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On modular lattices generated by $\mathbf{1}+\mathbf{2}+\mathbf{2}$

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1. Main results

One approach to classification problems of modular lattices is to determine all subdirectly irreducible modular lattices which are generated by a homomorphic image of a given poset (or, more generally, of a given partial lattice). In Wille [10], it is proved for a finite poset P not containing any subset isomorphic to

$\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$: ∘ ∘ ∘ ∘ or $\mathbf{1}+\mathbf{2}+\mathbf{2}$: ∘ ⫶ ⫶

that a subdirectly irreducible modular lattice generated by a homomorphic image of P is either a two-element lattice D_2 or a five-element non-distributive lattice M_3. Each of the 'critical' posets $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}$ generates infinitely many subdirectly irreducible modular lattices which are not isomorphic (s. Herrmann [5] and Herrmann and Huhn [6]).

In this note we analyse subdirectly irreducible modular lattices generated by $\mathbf{1}+\mathbf{2}+\mathbf{2}$ to such an extend that a complete list of subdirectly irreducibles can be described for the variety $\mathcal{N}$ generated by all lattices of normal subgroups of groups and for the variety $\mathcal{C}$ generated by all complemented modular lattices. As a byproduct we get that every subdirectly irreducible modular lattice generated by $\mathbf{1}+\mathbf{2}+\mathbf{2}$ has already four generators.

THEOREM. *Let M be a subdirectly irreducible modular lattice generated by elements a, b, c, d, e with $b \leqslant c$ and $d \leqslant e$; furthermore, let M be non-isomorphic to D_2 and M_3. Then the elements $0, b, c \wedge e, d, c, b \vee d, e$, and 1 form an eight-element boolean sublattice of M and either*

(∗) *a is a common complement of b, $c \wedge e$, and d in M or*

(∗∗) *a is a common complement of c, $b \vee d$, and e in M.*

In the terminology of Herrmann and Huhn [6] the theorem says that M is generated by a 3-frame or its dual. Thus, applying the main results of Herrmann and Huhn [6], we get the following two corollaries.

COROLLARY 1. *The following is, up to isomorphism, a complete list of all subdirectly irreducible lattices in $\mathcal{N}$ which are generated by elements a, b, c, d, e with $b \leqslant c$ and $d \leqslant e$:*

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(1) D_2, M_3;

(2) *the rational projective plane* $L((\mathbf{Q}^3)_{\mathbf{Q}})$;

(3) *the subgroup lattices* $L(C^3_{p^k})$ *of the third power of the cyclic groups* C_{p^k} *of prime power order;*

(4) *for any prime p the lattice of all those subgroups of the third power* $C^3_{p^\infty}$ *of the p-Prüfer-group which are the solution sets of finite systems of linear equations with integer coefficients*;

(5) *the duals of the lattices under* (4).

Moreover, we may assume that $b=\{(x,0,0)\mid x\in A\}$, $c=\{(x,y,0)\mid x,y\in A\}$, $d=\{(0,0,z)\mid z\in A\}$, $e=\{(0,y,z)\mid y,z\in A\}$ *in the cases* (2)–(5) *with* $A=\mathbf{Q}$, C_{p^k}, C_{p^∞}, *and* Z_p, *resp., and either* $a=\{(x,x,x)\mid x\in A\}$ *or* $a=\{(x,x+z,z)\mid x,z\in A\}$.

COROLLARY 2. *A subdirectly irreducible lattice M in* $\mathscr{C}$ *generated by elements* a,b,c,d,e *with* $b\leqslant c$ *and* $d\leqslant e$ *is either* D_2, M_3 *or a projective plane; if M is an arguesian projective plane, M has to be a plane over a prime field.*

The proof of the Theorem will be divided into two parts: First, we derive a list of relations on a, b, c, d, e valid in both cases (*) and (**); secondly, we use these relations to get a subdirect decomposition in a factor with relations (*) and another with relations (**). Then, as M is subdirectly irreducible, one projection must be an isomorphism, i.e. M satisfies either (*) or (**).

2. Relations for M

As in Wille [9, 10], the essential tools will be the following two lemmas:

D_2-LEMMA. *Let S be a subdirectly irreducible modular lattice generated by the union of two finite subsets* E_0 *and* E_1. *Then* $S\not\cong D_2$ *implies*

$$\sup E_0\geqslant\inf E_1.$$

M_3-LEMMA. *Let S be a subdirectly irreducible modular lattice generated by the union of five finite subsets* E_0, E_2, E_3, E_0 *and* E_1, *where* E_2, E_3, *and* E_5 *are not empty; furthermore, let* $\bar{e}_i:=\sup\bigcup(E_j\mid i \text{ divides } j)$ *and* $\underline{e}_i:=\inf\bigcup(E_j\mid j \text{ divides } i)$ *for* $i\in\{2,3,5\}$. *Then* $S\not\cong M_3$ *implies*

$$(\bar{e}_2\wedge\bar{e}_3)\vee(\bar{e}_2\wedge\bar{e}_5)\vee(\bar{e}_3\wedge\bar{e}_5)\geqslant(\underline{e}_2\vee\underline{e}_3)\wedge(\underline{e}_2\vee\underline{e}_5)\wedge(\underline{e}_3\vee\underline{e}_5).$$

By the D_2-Lemma, we get the following relations for the generators a, b, c, d, e of the subdirectly irreducible modular lattice M ($b\leqslant c$, $d\leqslant e$):

$$\begin{array}{ll} a\wedge b=a\wedge d=b\wedge d=0 & a\vee c=a\vee e=c\vee e=1\\ a\wedge c\leqslant b\vee e & a\vee b\geqslant c\wedge d\\ a\wedge e\leqslant c\vee d & a\vee d\geqslant b\wedge e\\ c\wedge e\leqslant a\vee b\vee d & b\vee d\geqslant a\wedge c\wedge e. \end{array}\qquad(1)$$

Proof. $b \leqslant c$ and $d \leqslant e$ implies $a \wedge b \wedge d = 0$. Since $a \geqslant b \wedge d$ ($E_0 = \{a\}$, $E_1 = \{b, c, d, e\}$), we get $b \wedge d = 0$. Because of $d \geqslant a \wedge b \wedge e$ ($E_0 = \{d\}$, $E_1 = \{a, b, c, e\}$) and $e \geqslant a \wedge b$ ($E_0 = \{d, e\}$, $E_1 = \{a, b, c\}$), we have $d \geqslant a \wedge b$; hence $a \wedge b = 0$ and, by symmetry, $a \wedge d = 0$. Dually, we get $a \vee c = a \vee e = c \vee e = 1$. The other relations are direct consequences of the D_2-Lemma.

In the following we apply several times the M_3-Lemma to get a homomorphism α from the third power of the three-element chain $D_3 := \{0, 1, 2\}$ into M such that $b = \alpha 200$, $c = \alpha 220$, $d = \alpha 002$ and $e = \alpha 022$ (then it immediately follows that the elements $0, b, c \wedge e, d, c, b \vee d, e$, and 1 form a boolean sublattice of M).

$$a \wedge c \wedge e = 0. \tag{2}$$

Proof. It results from the M_3-Lemma with $E_0 = \emptyset$, $E_2 = \{a\}$, $E_3 = \{b\}$, $E_5 = \{d\}$, and $E_1 = \{c, e\}$ that

$$(a \wedge b) \vee (a \wedge d) \vee (b \wedge d) \geqslant ((a \wedge c \wedge e) \vee (b \wedge e)) \wedge \\ \wedge ((a \wedge c \wedge e) \vee (c \wedge d)) \wedge ((b \wedge e) \vee (c \wedge d));$$

from this we get by (1) and the modular law

$$0 \geqslant a \wedge c \wedge e \wedge ((b \wedge e) \vee (c \wedge d)) = a \wedge c \wedge e \wedge (b \vee d) = a \wedge c \wedge e.$$

$$b \wedge e = 0 = c \wedge d. \tag{3}$$

Proof. By the M_3-Lemma ($E_0 = \emptyset$, $E_2 = \{a\}$, $E_3 = \{b, c\}$, $E_5 = \{d\}$, $E_1 = \{e\}$), we have

$$(a \wedge c) \vee (a \wedge d) \vee (c \wedge d) \geqslant ((a \wedge e) \vee (b \wedge e)) \wedge ((a \wedge e) \vee d) \wedge ((b \wedge e) \vee d);$$

it follows by (1), (2) and the modular law

$$0 = b \wedge d = b \wedge (d \vee (a \wedge c \wedge e)) = b \wedge e \wedge (d \vee (a \wedge c)) \geqslant \\ \geqslant b \wedge e \wedge ((a \wedge c) \vee (a \wedge d) \vee (c \wedge d)) \geqslant b \wedge e \wedge ((a \wedge e) \vee d) = b \wedge e \wedge (a \vee d) = b \wedge e,$$

hence $b \wedge e = 0$. $c \wedge d = 0$ follows by symmetry.

CLAIM 1. There exists a unique homomorphism α from the distributive lattice $(D_3)^3$ into M with $\alpha 000 = 0$, $\alpha 100 = b \wedge (a \vee d)$, $\alpha 200 = b$, $\alpha 010 = c \wedge (a \vee b) \wedge e \wedge (a \vee d)$, $\alpha 020 = c \wedge e$, $\alpha 001 = d \wedge (a \vee b)$ and $\alpha 002 = d$.

Proof. The elements 0, $b \wedge (a \vee d)$, b, $c \wedge (a \vee b) \wedge e \wedge (a \vee d)$, $c \wedge e$, $d \wedge (a \vee b)$, and d are contained in the sublattice D of M which is generated by the two chains $\{b \wedge (a \vee d), b, c \wedge (a \vee b), c\}$ and $\{d \wedge (a \vee b), d, e \wedge (a \vee d), e\}$. By Birkhoff [2], Theorem

III.9, D is distributive. If we take the conditions on α in Claim 1 as a definition of a map, we get a meet-homomorphism from the join-irreducible elements of $(D_3)^3$ into the distributive sublattice D by (3). Since the meet of join-irreducible elements is again join-irreducible in the distributive lattice $(D_3)^3$, the meet-homomorphism can be uniquely extended to a homomorphism from $(D_3)^3$ into D (s.e. Balbes and Horn [1], Theorem 4).

Claim 1 has the following dualization:

CLAIM 1*. There exists a unique homomorphism α^* from the distributive lattice $(D_3)^3$ into M with $\alpha^*222=1$, $\alpha^*221=c\vee(a\wedge e)$, $\alpha^*220=c$, $\alpha^*212=b\vee(a\wedge c)\vee d\vee(a\wedge e)$, $\alpha^*202=b\vee d$, $\alpha^*122=e\vee(a\wedge c)$, and $\alpha 022=e$.

$$c\wedge e\wedge(a\vee b)=c\wedge e\wedge(a\vee d). \tag{4}$$

Proof. By the M_3-Lemma ($E_0=\{b\}$, $E_2=\{a\}$, $E_3=\{c\}$, $E_5=\{d\}$, $E_1=\{e\}$), we have

$$((a\vee b)\wedge c)\vee((a\vee b)\wedge(b\vee d))\vee(c\wedge(b\vee d))\geqslant$$
$$\geqslant((a\wedge e)\vee(c\wedge e))\wedge((a\wedge e)\vee d)\wedge((c\wedge e)\vee d).$$

Using (3) and the modular law, we get

$$((a\vee b)\wedge c)\vee((a\vee b)\wedge(b\vee d))\vee(c\wedge(b\vee d))=$$
$$=(a\wedge c)\vee b\vee((a\vee b)\wedge(b\vee d))\vee b=(a\vee b)\wedge(b\vee d\vee(a\wedge c))$$

and, by duality and symmetry,

$$((a\wedge e)\vee(c\wedge e))\wedge((a\wedge e)\vee d)\wedge((c\wedge e)\vee d)=(a\wedge e)\vee(e\wedge c\wedge(a\vee d));$$

hence

$$c\wedge e\wedge(a\vee b)=c\wedge e\wedge(a\vee b)\wedge(d\vee(c\wedge(a\vee b))=c\wedge e\wedge(a\vee b)\wedge(b\vee d\vee(a\wedge c))\geqslant$$
$$\geqslant c\wedge e\wedge((a\wedge e)\vee(e\wedge c\wedge(a\vee d))\geqslant c\wedge e\wedge(a\vee d).$$

By symmetry we also get $c\wedge e\wedge(a\vee b)\leqslant c\wedge e\wedge(a\vee d)$.

$$b\wedge(e\vee(a\wedge c))=b\wedge(a\vee d) \tag{5}$$

Proof. By the M_3-Lemma ($E_0=\emptyset$, $E_2=\{a\}$, $E_3=\{b, c\}$, $E_5=\{d, e\}$, $E_1=\emptyset$) we have

$$(a\wedge c)\vee(a\wedge e)\vee(c\wedge e)\geqslant(a\vee b)\wedge(a\vee d)\wedge(b\vee d);$$

it follows

$$(a\wedge c)\vee e\geqslant b\wedge(a\vee d),$$

hence

$$b \wedge (e \vee (a \wedge c)) \geqslant b \wedge (a \vee d).$$

Conversely, using (4), we get

$$\begin{aligned} b \wedge (a \vee d) \geqslant b \wedge c \wedge ((a \wedge c) \vee (a \wedge e) \vee d) = b \wedge ((a \wedge c) \vee (c \wedge e \wedge (a \vee d))) = \\ = b \wedge ((a \wedge c) \vee (c \wedge e \wedge (a \vee b))) = b \wedge c \wedge ((a \wedge c) \vee (e \wedge (a \vee b))) = \\ = b \wedge (a \vee b) \wedge (e \vee (a \wedge c)) = b \wedge (e \vee (a \wedge c)). \end{aligned}$$

CLAIM 2. $\alpha = \alpha^*$.

Proof. The assertion follows from the fact that α and α^* coincide on the generating subset $\{100, 200, 010, 020, 001, 002\}$ of $(D_3)^3$:

$\alpha^*200 = \alpha^*220 \wedge \alpha^*202 = c \wedge (b \vee d) = b \vee (c \wedge d) = b = \alpha 200$	by (3)
$\alpha^*002 = d = \alpha 002$	by symmetry
$\alpha^*020 = \alpha^*220 \wedge \alpha^*022 = c \wedge e = \alpha 020,$	
$\alpha^*100 = \alpha^*122 \wedge \alpha^*200 = (e \vee (a \wedge c)) \wedge b = b \wedge (a \vee d) = \alpha 100$	by (5)
$\alpha^*001 = d \wedge (a \vee b) = \alpha 001$	by symmetry
$\alpha^*010 = \alpha^*212 \wedge \alpha^*020 = (b \vee (a \wedge c) \vee d \vee (a \wedge e)) \wedge c \wedge e$	
$= (b \vee (a \wedge c) \vee d) \wedge c \wedge e$	by the dual of (4)
$= (((a \vee b) \wedge c) \vee (d \wedge c)) \wedge e$	
$= (a \vee b) \wedge c \wedge e$	by (3)
$= c \wedge (a \vee b) \wedge e \wedge (a \vee d)$	by (4)
$= \alpha 010$	

$$(a \vee b) \wedge (a \vee d) = a \vee \alpha 100 = a \vee \alpha 010 = a \vee \alpha 001 = a \vee \alpha 111. \tag{6}$$

Proof. By the modular law, we have $a \vee \alpha 100 = a \vee (b \wedge (a \vee d)) = (a \vee b) \wedge (a \vee d)$ and symmetrically $a \vee \alpha 001 = (a \vee b) \wedge (a \vee d)$; furthermore, $a \vee \alpha 010 = a \vee (c \wedge (a \vee b) \wedge e \wedge (a \vee d)) = (a \vee (c \wedge e)) \wedge (a \vee b) \wedge (a \vee d) = (a \vee b) \wedge (a \vee d)$, since $a \vee (c \wedge e) = a \vee (a \wedge c) \vee (c \wedge e) = a \vee (c \wedge (e \vee (a \wedge c))) \geqslant a \vee (b \wedge (a \vee d)) = (a \vee b) \wedge (a \vee d)$ by (5) and the modular law.

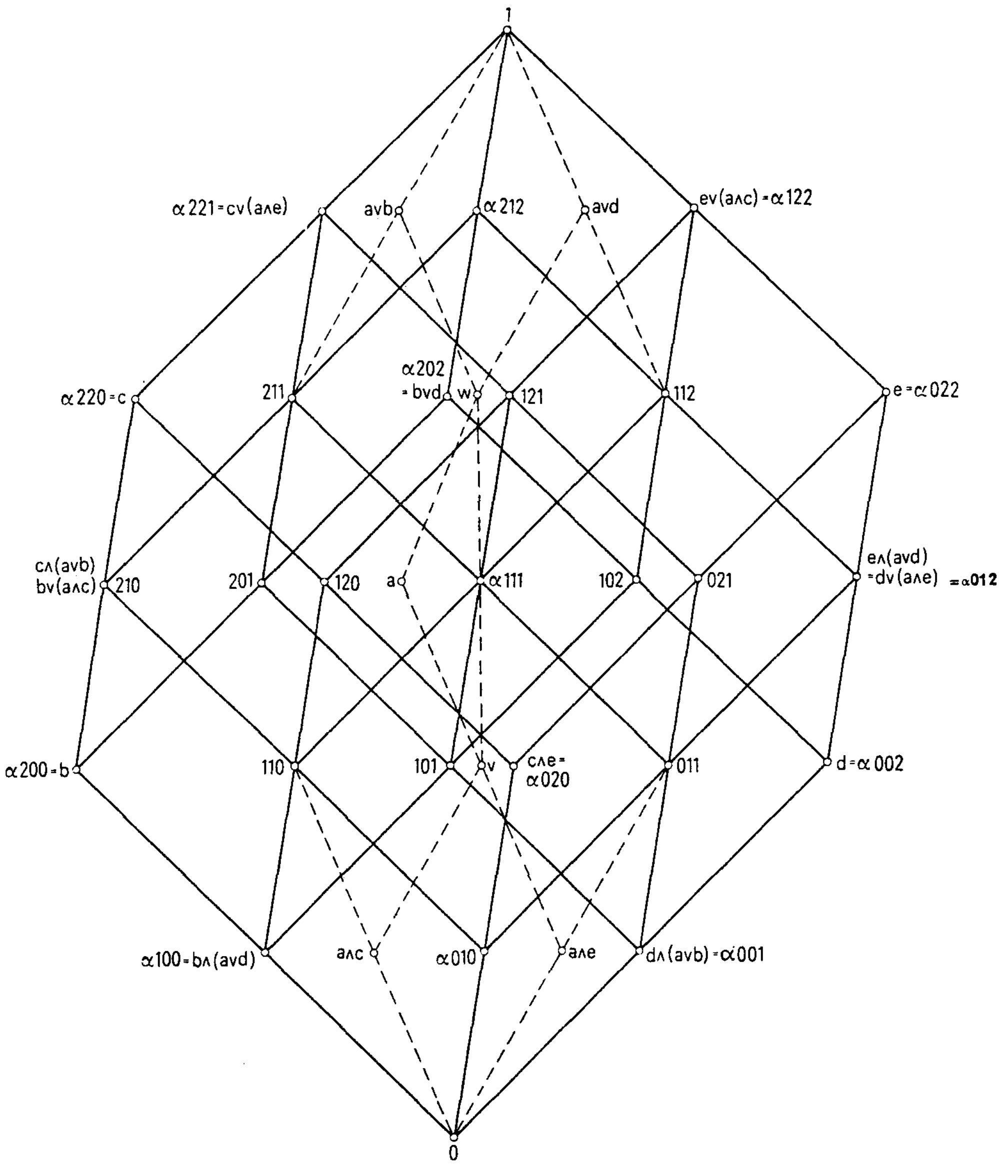
CLAIM 3. $v := (a \wedge c) \vee (a \wedge e)$ is a common complement of $\alpha 100$, $\alpha 010$, and $\alpha 001$ in the interval $[0, \alpha 111]$.

Proof. By (6), the dual of (6), and Claim 2, we get for $x \in \{100, 010, 001\}$ $(a \wedge c) \vee (a \wedge e) \vee \alpha x = (a \wedge \alpha 111) \vee \alpha x = (a \vee \alpha x) \wedge \alpha 111 = (a \vee b) \wedge (a \vee d) \wedge \alpha 111 = \alpha 111$; furthermore, $((a \wedge c) \vee (a \wedge e)) \wedge \alpha 100 = ((a \wedge c) \vee (a \wedge e)) \wedge b \wedge (a \vee d) \leqslant a \wedge b = 0$ by (1), symmetrically $(a \wedge c) \vee (a \wedge e) \wedge \alpha 001 = 0$, and finally $(a \wedge c) \vee (a \wedge e) \wedge \alpha 010 = ((a \wedge c) \vee (a \wedge e)) \wedge c \wedge (a \vee b) \wedge e \wedge (a \vee d) \leqslant a \wedge c \wedge e = 0$ by (2).

Claim 3 has the following dualization ($\alpha=\alpha^*$ by Claim 2!):

CLAIM 3*. $w:=(a\vee b)\wedge(a\vee d)$ is a common complement of $\alpha 221$, $\alpha 212$, and $\alpha 122$ in the interval $[\alpha 111, 1]$.

The relations proved in this section may be visualized by the following diagram:



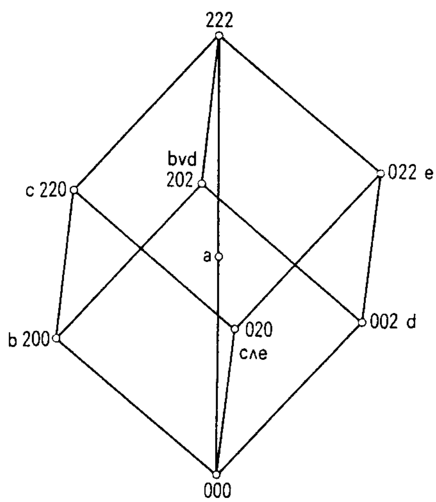
3. The subdirect decomposition

In this section we describe M as the union of certain sublattices of M. This leads to an isomorphism from M onto a subdirect product of the intervals $[0, \alpha 111]$ and $[\alpha 111, 1]$. For the first step we use the following lemma which is essentially out of Herrmann [4].

LEMMA. *Let $\notin$ be a lattice of finite length and let σ and γ be a join-homomorphism and a meet-homomorphism, resp., from S into a lattice L with certain bounded sublattices L_x $(x \in S)$ such that σx and γx are the lower bound and the upper bound of L_x, resp.; furthermore, let $L_x \cap L_y$ be a (non-empty) filter in L_x and an ideal in L_y if y covers x in S. Then $\bigcup (L_x \mid x \in S)$ is a sublattice of L.*

Proof. For $z_i \in L_{x_i}$ $(i=0, 1)$ we have to show that $z_0 \vee z_1$ is again in the union of the L_x. This is trivial for $x_0 = x_1$. If $x_0 \neq x_1$, then w.l.o.g. $x_1 < x_0 \vee x_1$. Let x_2 be the cover of x_1 in a maximal chain from x_1 to $x_0 \vee x_1$ of minimal length. By assumption, $z_2 := z_1 \vee \sigma x_2 \in L_{x_2}$. Therefore, we can conclude by induction that $z_0 \vee z_1 = z_0 \vee \sigma x_0 \vee z_1 \vee \sigma x_1 = z_0 \vee z_1 \vee \sigma(x_0 \vee x_1) = z_0 \vee z_1 \vee \sigma x_2 = z_0 \vee z_2 \in L_{x_0 \vee x_2} = L_{x_0 \vee x_1}$. Dually, we get $z_1 \wedge z_2 \in L_{x_0 \wedge x_1}$.

Let S be the lattice described by the following diagram:



We define $\hat{\alpha}: S \to M$, $\sigma: S \to M$, and $\gamma: S \to M$ by

$$\hat{\alpha} a := a \quad \text{and} \quad \hat{\alpha} x := \alpha x \quad \text{for} \quad x \in S \setminus \{a\},$$
$$\sigma x := \hat{\alpha} x \wedge \alpha 111 \quad \text{for} \quad x \in S,$$
$$\gamma x := \hat{\alpha} x \vee \alpha 111 \quad \text{for} \quad x \in S.$$

σ and γ are a join-homomorphism and a meet-homomorphism, resp., from S into M by Claim 1, 1*, 2, 3, and 3*. Let M_x $(x \in S)$ be the sublattice of M generated by $\{\hat{\alpha}x\} \cup [\sigma x, \alpha 111] \cup [\alpha 111, \gamma x]$.

$$z = (z \wedge \alpha 111) \vee (z \wedge \hat{\alpha}x) \quad \text{for} \quad x \in S \quad \text{and} \quad z \in M_x. \tag{7}$$

Proof. By modularity, M_x $(x \in S)$ is generated by $[\sigma x, \alpha 111] \cup [\sigma x, \hat{\alpha}x]$. Thus, Theorem III.15 in Birkhoff [2] can be applied which proves the assertion

CLAIM 4. $M = \bigcup (M_x \mid x \in S)$.

Proof. Using (7), it can be easily seen that S, M, σ, γ, and the M_x $(x \in S)$ fullfil the assumptions of the Lemma. Therefore, $\bigcup (M_x \mid x \in S)$ is a sublattice of M, which contains the generators a, b, c, d, and e of M by definition; hence $M = \bigcup (M_x \mid x \in S)$.

$$\alpha 111 \wedge (z_0 \vee z_1) = (\alpha 111 \wedge z_0) \vee (\alpha 111 \wedge z_1) \quad \text{for} \quad z_0, z_1 \in M. \tag{8}$$

Proof. By Claim 4, there are $x_i \in S$ with $z_i \in M_{x_i}$ $(i = 0, 1)$. It follows:

$$\begin{aligned}
\alpha 111 \wedge (z_0 \vee z_1) &= \alpha 111 \wedge ((z_0 \wedge \alpha 111) \vee (z_0 \wedge \hat{\alpha}x_0) \vee (z_1 \wedge \alpha 111) \vee (z_1 \wedge \hat{\alpha}x_1)) \\
& \hspace{20em} \text{by (7)} \\
&= (z_0 \wedge \alpha 111) \vee (z_1 \wedge \alpha 111) \vee (\alpha 111 \wedge ((z_0 \wedge \hat{\alpha}x_0) \vee (z_1 \wedge \hat{\alpha}x_1))) \\
&\leqslant (z_0 \wedge \alpha 111) \vee (z_1 \wedge \alpha 111) \vee (\alpha 111 \wedge (\hat{\alpha}x_0 \vee \hat{\alpha}x_1)) \\
&= (z_0 \wedge \alpha 111) \vee (z_1 \wedge \alpha 111) \vee (\alpha 111 \wedge \hat{\alpha}x_0) \vee (\alpha 111 \wedge \hat{\alpha}x_1) \\
& \hspace{20em} \text{by section 2} \\
&= (z_0 \wedge \alpha 111) \vee (z_1 \wedge \alpha 111) \vee \sigma x_0 \vee \sigma x_1 \\
&= (\alpha 111 \wedge z_0) \vee (\alpha 111 \wedge z_1).
\end{aligned}$$

The other inequality is always fullfilled.

CLAIM 5. The mapping $\psi : M \to [0, \alpha 111] \times [\alpha 111, 1]$ with $\psi z = (z \wedge \alpha 111, z \vee \alpha 111)$ is an isomorphism from M onto a subdirect product of $[0, \alpha 111]$ and $[\alpha 111, 1]$.

Proof. The assertion is a direct consequence of (8), Satz I.1.3 and Satz I.3.1 in Maeda [7].

Since M is subdirectly irreducible, the composition of ψ with one projection of $[0, \alpha 111] \times [\alpha 111, 1]$ has to be an isomorphism. Since $\psi a = (v, w)$, $\psi b = (\alpha 100, \alpha 211)$, $\psi c = (\alpha 110, \alpha 221)$, $\psi d = (\alpha 001, \alpha 112)$, and $\psi e = (\alpha 011, 122)$ by section 2, it follows from Claim 3 and 3* that M satisfies (*) or (**). Claim 1, 1*, and 2 result that the elements 0, b, $c \wedge e$, d, c, $b \vee d$, e, and 1 form a boolean sublattice of M. This boolean sublattice must consist of eight elements, since the intervals $[0, b]$, $[0, c \wedge e]$, and

$[0, d]$ are projective by the claims of section 2. Thus, the proof of the Theorem is finished

Appendix

In section 2 it has been proved, actually, that any elements a, b, c, d, e in a modular lattice which satisfy the relations $b \leqslant c$ and $d \leqslant e$ and the relations given by the D_2- and M_3-Lemma also satisfy the relations expressed in the first diagram, i.e. the relations stated in claims 1–3*. This result has been applied by W. Poguntke [8] to get the classification of the indecomposable S-spaces for $S = \mathbf{1}+\mathbf{2}+\mathbf{2}$ in a more lattice-theoretical way as it is done by other authors (cf. Gabriel [3]).

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