*x \neq 1$ in S for $x = (r, s)$.

Problem 1. *Is every simple directly finite $*$-regular ring representable?*

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