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On the Word Problem for the Modular Lattice with Four Free Generators

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Dedekind [4] introduced modular lattices as an abstraction of systems of submodules of modules and determined the free modular lattice FM(3) on three generators which has 28 elements. Birkhoff [2, 3], who showed that FM(4) is infinite and contains even infinite chains, raised the question of the word problem for the FM(n)'s. Finitely presented modular lattices with unsolvable word problems have been constructed by Hutchinson [14] and Lipshitz [17], even with five generators by Hutchinson [15]. Then, Freese [7] proved that FM(n) has an unsolvable word problem for $n \geqq 5$.
Here, we deal with the remaining case $n=4$. Gel'fand and Ponomarev [9] found generating quadruples for all finite dimensional rational projective geometries. Freese [8] pointed out that there are uncountably many simple modular lattices with four generators. A way of generating subgroup lattices of finite primary abelian groups with at least three independent maximal cycles has been indicated in [12]. The latter and the Hall and Dilworth [10] gluing technique are the basis for our main result.

Theorem 1. *The word problem for the modular lattice with four free generators is recursively unsolvable.*

The proof is by reduction to the word problem for two-generator groups. We use the interpretation of rings (group rings, of course, here) in modular lattices by the von Neumann ring construction. The crucial tool for the result on free generators is Freese's [7] ingenious device for forcing group relations.

In the first two sections we present the concepts of frames and skew frames needed for this interpretation as well as the results which allow the retraction into FM(4). The reader interested in a finitely presented lattice, only, may skip the latter and concentrate on the definitions of the particular lattice words. These are modelled according to the basic example constructed in Sect. 4. It might be a good advice to compare the meaning of the lattice words at each step – indeed, this is the essence of the proof of Lemma 18 below.

1. Frames and Coordinates

Elements a_i, c_{ij} $(1 \leqq i \neq j \leqq n)$ of a modular lattice M are said to form a *frame* Φ of order n if for all distinct i, j, k

$$a_i \sum (a_j \mid j \neq i) = 0, \qquad a_i c_{ij} = 0, \qquad a_i + c_{ij} = a_i + a_j,$$

$$c_{ij} = c_{ji}, \qquad c_{ik} = (c_{ij} + c_{jk})(a_i + a_k),$$

where 0 denotes the infimum of Φ. Clearly, the a_i with a suitable choice of the c_{ij}'s determine the frame, already. In particular, a_i, c_{1i} $(1 \leqq i \leqq n)$ satisfying the relations of the first three types can be completed to a frame in a unique way.

The *coordinate domain* $R_{ij} = R_{ij}(\Phi)$ consists of all x in the interval $[0, a_i + a_j]$ with $x a_j = 0$ and $x + a_j = a_i + a_j$. If $n \geqq 4$ then each R_{ij} carries the structure of an associative ring $(R_{ij}, \oplus, \otimes)$ with zero a_i and unit c_{ij}. Namely, independent of the choice of $k \neq i, j$,

$$x \oplus y = ((x + c_{ik})(a_j + a_k) + (y + a_k)(a_j + c_{ik}))(a_i + a_j),$$

$$x \otimes y = ((x + c_{jk})(a_i + a_k) + (y + c_{ik})(a_j + a_k))(a_i + a_j).$$

The rings R_{ij} and R_{ik} are isomorphic with x being mapped onto $(x + c_{jk})(a_i + a_k)$. All this was shown by von Neumann [18] for complemented and extended by Artmann [1] to general M – see Freese [6], too.

By the *coordinate ring* $R(\Phi)$ of Φ we shall mean $R_{12}(\Phi)$. Frames are an important tool in the equational theory of modular lattices since they are projective configurations as has been shown by Huhn [13]. Additional relations can be forced by Freese's technique of modification [5].

For a fixed index k let b_k with $0 \leqq b_k \leqq a_k$ be given. Define $b_i = a_i(b_k + c_{ik})$ for $i \neq k$ and $b = \sum (b_i \mid i \leqq n)$. This yields two new frames $\Phi_b : b_i, b_{ij}$ with $b_{ij} = b c_{ij}$ and $\Phi^b : a_i + b, c_{ij} + b$. We say that Φ_b arises from Φ by *reduction* with b_k or b. Note that $b_i = a_i b$ and $b_{ij} = c_{ij}(b + a_j)$. Reduction with $b_l = a_l(b_k + c_{kl})$ yields the same result as that with b_k.

Observe that for $b_k \leqq d_k$ one has $(\Phi_d)_b = \Phi_b$, $(\Phi^b)^d = \Phi^d$, and $(\Phi_d)^b = (\Phi^b)_d$. In particular, any iterated modification can be obtained in two steps.

Now, let Φ be a frame of order $n \geqq 4$. An element g of the coordinate ring $R(\Phi)$ is invertible if and only if $a_1 + g = a_1 + a_2$ and $a_1 g = 0$ – and the inverse is given by a lattice polynomial in Φ, cf. Freese [6, 2.3]. An invertible element of $R(\Phi)$ is *stable* if $g + b_1 = g + b_2$ for every b_1 with $0 \leqq b_1 \leqq a_1$ and $b_2 = a_2(b_1 + c_{12})$. Observe that this definition involves a_1, a_2, c_{12} and the lattice interval $[0, a_1 + a_2]$ only, but not the ring structure of $R(\Phi)$. The importance of this concept lies in the following result of Freese [7].

Lemma 2. *Let z_1, z_2 be stable elements of $R(\Phi)$ and consider any modification Φ^b of Φ with $0 \leqq b_1 \leqq a_1$. Then the elements $z_i + b$ are stable in $R(\Phi^b)$ and for every two variable group word w one has $w(z_1, z_2) + b = w(z_1 + b, z_2 + b)$ where the first is evaluated in $R(\Phi)$ and the second in $R(\Phi^b)$.*

Lemma 3. *If g is stable in $R(\Phi)$ and Φ_d arises from Φ by reduction with $d_1 \leqq a_1$ then gd is stable in $R(\Phi_d)$. If Φ has finite characteristic then invertible integers are stable in $R(\Phi)$. Products and inverses of stable elements are stable.*

Proof. $d_1+g\geqq d_2$ since g is stable, whence $d_1+dg=d_1+d_2$. Now, let $b_1\leqq d_1$. Then $b_1+dg=d(b_1+g)=b_2+dg$. The next claim is Lemma 1.4 in [7]. Of course, we may speak of stable elements in R_{ij}, too. Then, stability is preserved under the projective isomorphisms between the R_{ij}'s and the formulas for products and inverses [6] show that these operations preserve stability.

Here, too, to build up stable elements we use lattice words related to integers. The following property of these words will enable us to force the relations needed for stability. A lattice word w in variables x_i, x_{ij} is called *invariant* (for the frame Φ) if for every modification of Φ with $b_k\leqq a_k$ one has $bw(\Phi)=w(\Phi_b)$ where $w(\Phi)$ denotes the element of M obtained by substitution of the frame elements. Also, we write $w=w(\Phi)$. We say that w is *i-invariant* if in addition $wb=w(b_i+\sum(a_j\,|\,j\neq i))$. a_i and c_{ij} are i-invariant.

The following proposition is a direct consequence of the fact that if $w\leqq a_i+a_j$ then

$$wb+b_j=b(w(b_i+a_j)+a_j)=b(w+a_j)(b_i+a_j)=b_i+b_j.$$

Proposition 4. *If w is i-invariant in $R_{ij}(\Phi)$ then for any $b_k\leqq a_k$ the element wb is i-invariant in $R(\Phi_b)$.*

Now, define for a given frame Φ of order $n\geqq 3$

$$c_{123}=(c_{12}+a_3)(c_{13}+a_2),\qquad r\oplus 1=((r+a_3)(c_{13}+a_2)+c_{23})(a_1+a_2),$$

$$\sigma s=(s+c_{123})(a_2+a_3),\qquad r\boxplus s=((r+a_3)(c_{13}+a_2)+\sigma(s\oplus 1))(a_1+a_2).$$

Lemma 5. *If r and s are 1-invariant in R_{12} then $r\oplus 1$ and $r\boxplus s$ are 1-invariant in R_{12} and σs is 3-invariant in R_{32}.*

Proof. That $r\oplus 1$ is invariant in R_{12} has been shown in [5]. Moreover, we have by the 1-invariance of r and c_{13} for any $b_k\leqq a_k$

$$\begin{aligned}(r+a_3)(c_{13}+a_2)(b_1+a_2+a_3)&=(rb+a_3)(b_{13}+a_2)\\&=(rb+a_3(b+a_2))(b_{13}+a_2(b+a_3))\\&=(rb+b_3)(b_{13}+b_2)\leqq b\end{aligned}$$

whence by the 3-invariance of c_{23}

$$(r\oplus 1)(b_1+a_2)=((rb+b_3)(b_{13}+b_2)+c_{23})(b_1+a_2)=(r\oplus 1)(\Phi_b).$$

Thus, $r\oplus 1$ is 1-invariant in R_{12}. Easily, one gets

$$\sigma s+a_2=(s+c_{123}+a_2)(a_2+a_3)=(a_1+a_2+c_{123})(a_2+a_3)=a_2+a_3,$$

$$(a_1+a_2+b_3)c_{123}=bc_{123}=c_{123}(\Phi_b).$$

Now, the 3-invariance of σs follows from

$$(a_2+b_3)\sigma s=(s+bc_{123})(a_2+b_3)=(s(b+a_2)+bc_{123})(a_2+b_3)=\sigma s(\Phi_b)$$

and letting $b=0$ one sees that σs is in R_{32}. The 1-invariance of $r\boxplus s$ in R_{12} is derived in the same way as that of $r\oplus 1$ using the 3-invariance of $\sigma(s\oplus 1)$.

For the next definition we need some preparation.

Lemma 6. *Let $p \geqq 3$ be a prime and $q = p^2$. Then the additive group of integers modulo q equipped with the partial operations*

$$r \mapsto r \oplus 1 = r + 1, \quad (r, s) \mapsto r \boxplus s = r + s, \quad r \not\equiv -1, \quad s \not\equiv 0 \bmod p$$

is generated by the zero 0.

Proof. Let X be a closed set containing 0 and $x > 0$ a minimal representant not in X. Then $x \equiv 0 \bmod p$. On the other hand $1 \in X$, whence $x - 1 \in X$. Since $x - 1 \not\equiv 0, -1 \bmod p$ we have $x = x - 1 + 1$ in X, a contradiction.

Now, fix a prime $p \geqq 3$, $q = p^2$, and an expression $\tilde{q}$ in $\oplus 1$ and $\boxplus$ such that $\tilde{q}(0) \equiv -1 \bmod q$. Let $\bar{q}$ be the corresponding lattice word built up from a_1 by means of $\oplus 1$ and $\boxplus$. Then $\bar{q}$ is 1-invariant in R_{12} by Lemma 5. We define

$$\boxminus 1_{32} = (a_1 + c_{123})(a_2 + a_3), \qquad r \boxminus 1 = ((r + a_1)(c_{13} + a_2) + \bar{q})(a_2 + a_3),$$

$$p_{32} = (\ldots(\boxminus 1_{32} \boxminus 1)\ldots) \boxminus 1, p \text{ minuses}.$$

Lemma 7. *p_{32} is 3-invariant in R_{32}.*

Proof. $\boxminus 1_{32}$ is 3-invariant in R_{32}, obviously. Hence it suffices to show that with r also $r \boxminus 1$ is 3-invariant in R_{32}. This, we derive as in the proof of Lemma 5 from

$$\bar{q}(a_2 + c_{13} + b_3) = \bar{q}(a_2 + c_{13} + b_1) = \bar{q}(b_1 + a_2 + (a_1 + a_2)c_{13}) = \bar{q}b,$$

$$(r + a_1)(c_{13} + a_2)(a_2 + b) = (rb + b_1)(b_{13} + b_2).$$

2. Skew Frames and Stability

If $\Phi : a_i, c_{ij}$ and $\Phi : a'_i, c'_{ij}$ are frames of order 3 and 4, respectively, such that the subframe a'_i, c'_{ij} $(i, j \leqq 3)$ of Φ' arises from Φ by reduction with any a'_i and if $a'_4 \sum a_i = 0$ then we say that Φ', Φ is a *skew frame* of type (4, 3). Notice that for $b'_1 \leqq a'_1 \leqq b_1 \leqq a_1$ the modification $\Phi'_{b'}, \Phi_b$ yields again a skew frame of type (4, 3).

Lemma 8. *The modular lattice freely generated by a skew frame of type* (4, 3) *is a projective modular lattice. It has four generators.*

Proof. Let F be a modular lattice, φ a homomorphism of F onto M, and $\mathbf{\Phi}', \mathbf{\Phi}$ a skew frame of type (4, 3) in M. Choose A, B in F with $\varphi B = \mathbf{a}'_4$ and $\varphi A = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$. Then $\varphi AB = 0$. Now, choose a frame $\tilde{\Phi} : \tilde{a}_i, \tilde{c}_{ij}$ of order 3 in $[AB, A]$ which is mapped onto $\mathbf{\Phi}$, cf. [5, 1.6]. Write $u = \tilde{a}_1 \tilde{a}_2$, $v = \sum \tilde{\Phi}$ and choose x in $[u, \tilde{a}_3 + B]$ with $\varphi x = \mathbf{c}'_{34}$. Let $b_3 = \tilde{a}_3 x$, $y = (x + \tilde{a}_3)(b_3 + B)$, $b_4 = x(b_3 + B)$, $z = \tilde{a}_3(x + y) + b_4$. Then x, y, z yield an M_3 with $xy = b_4$ and $\varphi b_4 = \varphi b_3 = 0$, $\varphi y = \mathbf{a}'_4$, $\varphi z = \mathbf{a}'_3$. Let $\tilde{\Phi}^b$ arise from $\tilde{\Phi}$ by reduction with b_3. Now,

$$v(b + b_4) = b + vx(b_3 + B) = b + xb_3 = b,$$

$$(v + b_4)(x + y + b) = b + b_4 + v(x + y) = b + b_4 + \tilde{a}_3(x + y) = b + z,$$

$$(b + b_4)(x + y) = b_4 + b(x + y) = b_4 + b\tilde{a}_3 = b_4 + b_3 = b_4.$$

Thus, $\tilde{\Phi}^b + b_4$ forms a frame $\Phi : a_i, c_{ij}$ of order 3 mapped onto Φ. Also $a'_4 = y + b$ $c'_{34} = x + b$, $a'_3 = z + b$ yield an M_3 with bottom $b + b_4$, and $(a'_3 + a'_4) \sum \Phi = a'_3 \leqq a_3$

which is mapped onto $\mathbf{a}_4', \mathbf{c}_{34}', \mathbf{a}_3'$. Let a_i', c_{ij}' $(i,j \leqq 3)$ arise from Φ by reduction with a_3'. Then, one gets a skew frame Φ', Φ of type (4,3) mapped onto $\mathbf{\Phi}', \mathbf{\Phi}$ under φ.

As generators we may choose

$$a = c_{23}, \quad b = a_1 + a_2, \quad c = a_3 + a_4', \quad d = c_{13} + c_{24}'.$$

Namely,

$$a_3 = (a+b)c, \quad a_2 = (a+a_3)b, \quad a_1 = (a_3+d)b, \quad c_{13} = (a_1+a_3)d,$$
$$a_4' = (d+a_2)c, \quad c_{24}' = (a_2+a_4')d, \quad a_2' = a_2(a_4'+c_{24}').$$

We say that a skew of type (4,3) has *characteristic* $p \times p$ if

$$p \otimes c_{12}' = a_1', \quad p_{32} \geqq a_3', \quad a_3 + p_{32} = a_2' + p_{32} = a_2' + a_3.$$

Here, p is a prime, $p \geqq 3$, and p_{32} is defined as in Lemma 7 depending on the choice of the lattice term $\tilde{q}$. Also, $p \otimes c_{12}'$ is $\oplus 1$ p-times applied to a_1' in Φ'. Thus Φ' has characteristic p in the sense of Freese [5]. These relations describe a projective configuration. More precisely we have the following.

Lemma 9. *Let F be a modular lattice, φ a homomorphism of F onto M, and Φ', Φ a skew frame of type* (4,3) *in F such that its image $\mathbf{\Phi}', \mathbf{\Phi}$ in M has characteristic $p \times p$. Then there are b_1' and b_1 in F with $0 \leqq b_1' \leqq a_1' \leqq b_1 \leqq a_1$ such that $\Phi_{b'}', \Phi_b$ has characteristic $p \times p$ and is mapped onto $\mathbf{\Phi}', \mathbf{\Phi}$ under φ.*

Proof. Let

$$b_1' = a_1'(a_3' p_{32} + c_{13}')(a_2'(a_3 + p_{32}) + c_{12}')(p \otimes c_{12}').$$

Then Φ' has characteristic p by the invariance of $p \otimes c_{12}'$ and one gets $b_3' \leqq p_{32}$ and $b_2' \leqq a_3 + p_{32}$. Now, let $b_3 = a_3(p_{32} + b_2')$. Then $b_3 \geqq b_3'$ whence $b_i \geqq b_i'$ for all $i \leqq 3$. By the invariance of p_{32} (Lemma 7) one has $p_{32}(\Phi_b) = bp_{32}$ and derives

$$b_2' + bp_{32} = b(b_2' + p_{32}) \geqq b_3,$$
$$b_3 + bp_{32} = b(a_3(p_{32} + b_2') + p_{32}) = b(a_3 + p_{32})(p_{32} + b_2') \geqq b_2',$$
$$bp_{32} \leqq (a_3(p_{32} + b_2') + a_2)p_{32} \leqq a_3 + a_2(p_{32} + b_2') = a_3 + b_2'.$$

It follows $bp_{32} \leqq b_3 + b_2'$ and the intended characteristic of $\Phi_{b'}', \Phi_b$. Of course, one has $\varphi b_1' = \mathbf{a}_1'$ and $\varphi b_3 = \mathbf{a}_3$.

For the remainder of this section let Φ', Φ a skew frame of type (4,3) and characteristic $p \times p$ (p a fixed odd prime) in a modular lattice. Denote by $\tilde{\Phi}$ the frame arising from Φ by lower reduction with $\tilde{a}_3 = a_3 p_{32}$. Let $-1_{12}'$ the negative unit in the ring $R(\Phi')$ and

$$p_{12} = (p_{32} + c_{13})(a_1 + a_2'),$$
$$p_{21} = (((p_{32} + -1_{12}')(a_1 + a_3) + a_2)(a_1 + c_{23}) + a_3)(a_1' + a_2).$$

Given $b_1' \leqq a_1'$ let $b_3 = a_3(b_2' + p_{32})$ and Φ_b the associated frame.

Lemma 10.

$$p_{12}\in R_{12}, \quad p_{21}\in R_{21}, \quad a_1p_{12}=\tilde{a}_1, \quad a_2p_{21}=\tilde{a}_2,$$
$$b_1+p_{12}\geqq b'_2, \quad b'_2+p_{12}\geqq b_1, \quad b_2+p_{21}\geqq b'_1, \quad b'_1+p_{21}\geqq b_2.$$

Proof. We consider the claims for p_{21}, only, leaving p_{12} to the reader. Since

$$(a_1+c_{23})(a_1+a_3)=a_1, \quad a_3a_1=0, \quad a_2(a_1+a_3)=0$$

we get by modularity

$$p_{21}a_1=(p_{32}+-1'_{12})a_1=(p_{32}(a_1+a_2)+-1'_{12})a_1=-1'_{12}a_2=0.$$

Similarily, due to

$$(a_1+c_{23})(a_2+a_3)=c_{23}, \quad (a_1+a_3)(a_2+c_{23})=a_3,$$
$$(p_{32}+-1'_{12})a_3=(p_{32}+-1'_{12}(a'_2+a_3))a_3=p_{32}a_3=\tilde{a}_3$$

we get

$$p_{21}a_2=(((p_{32}+-1'_{12})a_3+a_2)c_{23}+a_3)a_2=\tilde{a}_2.$$

On the other hand by definition

$$p_{32}+-1'_{12}+b'_1=p_{32}+b'_2+-1'_{12}\geqq b_3,$$
$$p_{21}+b'_1\geqq((b_3+a_2)c_{23}+a_3)(a'_1+a_2)=(b_{23}+a_3)a_2=b_2$$

since b'_1 sneaks in by modularity. Finally, we derive

$$p_{32}+-1'_{12}+b_3\geqq b'_2+-1'_{12}\geqq b'_1,$$
$$p_{21}+b_2=(((p_{32}+-1'_{12}+b_3)(a_1+a_3)+a_2+b_{23})(a_1+c_{23})+a_3+b_2)(a'_1+a_2)\geqq b'_1.$$

Taking $b'_1=a'_1$ we get $p_{21}\in R_{21}$.

Let

$$c_{24}=((((p_{12}+c'_{24})(a_1+a'_4)+a_2)(c_{12}+a'_4)+a_1)(a_2+a'_4)).$$

Lemma 11.

$$a_2c_{24}=\tilde{a}_2=(\tilde{a}_2+a'_4)c_{24}, \quad b'_4+c_{24}\geqq b_2, \quad b_2+c_{24}\geqq b'_4.$$

Proof. By Lemma 10 we have

$$((p_{12}+c'_{24})(a_1+a'_4)+a_2)a_1=(p_{12}+c'_{24})a_1=p_{12}a_1=\tilde{a}_1$$

hence using

$$(c_{12}+a'_4)(\tilde{a}_1+a_2)=\tilde{c}_{12}, \quad (a_1+a'_4)(a_2+\tilde{a}_1)=\tilde{a}_1,$$
$$p_{12}(\tilde{a}_1+a'_2+a'_4)=p_{12}(a_1p_{12}+a'_2)=\tilde{a}_1$$

we derive

$$c_{24}a_2=(((\tilde{a}_1+c'_{24})(\tilde{a}_1+a'_4)+\tilde{a}_2)(\tilde{c}_{12}+a'_4)+\tilde{a}_1)\tilde{a}_2=\tilde{a}_2.$$

Now, $(\tilde{a}_2+a'_4)c_{24}=\tilde{a}_2$ follows from

$$c_{24}a'_4=(p_{12}+c'_{24})a'_4=(p_{12}(a'_2+a'_4)+c'_{24})a'_4=c'_{24}a'_4=0$$

which we get using

$$a_1(a_2+a'_4)=a_2(a_1+a'_4)=a_2p_{12}=0.$$

On the other hand we have $c_{12}+b_2\geqq b_1$ and by Lemma 10

$$p_{12}+c'_{24}+b_1\geqq b'_2+c'_{24}\geqq b'_4$$

whence $c_{24}+b_2$ equals

$$(((p_{12}+c'_{24}+b_1)(a_1+a'_4)+a_2+b_1)(c_{12}+a'_4+b_2)+a_1+b_2)(a_2+a'_4)\geqq b'_4.$$

Similarly, we get $c_{24}+b'_4\geqq b_2$ from

$$p_{12}+c'_{24}+b'_4=p_{12}+b'_2+c'_{24}\geqq b_1, \qquad c_{12}+b_1\geqq b_2.$$

Let $h_{14}=(p_{21}+c_{24})(a'_1+a'_4)$, $h_{12}=(h_{14}+c'_{24})(a'_1+a'_2)$.

Lemma 12. *h_{12} is a stable element of $R(\Phi')$.*

Proof. Since the passage from h_{14} to h_{12} is a familiar change of coordinate domain in a 4-frame it suffices to show

$$a'_1h_{14}=a'_4h_{14}=0, \qquad b'_1+h_{14}\geqq b'_4, \qquad b'_4+h_{14}\geqq b'_1.$$

Using Lemmas 10 and 11 one derives

$$\begin{aligned} a'_1h_{14}&=(p_{21}+c_{24}(a'_1+a_2))a'_1=(p_{21}+\tilde{a}_2)a'_1\\ &=(p_{21}(a'_1+\tilde{a}_2)+\tilde{a}_2)a'_1=\tilde{a}_2a'_1=0,\\ a'_4h_{14}&=(p_{21}(a_2+a'_4)+c_{24})a'_4=(\tilde{a}_2+c_{24})a'_4\\ &=(\tilde{a}_2+c_{24}(\tilde{a}_2+a'_4))a'_4=\tilde{a}_2a'_4=0. \end{aligned}$$

The proof is completed with

$$p_{21}+c_{24}+b'_1\geqq b_2+c_{24}\geqq b'_4, \qquad p_{21}+c_{24}+b'_4\geqq p_{21}+b_2\geqq b'_1.$$

Because of Lemma 3 we conclude the following.

Corollary 13. *$g_1^*=-h_{12}^{-1}$ is a stable element of $R(\Phi')$.*

For the definition of the second stable element we use symmetry. Let

$$\bar{a}_1=a_1, \quad \bar{a}_2=a_2, \quad \bar{a}_3=c_{13}, \quad \bar{c}_{12}=c_{12}, \quad \bar{c}_{13}=a_3,$$
$$\bar{a}'_1=a'_1, \quad \bar{a}'_2=a'_2, \quad \bar{a}'_3=c'_{13}, \quad \bar{a}'_4=c'_{24}, \quad \bar{c}'_{12}=c'_{12}, \quad \bar{c}'_{13}=a'_3, \quad \bar{c}'_{24}=a'_4.$$

Then, this determines a skew frame $\bar{\Phi}', \bar{\Phi}$ of type (4, 3), obviously, and every skew frame can be understood as arising in this way from a suitable one (actually, this transition is an involution). Also, for $b'_1\leqq a'_1\leqq b_1\leqq a_1$ this transition is compatible with the formation of reduced frames. Moreover, since stability is a property refering to the interval $[0, a_1+a_2]$, only, it means the same for the frames Φ' and $\bar{\Phi}'$.

In particular, defining g_2^* in terms of $\bar{\Phi}', \bar{\Phi}$ the same way as g_1^* in terms of Φ', Φ we get the following.

Corollary 14. *If $\bar{\Phi}', \bar{\Phi}$ has characteristic $p \times p$ then g_2^* is a stable element of $R(\bar{\Phi}')$ and $R(\Phi')$ as well.*

3. Gluing Constructions

The construction of pathological modular lattices via gluing dates back to Dilworth and Hall [10]. A generalization of their method has been studied in [11]. Let S be a lattice of finite length and for each x in S let L_x be a bounded modular lattice. For all pairs x, y in S with $x \leqq y$ let φ_{yx} be empty or an isomorphism of a principal filter of L_x onto a principal ideal of L_y. Let φ_{yx} be nonempty if y covers x and φ_{xx} the identity on L_x. Assume that $\varphi_{yx} = \varphi_{yz} \circ \varphi_{zx}$ for all $x \leqq z \leqq y$ in S and that for all x, y in S the following identities for image and domain are satisfied:

$$\operatorname{im} \varphi_{(x+y)x} \cap \operatorname{im} \varphi_{(x+y)y} = \operatorname{im} \varphi_{(x+y)(xy)}, \tag{1}$$

$$\operatorname{dom} \varphi_{x(xy)} \cap \operatorname{dom} \varphi_{y(xy)} = \operatorname{dom} \varphi_{(x+y)(xy)}. \tag{2}$$

Then, S, the L_x, and φ_{yx} are said to form *an S-connected system.* Make the L_x disjoint and identify two elements a, b of the union of the L_x, x in S, if there are x, y, z with $\varphi_{zx} a = \varphi_{zy} b$. Clearly, no two elements of a single L_x are identified. Thus, we can think of $L = \bigcup \{L_x \mid x \in S\}$ as union of the L_x with overlaps given by the φ_{yx}. Define on L, $a \leqq b$ if and only if there are x_i in S, a_i in $L_{x_i} \cap L_{x_{i+1}}$ such that $a = a_0$, $b = a_n$, and $a_i \leqq a_{i+1}$ in L_{x_i} for $0 \leqq i < n$.

Proposition 15. *Under the above hypotheses L with the relation $\leqq$ is a modular lattice having the L_x as interval sublattices such that $0_x + 0_y = 0_{x+y}$ and $1_x 1_y = 1_{xy}$ for the bounds of the L_x. If a is in L_x, b in L_y and $x \prec x_1 \prec x_2 \prec \ldots \prec x+y$, $y \prec y_1 \prec y_2 \prec \ldots \prec x+y$ then*

$$a+b = (((a +_x 0_{x_1}) +_{x_1} 0_{x_2}) + \ldots + 0_{x+y}) +_{x+y} ((b +_y 0_{y_1}) +_{y_1}) +_{y_1} 0_{y_2}) + \ldots + 0_{x+y}).$$

For meets the dual formula applies. L will be called the S-glued sum of the L_x (with identification maps φ_{yx}).

The proof of the fact that L is a lattice can be taken from [11, Satz 4.2]: We see that L is an "S-glued system" of the L_x and check that the proof of Satz 2.1 carries over. The only point needing some consideration is that $\sup_L(a, b) = a +_z b$ for all a, b in L_z. We handle a special case, first. Let $z \leqq x, y$ and $0_x, 0_y$ in L_z. Then 0_x and 0_y belong to L_{xy}, too, and we have $0_x +_z 0_y = 0_x +_{xy} 0_y$ in $L_x \cap L_y = L_{x+y} \cap L_{xy}$. Hence $0_x +_z 0_y \geqq 0_{x+y}$ and 0_{x+y} is in L_z. Therefore, $0_x +_z 0_y = 0_{x+y} = \sup_L(0_x, 0_y)$. Now, we show by inverse order induction in S that $e \geqq a, b$ implies $e \geqq a +_z b$ for a, b in L_z. If e is in L_z, too, this follows from $e \geqq_z a$ and $e \geqq_z b$. Otherwise, there are $x, y > z$ and a_1 in L_x, b_1 in L_y with $a \leqq_z a_1 \leqq e$ and $b \leqq_z b_1 \leqq e$. Consequently, $0_x \leqq e$, $0_y \leqq e$, and $e \geqq 0_x + 0_y =: c \in L_x \cap L_y$ by the above. By induction $e \geqq a_1 +_x c$ and $e \geqq b_1 +_y c$. Since both elements are in $L_{x+y} \cap L_z$ we have by induction, again, that $e \geqq (a_1 +_x c) +_z (b_1 +_y c) \geqq a_1 +_z c +_z b_1 \geqq a +_z b$.

To derive modularity we show by order induction on $x+y+z$ that $x \leqq y$, $a \in L_x$, $b \in L_y$, $c \in L_z$, and $a \leqq b$ jointly imply $b(a+c) = a+bc$. Let us consider the special case $x = y \geqq z$, $a = 0_y$, first. For $y = z$ this case is trivial. Let $y \succ w \geqq z$. Then $1_w \geqq 0_y$ and $1_w \geqq c$ whence $b(0_y + c) = b1_w(0_y + c) = 0_y + b1_w c = 0_y + bc$ by induction. Now, for the general case let $b(a+c) \neq a+bc = a+b(a+c)c$. In view of $a+c \leqq 1_x + 1_z \leqq 1_{x+z}$ we may assume $b \leqq a+c$ and $y \leqq x+z$. If $y = x+z$ then $b(a+c) = b(a+0_{x+z}+c) = a+0_{x+z}+b(0_{x+z}+c) = a+0_{x+z}+bc = a+bc$ by the modularity of L_y and the special case. Otherwise, let $y \leqq w \prec x+z$. Then 1_w belongs to L_{x+z} and $b(a+c) = b1_w(a+c) = b(a+1_w c) = a+b1_w c = a+bc$ with the preceding case and induction.

If the lattice S is modular, itself, we need the maps φ_{yx} to be defined for $x \prec y$, only. The axioms are (1) and (2) from above with $xy \prec x$, $y \prec x+y$ and $\varphi_{(x+y)(xy)} := \varphi_{(x+y)x} \circ \varphi_{x(xy)} = \varphi_{(x+y)y} \circ \varphi_{y(xy)}$. This is called a *locally S-connected system*. An S-connected system is derived by $\varphi_{yx} = \varphi_{yx_n} \circ \ldots \circ \varphi_{x_1 x}$ for $x \prec x_1 \prec \ldots \prec x_n \prec y$.

Z_p = integers mod p. Read $Z_p x = Z_q px$ and $Rx = Qpx$. Cf comments on last page.

4. Representation of Two-Generator Groups

Theorem 16. *For every group G with generators g_1, g_2 there is a modular lattice L generated by a skew frame Φ', Φ of type (4, 3) and characteristic $p \times p$, $p \geqq 3$ any prime, such that G can be embedded as a subgroup into the coordinate ring $R(\Phi')$ with g_i mapped onto the lattice element g_i^* as defined in Corollaries 13 and 14.*

Proof. Let $q = p^2$ and Z_q the integers modulo q. Let Z_p the ideal pZ_q of Z_q, A the abelian group $Z_q \times Z_q \times Z_q \times Z_p$ and $L(A)$ its lattice of subgroups. Observe that Z_p is isomorphic to Z_q/Z_p. Let R and Q the group rings with coefficients in Z_p and Z_q, respectively. Then R can be understood as the ideal pQ of Q and $B = Q \times Q \times Q \times R$ as a left Q-module. If we identify r from Z_q with re in Q (e the neutral element of G) then Z_q becomes a central subring of Q, $e = 1$ the unit of Q, and A a subgroup of B.

For a subgroup U of A let εU be the Q-submodule generated by U. (Actually, εU can be determined as $Q \otimes_{Z_q} U$ since Q is flat as a Z_q-module.) Observe that if U is defined by a system Γ of linear identities with coefficients in Z_q then εU is defined by the same Γ in B. Namely, let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in B with $\alpha_i = \sum (k_{g,i} g \mid g \in G)$ satisfy Γ. If $\sum_i r_i x_i = 0$ is an identity from Γ, then for each g in G we have $\sum_i r_i k_{g,i} g = 0$ since G is an independent subset of Q. Multiplication with g^{-1} from the left yields $\sum_i r_i k_{g,i} e = 0$. Thus, $(k_{g,1}, \ldots, k_{g,4})$ belongs to U and $(k_{g,1} g, \ldots, k_{g,4} g)$ to εU for all g in G. Therefore, $(\alpha_1, \ldots, \alpha_4)$ lies in εU. It follows that ε is a meet preserving map of $L(A)$ into the lattice $L(B_0)$ of Q-submodules of B. Preserving joins, obviously, ε is a lattice homomorphism and one to one since its domain is a simple lattice.

Let S be the interval sublattice $[0, \mathbb{Z}_p^3 \times 0]$ of $L(A)$ and define for x in S the element x^* of $L(A)$ as the join of all covers of x. Then, the intervals $[x, x^*]$ are canonically isomorphic to $L(\mathbb{Z}_p^4)$ and $\{x^* \mid x \in S\}$ is the sublattice $[\mathbb{Z}_p^4, A]$. In the terminology of [11] S is the skeleton of $L(A)$.

Now, for x in S let M_x be the interval sublattice $[\varepsilon x, \varepsilon x^*]$ of $L(B)$ and M the union of all M_x, x in S. Then M is a sublattice of $L(B)$ and each M_x canonically isomorphic to the lattice $L(R_R^4)$ of R-submodules of the free R-module R^4. In particular, we can view M as the S-glued sum of the lattices M_x.

Two of the components M_x are of particular interest. Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, and $e_4 = (0,0,0,p)$ denote the canonical spanning vectors of B and consider the point $u = \mathbb{Z}_p e_3$ of the projective plane S. Then $\varepsilon u = \mathbb{R} e_3 = : U$ and M_u is isomorphic to the Q-submodule lattice of the factor module C_u/U where $C_u = \mathbb{R} e_1 + \mathbb{R} e_2 + Qe_3 + Qe_4$ is defined by the linear identities $px_1 = px_2 = 0$. Using the isomorphism $Q/pQ = Q/\mathbb{R} \cong R$ we may consider C_u/U as free R-module with generators $pe_1 + U$, $pe_2 + U$, $e_3 + U$, and $e_4 + U$ and $\mathbb{R}^4/U$ as a rank 3 free submodule. Therefore, we have an R-linear map on $\mathbb{R}^4/U$ which maps $pe_2 + U$ onto itself, $e_4 + U$ onto $g_1 e_4 + U$, and $pe_1 + U$ onto $g_1^{-1} pe_1 + U$, g_1 the first generator of G. Let φ_{u0} be the induced automorphism of the interval $[U, \mathbb{R}^4]$ of $M_u \cap M_0$. For $y \succ u$ in S we define an isomorphism φ_{yu} of $[\varphi_{u0}\varepsilon y, C_u]$ onto $[\varepsilon y, C_u]$ which is φ_{u0}^{-1} on $[\varphi_{u0}\varepsilon y, \mathbb{R}^4]$ and identity on $[W, C_u]$ where $W = \mathbb{R} e_1 + \mathbb{R} e_2 + \mathbb{R} e_3 = \varepsilon 1_S$ is the image of the top of S and $\mathbb{R}^4 = \varepsilon 0^*$ the top of the lowest interval. For given y let $Y = \varepsilon y$ and $X = \varphi_{u0}\varepsilon y$ and define φ_{yu} as the lattice isomorphism induced by the R-linear map of C_u/X onto C_u/Y which is given by

$$\begin{aligned}
&e_4 + X \mapsto g_1^{-1} e_4 + Y \\
&pe_1 + X \mapsto g_1 pe_1 + Y \quad \text{but,} \quad pe_2 + X \mapsto pe_2 + Y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad y = u + \mathbb{Z}_p e_1 \\
&e_3 + X \mapsto g_1^{-1} e_3 + Y \quad \text{but,} \quad e_3 + pe_2 + X \mapsto g_1^{-1}(e_3 + pe_2) + Y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad y = u + \mathbb{Z}_p(e_1 + e_2).
\end{aligned}$$

Next, consider the point $v = \mathbb{Z}_p(e_1 - e_3)$ of S and $V := \varepsilon v = \mathbb{R}(e_1 - e_3)$. Again, M_v is the submodule lattice of the free R-module C_v/V where $C_v = \mathbb{R} e_1 + \mathbb{R} e_2 + Q(e_1 - e_3) + Q(pe_2 - e_4)$ is defined by the linear identities $px_2 = p(x_1 + x_3) = 0$. Let φ_{v0} be the automorphism of $[V, \mathbb{R}^4]$ induced by the R-linear map on $\mathbb{R}^4/V$ which maps $pe_2 + V$ onto itself, $pe_1 + V$ onto $g_2^{-1} pe_1 + V$, and $pe_2 - e_4 + V$ onto $g_2(pe_2 - e_4) + V$, g_2 the second generator of G.

For $y \succ v$ in S define an isomorphism φ_{yv} of $[\varphi_{v0}\varepsilon y, C_v]$ onto $[\varepsilon y, C_v]$ which is φ_{v0}^{-1} on $[\varphi_{v0}\varepsilon y, \mathbb{R}^4]$ and identity on $[W, C_v]$. For given y let $Y = \varepsilon y$ and $Z = \varphi_{v0}\varepsilon y$ and define φ_{yv} as the lattice isomorphism induced by the R-linear map of C_v/Z onto C_v/Y given by

$$\begin{aligned}
&pe_2 - e_4 + Z \mapsto g_2^{-1}(pe_2 - e_4) + Y \\
&pe_1 + Z \mapsto g_2 pe_1 + Z \qquad \text{but,} \quad pe_2 + Z \mapsto pe_2 + Y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad y = u + v \\
&e_3 - e_1 + Z \mapsto g_2^{-1}(e_3 - e_1) + Y \quad \text{but,} \quad e_3 + pe_2 - e_1 + Z \mapsto g_2^{-1}(e_3 + pe_2 - e_1) + Y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad y = v + \mathbb{Z}_p(e_1 + e_2).
\end{aligned}$$

With the φ_{yx} from above and φ_{yx} the identity on $[\varepsilon y, \varepsilon x^*]$ for $y \succ x$, else, we have new identification maps on the M_x which give rise to a new locally S-connected system. Let K be the modular lattice resulting due to Proposition 15.

Observe that, leaving u and v aside, the gluing maps of K are just those of M, namely identities. Therefore, we can consider $P = \bigcup\{M_x \mid x \in S, x \neq u, v\}$ as a subset of both K and M. Also, when computing a join or meet in M or K by use of the formula in Proposition 1 we can bypass u and v provided that the inputs and the results belong to P. Thus, the outcome is the same for M and K which means that M and K induce the same partial lattice structure on P. In particular, when speaking about an element of K it suffices to specify it by a submodule of B and to indicate whether it is understood as an element of P, M_u, or M_v. The defining identities of C_u and C_v provide a quick method to show that an element is in P.

We use $\cap$ to denote meets in K where this seems to be more readable. The following observation will be helpful when calculating joins. Let $x = \underline{Z}_p(re_1 + se_2 + te_3)$ be a point in S excluding $r \equiv s \bmod p$.

Consider the elements $\varepsilon x \in M_0$ and $Qe_3 \in M_u$. Then

$$\varepsilon x +_K Qe_3 = \underline{R}(re_1 + se_2) + Qe_3 \in M_{x+u} \subseteq P.$$

Namely, $\varepsilon x +_K Qe_3 \geq \underline{R}e_3 \in M_0 \cap M_u$ and $x + u \neq u + \underline{Z}_p(e_1 + e_2)$ whence

$$\varphi_{u0}(\varepsilon x +_{M_0} \underline{R}e_3) = \underline{R}(g_1^{-1} re_1 + se_2) + \underline{R}e_3 = \varphi_{u0}\varepsilon(x+u) \in M_u,$$

$$\varphi_{(x+u)u}(\varphi_{u0}\varepsilon(x+u) +_{M_u} Qe_3) = \varepsilon(x+u) + Qe_3 = \underline{R}(re_1 + se_2) + Qe_3 \in M_{x+u}.$$

Similarily, we have for $Q(e_1 - e_3) \in M_v$

$$\varepsilon x +_K Q(e_1 - e_3) = \underline{R}(re_1 + se_2 + te_3) + Q(e_1 - e_3) \in M_{x+v} \subseteq P.$$

In particular,

$$Qe_1 +_K Qe_3 = Qe_1 +_K Q(e_1 - e_3) = Qe_3 +_K Q(e_1 - e_3) = Qe_1 + Qe_3 \in P.$$

Let Φ', Φ consist of the

$$a_i = Qe_i, \quad c_{ij} = Q(e_i - e_j),$$

$$a_i' = \underline{R}e_i, \quad c_{ij}' = \underline{R}(e_i - e_j), \quad a_4' = Qe_4, \quad c_{i4}' = c_{4i}' = Q(pe_i - e_4)$$

$(i, j \leqq 3)$ all of course considered as elements of P except $a_3 = Qe_3 \in M_u$ and $c_{13} = Q(e_1 - e_3) \in M_v$.

Lemma 17. *Φ', Φ is a skew frame of type $(4,3)$ in K and G is embedded into $R(\Phi')$ such that $g_i \mapsto \underline{R}(e_1 - g_i e_2)$.*

Proof. $a_2(a_1 + a_3) = 0$ and $a_4'(a_1 + a_2 + a_3) = 0$ which means that the a_1, a_2, a_3, a_4' are independent. The relations $a_i c_{ij} = 0$ are obvious. In $a_i + c_{ij} = a_i + a_j$ only the case

$i=3$ deserves a closer look – at the above observation. Moreover,

$$\begin{aligned}(c_{12}+c_{23})(a_1+a_3) &= (c_{12}+c_{23})\cap C_v\cap(a_1+a_3)\\ &= (Q(e_1-e_3)+_{M_v}\underline{R}(e_1-e_2))\cap_{M_v}(\underline{R}e_1+_{M_v}Q(e_1-e_3))\\ &= Q(e_1-e_3)\in M_v.\end{aligned}$$

Thus, Φ is a frame. Now, Φ' is the canonical frame of the lattice $M_0 \cong L(R_R^4)$ whence $r\mapsto \underline{R}(e_1-re_2)$ is an isomorphism of R onto $R(\Phi')$ – see Artmann [1]. Also, the first three dimensions of Φ' arise from Φ by intersection with $\sum\Phi'$.

Lemma 18. *Φ', Φ has characteristic $p\times p$ and the words g_i^* defined in Corollaries 13 and 14 take value $\underline{R}(e_1-g_ie_2)$ in K.*

Proof. We have $c_{123}=Q(e_1-e_2-e_3)$ in K and for $r\not\equiv -1$, $s\not\equiv 0 \bmod p$

$$Q(e_1-re_2)\oplus 1=Q(e_1-(r+1)e_2),\qquad \sigma Q(e_1-(s+1)e_2)=Q(e_3-se_2),$$
$$Q(e_1-re_2)\boxplus Q(e_1-se_2)=Q(e_1-(r+s)e_2).$$

In particular, we get $\tilde{q}=Q(e_1+e_2)$. Also $\boxminus 1_{32}=Q(e_3+e_2)$ and

$$Q(e_3+re_2)\boxminus 1=Q(e_3+(r+1)e_2)$$

for $r\not\equiv 0,-1 \bmod p$. For $r=p-1$ one has $Q(e_3+re_2)\boxminus 1\subseteq C_u$ and

$$\begin{aligned}Q(e_3+re_2)\boxminus 1 &= (\underline{R}(e_1+e_2)+_M Q(e_3+pe_2))\cap_K(\underline{R}e_2+Qe_3)\\ &= (\underline{R}(e_1+e_2)+_{M_u}Q(e_3+pe_2))\cap_{M_u}(\underline{R}e_2+Qe_3)=Q(e_3+pe_2)\in M_u\end{aligned}$$

due to the definition of φ_{yu} for $y=u+\underline{Z}_p(e_1+e_2)$. Hence

$$p_{32}=Q(e_3+pe_2)\in M_u$$

and it is clear that Φ', Φ has characteristic $p\times p$. Since

$$(Qe_3+pe_2)+_K\underline{R}(e_1+e_2))\cap_K(Qe_1+Qe_3)=Q(e_3-pe_1)\in M_u,$$
$$Q(e_3-pe_1)+_K Qe_2=Q(g_1^{-1}e_3-g_1pe_1)+Qe_2=Q(e_3+g_1^2pe_1)+Qe_2\in P,$$
$$(Q(e_3+g_1^2pe_1)+Qe_2)\cap_P(Qe_1+Q(e_2-e_3))=Q(e_3-g_1^2pe_1-e_2)\in P,$$
$$\begin{aligned}&(Qe_3-g_1^2pe_1-e_2)+_K Qe_3)\cap_K(\underline{R}e_1+Qe_2)\\ &=(Q(e_3-g_1^2pe_1-e_2)+_M Qe_3)\cap_M(\underline{R}e_1+Qe_2)=Q(g_1^2pe_1+e_2)\in P\end{aligned}$$

it follows

$$p_{21}=Q(g_1^2pe_1+e_2)\in P.$$

On the other hand

$$\varphi_{(u+v)u}(Q(e_3+pe_2)+_{M_u}\underline{R}e_1)=Q(g_1^{-1}e_3+pe_2)+\underline{R}e_1=Q(e_3+g_1pe_2)+\underline{R}e_1\in P,$$
$$(Q(e_3+pe_2)+_K Q(e_1-e_3))\cap_K(Qe_1+Re_2)=Q(e_1+g_1pe_2)$$

whence we get

$$p_{12} = Q(e_1 + g_1 p e_2) \in P.$$

Now, with an easy calculation in P we derive

$$c_{24} = Q(e_2 - g_1 e_4), \quad h_{14} = Q(g_1 p e_1 + e_4), \quad h_{12} = \underline{R}(e_1 + g_1^{-1} e_2)$$

and calculation in $R(\Phi')$ yields $g_1^* = \underline{R}(e_1 - g_1 e_2)$. The symmetry alluded to in Corollary 14 has been built into the lattice K and it takes g_1 to g_2. Hence, $g_2^* = \underline{R}(e_1 - g_2 e_2)$ in K.

Thus, choosing L as the sublattice of K generated by Φ', Φ the proof of Theorem 16 is completed: g_1^* and g_2^* and their inverses belong to K and so does the subgroup they generate – which is isomorphic to G.

5. Unsolvability

Novikov's theorem and HNN-extensions [16, IV, 3.1, 7.2] yield a group $\mathbf{G}$ with generators $\mathbf{g}_1, \mathbf{g}_2$ and presentation $\mathbf{w}_1 = \ldots = \mathbf{w}_m = 1$ such that the word problem for $\mathbf{G}$ is recursively unsolvable. Let L with skew frame $\mathbf{\Phi}', \mathbf{\Phi}$ associated with $\mathbf{G}$ according to Theorem 16. In particular, $\mathbf{G}$ is isomorphic to a subgroup of the coordinate ring $R(\mathbf{\Phi}')$ with $\mathbf{g}_i \mapsto g_i^*(\mathbf{\Phi}', \mathbf{\Phi})$ where the g_i^* are the expressions for stable elements considered in Corollaries 13 and 14. For simplicity, let us identify $\mathbf{G}$ with its image in $R(\mathbf{\Phi}')$.

Now, let F be the modular lattice freely generated by a skew frame Φ', Φ of type (4, 3). By Lemma 8 it suffices to show that F has an unsolvable word problem. If one is interested in the construction of a finitely presented four generated modular lattice with unsolvable word problem, only, one may assume that F already satisfies all the relations which we force by modifications in the sequel.

According to Lemma 9 there are b_1 and b'_1 in F with $b'_1 \leqq a'_1 \leqq b_1 \leqq a_1$ such that $\Phi'_{b'}, \Phi_b$ is of characteristic $p \times p$ and $\alpha b'_1 = \mathbf{a}'_1$, $\alpha b_1 = \mathbf{a}_1$ where α is the canonical homomorphism of F onto L. By Corollary 13 the element $\tilde{g}_1 = g_1^*(\Phi'_{b'}, \Phi_b)$ is stable in $R(\Phi'_{b'})$ and by Lemma 18 $\alpha\tilde{g}_1 = \mathbf{g}_1$ in L.

We pass to the skew frame $\bar{\Phi}'_{b'}, \bar{\Phi}_b$ reflecting the basic symmetry with $\tilde{g}_1$ still stable in $R(\bar{\Phi}'_{b'})$. Again by Lemma 9 there are d_1 and d'_1 in F with $d'_1 \leqq a'_1 \leqq d_1 \leqq a_1$ such that $\bar{\Phi}'_{d'}, \bar{\Phi}_d$ has characteristic $p \times p$ and $\alpha d_1 = \mathbf{a}_1$, $\alpha d'_1 = \mathbf{a}'_1$. Then $g_1 = d'\tilde{g}_1$ is stable in $R(\bar{\Phi}'_{d'})$ by Lemma 3 and $g_2 = g_2^*(\bar{\Phi}'_{d'}, \bar{\Phi}_d)$ is stable in $R(\bar{\Phi}'_{d'})$ by Corollary 14. Hence we have g_i stable in $R(\Phi'_{d'})$ and $\alpha g_i = \mathbf{g}_i$ for $i = 1, 2$. Also, $\Phi'_{d'}$ is mapped onto $\mathbf{\Phi}'$ under α. To simplify notation write Φ' for $\Phi'_{d'}$.

Now, following Freese [7] by the use of Lemma 3 we may modify the frame Φ' to force the relations of $\mathbf{G}$. Namely, if w_i is the element of $R(\Phi')$ corresponding to $\mathbf{w}_i$ we may pass to the frame Φ'^b where $b_1 = a'_1(c'_{12} + w_i)$. We have $\alpha(\Phi'^b) = \mathbf{\Phi}'$, $\alpha(g_i + b) = \mathbf{g}_i$ since $\mathbf{c}'_{12} = \mathbf{w}_i$ in $R(\mathbf{\Phi}')$ and $\alpha(b) = 0$. Moreover, w_i being invertible,

$$w_i + b = w_i + b_1 + b = (a'_1 + w_i)(c'_{12} + w_i) + b \geqq c'_{12} + b, \qquad w_i + b = c'_{12} + b$$

since these are complements of a'_2 in $[0, \sum \Phi']$. Thus, by Lemma 3 the relation $w_i = 1$ is satisfied in (Φ'^b). Also, the $g_i + b$ are stable and the relations $w_j = 1$ which were valid in $R(\Phi')$ are valid in $R(\Phi'^b)$, too. We end up with a frame, again called Φ', and invertible elements g_1, g_2 in the coordinate ring $R(\Phi')$ which satisfy all the defining relations of the group **G** and are mapped onto the generators $\mathbf{g}_1, \mathbf{g}_2$ under the lattice homomorphism α. Let G be the subgroup of $R(\Phi')$ generated by g_1 and g_2. Since the group operations can be expressed by lattice polynomials in Φ' it follows that α maps G onto **G** and is, indeed, an isomorphism. Thus, deciding a relation in **G** amounts to deciding a relation in the lattice F.

Theorem 19. *Let P be a partially ordered set containing one of* $1+1+1+1$, $2+2+2$, *and* $1+2+3$. *Then the modular lattice* $\mathrm{FM}(P)$ *freely generated by P has an unsolvable word problem.*

Here, $1+1+1+1$ means the 4-element antichain, $2+2+2$ the disjoint union of three 2-element chains, and $1+2+3$ the disjoint union of three chains having one, two, and three elements. The latter is of finite representation type in the sense of Nazarova [19]. In particular, it follows that $\mathrm{FM}(P)$ has unsolvable word problem for each P of infinite representation type.

Proof. If Q is contained in P then $x \mapsto \sup(q \in Q, q \leqq x)$ provides a monotone map of P onto the generating set of $\mathrm{FM}(Q)$. Hence it suffices to consider P being one of the above posets. It remains to show that every skew frame of type (4, 3) can be generated by posets $2+2+2$ and $1+2+3$. For the first let

$$a=a_1, \quad b=a_1+a_2, \quad c=a_3, \quad d=a_2+a_3+a'_4,$$
$$f=c_{13}+c_{23}+a'_4, \quad e=f(c'_{14}+a_3).$$

Then we get the skew frame from

$$a_2=bd, \quad c_{13}=(a_1+a_3)f, \quad c_{23}=(a_2+a_3)f,$$
$$a'_4=(e+c_{13})d, \quad c'_{14}=(a_1+a'_4)(e+a_3).$$

On the other hand let

$$a=a_1, \quad b=a_1+a_2, \quad c=a_1+a_2+a'_4, \quad d=a_3,$$
$$e=a_2+a_3+a'_4, \quad f=c'_{34}+(c_{12}+a_3)(c_{23}+a_1)$$

and derive

$$a_2=be, \quad c_{23}=(f+a)(a_2+d), \quad c_{13}=(f+d)b, \quad a'_4=(f+d)ce, \quad c'_{34}=ef.$$

Remark. Our method does not apply to Arguesian lattices. Indeed, as Douglas Pickering and Ralph Freese pointed out to me one has $x\oplus(1\oplus 1) \neq (x\oplus 1)\oplus 1$ for $x=Q(e_1-(p-1)e_2)$ in the Day-Pickering coordinate construction for L from Theorem 16.

References

1. Artman, B.: On coordinates in modular lattices. Illinois J. Math. **12**, 626–648 (1968)
2. Birkhoff, G.: On the combination of subalgebras. Proc. Cambridge Phil. Soc. **29**, 441–464 (1933)
3. Birkhoff, G.: Lattice theory. Providence: Am. Math. Soc. Colloquium Publ. 1940

4. Dedekind, R.: Über die von drei Moduln erzeugte Dualgruppe. Math. Ann. **53**, 236–271 (1900)
5. Freese, R.: The variety of modular lattices is not generated by its finite members. Trans. Am. Math. Soc. **255**, 277–300 (1979)
6. Freese, R.: Projective geometries as projective modular lattices. Trans. Am. Math. Soc. **251**, 329–342 (1979)
7. Freese, R.: Free modular lattices. Trans. Am. Math. Soc. **261**, 81–91 (1980)
8. Freese, R.: Some order theoretic questions about free lattices and free modular lattices. In: Ordered sets, pp. 355–377. Rival, I. (ed.). Dordrecht: Reidel 1982
9. Gel'fand, I.M., Ponomarev, V.A.: Free modular lattices and their representations (in Russian). Usp. Math. Nauk. **29**, 3–58 (1974)
10. Hall, M., Dilworth, R.P.: The embedding problem for modular lattices. Ann. Math. **45**, 450–456 (1944)
11. Herrmann, C.: *S*-verklebte Summen von Verbänden. Math. Z. **130**, 255–274 (1973)
12. Herrmann, C.: On elementary Arguesian lattices with four generators. Algebra Universalis (to appear)
13. Huhn, A.: Schwach distributive Verbände. I. Acta Sci. Math. **33**, 297–305 (1972)
14. Hutchinson, G.: Recursively unsolvable word problems for modular lattices and diagram-chasing. J. Algebra **26**, 385–399 (1973)
15. Hutchinson, G.: Embedding and unsolvability theorems for modular lattices. Algebra Universalis **7**, 47–84 (1977)
16. Lyndon, R.C., Schupp, P.E.: Combinatorial group theory. Berlin, Heidelberg, New York: Springer 1977
17. Lipshitz, L.: The undecidability of the word problem for projective geometries and modular lattices. Trans. Am. Math. Soc. **193**, 171–180 (1974)
18. v. Neumann, J.: Continuous geometry. Princeton, NJ: Princeton University Press 1960
19. Nazarova, L.A.: Partially ordered sets with an infinite number of indecomposable representations. In: Representations of algebras. Ottawa 1974. Dlab, V., Gabriel, P. (eds.). Lecture Notes in Mathematics, Vol. 488. Berlin, Heidelberg, New York: Springer 1975

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Comments on Section 4. pQ is the annihilator of pQ in Q and Q/pQ isomorpic to R - extending the isomorphism of Z_q /pZ onto Z_p. Thus, pQ is a faithful R-module, canonically. .

Moreover, for any Q-submodules X \subseteq Y of B and $X+Qx_1$,...$X+Qx_m$ independent in the lattice [X,Y] with sum Y and such that $p(X+Qx_i)=X$ (i.e. independent atoms in [X,Y] spanning Y) one has [X,Y] canonically isomorphic to the submodule lattice of a free R-module with m generators.