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RINGS OF QUOTIENTS OF FINITE AW^*-ALGEBRAS: REPRESENTATION AND ALGEBRAIC APPROXIMATION

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We show that Berberian's $*$-regular extension of a finite AW^*-algebra admits a faithful representation, matching the involution with adjunction, in the $\mathbb{C}$-algebra of endomorphisms of a closed subspace of some ultrapower of a Hilbert space. It also turns out that this extension is a homomorphic image of a regular subalgebra of an ultraproduct of matrix $*$-algebras $\mathbb{C}^{n\times n}$.

INTRODUCTION

Goodearl and Menal [1, Thm. 1.6] have shown that any C^*-algebra C is a homomorphic image of a residually finite-dimensional C^*-algebra B. Moreover, if C is separable, then B is a subdirect product of matrix algebras $\mathbb{C}^{n\times n}$. The prime objective of the present paper is to show that we can always choose B as a subalgebra of an ultraproduct of algebras $\mathbb{C}^{n\times n}$ and also to generalize the result to algebras represented in any inner product space. Ultraproducts here are those defined in model theory.

Another main objective is to extend this kind of algebraic approximation, with $*$-regular B, to $*$-regular algebras of quotients. Such algebras have been constructed by Berberian [2] (analyzed by

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Hafner [3], Pyle [4], and Berberian [5]), generalizing the Murray and von Neumann [6] $*$-regular algebra of unbounded operators associated with a finite von Neumann algebra factor. In a more general setting, these have been taken up by Handelman [7] and Ara and Menal [8]. The results mentioned are summarized in the following theorem (details are given in Sec. 6 below; see also [9, Thm. 2.3; 10, Prop. 21.2]).

THEOREM 1. Let A be a finite Rickart C^*-algebra. Then A admits a classical ring $Q(A)$ of right quotients. The involution and the structure of a $\mathbb{C}$-algebra on A extend uniquely to $Q(A)$, turning $Q(A)$ into an $*$-regular $\mathbb{C}$-algebra. Moreover, A and $Q(A)$ have the same projections. If, in addition, A is an AW^*-algebra, then $Q(A)$ is the maximal ring of right quotients of A.

The main result of the present paper is the following:

THEOREM 2. Let A and $Q(A)$ be as in Theorem 1.

(i) There are an inner product space $\hat{V}_{\hat{\mathbb{C}}}$, which is an ultrapower of a Hilbert space $V_{\mathbb{C}}$, a closed $\hat{\mathbb{C}}$-linear subspace U of $\hat{V}$, and a $\mathbb{C}$-algebra embedding ι of $Q(A)$ into the endomorphism algebra of $U_{\hat{\mathbb{C}}}$ such that $\iota(r^*)$ is the adjoint of $\iota(r)$ for any $r \in Q(A)$.

(ii) For $\mathbb{C}$-algebras with involution and pseudoinversion, $Q(A)$ is a homomorphic image of a subalgebra of an ultraproduct of algebras $\mathbb{C}^{n\times n}$.

(iii) The ortholattice $\mathbb{L}(A)$ of projections of A is a homomorphic image of a subortholattice of an ultraproduct of projection ortholattices of algebras $\mathbb{C}^{n\times n}$.

Relevant concepts are explained below; the proof of the theorem is given in Sec. 6. In (i), in particular, $\hat{\mathbb{C}}$ is an ultrapower of $\mathbb{C}$ and the scalar product on $\hat{V}_{\hat{\mathbb{C}}}$ is obtained from that of $V_{\mathbb{C}}$ via an ultrapower construction. Moreover, if A is separable then $V_{\mathbb{C}}$ can also be chosen separable.

The method of proof, of some interest in itself, is a representation of $Q(A)$ within a suitable inner product space, which is in fact a closed subspace of an ultrapower of the Hilbert space V in which A is represented due to the Gelfand–Naimark–Segal construction (GNS-construction for short). Such a representation for $Q(A)$ is obtained if we consider $Q(A)$ as a homomorphic image of some abstract algebraic structure, mimicking an algebra of unbounded operators. This, in turn, is used to reveal $Q(A)$ as a homomorphic image of a subalgebra of a sufficiently saturated elementary extension $\hat{T}$ of the algebra (with unit) T of endomorphisms of V generated by those having finite-dimensional image. The algebra $\hat{T}$ can be obtained as an ultrapower of T and admits a representation in an ultrapower of V.

1. INNER PRODUCT SPACES, $*$-REGULAR RINGS, AND PROJECTIONS

An $*$-*ring* is a ring R (associative with unit) endowed with an *involution*, i.e., an antiautomorphism $x \mapsto x^*$ of order 2. We will consider representations of Λ-algebras R with involution, where Λ is a commutative $*$-ring, within inner product spaces V_F. To define such, we have to assume that F is also a Λ-algebra and that involutions are related in a proper way. An adequate concept is that of an $*$-Λ-algebra, i.e., a Λ-algebra R which is an $*$-ring such that $1_\Lambda r = r$ and $(\lambda r)^* = \lambda^* r^*$ for all $\lambda \in \Lambda$ and $r \in R$. For $*$-Λ-algebras, the concepts of a homomorphism and of a subalgebra will refer to both the Λ-algebra structure and the involution. Note that C^*-algebras are (rather special) $*$-$\mathbb{C}$-algebras.

Our main interest here is the case where Λ is an $*$-ring $\mathbb{C}$ of complex numbers with conjugation and F is an elementary extension of $\mathbb{C}$. However, no extra effort is needed if we allow F to be any $*$-Λ-algebra which is a division ring. We say that a (right) F-vector space V_F is an *inner product space* if it is endowed with a *scalar product* $(x, y) \mapsto \langle x \mid y\rangle$ which is an anisotropic (or totally regular) sesquilinear form, Hermitian with respect to the involution (cf. [11]). Basic concepts and results for unitary spaces extend naturally to inner product spaces. In particular, any F-linear subspace U is an inner product space U_F with the induced scalar product. Let π_U denote the orthogonal projection onto U if it exists, for example, if $\dim U < \infty$.

The endomorphisms φ of V_F form a Λ-algebra, where $(\lambda\varphi)(v) = \varphi(v)\lambda$ for $\lambda \in \Lambda$ and $v \in V$. Notice that the action of Λ on V is defined by the equality $v\lambda = v(\lambda 1_F)$. Endomorphisms of V_F admitting an adjoint φ^* with respect to the scalar product form a subalgebra $\mathsf{End}^*_\Lambda(V_F)$ in the Λ-algebra of all endomorphisms, which is also an $*$-Λ-algebra; indeed, $\lambda^*\varphi^*$ is the adjoint of $\lambda\varphi$. If φ is an endomorphism with $\dim \mathsf{im}\varphi < \infty$ then $\varphi \in \mathsf{End}^*_\Lambda(V_F)$ and $\dim \mathsf{im}\varphi^* = \dim(\ker \varphi)^\perp = \dim \mathsf{im}\varphi$.

A *representation* of an $*$-Λ-algebra R within an inner product space V_F is a homomorphism $\varepsilon\colon R \to \mathsf{End}^*_\Lambda(V_F)$ or, more conveniently, a unitary R-F-bimodule ${}_RV_F$ such that $(\lambda r)v = r(v\lambda)$ and $\langle rv \mid w\rangle = \langle v \mid r^*w\rangle$ for all $r \in R$, $\lambda \in \Lambda$, and $v, w \in V$; here $rv = \varepsilon(r)(v)$. In accordance with this, we denote the action of an endomorphism φ by φv, and the composition of endomorphisms by $\psi\varphi$.

Our main concern will be *faithful* representations, i.e., representations ${}_RV_F$ such that $rv = 0$ for all $v \in V$ iff $r = 0$. If a faithful representation exists, then we say that R is *representable* within V_F. The Gelfand–Naimark–Segal construction (cf. [12, Sec. 62]) yields the following:

PROPOSITION 3. Any (separable) C^*-algebra is representable within a (separable) Hilbert space (as an algebra of bounded operators).

There are two approaches to $*$-regular rings. In the first approach, an ideal I of a ring is said to be (*von Neumann*) *regular* if, for any $a \in I$, there is $x \in I$ for which $axa = a$; such an element x is called a *quasi-inverse* of a (cf. [13]). We recall the following:

PROPOSITION 4 [14, Lemma 1.3]. A ring R is regular if and only if it admits a regular ideal I such that R/I is regular. Any ideal of a regular ring is regular.

An $*$-ring R is *proper* if $r^*r = 0$ implies $r = 0$ for all $r \in R$. Within any $*$-ring R, a^+ is a *Moore–Penrose pseudoinverse* (or a *Rickart relative inverse*) of a if

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a.$$

PROPOSITION 5. An $*$-ring R is proper and regular if and only if any $a \in R$ admits a pseudoinverse $a^+ \in R$. In this case a^+ is uniquely determined by a.

Proposition 5 is well known (see, e.g., [15, XII, Satz 2.4; 16, Prop. 88; 17, Lemma 4]). This allows an $*$-*regular ring* R to be defined as an $*$-ring with an additional operation $a \mapsto a^+$ under which a^+ is a pseudoinverse of a. If R is also an $*$-Λ-algebra, then we speak of an $*$-*regular* Λ-*algebra*. The concepts of subalgebras and homomorphisms of $*$-regular algebras concern pseudoinversion too. However, when speaking about representations, only the $*$-Λ-algebra structure matters. With $\Lambda = \mathbb{Z}$ we may subsume all $*$-regular rings. The equivalence between the two concepts of $*$-regularity extends beyond consideration of single algebras.

LEMMA 6. Let R and T be $*$-regular Λ-algebras, S an $*$-Λ-subalgebra of T, and $f\colon S \to R$ a surjective homomorphism (of $*$-Λ-algebras) such that the ideal $\mathsf{ker} f$ of S is regular. Then S is closed under pseudoinversion in T and $f\colon S \to R$ preserves pseudoinversion; i.e., in the language of $*$-regular Λ-algebras, S is a subalgebra of T and f is a homomorphism.

Proof. The algebra S, being an $*$-subring of T, is proper, and it is regular by Prop. 4. Therefore, S is $*$-regular by Prop. 5. The uniqueness of a pseudoinverse in T implies that S is closed under pseudoinversion. The uniqueness of a pseudoinverse in R implies that f preserves pseudoinversion. □

PROPOSITION 7. Let V_F be an inner product space. The set

$$\mathsf{End}^*_{\Lambda f}(V_F) = \{\varphi + \lambda \mathsf{id} \mid \lambda \in F,\ \varphi \in \mathsf{End}^*_\Lambda(V_F),\ \dim \mathsf{im}\varphi < \infty\}$$

forms a subalgebra of $\mathsf{End}^*_\Lambda(V_F)$ which is an $*$-regular Λ-algebra. In particular, if V_F is finite-dimensional, then $\mathsf{End}^*_\Lambda(V_F)$ is an $*$-regular Λ-algebra.

Proof. If $\dim \mathsf{im}\varphi < \infty$, then the subspace $U = (\ker \varphi)^\perp$ is finite-dimensional, which implies that $V = U \oplus U^\perp$ and $\varphi|_U$ is an isomorphism of U onto $W = \mathsf{im}\varphi$. Since U and W are finite-dimensional, the inverse $\psi\colon W \to U$ of $\varphi|_U$ has an adjoint ψ^*. Thus $\pi_W \psi^* \pi_U$ has $\pi_U \psi \pi_W$ as adjoint and belongs to $\mathsf{End}^*_{\Lambda f}(V_F)$. Moreover, $\varphi = \varphi \pi_U \psi \pi_W \varphi$. Therefore, $I = \{\varphi \in \mathsf{End}^*_\Lambda(V_F) \mid \dim \mathsf{im}\varphi < \infty\}$ is a regular ideal of $\mathsf{End}^*_{\Lambda f}(V_F)$. If $\dim V_F < \infty$ then $I = V_F$. Otherwise, since $\mathsf{End}^*_{\Lambda f}(V_F)/I$ is isomorphic to F, Proposition 6 applies to prove that $\mathsf{End}^*_{\Lambda f}(V_F)$ is $*$-regular. □

The following is granted by the Gram–Schmidt orthonormalization process.

PROPOSITION 8. Let V_F be an inner product space such that $\dim V_F = n < \infty$, and for any $\lambda, \mu \in F$, there is $\nu = \nu^* \in F$ for which $\lambda^*\lambda + \mu^*\mu = \nu^2$. Then $\mathsf{End}^*_\Lambda(V_F)$ is isomorphic to the matrix algebra $F^{n\times n}$ with involution $A = (a_{ij}) \mapsto A^*$, where A^* is the transpose of (a^*_{ij}).

An element e of an $*$-ring is a *projection* if $e = e^2 = e^*$. We observe that any projection is its own pseudoinverse and that $e = aa^+$ and $f = a^+a$ are projections if a^+ is a pseudoinverse of a.

PROPOSITION 9. The equality $ap = 0$ implies $(a^+)^*p = 0$ and $a^*p = 0$ implies $a^+p = 0$ for any $*$-regular ring R, any $a \in R$, and any projection $p \in R$.

Proof. For $e = aa^+$ and $f = a^+a$, $ap = 0$ entails $fp = a^+ap = 0$, whence $pf = (fp)^* = 0$. Thus $pa^+ = pa^+aa^+ = pfa^+ = 0$, and so $(a^+)^*p = 0$. From $a^*p = 0$, we obtain $pa = (a^*p)^* = 0$, i.e., $pe = paa^+ = 0$. Consequently, $ep = 0$ and $a^+p = a^+aa^+p = a^+ep = 0$. □

PROPOSITION 10 [16, Chap. 2; 10, Sec. 1]. If R is a regular ring (not necessarily with unit), then principal left ideals Ra form a (complemented) sublattice $\bar{L}(R)$ of the lattice of all left ideals. In addition, for any $a \in R$, there is an idempotent $e \in R$ such that $Ra = Re$.

Projections of a $*$-regular Λ-algebra R form an *ortholattice* $\mathbb{L}(R)$ where the partial order is given by the rule

$$e \leq f \Longleftrightarrow fe = e \Longleftrightarrow ef = e.$$

The least and greatest elements are equal to respectively 0 and 1 in R. The *orthocomplement* is $e' = 1 - e$; more exactly, e' is a complement of e, $e'' = e$, and $e \leq f$ if and only if $f' \leq e'$. Join (supremum) and meet (infimum) are defined by the equalities

$$e \cup f = f + (e(1-f))^{+}e(1-f),\ e \cap f = (e' \cup f')'.$$

The map $e \mapsto Re$ is an isomorphism from $\mathbb{L}(R)$ onto $\bar{L}(R)$.

As shown by von Neumann, any modular ortholattice with a certain type of coordinate system is isomorphic to $\mathbb{L}(R)$ for some $*$-regular algebra R with a system of matrix units. In this sense, we have equivalent structures. Nonetheless, $*$-regular rings appear much better suited for the present discussion.

PROPOSITION 11. If S is a subalgebra of an $*$-regular Λ-algebra R, then $\mathbb{L}(S)$ is a subortholattice of $\mathbb{L}(R)$. If $\varphi\colon R \to S$ is a homomorphism, then its restriction ψ to $\mathbb{L}(R)$ is a homomorphism into $\mathbb{L}(S)$; if φ is surjective, then ψ is also surjective.

Proof. In view of Proposition 10, it suffices to state that φ is surjective. In fact, let e be a projection in S. Choose its arbitrary preimage $a \in R$ under φ. Then aa^{+} is a projection and $\varphi(aa^{+}) = ee^{+} = e^2 = e$. $\square$

2. CONCEPTS FROM MODEL THEORY

For a fixed commutative $*$-ring Λ with unit, we are going to consider the classes of all $*$-Λ-algebras and of all $*$-regular Λ-algebras. Their members are viewed as 1-sorted algebraic structures where each $\lambda \in \Lambda$ determines a unary operation $x \mapsto \lambda x$. Moreover, besides this and the ring structure (note that additive inversion is not required since it can be captured via $-x = (-1_{\Lambda})x$), for both types of algebras, we also have a unary operation of involution, while for $*$-regular Λ-algebras, we consider in addition a unary operation of pseudoinversion. Ortholattices are treated in a signature containing the binary operations of join and meet, and also a unary operation of orthocomplementation. Since the three classes mentioned can be defined by identities, they are closed under taking direct products, subalgebras, and homomorphic images.

Let $\mathcal{U}$ be an ultrafilter over a set I. In view of the explicit definition of $\mathbb{L}(R)$ in terms of R (Prop. 10), the following holds:

PROPOSITION 12. $\mathbb{L}\left(\prod\limits_{i\in I} R_i/\mathcal{U}\right) = \prod\limits_{i\in I} \mathbb{L}(R_i)/\mathcal{U}$ for any $*$-regular Λ-algebras R_i, $i \in I$.

We will also have to use ultraproducts of inner product spaces V_F and representations ${}_RV_F$, but the presence of scalar products excludes viewing them as 1-sorted algebraic structures. The most convenient way is to treat an inner product space V_F as a 2-sorted algebraic structure with sorts V and F, endowed with group operations and $*$-Λ-algebra operations, respectively. Moreover, there are two binary operations $(v,\alpha) \mapsto v\alpha \in V$ and $(u,v) \mapsto \langle u \mid v\rangle \in F$, where $u, v \in V$ and $\alpha \in F$. In dealing with a representation ${}_RV_F$, we treat the $*$-Λ-algebra R as a third sort and, in addition, a binary operation $(r,v) \mapsto rv \in V$ for $r \in R$ and $v \in V$.

Concepts such as a homomorphism, a subalgebra, a direct product, and an ultraproduct can be generalized to many-sorted algebraic structures in an obvious manner. All constructions are built sortwise; i.e., the sorts of a direct product (an ultraproduct, etc.) of A_i, $i \in I$, are direct products (ultraproducts, etc.) of the sorts of A_i, $i \in I$. Of course, at the price of undue technical complications, we could also consider many-sorted structures as 1-sorted relational structures.

Given a formula φ in a fixed signature, a structure A, and elements $a_1, \ldots, a_n$ of A matching the sorts of the free variables $x_1, \ldots, x_n$ occurring in φ, the validity of φ in A under the substitution $x_i \mapsto a_i$ is defined by the same inductive approach as in the 1-sorted case and is denoted by

$$A \models \varphi(a_1, \ldots, a_n).$$

PROPOSITION 13. Let $\mathcal{U}$ be an ultrafilter over a set I. The following statements hold:

(1) Suppose Λ is a commutative $*$-ring, R_i and F_i are $*$-Λ-algebras, $(V_i)_{F_i}$ is an inner product space, and ${}_{R_i}(V_i)_{F_i}$ is a faithful representation for all $i \in I$. Then $V_F = \prod_{\mathcal{U}}(V_i)_{F_i}$ is an inner product space and ${}_RV_F = \prod_{\mathcal{U}} {}_{R_i}(V_i)_{F_i}$ is a faithful representation, where $F = \prod_{\mathcal{U}} F_i$ and $R = \prod_{\mathcal{U}} R_i$.

(2) For an $*$-Λ-algebra F and a natural number n, the ultrapower $(F^{n\times n})^I/\mathcal{U}$ of matrix $*$-Λ-algebras is isomorphic to the matrix $*$-Λ-algebra $(F^I/\mathcal{U})^{n\times n}$.

Proof. (i) Is an obvious consequence of the Łoś theorem. The isomorphism in (ii) is given by the rule

$$\big[(a_i^{jk})_{j,k=1,\ldots,n} \mid i \in I\big] \mapsto \big([a_i^{jk} \mid i \in I]\big)_{k,j=1,\ldots,n}. \ \square$$

PROPOSITION 14. Any elementary extension of a representation ${}_RV_F$ is again a representation, ${}_{\hat{R}}\hat{V}_{\hat{F}}$, where $\hat{F}$, $\hat{V}_{\hat{F}}$, and $\hat{R}$ are elementary extensions of F, V_F, and R, respectively.

In the proof of our central result, we have to apply a concept of saturated structures. We will present here a very weak form just sufficient for our purposes. Considering a fixed structure A, we add a new constant symbol $\underline{a}$, called a *parameter*, for each $a \in A$. In what follows, $\Sigma(x)$ is a set of formulas with one free variable x in this extended language. Given an embedding $h\colon A \to B$, we say that B is *modestly saturated over* A *via* h if any set $\Sigma(x)$ of formulas with parameters from A which is finitely realized in A (where $\underline{a}$ is interpreted as $\underline{a}^A = a$) is realized in B (where $\underline{a}$ is interpreted as $\underline{a}^B = h(a)$).

PROPOSITION 15 [18, Cor. 4.3.14]. Every structure A admits an elementary embedding h into some structure B which is modestly saturated over A via h. We can take B to be an ultrapower of A and take h to be the canonical embedding. Identifying a with $h(a)$, we may assume that B is an elementary extension of A.

3. REPRESENTATIONS OF ALGEBRAS WITH INVOLUTION

THEOREM 16. Let an $*$-Λ-algebra R have a faithful representation in an inner product space V_F. Then R is a homomorphic image of a subalgebra S of an ultraproduct of $\mathsf{End}^*_\Lambda(U_F)$ with U ranging over finite-dimensional subspaces of V_F. Moreover, if R is ($*$-)regular, then so is S.

Recall that in the $*$-regular case, all algebraic constructions also refer to pseudoinversion; in particular, only $*$-regular $*$-Λ-subalgebras are admitted.

The **proof** is a variation of the approach in [19, 20].

Choose any set I of finite-dimensional subspaces of V_F such that every finite-dimensional subspace W of V_F is contained in some member of I. Given a basis B for the vector space V_F, as I we may take, for example, the set of all subspaces U of V_F spanned by finite subsets of B. If the basis B is countable and enumerated, then we may choose I consisting of all subspaces spanned by initial segments of B. For $U \in I$, we set $U^+ = \{W \in I \mid U \subseteq W\}$. Notice that $U_1^+ \cap U_2^+ = (U_1 + U_2)^+$. Therefore, there is an ultrafilter $\mathcal{U}$ on I such that $U^+ \in \mathcal{U}$ for all $U \in I$. For simplicity, let R_U denote the $*$-regular Λ-algebra $\mathsf{End}^*_\Lambda(U_F)$. Consider a direct product $T = \prod_{U \in I} R_U$ and an ultraproduct $\hat{T} = \prod_{U \in I} R_U/\mathcal{U}$. The elements of T and $\hat{T}$ will be denoted $\sigma = (\sigma_U \mid U \in I)$ and $[\sigma]$, respectively. We first relate T to ${}_RV_F$.

Let $\sigma \in T$, $r \in R$, and $U_0 \in I$. We say that $J \in \mathcal{U}$ *witnesses* $\sigma \sim r$ for U_0 if $J \subseteq U_0^+$ and

$$\sigma_U v = rv,\ \sigma_U^* v = r^* v \text{ for all } U \in J \text{ and all } v \in U_0.$$

Note that if J witnesses $\sigma \sim r$ for $U_0 \in I$, then J witnesses $\sigma \sim r$ for any $U_1 \subseteq U_0$. We put $\sigma \sim r$ if for any $U_0 \in I$ there is $J \in \mathcal{U}$ witnessing $\sigma \sim r$ for U_0. Let

$$S = \{\sigma \in T \mid \sigma \sim r \text{ for some } r \in R\}.$$

We prove several auxiliary assertions. Let $\sigma, \tau \in T$ and $r, r_0, r_1, s \in R$.

ASSERTION 17. There is a well-defined map $g\colon S \to R$, $g(\sigma) = r$, where $\sigma \sim r$.

Proof. We have to show that $\sigma \sim r_0$ and $\sigma \sim r_1$ imply $r_0 = r_1$. Consider an arbitrary vector $v \in V$ and a set $U_0 \in I$ containing v. Let J_i witness $\sigma \sim r_i$ for U_0, $i < 2$. Then $J = J_1 \cap J_2$ also witnesses $\sigma \sim r_i$ for U_0, $i < 2$, whence $r_0 v = \sigma_{U_0} v = r_1 v$. This shows that $r_0 v = r_1 v$ for all $v \in V$. Since ${}_RV_F$ is a faithful representation, we conclude that $r_0 = r_1$. $\square$

ASSERTION 18. In the language of $*$-Λ-algebras, S is a subalgebra of T and $g\colon S \to R$ is a homomorphism.

Proof. Let $\sigma \sim r$, $\tau \sim s$, and $\lambda \in \Lambda$. We need to show that

$$(1)\ \sigma^* \sim r^*,\ (2)\ \lambda\sigma \sim \lambda r,\ (3)\ \tau + \sigma \sim s + r,\ (4)\ \tau\sigma \sim sr.$$

Note that (1) is obvious by definition. To prove (2)-(4), we consider $U_0 \in I$ and choose $J \in \mathfrak{U}$ witnessing $\sigma \sim r$ and also $K \in \mathfrak{U}$ witnessing $\tau \sim s$ for U_0. Then $J \cap K \in \mathfrak{U}$ witnesses both $\sigma \sim r$ and $\tau \sim s$ for U_0. Applying linearity, we see that for any $U \in J \cap K$ and any $v \in U_0$, the following equalities hold:

$$
\begin{aligned}
&(\lambda\sigma_U)v = (\sigma_U v)\lambda = (rv)\lambda = (\lambda r)v,\\
&(\lambda\sigma)^*_U v = (\sigma^*_U v)\lambda^* = (r^* v)\lambda^* = (\lambda^* r^*)v = (\lambda r)^* v,\\
&(\sigma+\tau)_U v = \sigma_U v + \tau_U v = rv + sv = (r+s)v,\\
&(\sigma+\tau)^*_U v = \sigma^*_U v + \tau^*_U v = r^* v + s^* v = (r^* + s^*)v = (r+s)^* v.
\end{aligned}
$$

Therefore, $J \cap K$ witnesses $\lambda\sigma \sim \lambda r$ and $\sigma + \tau \sim r + s$ for U_0; i.e., items (2) and (3) hold. To prove (4), we choose $U_1 \in I$ so that $U_1 \supseteq rU_0$ and choose $U_2 \in I$ so that $U_2 \supseteq s^* U_0$. Let $J_0 \in \mathfrak{U}$ witness $\tau \sim s$ for U_1 and $J_1 \in \mathfrak{U}$ witness $\sigma \sim r$ for U_2. Then $J' = J \cap K \cap J_0 \cap J_1 \in \mathfrak{U}$, and for any $U \in J'$ and any $v \in U_0$, we have

$$
\begin{aligned}
&(\tau\sigma)_U v = \tau_U(\sigma_U v) = \tau_U(rv) = s(rv) = (sr)v,\\
&(\tau\sigma)^*_U v = \sigma^*_U(\tau^*_U v) = \sigma^*_U(s^* v) = r^*(s^* v) = (r^* s^*)v = (sr)^* v;
\end{aligned}
$$

i.e., J' witnesses $\tau\sigma \sim sr$ for U_0. □

ASSERTION 19. The map $g\colon S \to R$ is surjective.

Proof. For $r \in R$, denote by $\varphi = \varepsilon(r)$ the corresponding endomorphism of V_F. For $U \in I$, put

$$\sigma_U = \pi_U \varphi|_U = \pi_U \varphi \pi_U|_U \in R_U$$

and $\sigma = (\sigma_U \mid U \in I)$, where π_U is the orthogonal projection of V onto U. Notice that $\sigma^*_U = \pi_U \varphi^* \pi_U|_U = \pi_U \varphi^*|_U$. For $U_0 \in I$ fixed, $U_1 \in I$ is chosen so that $U_1 \supseteq U_0 + rU_0 + r^* U_0$. Let $J = U_1^+ \in \mathfrak{U}$. For any $U \in J$ and any $v \in U_0$, we have $rv, r^* v \in U$ and

$$
\begin{aligned}
&\sigma_U v = \pi_U\big(\varphi(\pi_U v)\big) = \pi_U(\varphi v) = \pi_U(rv) = rv,\\
&\sigma^*_U v = \pi_U\big(\varphi^*(\pi_U v)\big) = \pi_U(\varphi^* v) = \pi_U(r^* v) = r^* v.
\end{aligned}
$$

Therefore, J witnesses $\sigma \sim r$ for U_0; consequently, $g(\sigma) = r$. □

We put

$$\hat{S} = \{[\sigma] \mid \sigma \in S\}.$$

ASSERTION 20. In the language of $*$-Λ-algebras, $\hat{S}$ is a subalgebra of $\hat{T}$, while $f\colon \hat{S} \to R$, $f([\sigma]) = r$, where $\sigma \sim r$, is a well-defined surjective homomorphism.

Proof. In view of Assertion 18, it suffices to show that $[\tau] = [\sigma]$ and $\sigma \sim r$ imply $\tau \sim r$. Let $K = \{U \in I \mid \sigma_U = \tau_U\}$; in particular, $K \in \mathfrak{U}$. Given $U_0 \in I$, let J witness $\sigma \sim r$ for U_0. Then $J \cap K$ witnesses $\tau \sim r$ for U_0. □

For any $U_0 \in I$, we put $\chi^{U_0} = (\chi_U^{U_0} \mid U \in I) \in T$, where $\chi_U^{U_0}$ is the orthogonal projection of U onto U_0 if $U_0 \subseteq U$ and $\chi_U^{U_0} = 0$ otherwise.

ASSERTION 21. We have $[\sigma] \in \mathsf{ker} f$ if and only if

$$(*) \qquad [\sigma] \cdot [\chi^{U_0}] = [\sigma^*] \cdot [\chi^{U_0}] = 0 \text{ for all } U_0 \in I.$$

Proof. By definition, $[\sigma] \in \mathsf{ker} f$ iff $\sigma \sim 0$. Thus let $\sigma \sim 0$ and $U_0 \in I$. Choose $J \in \mathfrak{U}$ witnessing $\sigma \sim 0$ for U_0. Then for all $U \in J$ and all $v \in U_0$ we have $\sigma_U v = 0 = \sigma_U^* v$, which implies $\sigma_U \chi_U^{U_0} = 0 = \sigma_U^* \chi_U^{U_0}$. Since $J \in \mathfrak{U}$, $(*)$ holds.

Conversely, assume that $(*)$ holds and $U_0 \in I$. This means that $K = \{U \in I \mid \sigma_U \chi_U^{U_0} = 0 = \sigma_U^* \chi_U^{U_0}\} \in \mathfrak{U}$. Then $J = K \cap U_0^+ \in \mathfrak{U}$ witnesses $\sigma \sim 0$ for U_0. □

ASSERTION 22. The ideal $\mathsf{ker} f$ is regular.

Proof. By Proposition 7, R_U is $*$-regular for any $U \in I$; so the ultraproduct $\hat{T}$ is also $*$-regular. Note that $[\chi^{U_0}] \in \hat{T}$ is a projection for any $U_0 \in I$ since $\chi_U^{U_0}$ is a projection in R_U for any $U \in I$. Now let $[\sigma] \in \mathsf{ker} f$ and $[\tau]$ be its pseudoinverse in $\hat{T}$. Then Assertion 21 and Proposition 9 imply that

$$[\tau] \cdot [\chi^{U_0}] = [\tau^*] \cdot [\chi^{U_0}] = 0 \text{ for all } U_0 \in I,$$

which, in view of Assertion 21, yields $[\tau] \in \mathsf{ker} f$. □

Now we complete the proof of the theorem. The first statement for $*$-Λ-algebras follows immediately from Assertion 20. In the $*$-regular setting, it suffices to apply Assertion 22 and Lemma 6. □

Remark 23. Assume that V_F has a countable orthonormal basis $v_0, v_1, v_2, \dots$ (which is the case if, e.g., $\dim V_F = \omega$ and F satisfies the hypothesis of Prop. 8). Let I consist of subspaces U_n spanned by $\{v_0, \dots, v_n\}$, $n < \omega$, and let $\mathfrak{U}$ extend the cofinite filter. Then $\mathsf{End}^*_\Lambda\big((U_n)_F\big)$, $n < \omega$, can be uniformly viewed as matrix algebras $F^{n \times n}$. In this event $\mathsf{ker} f$ is composed of elements of the form $[A_n \mid n < \omega]$ where for any $m < \omega$, there is $J \in \mathfrak{U}$ such that the first m rows and m columns of A_n consist of 0's provided that $U_n \in J$ (see [19]).

COROLLARY 24. Any C^*-algebra is a homomorphic image of a subalgebra of an ultraproduct of algebras $\mathbb{C}^{n \times n}$, $n < \omega$.

Proof. We apply Theorem 16 to a representation created by the GNS-construction and observe that $\mathsf{End}^*_\Lambda(U_\mathbb{C}) \cong \mathbb{C}^{n \times n}$ if U is a finite-dimensional subspace of a unitary space (cf. Prop. 8). □

A method for obtaining representations of homomorphic images (elaborating on [20, Thm. 3.8]) is given in

PROPOSITION 25. For any regular $*$-Λ-algebra R having a faithful representation within an inner product space V_F, there is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that, for any regular ideal $I = I^*$, R/I admits a faithful representation within some closed subspace of $\hat{V}_{\hat{F}}$.

Proof. According to Proposition 15, there is an ultrapower ${}_{\hat{R}}\hat{V}_{\hat{F}}$ of the faithful representation ${}_RV_F$ which is modestly saturated over ${}_RV_F$ via the canonical embedding. Then $\hat{V}$ is an R-module since R is canonically embedded into $\hat{R}$ and

$$U = \{v \in \hat{V} \mid av = 0 \text{ for all } a \in I\} = \bigcap_{a \in I} (a\hat{V})^{\perp}$$

is a closed subspace of $\hat{V}_{\hat{F}}$ and a left (R/I)-module. Moreover, the equality $I = I^*$ implies that

$$\langle (r + I)v \mid w\rangle = \langle v \mid (r^* + I)w\rangle \text{ for all } v, w \in U,$$

which proves that ${}_{R/I}U_{\hat{F}}$ is a representation of R/I.

We argue that this representation is faithful; i.e., for any $a \notin I$, there is $v \in U$ such that $av \neq 0$. Since $v \in U$ means that $bv = 0$ for all $b \in I$, we need to show that the set

$$\Sigma(x) = \{\underline{a}x \neq 0\} \cup \{\underline{b}x = 0 \mid b \in I\}$$

of formulas with parameters from $\{a\} \cup I$ and a free variable x of type V is satisfiable in ${}_{\hat{R}}\hat{V}_{\hat{F}}$. Due to modest saturation, it suffices to state that for any $b_1, \ldots, b_n \in I$, there is $v \in V$ such that $av \neq 0$ and $b_i v = 0$ for all $i \in \{1, \ldots, n\}$. By virtue of Proposition 10 and the fact that I is regular, there is an idempotent $e \in I$ for which $Ie = \sum\limits_{i=1}^{n} Ib_i$; in particular, $b_i e = b_i$ and $b_i v = 0$ for any $i \in \{1, \ldots, n\}$ and any $v \in V$ with $ev = 0$. Thus we need only show that there is $v \in V$ such that $ev = 0$ but $av \neq 0$.

Assume the contrary, namely, that $ev = 0$ implies $av = 0$ for any $v \in V$. For $w \in V$, put $v = (1 - e)w$. We have $ev = 0$, so $av = 0$ by our assumption. Thus $0 = av = a(1 - e)w$ for all $w \in V$, and since ${}_RV_F$ is a faithful representation, we obtain $a(1 - e) = 0$. In this case $a = ae \in I$, which clashes with the choice of a. □

4. ALGEBRAS OF GENERALIZED OPERATORS

Given a commutative $*$-ring Λ, a *pre-$*$-Λ-algebra* is a set R endowed with binary operations $+$ and $\cdot$, constants 0_R and 1_R, a unary operation written $r \mapsto \lambda r$ for each $\lambda \in \Lambda$, and a symmetric binary relation $\bowtie$ such that for any $r \in R$ there is $r^* \in R$ with $r \bowtie r^*$, and for all $r, r^*, s, s^* \in R$ and all $\lambda \in \Lambda$, the following hold:

(a) $r \bowtie r^*$ and $s \bowtie s^*$ imply $r + s \bowtie r^* + s^*$;

(b) $r \bowtie r^*$ and $s \bowtie s^*$ imply $r \cdot s \bowtie s^* \cdot r^*$;

(c) $r \bowtie r^*$ implies $\lambda r \bowtie \lambda^* r^*$;

(d) $0_R \bowtie 0_R$ and $1_R \bowtie 1_R$.

An *action* of R on an inner product space V_F, where F is an $*$-Λ-algebra, assigns each $r \in R$ a linear subspace $\mathsf{dom} r$ of V_F and an F-linear map $\mathsf{dom} r \to V$ written $v \mapsto rv$. Thus, in particular, for all $v, w \in V$, $r \in R$, and $\alpha \in F$, the following hold:

(e) if $u, v \in \mathsf{dom} r$, then $v + w \in \mathsf{dom} r$ and $r(v + w) = rv + rw$;

(f) if $u \in \mathsf{dom} r$, then $u\alpha \in \mathsf{dom} r$ and $r(v\alpha) = (rv)\alpha$.

We write ${}_RV_F$ for an inner product space V with right action of F and left action of R. Consider ${}_RV_F$ as a 3-sorted structure with sorts V, F, and R; the action of R on V is defined by the ternary relation

$$\{(r,v,w)\in R\times V\times V \mid r\in R,\ v\in \mathsf{dom}r,\ w=rv\}.$$

Put $rX=\{rv\mid v\in X\}$ and $r^{-1}(X)=\{v\in\mathsf{dom}r\mid rv\in X\}$ for any $X\subseteq V$.

Furthermore, we assume that there is a downward directed set $\mathcal{D}$ of linear subspaces of V_F. We say that the action of R on V_F is $\mathcal{D}$-*supported* if, for all $r,s\in R$ and all $\lambda\in\Lambda$, the following hold:

(i) $D\subseteq\mathsf{dom}r$ for some $D\in\mathcal{D}$;

(ii) for any $D\in\mathcal{D}$, there is $D'\in\mathcal{D}$ such that $D'\subseteq r^{-1}(D)$;

(iii) there is $D\in\mathcal{D}$ such that $D\subseteq\mathsf{dom}r\cap\mathsf{dom}s\cap\mathsf{dom}(r+s)$ and $(r+s)v=rv+sv$ for all $v\in D$;

(iv) $0_Rv=0$ and $1_Rv=v$ for all $v\in V$;

(v) there is $D\in\mathcal{D}$ such that $D\subseteq\mathsf{dom}(r\cdot s)\cap s^{-1}(\mathsf{dom}r)\cap\mathsf{dom}s$ and $(r\cdot s)v=r(sv)$ for all $v\in D$;

(vi) there is $D\in\mathcal{D}$ such that $D\subseteq\mathsf{dom}r\cap\mathsf{dom}(\lambda r)$ and $(\lambda r)v=(rv)\lambda$ for all $v\in D$;

(vii) if $r\bowtie r^*$, then there is $D\in\mathcal{D}$ such that $D\subseteq\mathsf{dom}r\cap\mathsf{dom}r^*$ and $\langle rv\mid w\rangle=\langle v\mid r^*w\rangle$ for all $v,w\in D$.

For a $\mathcal{D}$-supported action of R on V_F, we define a binary relation $\approx_{\mathcal{D}}$ on R by setting

$r\approx_{\mathcal{D}} s$ iff for any $r^*\bowtie r$ and any $s^*\bowtie s$ there is $D\in\mathcal{D}$ such that $r\approx_{\mathcal{D}} s$ is *witnessed by* D *under the given condition*: namely, $D\subseteq\mathsf{dom}r\cap\mathsf{dom}r^*\cap\mathsf{dom}s\cap\mathsf{dom}s^*$ and $rv=sv$ and $r^*v=s^*v$ for all $v\in D$.

We speak of a Λ-*algebra of generalized operators* on V_F and denote it by $({}_RV_F;\mathcal{D})$ if, in addition, the following holds:

(viii) if $r\bowtie t$ and $s\bowtie t$, then $r\approx_{\mathcal{D}} s$ for all $r,s,t\in R$.

LEMMA 26. If $({}_RV_F;\mathcal{D})$ is a Λ-algebra of generalized operators, then $\approx_{\mathcal{D}}$ is a congruence with respect to the operations defined on R. Moreover, for any $r,r^*,s,s^*\in R$ such that $r\bowtie r^*$ and $s\bowtie s^*$, $r\approx_{\mathcal{D}} s$ implies $r^*\approx_{\mathcal{D}} s^*$.

Proof. It is straightforward to verify that $\approx_{\mathcal{D}}$ is reflexive and symmetric. We show that $\approx_{\mathcal{D}}$ is transitive. Let $r\approx_{\mathcal{D}} s$, $s\approx_{\mathcal{D}} t$ and $r\bowtie r^*$, $t\bowtie t^*$. Then there is $s^*\in R$ for which $s\bowtie s^*$. Let $D_0\in\mathcal{D}$ witness $r\approx_{\mathcal{D}} s$ under the conditions $r\bowtie r^*$ and $s\bowtie s^*$, and let $D_1\in\mathcal{D}$ witness $s\approx_{\mathcal{D}} t$ under the conditions $s\bowtie s^*$ and $t\bowtie t^*$. Since $\mathcal{D}$ is directed downward, there is $D\in\mathcal{D}$ such that $D\subseteq D_0\cap D_1$. Then D witnesses $r\approx_{\mathcal{D}} t$ under the conditions $r\bowtie r^*$ and $t\bowtie t^*$.

For any $\lambda\in\Lambda$, we verify that $\approx_{\mathcal{D}}$ respects the unary operation $\lambda\cdot$. Suppose that $r\approx_{\mathcal{D}} s$ for some $r,s\in R$. To prove that $\lambda r\approx_{\mathcal{D}}\lambda s$, let $t\bowtie\lambda r$ and $u\bowtie\lambda s$. Let also $D'\subseteq\mathsf{dom}r\cap\mathsf{dom}r^*\cap\mathsf{dom}s\cap\mathsf{dom}s^*$ witness $r\approx_{\mathcal{D}} s$ under the conditions $r\bowtie r^*$ and $s\bowtie s^*$. Then, in view of (c) and (viii), $t\approx_{\mathcal{D}}\lambda^*r^*$ and $u\approx_{\mathcal{D}}\lambda^*s^*$; in particular, there are $D_{0r},D_{0s}\in\mathcal{D}$ such that $tv=(\lambda^*r^*)v$, for all $v\in D_{0r}$, and $uv=(\lambda^*s^*)v$ for all $v\in D_{0s}$. Moreover, according to (vi), there are $D_{1r},D_{2r},D_{1s},D_{2s}\in\mathcal{D}$ for which the following hold:

$$
\begin{gathered}
D_{1r} \subseteq \mathsf{dom}r \cap \mathsf{dom}(\lambda r) \text{ and } (\lambda r)v = (rv)\lambda \text{ for all } v \in D_{1r};\\
D_{2r} \subseteq \mathsf{dom}r^* \cap \mathsf{dom}(\lambda^* r^*) \text{ and } (\lambda^* r^*)v = (r^* v)\lambda^* \text{ for all } v \in D_{2r};\\
D_{1s} \subseteq \mathsf{dom}s \cap \mathsf{dom}(\lambda s) \text{ and } (\lambda s)v = (sv)\lambda \text{ for all } v \in D_{1s};\\
D_{2s} \subseteq \mathsf{dom}s^* \cap \mathsf{dom}(\lambda^* s^*) \text{ and } (\lambda^* s^*)v = (s^* v)\lambda^* \text{ for all } v \in D_{2s}.
\end{gathered}
$$

The set $\mathcal{D}$ is directed downward, so there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap \bigcap_{i<3} D_{ir} \cap \bigcap_{i<3} D_{is}$. For any $v \in D$, we then have

$$
\begin{aligned}
&(\lambda r)v = (rv)\lambda = (sv)\lambda = (\lambda s)v,\\
&tv = (\lambda^* r^*)v = (r^* v)\lambda^* = (s^* v)\lambda^* = (\lambda^* s^*)v = uv.
\end{aligned}
$$

Therefore, D witnesses $\lambda r \approx_{\mathcal{D}} \lambda s$ under the conditions $\lambda r \bowtie t$ and $\lambda s \bowtie u$.

The fact that $\approx_{\mathcal{D}}$ respects $+$ can be established in a similar (and even simpler) way using (a) and (iii). Now we prove that $\approx_{\mathcal{D}}$ respects $\cdot$. Let $r_0 \approx_{\mathcal{D}} s_0$ and $r_1 \approx_{\mathcal{D}} s_1$. To prove that $r_0 \cdot r_1 \approx_{\mathcal{D}} s_0 \cdot s_1$, assume $t \bowtie r_0 \cdot r_1$ and $u \bowtie s_0 \cdot s_1$. Also let $D_0 \subseteq \mathsf{dom}r_0 \cap \mathsf{dom}r_0^* \cap \mathsf{dom}s_0 \cap \mathsf{dom}s_0^*$ witness $r_0 \approx_{\mathcal{D}} s_0$ under the conditions $r_0 \bowtie r_0^*$ and $s_0 \bowtie s_0^*$, and let $D_1 \subseteq \mathsf{dom}r_1 \cap \mathsf{dom}r_1^* \cap \mathsf{dom}s_1 \cap \mathsf{dom}s_1^*$ witness $r_1 \approx_{\mathcal{D}} s_1$ under the conditions $r_1 \bowtie r_1^*$ and $s_1 \bowtie s_1^*$. Then, in view of (b) and (viii), $t \approx_{\mathcal{D}} r_1^* \cdot r_0^*$ and $u \approx_{\mathcal{D}} s_1^* \cdot s_0^*$; in particular, there are $D_{0r}, D_{0s} \in \mathcal{D}$ such that $tv = (r_1^* \cdot r_0^*)v$, for all $v \in D_{0r}$, and $uv = (s_1^* \cdot s_0^*)v$ for all $v \in D_{0s}$. Moreover, according to (v), there are $D_{1r}, D_{2r}, D_{1s}, D_{2s} \in \mathcal{D}$ for which the following hold:

$$
\begin{aligned}
D_{1r} \subseteq\, & \mathsf{dom}(r_0 \cdot r_1) \cap r_1^{-1}(\mathsf{dom}r_0) \cap \mathsf{dom}r_1\\
& \text{and } (r_0 \cdot r_1)v = r_0(r_1 v) \text{ for all } v \in D_{1r};\\
D_{2r} \subseteq\, & \mathsf{dom}(r_1^* \cdot r_0^*) \cap (r_0^*)^{-1}(\mathsf{dom}r_1^*) \cap \mathsf{dom}r_0^*\\
& \text{and } (r_1^* \cdot r_0^*)v = r_1^*(r_0^* v) \text{ for all } v \in D_{2r};\\
D_{1s} \subseteq\, & \mathsf{dom}(s_0 \cdot s_1) \cap s_1^{-1}(\mathsf{dom}s_0) \cap \mathsf{dom}s_1\\
& \text{and } (s_0 \cdot s_1)v = s_0(s_1 v) \text{ for all } v \in D_{1s};\\
D_{2s} \subseteq\, & \mathsf{dom}(s_1^* \cdot s_0^*) \cap (s_0^*)^{-1}(\mathsf{dom}s_1^*) \cap \mathsf{dom}s_0^*\\
& \text{and } (s_1^* \cdot s_0^*)v = s_1^*(s_0^* v) \text{ for all } v \in D_{2s}.
\end{aligned}
$$

In view of (ii), there are $D_0', D_1' \in \mathcal{D}$ for which $D_0' \subseteq r_1^{-1}(D_0)$ and $D_1' \subseteq (r_0^*)^{-1}(D_1)$. In particular, $r_1 D_0' \subseteq D_0$ and $r_0^* D_1' \subseteq D_1$. Since $\mathcal{D}$ is directed downward, there is $D \in \mathcal{D}$ such that

$$
D \subseteq D_0 \cap D_1 \cap D_0' \cap D_1' \cap \bigcap_{i<3} D_{ir} \cap \bigcap_{i<3} D_{is}.
$$

For any $v \in D$, we then have

$$
(r_0 \cdot r_1)v = r_0(r_1 v) = s_0(r_1 v) = s_0(s_1 v) = (s_0 \cdot s_1)v,
$$

$$tv = (r_1^* \cdot r_0^*)v = r_1^*(r_0^* v) = s_1^*(r_0^* v) = s_1^*(s_0^* v) = (s_1^* \cdot s_0^*)v = uv.$$

Therefore, D witnesses $r_0 \cdot r_1 \approx_{\mathfrak{D}} s_0 \cdot s_1$ under the conditions $r_0 \cdot r_1 \bowtie t$ and $s_0 \cdot s_1 \bowtie u$.

Finally, suppose $r \bowtie r^*$, $s \bowtie s^*$, and $r \approx_{\mathfrak{D}} s$. To prove compatibility with $\bowtie$, let $t \bowtie r^*$ and $u \bowtie s^*$. Then $r \approx_{\mathfrak{D}} t$ and $s \approx_{\mathfrak{D}} u$ by (viii). Since $\approx_{\mathfrak{D}}$ is transitive, $t \approx_{\mathfrak{D}} u$. Then there is $D \in \mathfrak{D}$ which witnesses $t \approx_{\mathfrak{D}} u$ under the conditions $t \bowtie r^*$ and $u \bowtie s^*$. Thus D witnesses $r^* \approx_{\mathfrak{D}} s^*$ under the same conditions. □

In Theorem 28, we will show that the factor structure $R/{\approx_{\mathfrak{D}}}$ is always an $*$-Λ-algebra.

PROPOSITION 27. If ${}_RV_F$ is a representation of the $*$-Λ-algebra R, then $({}_RV_F; \{V\})$ is a Λ-algebra of generalized operators where $r \bowtie s$ if and only if $s = r^*$. In this case $\approx_{\mathfrak{D}}$ is the equality relation.

THEOREM 28. Let $({}_RV_F; \mathfrak{D})$ be a Λ-algebra of generalized operators on an inner product space V_F. Then $R/{\approx_{\mathfrak{D}}}$ is an $*$-Λ-algebra. Moreover, if $R/{\approx_{\mathfrak{D}}}$ is $*$-regular, then it admits a faithful representation within a closed subspace of some ultrapower of V_F.

Remark 29. The proof of Theorem 28 is quite similar to that of Theorem 16, and we might conjecture that $R/{\approx_{\mathfrak{D}}}$ is a homomorphic image of a subalgebra of an ultraproduct of endomorphism algebras of finite-dimensional subspaces of V_F. A difficulty in proving this statement comes from the fact that no ultrafilter 'compatible' with $\mathfrak{D}$ is at hand. The conjecture turns out valid for $F = \mathbb{C}$ (cf. proof of Thm. 2) but is doubtful in general. Nonetheless, the proof of Theorem 16 (and Tyukavkin's ideas behind it) gives some intuition for the following:

Proof of Theorem 28. Let $T = \mathsf{End}^*_{\Lambda f}(V_F)$. In this case, besides the left action of R on V, we also have the left action of T on V. The resulting 4-sorted structure is denoted by ${}_{T,R}V_F$. In particular, T is an $*$-regular algebra by Proposition 15, while ${}_TV_F$ is a faithful representation of T. According to Proposition 15, ${}_{T,R}V_F$ admits a modestly saturated elementary extension ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$. By Proposition 14, $\hat{T}$ is an $*$-regular algebra and ${}_{\hat{T}}\hat{V}_{\hat{F}}$ is its faithful representation. For $\sigma \in \hat{T}$ and $r \in R$, put

$\sigma \sim r$ if, for any $r^* \bowtie r$ in R, there is $D \in \mathfrak{D}$, $D \subseteq \mathsf{dom} r \cap \mathsf{dom} r^*$, such that $\sigma v = rv$ and $\sigma^* v = r^* v$ for all $v \in D$.

In this case we say that D *witnesses* $\sigma \sim r$ *under the condition* $r \bowtie r^*$. Put

$$S = \{\sigma \in \hat{T} \mid \sigma \sim r \text{ for some } r \in R\}.$$

Recall that the relation $\approx_{\mathfrak{D}}$ on R defined for an algebra of generalized operators is a congruence relation by Lemma 26. Let $[r] = \{s \in R \mid s \approx_{\mathfrak{D}} r\}$.

ASSERTION 30. The map $g\colon S \to R/{\approx_{\mathfrak{D}}}$, $g(\sigma) = [r]$, where $\sigma \sim r$, is well defined.

Proof. Let D_r and D_s witness $\sigma \sim r$ and $\sigma \sim s$ under the conditions $r \bowtie r^*$ and $s \bowtie s^*$, respectively. Since $\mathfrak{D}$ is directed downward, there is $D \in \mathfrak{D}$ such that $D \subseteq D_r \cap D_s$. For any $v \in D$, we have $rv = \sigma v = sv$ and $r^* v = \sigma^* v = s^* v$, whence $r \approx_{\mathfrak{D}} s$. □

ASSERTION 31. In the language of $*$-Λ-algebras, S is a subalgebra of $\hat{T}$ and $g\colon S \to R/{\approx_{\mathcal{D}}}$ is a homomorphism.

Proof. Assume $\sigma \sim r$ and $\tau \sim s$. In view of Lemma 26, it suffices to show that

$$(1)\, \sigma^* \sim r^*,\ (2)\, \lambda\sigma \sim \lambda r,\ (3)\, \tau + \sigma \sim s + r,\ (4)\, \tau\sigma \sim s \cdot r.$$

Let $D_r \in \mathcal{D}$ witness $\sigma \sim r$ under the condition $r \bowtie r^*$.

(1) We consider any $t \in R$ with $r^* \bowtie t$. By (viii), we have $t \approx_{\mathcal{D}} r$, which is witnessed by some $D' \in \mathcal{D}$ under the conditions $t \bowtie r^*$ and $r \bowtie r^*$. As $\mathcal{D}$ is directed downward, there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap D_r$. Then D witnesses $\sigma^* \sim r^*$ under $r^* \bowtie t$ since $\sigma^* v = r^* v$ and $(\sigma^*)^* v = \sigma v = rv = tv$ for all $v \in D$.

(2) We consider any t with $\lambda r \bowtie t$. By (c), we have $\lambda r \bowtie \lambda^* r^*$. This, in view of (viii), yields $t \approx_{\mathcal{D}} \lambda^* r^*$, which is witnessed by some $D_0 \in \mathcal{D}$. According to (vi), there is $D_1 \in \mathcal{D}$ such that $D_1 \subseteq \mathsf{dom} r \cap \mathsf{dom}(\lambda r)$ and $(\lambda r)v = (rv)\lambda$ for all $v \in D_1$. Since $\mathcal{D}$ is directed downward, there is $D \in \mathcal{D}$ with $D \subseteq D_0 \cap D_1 \cap D_r$. For all $v \in D$,

$$\begin{aligned}
&(\lambda\sigma)v = (\sigma v)\lambda = (rv)\lambda = (\lambda r)v,\\
&(\lambda\sigma)^* v = (\lambda^* \sigma^*)v = (\sigma^* v)\lambda^* = (r^* v)\lambda^* = (\lambda^* r^*)v = tv;
\end{aligned}$$

i.e., D witnesses relation (2) under the condition $\lambda r \bowtie t$.

(3), (4) We assume that $D_s \in \mathcal{D}$ witnesses $\tau \sim s$ under the condition $s \bowtie s^*$. Let $s + r \bowtie t$ for some $t \in R$. According to (a), we also have $s + r \bowtie s^* + r^*$, whence $s^* + r^* \approx_{\mathcal{D}} t$ by (viii). Let D' witness the last relation under $t \bowtie s + r$ and $s^* + r^* \bowtie s + r$. By virtue of (iii), there are $D_0, D_1 \in \mathcal{D}$ such that $D_0 \subseteq \mathsf{dom} s \cap \mathsf{dom} r \cap \mathsf{dom}(s + r)$, $D_1 \subseteq \mathsf{dom} s^* \cap \mathsf{dom} r^* \cap \mathsf{dom}(s^* + r^*)$, and $sv + rv = (s + r)v$ for all $v \in D_0$, while $s^* v + r^* v = (s^* + r^*)v$ for all $v \in D_1$. As $\mathcal{D}$ is directed downward, there is $D \in \mathcal{D}$ with $D \subseteq D' \cap D_0 \cap D_1 \cap D_r \cap D_s$. For all $v \in D$,

$$\begin{aligned}
&(\tau + \sigma)v = \tau v + \sigma v = sv + rv = (s + r)v,\\
&(\tau + \sigma)^* v = (\tau^* + \sigma^*)v = \tau^* v + \sigma^* v = s^* v + r^* v = (s^* + r^*)v = tv;
\end{aligned}$$

i.e., D witnesses relation (3) under the condition $s + r \bowtie t$.

Let $s \cdot r \bowtie t$ for some $t \in R$. According to (b), we also have $s \cdot r \bowtie r^* \cdot s^*$, whence $r^* \cdot s^* \approx_{\mathcal{D}} t$ by (viii). Let D' witness the last relation subject to the conditions $t \bowtie s \cdot r$ and $r^* \cdot s^* \bowtie s \cdot r$. By virtue of (v), there are $D_0, D_1 \in \mathcal{D}$ such that:

$$D_0 \subseteq \mathsf{dom}(s \cdot r) \cap r^{-1}(\mathsf{dom} s) \cap \mathsf{dom} r;$$

$$D_1 \subseteq \mathsf{dom}(r^* \cdot s^*) \cap (s^*)^{-1}(\mathsf{dom} r^*) \cap \mathsf{dom} s^*$$

and

$$(s \cdot r)v = s(rv) \text{ for all } v \in D_0;$$

$$(r^* \cdot s^*)v = r^*(s^*v) \text{ for all } v \in D_1.$$

Since $\mathcal{D}$ is directed downward, there is $D' \in \mathcal{D}$ with $D' \subseteq D_r \cap D_s \cap D_0 \cap D_1$. Moreover, according to (ii), there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap r^{-1}(D') \cap (s^*)^{-1}(D')$. Therefore, $D \subseteq D_r$ and $rD \subseteq D_s$. Similarly, $D \subseteq D_s$ and $s^*D \subseteq D_r$.

Thus, for all $v \in D$,

$$(\tau\sigma)v = \tau(\sigma v) = \tau(rv) = s(rv) = (s \cdot r)v,$$
$$(\tau\sigma)^*v = (\sigma^*\tau^*)v = \sigma^*(\tau^*v) = \sigma^*(s^*v) = r^*(s^*v) = (r^* \cdot s^*)v = tv;$$

i.e., D witnesses relation (4) under the condition $s \cdot r \bowtie t$.

Obviously, $0 \sim 0_R$ and $1 \sim 1_R$ by (iv). □

ASSERTION 32. The map g is surjective.

Proof. Let $r \bowtie r^*$ in R. By (vii), there is $D \in \mathcal{D}$ such that $D \subseteq \mathsf{dom} r \cap \mathsf{dom} r^*$ and $\langle x \mid r^*y\rangle = \langle rx \mid y\rangle$ for all $x, y \in D$. We show that there is $\sigma \in \hat{T}$ for which $\sigma v = rv$ and $\sigma^* v = r^* v$ with all $v \in D$. Let $v_1, \dots, v_n \in D$ and U be the subspace of V_F spanned by $v_1, \dots, v_n$. Consider the finite-dimensional subspace $W = U + rU + r^*U$ of V and the F-linear maps

$$\varphi_0 \colon U \to W,\ \varphi_0 v = rv, \text{ and } \psi_0 \colon U \to W,\ \psi_0 v = r^*v.$$

In particular, $\langle x \mid \psi_0 y\rangle = \langle \varphi_0 x \mid y\rangle$ for all $x, y \in U$. Choose a basis $u_1, \dots, u_k$ for U and extend it to a basis $u_1, \dots, u_m$ for W. There are unique maps $\varphi, \psi \in \mathsf{End}^*_\Lambda(W_F)$ such that:

$$\langle \varphi u_i \mid u_j\rangle = \langle u_i \mid \psi u_j\rangle = \begin{cases} \langle u_i \mid \psi_0 u_j\rangle & \text{for } j \leqslant k; \\ \langle \varphi_0 u_i \mid u_j\rangle & \text{for } i \leqslant k; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\psi = \varphi^*$, $\varphi|_U = \varphi_0$, and $\psi^*|_U = \psi_0$. If we consider the orthogonal projection $\rho = \pi_W$ we obtain $\rho \in T$. Moreover,

$$\sigma = \rho\varphi\rho \in T, \quad \sigma^* = \rho\psi\rho \in T; \quad \sigma v = \varphi_0 v = rv, \quad \sigma^* v = \psi_0 v = r^* v$$

for all $v \in U$. Now, consider a set of formulas of the form

$$\Sigma(\xi) = \{(\xi \underline{v} = \underline{r}\, \underline{v}) \ \&\ (\xi^* \underline{v} = \underline{r}^*\, \underline{v}) \mid v \in D\},$$

where ξ is a variable of sort T. If $\Psi(\xi) \subseteq \Sigma(\xi)$ is finite, then only finitely many parameters $\underline{v}_i, v_i \in D$, occur in $\Psi(\xi)$ and, as shown above, there is $\sigma \in T$ for which $\Psi(\sigma)$ holds in ${}_RV_F$. Since ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$ is modestly saturated over ${}_{T,R}V_F$, there is $\sigma \in \hat{T}$ such that $\Sigma(\sigma)$ holds in ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$; i.e. $\sigma v = rv$ and $\sigma^* v = r^* v$ for all $v \in D$. Hence $\sigma \sim r$. □

ASSERTION 33. For any $v \in V$, there is a unique projection $\hat{\pi}_v \in \hat{T}$ such that $\hat{\pi}_v v = v$, and for any $w \in \hat{V}$, there is $\lambda \in \hat{F}$ with $\hat{\pi}_v w = \lambda v$.

Proof. There is a unique projection $\pi_v \in T$ for which $\pi_v v = v$ and $\mathsf{im}\pi_v$ is the subspace spanned by v—namely, the orthogonal projection onto the subspace spanned by v. The required result now follows from the fact that ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$ is an elementary extension of ${}_{T,R}V_F$. $\square$

ASSERTION 34. We have $\sigma \in \ker g$ if and only if for any $t \bowtie 0_R$ there is a set $D \in \mathcal{D}$ such that $\sigma\hat{\pi}_v = 0 = \sigma^*\hat{\pi}_v$ for all $v \in D$, where $\hat{\pi}_v$ is as in Assertion 33.

Proof. Assume that $\sigma \in \ker g$ and $t \bowtie 0_R$. Then $\sigma \sim 0_R$ is witnessed by some $D_0 \in \mathcal{D}$ under the condition $0_R \bowtie t$. On the other hand, according to (d) and (viii), we have $t \approx_{\mathcal{D}} 0_R$. Let the last relation be witnessed by $D_1 \in \mathcal{D}$ under the conditions $0_R \bowtie 0_R$ and $t \bowtie 0_R$. Then there is $D \in \mathcal{D}$ such that $D \subseteq D_0 \cap D_1$ and

$$(*) \qquad \sigma v = 0_R v = tv = \sigma^* v \text{ for all } v \in D.$$

Since $0_R v = 0$, $(*)$ is equivalent to

$$(**) \qquad \sigma\hat{\pi}_v = 0 = \sigma^*\hat{\pi}_v \text{ for all } v \in D.$$

Conversely, consider any $t \bowtie 0_R$ and assume that $(**)$ holds for some $D \in \mathcal{D}$. By (d) and (viii), $t \approx_{\mathcal{D}} 0_R$ is witnessed by some $D_0 \in \mathcal{D}$. Then $\sigma \sim 0_R$ is witnessed by any $D' \in \mathcal{D}$ for which $D' \subseteq D \cap D_0$ provided that $t \bowtie 0_R$. $\square$

ASSERTION 35. The ideal $\mathsf{ker}g$ is regular.

Proof. Since T is $*$-regular, $\hat{T}$ is also $*$-regular. Therefore, any $\sigma \in \mathsf{ker}g$ has a pseudoinverse σ^+ in $\hat{T}$. In view of Assertion 34 and Proposition 9, $\sigma \in \mathsf{ker}g$ implies $\sigma^+ \in \mathsf{ker}g$. $\square$

We come back to the proof of Theorem 28. The first statement of the theorem follows from Assertions 31 and 32. If $\mathsf{im}g = R/{\approx_{\mathcal{D}}}$ is $*$-regular, then so is S by Lemma 6. Moreover, since T is faithfully represented in V_F, $\hat{T}$ is faithfully represented in $\hat{V}_{\hat{F}}$ by Prop. 14. Therefore, the substructure S is also faithfully represented in $\hat{V}_{\hat{F}}$, while the faithful representability of its homomorphic image $R/{\approx_{\mathcal{D}}}$ within a closed subspace U of an ultrapower $\tilde{V}_{\tilde{F}}$ of $\hat{V}_{\hat{F}}$ follows from Prop. 25. Finally, we observe that the modestly saturated extension ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$ can be chosen isomorphic to an ultrapower of ${}_{T,R}V_F$ in view of Prop. 15. In particular, $\tilde{V}_{\tilde{F}}$ is isomorphic to an ultrapower of V_F by Prop. 13(i). Composing the representation of $R/{\approx_{\mathcal{D}}}$ in $U_{\tilde{F}}$ with this isomorphism, we obtain a faithful representation of $R/{\approx_{\mathcal{D}}}$ in a closed subspace of an ultrapower of V_F. $\square$

5. RINGS OF QUOTIENTS

We refer the reader to [21, Chap. 3] for rings of quotients. Note, however, that we will deal with right quotients. Let A be a Λ-algebra. Denote by $\mathcal{I}_r(A)$ the set of all right ideals I in A, and by $\mathsf{Hom}(I, A)$ the set of all linear maps $f\colon I_A \to A_A$. A right ideal $I \in \mathcal{I}_r(A)$ is *dense* (in A) if, for any ideal $J \supseteq I$ in $\mathcal{I}_r(A)$ and any map $f \in \mathsf{Hom}(J, A)$, $f|I = 0$ implies $f = 0$.

A subset $\mathcal{E}$ of $\mathcal{I}_r(A)$ is a *set of supports* (for A) if the following hold:

(i) every member of $\mathcal{E}$ is dense (in A);

(ii) $A \in \mathcal{E}$ and $I \cap J \in \mathcal{E}$ for any $I, J \in \mathcal{E}$;

(iii) $f^{-1}(J) \in \mathcal{E}$ for any $I, J \in \mathcal{E}$ and any $f \in \mathsf{Hom}(I, A)$.

Note that (iii) applies in particular to the left multiplication $l_a\colon A \to A$, $l_a(x) = ax$, for any $a \in A$, including the special case $a = \lambda 1_A$, where $l_{\lambda 1_A}\colon x \mapsto \lambda x$. Furthermore, every ideal $I \in \mathcal{I}_r$ is invariant under λ, and we put $\lambda f = l_{\lambda 1_A} \circ f = f \circ (l_{\lambda 1_A} | I)$ for $f \in \mathsf{Hom}(I, A)$. The following is well known:

LEMMA 36. The set $\mathcal{E}_0$ of all dense right ideals in A is a set of supports for A.

Given a set $\mathcal{E}$ of supports, we define an *algebra $R(A, \mathcal{E})$ of abstract quotients over* $\mathcal{E}$ as follows:

$$R(A, \mathcal{E}) = \big\{ (f, I) \mid I \in \mathcal{E},\ f \in \mathsf{Hom}(I, A) \big\}.$$

Endow then $R(A, \mathcal{E})$ with operations of a pre-Λ-algebra, setting

$$\begin{gathered}
(f, I) + (g, J) = (f|K + g|K,\ K), \text{ where } K = I \cap J;\\
\lambda(f, I) = \big((\lambda f)|K,\ K\big), \text{ where } K = \lambda^{-1}(I);\\
(f, I) \cdot (g, J) = \big((f \circ g)|K,\ K\big), \text{ where } K = g^{-1}(I);\\
0_R = (0, A), \quad 1_R = (\mathsf{id}_A, A).
\end{gathered}$$

A binary relation $\equiv_{\mathcal{E}}$ on $R(A, \mathcal{E})$ is defined thus:

$$(f, I) \equiv_{\mathcal{E}} (g, J) \text{ if } f|K = g|K \text{ for some } K \in \mathcal{E} \text{ with } K \subseteq I \cap J.$$

The following facts either are well known or can be proved readily.

PROPOSITION 37. Let A be a Λ-algebra and $\mathcal{E}$ a set of supports for A. Then:

(i) $(f, I) \equiv_{\mathcal{E}} (g, J)$ if and only if $f|(I \cap J) = g|(I \cap J)$;

(ii) $\equiv_{\mathcal{E}}$ is an equivalence relation on $R(A, \mathcal{E})$, the factor structure with respect to which is denoted by $Q(A, \mathcal{E})$, and the canonical homomorphism by $\pi_{\mathcal{E}}$;

(iii) the map $\omega\colon A \to R(A, \mathcal{E})$, $\omega(a) = (l_a, A)$, is a Λ-algebra embedding and $\equiv_{\mathcal{E}}$ restricts to an identity relation on $\omega(A)$; in particular, $\pi_{\mathcal{E}} \circ \omega$ is a Λ-algebra embedding;

(iv) $(f, I) \cdot (l_a, A) \in \omega(A)$ if and only if $a \in I$, in which case $(f, I) \cdot (l_a, A) = (l_{f(a)}, A)$;

(v) $R(A, \mathcal{E})$ is a subalgebra of $R(A, \mathcal{E}_0)$ and $\equiv_{\mathcal{E}}$ is the restriction of $\equiv_{\mathcal{E}_0}$;

(vi) $Q(A, \mathcal{E}_0) = Q_{\max}(A)$, where $Q_{\max}(A)$ is the maximal ring of right quotients of A;

(vii) $\pi_{\mathcal{E}} \circ \omega$ embeds A into $Q_{\max}(A)$.

Recall that the set of projections of an $*$-ring is ordered by

$$e \le e' \text{ if } e'e = e, \text{ which is equivalent to } ee' = e.$$

THEOREM 38. Let A be an $*$-Λ-algebra, $\mathcal{E}$ be a set of supports for A, and $R = R(A, \mathcal{E})$ be such that:

(a) for any $I \in \mathcal{E}$, there is an upward directed set P_I of projections in I with $P_I A = \bigcup\limits_{e \in P_I} eA \in \mathcal{E}$;

(b) an involution on A extends to $Q(A, \mathcal{E})$;

(c) there is a faithful representation ε for A in the inner product space V_F.

Then there is an algebra $({}_R V_F; \mathcal{D})$ of generalized operators for which $R/{\approx_{\mathcal{D}}}$ and $Q(A, \mathcal{E})$ are isomorphic as $*$-Λ-algebras.

Proof. To any $I \in \mathcal{E}$ we assign a certain upward directed set P_I of projections which satisfies (a). The action of R on V_F is defined by setting

$$\mathsf{dom}(f, I) = D(I) = \bigcup_{e \in P_I} \mathsf{im}\varepsilon(e),$$

$$(f, I)v = \varepsilon(f(e))(v) \text{ if } v \in \mathsf{im}\varepsilon(e),\ e \in P_I.$$

Notice that $\varepsilon(e)(v) = v$ for any $v \in \mathsf{im}\varepsilon(e)$ since e is a projection.

ASSERTION 39. For any $(f, I) \in R$, the action of (f, I) is well defined.

Proof. Assume that $v \in \mathsf{im}\varepsilon(e) \cap \mathsf{im}\varepsilon(e')$ for some $e, e' \in P_I$. Since P_I is directed, there is a projection $e'' \in P_I$ such that $e''e = e$ and $e''e' = e'$. Then $\mathsf{im}\varepsilon(e), \mathsf{im}\varepsilon(e') \subseteq \mathsf{im}\varepsilon(e'')$ and

$$\begin{aligned}\varepsilon\big(f(e)\big)(v) &= \varepsilon\big(f(e''e)\big)(v) = \varepsilon\big(f(e'')e\big)(v) = \Big(\varepsilon\big(f(e')\big) \circ \varepsilon(e)\Big)(v)\\ &= \varepsilon\big(f(e'')\big)\big(\varepsilon(e)(v)\big) = \varepsilon\big(f(e'')\big)(v).\end{aligned}$$

Similarly, $\varepsilon\big(f(e')\big)(v) = \varepsilon\big(f(e'')\big)(v)$, i.e., $\varepsilon\big(f(e)\big)(v) = \varepsilon\big(f(e')\big)(v)$. $\square$

ASSERTION 40. For any $I \in \mathcal{E}$, the set $D(I)$ is an F-linear subspace of V_F, and

$$(f, I)\colon \mathsf{dom}(f, I) \to V$$

is an F-linear map for any $(f, I) \in R$.

Proof. Let $u, v \in D(I)$ and $\lambda \in F$. As P_I is directed, there is $e \in P_I$ such that $u, v \in \mathsf{im}\varepsilon(e)$. As $\mathsf{im}\varepsilon(e)$ is a subspace of V, we have $u + v, u\lambda \in \mathsf{im}\varepsilon(e)$. Using the fact that ε is a representation, we obtain

$$\begin{aligned}&(f, I)(u + v) = \varepsilon\big(f(e)\big)(u + v) = \varepsilon\big(f(e)\big)(u) + \varepsilon\big(f(e)\big)(v) = (f, I)u + (f, I)v,\\ &(f, I)(u\lambda) = \Big(\varepsilon\big(f(e)\big)(u)\Big)\lambda = \big((f, I)u\big)\lambda. \ \square\end{aligned}$$

For (f, I), $(g, J) \in R$, put

$$(f, I) \bowtie (g, J) \text{ if } \pi_{\mathcal{E}}(g, J) = \big(\pi_{\mathcal{E}}(f, I)\big)^*.$$

Since $\pi_{\mathcal{E}}$ is a pre-Λ-algebra homomorphism, and * is an involution on $Q(A, \mathcal{E})$, we conclude that conditions (a)-(c) in the definition of a pre-$*$-Λ-algebra hold. Condition (d) is satisfied in the obvious way. Put

$$\mathcal{D} = \{D(I) \mid I \in \mathcal{E}\}.$$

ASSERTION 41. For any (f, I), $(g, J) \in R$, the conditions $(f, I) \approx_{\mathcal{D}} (g, J)$ and $(f, I) \equiv_{\mathcal{E}} (g, J)$ are equivalent.

Proof. Assume that $(f, I) \approx_{\mathcal{D}} (g, J)$ is witnessed by $D(K)$ for an ideal $K \in \mathcal{E}$ under some conditions. Since $K \cap I \cap J \in \mathcal{E}$, we may assume that $K \subseteq I \cap J$. Then for any $e \in P_K$ and any $v \in \mathsf{im}\varepsilon(e)$ we have

$$\varepsilon\big(f(e)\big)(v) = (f, I)v = (g, J)v = \varepsilon\big(g(e)\big)(v).$$

Hence, for any $u \in V$, the following hold:

$$\begin{aligned}\varepsilon\big(f(e)\big)(u) &= \varepsilon\big(f(e^2)\big)(u) = \varepsilon\big(f(e)e\big)(u) = \varepsilon\big(f(e)\big)\big(\varepsilon(e)(u)\big)\\ &= \varepsilon\big(g(e)\big)\big(\varepsilon(e)(u)\big) = \varepsilon\big(g(e)e\big)(u) = \varepsilon\big(g(e^2)\big)(u) = \varepsilon\big(g(e)\big)(u).\end{aligned}$$

This, combined with the fact that ε is a faithful representation, yields $f(e) = g(e)$. Then for any $a \in eA$ we obtain

$$f(a) = f(ea) = f(e)a = g(e)a = g(ea) = g(a).$$

According to (a), $P_K A = \bigcup\limits_{e \in p(K)} eA$ is a dense right ideal. Since $f|P_K A = g|P_K A$, we conclude that $f|K = g|K$. Hence $(f, I) \equiv_{\mathcal{E}} (g, J)$.

Conversely, assume that $\pi_{\mathcal{E}}(f, I) = \pi_{\mathcal{E}}(g, J)$ and consider arbitrary (h_0, K_0), $(h_1, K_1) \in R$ for which $(h_0, K_0) \bowtie (f, I)$ and $(h_1, K_1) \bowtie (g, J)$ in R. By definition, this means that

$$\pi_{\mathcal{E}}(h_0, K_0) = \big(\pi_{\mathcal{E}}(f, I)\big)^* = \big(\pi_{\mathcal{E}}(g, J)\big)^* = \pi_{\mathcal{E}}(h_1, K_1).$$

Thus, for any ideal $K \in \mathcal{E}$ with $K \subseteq I \cap J \cap K_0 \cap K_1$, we have $f|K = g|K$ and $h_0|K = h_1|K$. Then $D(K)$ witnesses $(f, I) \approx_{\mathcal{D}} (g, J)$ under the given conditions. $\square$

ASSERTION 42. $({}_R V_F; \approx_{\mathcal{D}})$ Is an algebra of generalized operators.

Proof. For any $(f, I) \in R$, we have $D(I) = \mathsf{dom}(f, I)$, and so condition (i) in the definition of an algebra of generalized operators holds. If $(f, I) \in R$ and $D(J) \in \mathcal{D}$ for some $J \in \mathcal{E}$, then $K = f^{-1}(P_J A) \in \mathcal{E}$ by (a) and by condition (iii) in the definition of a set of supports for A. Let $v \in \mathsf{im}\varepsilon(e)$ for some $e \in P_K$. Since $K \subseteq I$, we obtain $f(e) \in P_J A$. According to (a), $f(e) = e' f(e)$ for some $e' \in P_J$. Therefore,

$$(f, I)v = \varepsilon\big(f(e)\big)(v) = \varepsilon\big(e' f(e)\big)(v) = \varepsilon(e')\Big(\varepsilon\big(f(e)\big)(v)\Big) \in \mathsf{im}\varepsilon(e') \subseteq D(J).$$

Hence $(f, I)D(K) \subseteq D(J)$ and condition (ii) holds.

Suppose that (f, I), $(g, J) \in R$ and $K = I \cap J$. Then $K \in \mathcal{E}$ and $D(K) \subseteq \mathsf{dom}(f, I) \cap \mathsf{dom}(g, J) \cap \mathsf{dom}\big((f, I) + (g, J)\big)$. Moreover, for any $e \in P_K$ and any $v \in \mathsf{im}\varepsilon(e)$, we have

$$\begin{aligned}\big((f, I) + (g, J)\big)v &= \varepsilon\big((f + g)(e)\big)(v) = \varepsilon\big(f(e) + g(e)\big)(v)\\ &= \Big(\varepsilon\big(f(e)\big) + \varepsilon\big(g(e)\big)\Big)(v) = \varepsilon\big(f(e)\big)(v) + \varepsilon\big(g(e)\big)(v)\\ &= (f, I)v + (g, J)v.\end{aligned}$$

Thus (iii) holds.

Consider (f, I), $(g, J) \in R$. As shown above, $K = g^{-1}(P_I A) \in \mathcal{E}$ and $(g, J)D(K) \subseteq D(I)$. Therefore, $D(K) \subseteq \mathsf{dom}(g, J) \cap (g, J)^{-1}\big(\mathsf{dom}(f, I)\big) = \mathsf{dom}\big((f, I) \cdot (g, J)\big)$. Moreover, by (a), for any $e \in P_K$ and any $v \in \mathsf{im}\varepsilon(e)$, there is $e' \in P_I$ with $g(e) = e'g(e)$. Thus, for any $v \in \mathsf{im}\varepsilon(e)$,

$$\begin{aligned}\big((f, I) \cdot (g, J)\big)v &= \varepsilon\big((f \circ g)(e)\big)(v) = \varepsilon\big(f\big(g(e)\big)\big)(v) = \varepsilon\big(f\big(e'g(e)\big)\big)(v)\\ &= \varepsilon\big(f(e')g(e)\big)(v) = \varepsilon\big(f(e')\big)\varepsilon\big(g(e)\big)(v) = (f, I)\big((g, J)v\big),\end{aligned}$$

and so (v) holds. It is clear that (iv) holds as well.

We verify (vi). Suppose $\lambda \in \Lambda$ and $K = \lambda^{-1}(I) \cap I$. Then $K \in \mathcal{E}$ by conditions (ii) and (iii) in the definition of a set of supports for A. For any $e \in P_K$, we have $\lambda e = \lambda e^2 = e(\lambda e)$. Thus, for any $v \in \mathsf{im}\varepsilon(e)$, the following hold:

$$\begin{aligned}\big(\lambda(f, I)\big)v &= \varepsilon\big((\lambda f)(e)\big)(v) = \varepsilon\big(f(\lambda e)\big)(v) = \varepsilon\big(f(e \cdot \lambda e)\big)(v)\\ &= \varepsilon\big(f(e) \cdot \lambda e\big)(v) = \varepsilon\big(f(e)\big)\varepsilon(\lambda e)(v) = \varepsilon\big(f(e)\big)\big(\varepsilon(e)(v)\lambda\big)\\ &= \varepsilon\big(f(e)\big)(v\lambda) = \varepsilon\big(f(e)\big)(v)\lambda = \big((f, I)v\big)\lambda.\end{aligned}$$

We verify (vii). Assume $(f, I) \bowtie (g, J)$. According to Proposition 37(iv), $\omega(f(e)) = (f, I) \cdot \omega(e)$ for any $e \in P_I$. By (b), the map $\pi_{\mathcal{E}}\omega$ is an $*$-homomorphism, whence

$$\begin{aligned}\pi_{\mathcal{E}}\omega\big(f(e)^*\big) &= \pi_{\mathcal{E}}\big(\omega(f(e))\big)^* = \big(\pi_{\mathcal{E}}(f, I) \cdot \pi_{\mathcal{E}}\omega(e)\big)^* = \pi_{\mathcal{E}}\omega(e)^* \cdot \big(\pi_{\mathcal{E}}(f, I)\big)^*\\ &= \pi_{\mathcal{E}}\omega(e) \cdot \pi_{\mathcal{E}}(g, J) = \pi_{\mathcal{E}}\big(\omega(e) \cdot (g, J)\big) = \pi_{\mathcal{E}}(l_e \circ g, J).\end{aligned}$$

Therefore, there is $K_0 \in \mathcal{E}$ such that $K_0 \subseteq J$, and for all $a \in K_0$,

$$e \cdot g(a) = (l_e \circ g)(a) = l_{f(e)^*}(a) = f(e)^* \cdot a.$$

Let $K = K_0 \cap I \cap J$. Then $K \in \mathcal{E}$ and $D(K) \subseteq D(I) \cap D(J) = \mathsf{dom}(f, I) \cap \mathsf{dom}(g, J)$. Suppose $u, v \in D(K)$. According to (a), there is a projection $e \in P_K$ such that $u, v \in \mathsf{im}\varepsilon(e)$. Since ε is a representation, the following hold:

$$\begin{aligned}\langle (f, I)u \mid v\rangle &= \langle \varepsilon\big(f(e)\big)(u) \mid v\rangle = \langle u \mid \varepsilon\big(f(e)\big)^*(v)\rangle = \langle u \mid \varepsilon\big(f(e)^*\big)(v)\rangle\\ &= \langle u \mid \varepsilon\big(f(e)^*\big)\varepsilon(e)(v)\rangle = \langle u \mid \varepsilon\big(f(e)^* \cdot e\big)(v)\rangle\\ &= \langle u \mid \varepsilon\big(e \cdot g(e)\big)(v)\rangle = \langle u \mid \varepsilon(e)\varepsilon\big(g(e)\big)(v)\rangle\\ &= \langle u \mid \varepsilon(e^*)\varepsilon\big(g(e)\big)(v)\rangle = \langle \varepsilon(e)(u) \mid \varepsilon\big(g(e)\big)(v)\rangle\\ &= \langle u \mid \varepsilon\big(g(e)\big)(v)\rangle = \langle u \mid (g, J)v\rangle.\end{aligned}$$

If $(g_0, J_0) \bowtie (f, I) \bowtie (g_1, J_1)$, then $\pi_{\mathcal{E}}(g_0, J_0) = \big(\pi_{\mathcal{E}}(f, I)\big)^* = \pi_{\mathcal{E}}(g_1, J_1)$, hence $(g_0, J_0) \equiv_{\mathcal{E}} (g_1, J_1)$, and so (viii) follows from Assertion 41. □

The above results show that $({}_RV_F; \mathcal{D})$ is an algebra of generalized operators. In view of Assertion 41, $R/{\approx_{\mathcal{D}}} \cong Q(A, \mathcal{E})$. □

6. PROOFS FOR THE MAIN RESULTS

The following concepts are borrowed from [22, 10]. Let A be a ring. For any $X \subseteq A$, put

$$\mathsf{Ann}_r(X) = \{a \in A \mid Xa = 0\}; \ \mathsf{Ann}_l(X) = \{a \in A \mid aX = 0\}.$$

We call these sets respectively the *right* annihilator and the *left annihilator of* X. An $*$-ring A is a *Baer* (*Rickart*) ring if, for any (singleton) subset X of A, there is a projection $e \in A$ such that $\mathsf{Ann}_r(X) = eA$. In this event the left annihilator of X is also generated by a projection. If, in addition, A is a C^*-algebra, then A is called an *AW^*-algebra* (a *Rickart C^*-algebra*, resp.). According to [10, 14.22, 14.24], this definition of an AW^*-algebra is equivalent to one given in [23]. Every von Neumann algebra is an AW^*-algebra. An $*$-ring A is said to be *finite* (in [10], such is referred to as $*$-finite) if $xx^* = 1$ implies $x^*x = 1$ for all $x \in A$. A ring A has *sufficiently many projections* if any proper right ideal contains a nonzero projection. We say that A satisfies the condition $LP \sim RP$ (written $LP \sim^* PR$ in [10]) if, for any element $x \in A$ and any projections $e, f \in A$ with $\mathsf{Ann}_r(x) = (1 - e)A$ and $\mathsf{Ann}_l(x) = A(1 - f)$, there is $y \in A$ for which $e = yy^*$ and $f = y^*y$.

A right ideal I of a ring A is *essential* or *large* in $J \supseteq I$ if $I \cap K \neq 0$ for any ideal $K \in \mathfrak{I}_r(A)$ such that $0 \neq K \subseteq J$. Obviously, if I is essential in A then any ideal $J \supseteq I$ is also essential in A. The following lemma is well known and its proof is straightforward.

LEMMA 43. Let $I, J \in \mathfrak{I}_r(A)$ be such that $I \subseteq J$ and J is essential in A. Then I is essential in J if and only if I is essential in A.

Recall that A is *nonsingular* if $\mathsf{Ann}_r(x)$ is essential iff $x = 0$. Moreover, for any nonsingular ring, the concept of being dense in A for a right ideal coincides with its being essential in A. Every Rickart $*$-ring is obviously nonsingular.

PROPOSITION 44. Let A and $Q(A)$ satisfy one of the following:

(i) A is a finite Rickart C^*-algebra and $Q(A)$ is its classical ring of right quotients;

(ii) A is an $*$-Λ-algebra, which is a finite Baer $*$-ring satisfying $LP \sim RP$ and having sufficiently many projections, and $Q(A)$ is its maximal ring of right quotients.

Then there is a set $\mathcal{E}$ of supports such that $Q(A, \mathcal{E})$ is isomorphic to $Q(A)$ and satisfies conditions (a) and (b) of Theorem 38, which turns $Q(A, \mathcal{E})$ into an $*$-regular Λ-algebra. Moreover, in both cases an involution on A extends uniquely to an involution on $Q(A)$; endowed with this involution, $Q(A)$ is $*$-regular.

Proof. (i) Following [7, p. 177], we assume that $\mathcal{E}$ consists of all right ideals $I \in \mathcal{E}_0$ for which there is a countable set $X \subseteq I$ with $\sum\limits_{x \in X} xA \in \mathcal{E}_0$. According to [7, Prop. 2.1] and Lemma 43, there is a countable orthogonal set $P \subseteq I$ of projections such that the right ideal $J = \sum\limits_{e \in P} eA$ is essential in A. Let

$$P_I = \{e_0 + \ldots + e_n \mid n < \omega, \ e_0, \ldots, e_n \in P\}.$$

Then P_I is a directed set of projections and $J = \bigcup_{e \in P_I} eA \in \mathcal{E}_0$. Therefore, Theorem 38(a) holds. Moreover, conditions (i)-(iii) in the definition of a set of supports are satisfied in view of [7, Lemmas 2.3, 2.4], and $Q(A, \mathcal{E})$ is regular by virtue of [7, Lemma 2.7]. Furthermore, in [7, Thm. 2.1], an $*$-regular subring R of $Q_H(A) = Q(A, \mathcal{E})$ was constructed in such a way that an involution on R extends an involution on A. Later, in [8, p. 129], it was shown that $R = Q_H(A)$ is the classical ring of right quotients of A. The uniqueness of the extension is obvious in this case.

(ii) Let $\mathcal{E} = \mathcal{E}_0$. According to Lemma 36, $\mathcal{E}_0$ is a set of supports. The proof in [4, Cor. 4.10] (see also [3, Lemma 5]), together with [4, Prop. 4.11], ensures that Theorem 38(a) holds. In view of [23; 3, Thm. 2], $Q(A, \mathcal{E}_0)$ is $*$-regular with involution extending an involution on A. The uniqueness of this extension follows from [10, Cors. 21.22, 21.27] (see also [9]). □

COROLLARY 45. Let A and $Q(A)$ be as in Proposition 44 and $\Lambda = F = \mathbb{C}$ be as in case (i). For any faithful representation ε of A within a Hilbert space $V_{\mathbb{C}}$, $Q(A)$ is isomorphic to an $*$-$\mathbb{C}$-algebra $R/{\approx_{\mathcal{D}}}$ for some $\mathbb{C}$-algebra $({}_R V_{\mathbb{C}}; \mathcal{D})$ of generalized operators on $V_{\mathbb{C}}$.

Proof of Theorem 2. (i) In view of the GNS-construction (Prop. 3), A has a faithful representation in some Hilbert space $V_{\mathbb{C}}$. By Corollary 45, $Q(A) \cong R/{\approx_{\mathcal{D}}}$ for some algebra $({}_R V_{\mathbb{C}}; \mathcal{D})$ of generalized operators. Theorem 28 provides a faithful representation for $({}_R V_{\mathbb{C}}; \mathcal{D})$ in some closed subspace U of an ultrapower $\hat{V}_{\hat{\mathbb{C}}} = V_{\mathbb{C}}^I/\mathcal{U}$.

(ii) By Theorem 16, $Q(A)$ is a homomorphic image of a subalgebra of an ultraproduct $\prod_{k \in K} \mathsf{End}^*_{\Lambda}\big((U_k)_{\hat{\mathbb{C}}}\big)/\mathcal{W}$, where $\dim U_k = n_k < \omega$ for all $k \in K$. In view of Propositions 8 and 13(ii), $\mathsf{End}^*_{\Lambda}\big((U_k)_{\hat{\mathbb{C}}}\big)$ is isomorphic to $(\mathbb{C}^{n_k \times n_k})^I/\mathcal{U}$ for any $k \in K$. Since all these algebraic constructions respect pseudoinversion, we are done.

(iii) Follows from (ii) and Props. 11 and 12. □

Finally, we recall some facts concerning finite AW^*-algebras. (As usual, such an algebra is denoted by A.) An $*$-regular extension $Q_B(A)$ for A was constructed in [2]. In [3, 4], it was shown that $Q_B(A)$, being a ring, is the maximal ring of right quotients of A. In this event, therefore, the conditions of case (ii) in Proposition 44 are satisfied. Indeed, according to [10, 14.31], A satisfies $LP \sim RP$ (see also [23, Thm. 5.2]). In view of [23, Lemma 2.2], A has sufficiently many projections.

On the other hand, in [5, proof of Thm. 10], it was observed that, for a finite AW^*-algebra, the construction of $Q_B(A)$ yields the classical ring of right quotients of A. We outline a proof in the present framework.

As noted in [8, p. 129], $Q_H(A)$ consists of all those elements x of the maximal ring $Q_M(A)$ of right quotients for which there is an orthogonal sequence of projections e_k such that $xe_k \in A$ for all k and $J = \sum_k e_k A$ is essential in A. The elements x of $Q_B(A)$ (which is $Q_M(A)$ as a ring) are represented by so-called operators with closure (OWCs), which, by definition, are sequences of the form (x_n, f_n), where $x_n \in A$ and $x_n f_m = x_m f_m$ and $x_n^* f_m = x_m^* f_m$ for all $m < n$. Here the sequence f_n forms a so-called strongly dense domain (SDD), i.e., an ascending chain of projections in A with join 1. Note that there is an orthogonal sequence e_n of projections with joins $f_n = \sum_{k \leqslant n} e_k$, whence $x_n e_k = x_n f_k e_k = x_k e_k$ and $x^* e_k = x^* e_k$ for $n \geqslant k$. From [2, proof of Thm. 2.1], it follows immediately that $xe_k \in Q_B(A)$ is represented by an OWC $(x_n e_k, g_n)$ which is equivalent to an OWC $(x_k e_k, h_n)$, where $h_n = 0$, for $n < k$, and $h_n = 1$ for $n \geqslant k$. Thus $xe_k \in A$ for any k. Therefore, to derive $Q_B(A) \subseteq Q_H(A)$, it suffices to show that $J = \sum\limits_k e_k A$ is essential in A. Consider a right ideal $K \neq 0$ in A. Then there is a nonzero projection $e \in K$ and the continuity of $\mathbb{L}(A)$ yields $e = e \cap \bigvee_k e_k = e \cap \bigvee_k f_k = \bigvee_k (f_k \cap e)$, whence $e = f_k \cap e = ef_k \in J \cap K$ for some k.

7. CONCLUSIONS

Obviously, our approach bears some similarities to the method developed in [25] for proving direct finiteness of the group ring $D(G)$ of a (sofic) group, where D is any division ring. The idea is to use ultralimits to construct a pseudorank function N on the direct product E of endomorphisms rings of D-vector spaces generated by finite subsets of G and to embed $D(G)$ into the continuous regular ring $E/\mathsf{ker}N$.

More specifically, we may ask to what extent we could replace, in the special case of von Neumann algebras, the model-theoretic ultraproducts by von Neumann algebra ultraproducts (see, e.g., [26]), and thus gain some insight into more serious problems concerning these. Note, however, that the saturation property of model-theoretic ultraproducts appears to be crucial for our approach.

There is a great variety of results on Baer $*$-rings satisfying certain conditions which imply $*$-regularity of maximal rings of quotients (see, e.g., [3-5, 9, 10]); (i) in Proposition 44 is one of them. In contrast, results on representations of $*$-rings within inner product spaces appear to be located at two extremes: the GNS-construction on the ‘continuous side’ and the results on rings with maximal right ideals (cf. [27]) on the ‘discrete side.’ It would be desirable to have results based on a weaker (lattice-theoretic) form of continuity.

In a subsequent work, we will use the results of the present paper for a detailed discussion of classes of $*$-algebras and modular ortholattices representable within inner product spaces over $*$-fields elementarily equivalent to $\mathbb{R}$ and $\mathbb{C}$, respectively, including solvability and complexity of certain decision problems for these classes. In particular, it will be shown that any algebra $Q(A)$ such as in Theorem 1 (as well as its projection ortholattice) has a decidable equational theory. This is one more indication that algebras of the form $Q(A)$ are very special members of the class of all $*$-regular rings. (Recall that $Q(A)$ is directly finite since it has a regular extension [7, 9].) The question remains open as to the extent to which direct finiteness is inherited by homomorphic images of subalgebras of (model-theoretic) ultraproducts of matrix $*$-algebras $\mathbb{C}^{n\times n}$.

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