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Linear representations of regular rings and complemented modular lattices with involution

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Dedicated to the memory of Tamás Schmidt

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Abstract. Faithful representations of regular $*$-rings and modular complemented lattices with involution within orthosymmetric sesquilinear spaces are studied within the framework of Universal Algebra. In particular, the correspondence between classes of spaces and classes of representable structures is analyzed; for a class $\mathcal{S}$ of spaces which is closed under ultraproducts and non-degenerate finite-dimensional subspaces, the class of representable structures is shown to be closed under complemented [regular] subalgebras, homomorphic images, and ultraproducts. Moreover, this class is generated by its members which are isomorphic to subspace lattices with involution [endomorphism $*$-rings, respectively] of finite-dimensional spaces from $\mathcal{S}$. Under natural restrictions, this result is refined to a 1-1-correspondence between the two types of classes.

1. Introduction

For $*$-rings, there is a natural and well-established concept of representation in a vector space V_F endowed with an orthosymmetric sesquilinear form: a homomorphism ε into the endomorphism ring of V_F such that $\varepsilon(r^*)$ is the adjoint of $\varepsilon(r)$. Famous examples of [faithful] representations are due to Gel'fand–Naimark–Segal

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(C^*-algebras in Hilbert space) and Kaplansky (primitive $*$-rings with a minimal right ideal), cf. [2, Theorem 4.6.6].

[Faithful] representability of $*$-regular rings within anisotropic inner product spaces has been studied by Micol [44] and used to derive results in the universal algebraic theory of these structures. For the $*$-regular rings of classical quotients of finite Rickart C^*-algebras (cf. Ara and Menal [1]), existence of representations has been established in [30]. For complemented modular lattices with involution $a \mapsto a'$ (CMILs for short), an analogue of the concept of representation is a lattice homomorphism ε, preserving the bounds 0 and 1, into the lattice of all subspaces such that $\varepsilon(a')$ is the subspace orthogonal to $\varepsilon(a)$ (cf. Niemann [46]). The latter has been considered in the context of synthetic orthogeometries in [22], continuing earlier work on anisotropic geometries and modular ortholattices [25–27]. Primary examples are atomic CMILs associated with irreducible desarguean orthogeometries and those CMILs which arise from lattices of principal right ideals of representable regular $*$-rings.

The [proofs of the] main results of these studies relate closure properties of a class $\mathcal{S}$ of spaces with closure properties of the class $\mathcal{R}$ of algebraic structures [faithfully] representable within spaces from $\mathcal{S}$. In particular, for a class $\mathcal{S}$ closed under ultraproducts and non-degenerate finite-dimensional subspaces, one has $\mathcal{R}$ closed under ultraproducts, homomorphic images, and regular [complemented, respectively] subalgebras. Moreover, with an approach due to Tyukavkin [50], it has been shown that $\mathcal{R}$ is generated, with respect to these operators, by the endomorphism $*$-rings [by the subspace lattices with involution $U \mapsto U^\perp$, respectively] of finite-dimensional spaces from $\mathcal{S}$ (cf. Theorem 11.3). Conversely, any class $\mathcal{R}$ of structures generated in this way has its members representable within $\mathcal{S}$.

The first purpose of the present paper is to extend these results to regular $*$-rings on one hand, to representations within orthosymmetric sesquilinear spaces on the other — thus allowing regular rings with an involution which may have $r^*r = 0$ for some $r \neq 0$, that is regular $*$-rings which are not $*$-regular; for example, $*$-rings associated with finite-dimensional spaces having some isotropic points. The second one is to give a more transparent presentation by dealing with types of classes naturally associated with representations in linear spaces. We call a class of structures $\mathcal{R}$ as above an $\exists$-*semivariety* of regular $*$-rings [CMILs] and we call $\mathcal{S}$ a *semivariety* of spaces. The quantifier '$\exists$' refers to the required existence of quasi-inverses [complements, respectively]. In this setting, the above-mentioned relationship between classes of spaces $\mathcal{S}$ and classes of representable structures $\mathcal{R}$ can be refined to a 1-1-correspondence (cf. Theorem 11.6). Also, we observe that $\mathcal{R}$ remains unchanged if $\mathcal{S}$ is enlarged by forming two-sorted substructures, corresponding to the subgeometries in the sense of [22], (cf. Theorem 11.3). We also provide a useful condition on $\mathcal{S}$ which implies that $\mathcal{R}$ is an $\exists$-*variety*, i.e. that $\mathcal{R}$ is also closed under direct products (see Proposition 12.1). For reference in later applications, e.g. to decidability results refining those of [23], we consider $*$-rings which are also algebras over a fixed commutative $*$-ring.

In the context of synthetic orthogeometries, the class $\mathcal{R}$ of representable structures is an $\exists$-variety if $\mathcal{S}$ is also closed under orthogonal disjoint unions. No such natural construction is available for sesquilinear spaces. The alternative, chosen by Micol [44], was to generalize the concept of faithful representation to residually faithful representation; thus, associating with any semivariety of spaces an $\exists$-variety of generalized representables. We derive these results in our more general setting (cf. Proposition 12.3).

We first present background material on sesquilinear spaces (Section 2), rings (Sections 3–4), and lattices (Sections 5,7). Synthetic orthogeometries are included (Section 6) to make use of the results in [22]. A key to results on representations is to view them as multi-sorted structures (Section 8). The Universal Algebra point of view and the class operators are introduced in Section 9. The basic reduction to finite dimensions is in Section 10, applications to correspondences between classes in Section 11. Section 12 relates these to Micol's more general concept of representation, cf. [44]. In Sections 8–12 results on rings and on lattices are presented in parallel. Proofs of the former do not depend on the latter. Though, the other way round, we have to use basic results on lattices of principal right ideals of regular rings.

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2. ε-Hermitian spaces

We first define the linear structures providing representations both for lattices and rings with involution. For any division ring F endowed with an anti-automorphism ν (we write λ^ν instead of $\nu(\lambda)$), we consider *sesquilinear spaces* which are [right] vector spaces V_F endowed with a *scalar product* or a *sesquilinear form* $\langle\ \mid\ \rangle\colon V\times V\to F$; that is, for all u, v, $w\in V$ and all λ, $\mu\in F$, one has

$$\langle u\mid v+w\rangle=\langle u\mid v\rangle+\langle u\mid w\rangle,\ \langle u+v\mid w\rangle=\langle u\mid w\rangle+\langle v\mid w\rangle,\ \langle u\lambda\mid v\mu\rangle=\lambda^\nu\langle u\mid v\rangle\mu.$$

Our basic reference is [20, Chapter I] (though, we use "sesquilinear space" in a more general meaning). Observe that from a right vector space V_F, one obtains a left vector space ${}_FV$ putting $\lambda v=v\lambda^{\nu^{-1}}$. By this, a sesquilinear form on V_F with respect to ν, as defined above, turns out a sesquilinear form on ${}_FV$, in the sense of [20], with respect to ν^{-1}. This gives access to results of [20] in the left vector space setting (in particular, the ε-hermitean forms of [20] will correspond to the ε-hermitian forms defined, below). Introductions to orthogonal geometry in infinite dimension are also given in [39], [12, Chapter 14], cf. [33, Chapter IV], [31, §1.21], [2, §4.6].

Given a second sesquilinear space $V'_{F'}$ with an anti-automorphism ν' and a sesquilinear form $\langle\ |\ \rangle'$, we have the following concepts relating it to the first: An *isomorphism* between the sesquilinear spaces V_F and $V'_{F'}$ is a bijection $\omega\colon V \to V'$ which is an α-semilinear map $V_F \to V'_{F'}$ for some isomorphism $\alpha\colon F \to F'$, such that $\alpha \circ \nu = \nu' \circ \alpha$ and $\langle \omega(v) \mid \omega(w)\rangle' = \alpha(\langle v \mid w\rangle)$ for all v, $w \in V$. The second space arises from the first by *scaling* with $\mu \in F$ if $F' = F$ as division rings and $V_F = V'_{F'}$ as vector spaces and if, in addition, $\mu \neq 0$, $\lambda^{\nu'} = \mu\lambda^{\nu}\mu^{-1}$ for all $\lambda \in F$ and $\langle u \mid v\rangle' = \mu\langle u \mid v\rangle$ for all $u, v \in V$. Finally, V_F and $V'_{F'}$ are *similar* if one arises from the other by composition of isomorphisms and scalings. It is easy to see that any similitude can be expressed as an isomorphism followed by a scaling.

Since we consider only one anti-automorphism ν on F and only one scalar product on V_F at a time, we use F to denote the division ring together with the anti-automorphism ν (and write λ^* instead of λ^{ν}) and V_F to denote the space endowed with the scalar product.

A sesquilinear space $V_F \neq 0$ is *non-degenerate* if $u = 0$ whenever $\langle u \mid v\rangle = 0$ for all $v \in V$ or $\langle v \mid u\rangle = 0$ for all $v \in V$. Any vector space V over a division ring F with anti-automorphism ν can be turned into a non-degenerate sesquilinear space: given a basis $(v_i \mid i \in I)$ and $0 \neq \delta_i \in F$, $i \in I$, define $\langle \sum_{i\in J} v_i\lambda_i \mid \sum_{i\in J} v_i\mu_i\rangle = \sum_{i\in J} \lambda_i^{\nu}\delta_i\mu_i$ for finite $J \subseteq I$ and λ_i, $\mu_i \in F$. Though, these examples are far from being exhaustive.

For $\varepsilon \in F$, V_F is *ε-hermitian* if $\langle v \mid u\rangle = \varepsilon \cdot \langle u \mid v\rangle^*$ for all u, $v \in V$; V_F is *hermitian* if it is 1-hermitian; V_F is *skew symmetric* if it is (-1)-hermitian and $\lambda^* = \lambda$ for all $\lambda \in F$; V_F is *alternate*, if $\langle v \mid v\rangle = 0$ for all $v \in V$ (observe that [2, §4.6] requires characteristic $\neq 2$). V_F is *anisotropic* if $\langle v \mid v\rangle \neq 0$ for all $v \in V$, $v \neq 0$.

Let $\mathsf{End}(V_F)$ denote the ring of all endomorphisms of the vector space V_F. For $\varphi, \psi \in \mathsf{End}(V_F)$ we say that ψ is an *adjoint* of φ if $\langle \varphi(u) \mid v\rangle = \langle u \mid \psi(v)\rangle$ for all u, $v \in V$. If V_F is non-degenerate, then any endomorphism φ has at most one adjoint ψ; if such ψ exists, we write $\psi = \varphi^*$. If φ^* and χ^* exist, then $(\chi \circ \varphi)^* = \varphi^* \circ \chi^*$. For vectors u, $v \in V$, we say that v is *orthogonal* to u and write $u \perp v$, if $\langle u \mid v\rangle = 0$. The space V_F is *orthosymmetric*, or *reflexive*, if $\perp$ is a symmetric relation. The anti-automorphism $\lambda \mapsto \lambda^*$ is an *involution* on F if it is of order 2, that is $(\lambda^*)^* = \lambda$ for all $\lambda \in F$.

Proposition 2.1. *The relations of orthogonality and adjointness are left unchanged under scaling; in particular, orthosymmetry is preserved under scaling. Consider a non-degenerate sesquilinear space V_F. The following are equivalent if $\dim V_F > 1$:*

(1) *the sesquilinear space V_F is orthosymmetric;*
(2) *the sesquilinear space V_F is ε-hermitian for some (unique) $\varepsilon \in F \setminus \{0\}$;*
(3) *up to scaling, V_F is either hermitian or skew-symmetric;*
(4) *the adjointness relation is symmetric on $\mathsf{End}(V_F)$;*
(5) *if φ^* exists then $\varphi^{**} = \varphi$.*

Furthermore, if V_F is ε-hermitian and non-degenerate, then $\lambda \mapsto \lambda^*$ is an involution on F. If V_F is alternate and non-degenerate then it is skew symmetric and F is commutative; moreover, any $V'_{F'}$ similar to V_F is alternate, too.

Proof. The first statement is obviously true. Now, assume that V_F is non-degenerate and that $\dim V_F > 1$. The following references are to [20, Chapter I]. (i) implies (ii) by Theorem 1 of §1.3. (ii) implies (iii) by (15) of §1.5. (iii) implies (i), obviously, thus proving pairwise equivalence of (i), (ii), and (iii).

Assuming (iii), symmetry of adjointness follows, easily. That, in turn, implies that $\varphi = \varphi^{**}$ for every $\varphi \in \mathsf{End}(V_F)$ having an adjoint. Thus, (iii) implies (iv) and (iv) implies (v).

Assuming (v), let $\lambda \mapsto \lambda^+$ denote the inverse of $\lambda \mapsto \lambda^*$. Given $u \in V$ such that $\mu := \langle u \mid u \rangle \neq 0$, consider two linear maps:

$$\varphi_u(v) = u\big(\langle v \mid u \rangle \mu^{-1}\big)^+ \text{ and } \psi_u(w) = u\mu^{-1}\langle u \mid w \rangle, \quad v,\ w \in V.$$

Observe that $\psi_u = \varphi_u^*$, whence by our hypothesis, $\varphi_u = \psi_u^*$. Moreover, φ_u and ψ_u are the projections onto uF associated with the decompositions $V = uF \oplus \{v \in V \mid \langle v \mid u \rangle = 0\}$ and $V = uF \oplus \{w \in V \mid \langle u \mid w \rangle = 0\}$, respectively. Now, $\varphi_u = \psi_u \circ \varphi_u$ and we get $\psi_u = \varphi_u^* = (\psi_u \circ \varphi_u)^* = \varphi_u^* \circ \psi_u^* = \psi_u \circ \varphi_u = \varphi_u$. Thus, under the proviso $\langle u \mid u \rangle \neq 0$, one has for any $x \in V$ that $\langle x \mid u \rangle = 0$ if and only if $\langle u \mid x \rangle = 0$.

It follows that $\langle v \mid w \rangle = 0$ is equivalent to $\langle w \mid v \rangle = 0$ unless $\langle v \mid v \rangle = \langle w \mid w \rangle = 0$. In the latter case, let $u = v + w$. If $\langle u \mid u \rangle = 0$ then $\langle v \mid w \rangle = -\langle w \mid v \rangle$; otherwise, $\langle v \mid w \rangle = 0$ iff $\langle v \mid u \rangle = 0$ iff $\langle u \mid v \rangle = 0$ iff $\langle w \mid v \rangle = 0$. This proves that (v) implies (i).

For non-degenerate ε-hermitian V_F, in order to prove that $\lambda \mapsto \lambda^*$ is an involution, by (15) of §1.5 we may assume $\varepsilon = \pm 1$. As (7) in §1.3 follows from (9), we have $(\lambda^*)^* = \varepsilon^{-1}\lambda\varepsilon = \lambda$. The alternate case is dealt with in (12) of §1.4. ∎

A sesquilinear space V_F, over a division ring F with involution, which is ε-hermitian for some ε and non-degenerate will be called *pre-hermitian.* In the sequel, we consider only pre-hermitian spaces. If V_F is, in addition, anisotropic, we also speak of an *inner product space.*

The following well-known facts are discussed in Section 7 in more detail. For vectors $u, v \in V$, we say that v is *orthogonal* to u and write $u \perp v$, if $\langle u \mid v\rangle = 0$. The *orthogonal* of $X \subseteq V$ is the subspace $X^\perp = \{v \in V \mid \forall u \in X.\ u \perp v\}$; observe that $Y^\perp \subseteq X^\perp$ if $X \subseteq Y$. A subspace U is *closed* if $U = U^{\perp\perp}$; in particular, any $X^\perp$ is closed. If U is a subspace with $\dim U = 1$ then $U^\perp = \ker f$ for the linear map $f\colon V \to F$ given as $f(w) = \langle v \mid w\rangle$ where $U = vF$; f is surjective since V_F is non-degenerate, whence $\dim V/U^\perp = 1$. Since $U^\perp = \bigcap_{v\in B} vF^\perp$ for any basis B of U, it follows that $\dim V/U^\perp \leqslant \dim U$ for any U with $\dim U < \omega$ (actually, equality holds and U is closed, see Propositions 7.1 and 5.2).

On any linear subspace U of V_F, one has the sesquilinear *subspace* U_F with the induced scalar product. When U_F is non-degenerate, U_F is pre-hermitian, too. A finite-dimensional subspace U_F of V_F is non-degenerate if and only if $U \cap U^\perp = 0$, if and only if $V = U \oplus U^\perp$ (as $\dim V/U^\perp \le \dim U$). We write in this case $U \in \mathbb{O}(V_F)$ and say that U is a *finite-dimensional orthogonal summand*; in particular, U is closed.

Proposition 2.2. *Every pre-hermitian space V_F is the directed union of the subspaces U_F, $U \in \mathbb{O}(V_F)$. Actually, for any finite-dimensional subspace $W \in \mathbb{L}(V_F)$ there is $U \in \mathbb{O}(V_F)$ such that $W \subseteq U$ and $\dim U \leqslant 2 \dim W$.*

Proof. This is [20, Chapter I, §5 Lemma 4], cf. [2, Remark 4.6.14]. Alternatively, one can apply [22, Theorem 1.2] to the "orthogeometry" $\mathbb{G}(V_F)$ associated with V_F and Proposition 7.1, below. ∎

For a subspace U_F of V_F, the linear subspace $\operatorname{rad} U = U \cap U^\perp$ is the *radical* of U_F. Defining $\langle v + \operatorname{rad} U \mid w + \operatorname{rad} U\rangle = \langle v \mid w\rangle$, the F-vector space $U/\operatorname{rad} U$ becomes a sesquilinear space $U_F/\operatorname{rad} U$ with respect to the given anti-automorphism of F. We call $U_F/\operatorname{rad} U$ a *subquotient space.*

Proposition 2.3. *Let V_F be a pre-hermitian space and let U_F be a subspace of V_F. Then $U_F/\operatorname{rad} U$ is non-degenerate; it is ε-hermitian if V_F is. The space $U_F/\operatorname{rad} U$ is isomorphic to any subspace W_F of V_F such that $U = W \oplus \operatorname{rad} U$.*

Proof. The map $w \mapsto w + \operatorname{rad} U$ establishes an isomorphism of sesquilinear spaces from W_F onto $U_F/\operatorname{rad} U$. ∎

3. Rings and algebras with involution

When mentioning rings, we always mean associative rings R possibly without unit. The principal right ideal aR generated by a equals $\{za \mid z \in \mathbb{Z}\} \cup \{ar \mid r \in R\}$. A $*$-*ring* is a ring R endowed with an *involution*; that is, an anti-automorphism $x \mapsto x^*$ of order 2; that is,

$$(r+s)^* = r^* + s^*, \quad (rs)^* = s^* r^*, \quad (r^*)^* = r \quad \text{for all } r, s \in R,$$

cf. [31, §1], [47, §2.13], [2, §4].

An element e of a $*$-ring R is a *projection*, if $e = e^2 = e^*$. A $*$-ring R is *proper* if $r^* r = 0$ implies $r = 0$ for all $r \in R$. Throughout this paper, let Λ be a commutative $*$-ring with unit. A $*$-Λ-*algebra* R is an associative (left) unital Λ-algebra, with unit 1 considered a constant, which is a $*$-ring such that

$$(\lambda r)^* = \lambda^* r^* \text{ for all } r \in R,\ \lambda \in \Lambda.$$

For example, involutive Banach algebras are $*$-$\mathbb{C}$-algebras. Unless stated otherwise, we consider the scalars $\lambda \in \Lambda$ as unary operations $r \mapsto \lambda r$ on R; in other words, we consider $*$-Λ-algebras as 1-sorted algebraic structures. The map $\lambda \mapsto \lambda 1$ is a $*$-ring homomorphism from Λ into the center of R; in view of this, denoting both involutions on R and on Λ by the same symbol * should not cause confusion; also, most arguments concerning the action of Λ are obvious and left to the reader.

An ideal I of a $*$-ring or a $*$-Λ-algebra R is a $*$-*ideal*, if $I = I^*$, where $I^* = \{r^* \mid r \in I\}$. We call R *strictly subdirectly irreducible* if the underlying ring is subdirectly irreducible, i.e. has a smallest non-zero ideal I; in this case, $I = I^*$. Similarly, R is *strictly simple* if 0 and R are the only ideals. In the $*$-ring literature, such $*$-rings are called 'simple', while simple $*$-rings are called '$*$-simple', cf. [3].

The [*right*] *socle* $\mathsf{Soc}(R)$ consists of all $a \in R$ such that aR is the sum of finitely many minimal right ideals; $\mathsf{Soc}(R)$ is an ideal of R. We say that a $*$-Λ-algebra is *atomic* if any non-zero right ideal contains a minimal one.

A ring R is [*von Neumann*] *regular* if for any $a \in R$, there is an element $x \in R$ such that $axa = a$; such an element is called a *quasi-inverse* of a. A $*$-ring R is $*$-*regular* if it is regular and proper. The reader interested in more details is referred to any of [4, 5, 16, 40, 45, 49]. Recall that for a vector space V_F over a division ring F, $\mathsf{End}(V_F)$ denotes the set of all endomorphisms of V_F.

Proposition 3.1.

(i) *For a vector space V_F, $\mathsf{End}(V_F)$ is a regular simple ring.*

(ii) *A ring R is regular if it admits a regular ideal I such that R/I is regular. Any ideal of a regular ring is regular.*
(iii) *A ring R is regular [a $*$-ring R is $*$-regular] if and only if for any $a \in R$ there is an idempotent [a (unique) projection, respectively] $e \in R$ such that $aR = eR$.*
(iv) *For any a, b in a regular ring R, there is an idempotent $e \in aR + bR$ such that $ea = a$ and $eb = b$.*
(v) *Homomorphic images and direct products of regular [$*$-regular] $*$-Λ-algebras are regular [$*$-regular] $*$-Λ-algebras.*

Proof. Statements (i)–(v) are well known, cf. [5, 1.26], [16, Lemma 1.3], [16, Theorem 1.7]. For the existence of projections, see [45, Part II Chapter IV Theorem 4.5] or [5, Proposition 1.13]. In (v), the claim for products is obvious; for homomorphic images, it follows from (iii). ■

In particular, any ideal of a $*$-regular ring is a $*$-ideal by Proposition 3.1(iii); thus subdirectly irreducibles [simples] are strictly subdirectly irreducible [strictly simple, respectively]. Call a $*$-Λ-algebra R *primitive* if the underlying ring is primitive, that is, admits a faithful irreducible module.

Proposition 3.2. *Every regular strictly subdirectly irreducible $*$-Λ-algebra R is primitive.*

Proof. Any regular ring is semi-simple, i.e. has zero radical, cf. [16, Corollary 1.2]. Hence the ring R is a subdirect product of primitive rings, cf. [33, Chapter I, §3, Theorem 1]. Being subdirectly irreducible, the ring R is therefore primitive, cf. the proof of [44, Corollary 3.4]. ■

4. Endomorphism rings

In the sequel, let F be a $*$-Λ-algebra, where the underlying ring of F is a division ring and V_F is a pre-hermitian space over F. By $\mathsf{End}(V_F)$, we also denote the unital Λ-algebra of V_F of all endomorphisms of the vector space V_F. By $\mathsf{End}^*(V_F)$, we denote the set of endomorphisms from $\mathsf{End}(V_F)$ having an adjoint. The following proposition is obvious in view of the definition of adjoints and by Proposition 2.1.

Proposition 4.1. *The set* $\mathsf{End}^*(V_F)$ *forms a Λ-subalgebra* $\mathsf{End}^*(V_F)$ *of* $\mathsf{End}(V_F)$ *which is a $*$-Λ-algebra with the involution $\varphi \mapsto \varphi^*$. If $V'_{F'}$ is similar to V_F then* $\mathsf{End}^*(V_F)$ *and* $\mathsf{End}^*(V'_{F'})$ *are isomorphic $*$-Λ-algebras.*

Observe that for $v \in V$, $\lambda \in \Lambda$, and $\varphi \in \mathsf{End}^*(V_F)$, one has

$$(\lambda\varphi)(v) := \varphi(v)\lambda, \quad (\lambda\varphi)^* = \lambda^*\varphi^*.$$

Also recall the well-known facts that for any $\varphi, \psi \in \mathsf{End}^*(V_F)$

$$(\mathsf{im}\,\varphi)^\perp = \mathsf{ker}\,\varphi^* \text{ is closed;}$$

$$\mathsf{im}\,\varphi \subseteq (\mathsf{im}\,\psi)^\perp \text{ if and only if } \varphi^* \circ \psi = 0.$$

Proposition 4.2. *For any subspace U of V_F, one has $V = U \oplus U^\perp$ if and only if there is a projection $\pi_U \in \mathsf{End}^*(V_F)$ such that $U = \mathsf{im}\,\pi_U$. Such a projection π_U is unique.*

The projection π_U in Proposition 4.2 is called the *orthogonal projection* onto U. Par abus de langage, π_U also denotes the induced epimorphism $V \to U$, while ε_U denotes the inclusion map $U \to V$. Observe that π_U and ε_U are adjoints of each other in the sense that

$$\langle \varepsilon_U(u) \mid v\rangle = \langle u \mid \pi_U(v)\rangle \quad \text{for all } u \in U,\ v \in V.$$

Moreover, the computational rules of $\mathsf{End}^*(V_F)$ yield, in particular, $(\varepsilon_U\varphi\pi_U)^* = \varepsilon_U\varphi^*\pi_U$ for any $\varphi \in \mathsf{End}^*(U_F)$. Finally, $\pi_U\varepsilon_U = \mathsf{id}_U$, while $\pi_U\varepsilon_U\pi_U = \pi_U$ and $U^\perp = \mathsf{ker}(\varepsilon_U\pi_U)$.

Let $\dim V_F = n < \omega$. We say that the bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ of V_F are a *dual pair of bases*, whenever $\langle v_i \mid w_i\rangle = 1$ for all $i \in \{1, \dots, n\}$ and $\langle v_i \mid w_j\rangle = 0$ for all $i \neq j$.

Proposition 4.3. *Let V_F be a pre-hermitian space and let* $\dim V_F = n < \omega$.

(i) *There is a dual pair of bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ of V_F. Moreover, for any $\varphi \in \mathsf{End}(V_F)$ with $\varphi(v_j) = \sum_i w_i a_{ij}$, $\varphi^* \in \mathsf{End}(V_F)$ exists and $\varphi^*(v_i) = \sum_j w_j a_{ij}^*$. In particular, $\mathsf{End}^*(V_F)$ contains all endomorphisms of V_F and $\mathsf{End}^*(V_F)$ is regular.*

(ii) *$\mathsf{End}^*(V_F)$ is ∗-regular if and only if V_F is anisotropic.*

(iii) *If $U_F \in \mathbb{O}(V_F)$ then $\mathsf{End}^*(U_F) \times \mathsf{End}^*(U_F^\perp)$ embeds into $\mathsf{End}^*(V_F)$; in particular, $\mathsf{End}^*(U_F)$ is a homomorphic image of a regular ∗-Λ-subalgebra of $\mathsf{End}^*(V_F)$.*

Proof. For the existence of dual bases, see [33, §IV.15] or [36, §II.6]. Straightforward and well-known calculations prove (i). Regularity of $\mathsf{End}^*(V_F)$ follows from Proposition 3.1(i). For (ii) see Proposition 4.4(vi), below. (iii) Let R consist of all $\varphi \in \mathsf{End}^*(V_F)$ which leave both U and $U^\perp$ invariant. As $R \cong \mathsf{End}^*(U_F) \times \mathsf{End}^*(U_F^\perp)$, all claims follow. ∎

We put $J(V_F) = \{\varphi \in \mathsf{End}^*(V_F) \mid \dim \mathsf{im}\, \varphi < \omega\}$. Cf. [2, Theorem 4.6.15] for the following.

Proposition 4.4. *Let V_F be a pre-hermitian space.*

(i) *The set $J(V_F)$ is an ideal and a strictly simple regular $*$-Λ-subalgebra of $\mathsf{End}^*(V_F)$ without unit.*

(ii) *The principal right ideals of $J(V_F)$ are in one-to-one correspondence with the finite-dimensional subspaces of V_F via the map $\varphi J(V_F) \mapsto \mathsf{im}\, \varphi$; moreover, $\varphi_0 J(V_F) \subseteq \varphi_1 J(V_F)$ is equivalent to $\mathsf{im}\, \varphi_0 \subseteq \mathsf{im}\, \varphi_1$ for any $\varphi_0, \varphi_1 \in J(V_F)$.*

(iii) *Let R be a subring of $\mathsf{End}(V)$ with $R \supseteq J(V_F)$. Then the minimal right ideals of R are of the form φR, where $\varphi \in J(V_F)$ is an idempotent such that $\varphi J(V_F)$ is a minimal right ideal of $J(V_F)$, that is, $\dim \mathsf{im}\, \varphi = 1$. In particular, R is atomic and $J(V_F) = \mathsf{Soc}(R)$ is its smallest non-zero ideal.*

(iv) *For any $\varphi_1, \ldots, \varphi_n \in J(V_F)$, there is $U \in \mathbb{O}(V_F)$ such that $\pi_U \varphi_i = \varphi_i = \varphi_i \pi_U$ for all $i \in \{1, \ldots, n\}$.*

(v) *The space V_F is alternate if and only if $J(V_F)$ does not contain a projection generating a minimal right ideal. If V_F is alternate then $\pi \circ \pi^* = 0 = \pi^* \circ \pi$ for any idempotent π with $\dim \mathsf{im}\, \pi = 1$.*

(vi) *The space V_F is anisotropic if and only if $\mathsf{End}^*(V_F)$ is proper if and only if $J(V_F)$ is proper.*

Proof. (i) Clearly, $J(V_F)$ is an ideal and a Λ-subalgebra of $\mathsf{End}^*(V_F)$ (without unit). Observe that $\pi_U \in J(V_F)$ for any $U \in \mathbb{O}(V_F)$ by Proposition 4.2. Moreover by Proposition 2.2, for any subspace W of V_F with $\dim W < \omega$, there exists $U \in \mathbb{O}(V_F)$ such that $W \subseteq U$.

Consider $\varphi \in J(V_F)$ and recall that the subspaces $\ker \varphi = (\mathsf{im}\, \varphi^*)^\perp$ and $\ker \varphi^* = (\mathsf{im}\, \varphi)^\perp$ are both closed. To prove that $\varphi^* \in J(V_F)$, choose $W \in \mathbb{O}(V_F)$ such that $W \supseteq \mathsf{im}\, \varphi = (\ker \varphi^*)^\perp$. Then $W^\perp \subseteq (\ker \varphi^*)^{\perp\perp} = \ker \varphi^*$, whence $\mathsf{im}\, \varphi^* = \varphi^*(W)$ is finite-dimensional. It follows that

($*$) For any $\varphi_1, \ldots, \varphi_n \in J(V_F)$, there is $U \in \mathbb{O}(V_F)$ such that $U \supseteq \mathsf{im}\, \varphi_i + \mathsf{im}\, \varphi_i^*$ for all $i \in \{1, \ldots, n\}$ and $\varphi_i(U) = \mathsf{im}\, \varphi_i$ and $\varphi_i^*(U) = \mathsf{im}\, \varphi_i^*$. In particular,
  (a) U is a finite-dimensional pre-hermitian space;
  (b) $V = U \oplus U^\perp$;
  (c) $U^\perp \subseteq \bigcap_i \ker \varphi_i \cap \ker \varphi_i^*$ and $\varphi_i = \pi_U \varphi_i \varepsilon_U \pi_U$;
  (d) $\pi_U \in J(V_F)$;
  (e) $\varepsilon_U \psi \pi_U \in J(V_F)$ and $(\varepsilon_U \psi \pi_U)^* = \varepsilon_U \psi^* \pi_U$ for any $\psi \in \mathsf{End}(U_F)$.

To prove that φ has a quasi-inverse in $J(V_F)$, choose for φ a subspace $U \in \mathbb{O}(V_F)$ according to ($*$). By Proposition 4.3(i), $\pi_U \varphi \varepsilon_U \in \mathsf{End}^*(U_F)$ has a quasi-inverse $\psi \in \mathsf{End}^*(U_F)$. We claim that $\chi = \varepsilon_U \psi \pi_U$ is a quasi-inverse of φ in $J(V_F)$. Indeed,

$\chi \in J(V_F)$ by (e) and $\varphi(v) = 0 = \chi(v)$ for any $v \in U^\perp$ by (c) and $\varphi\chi\varphi(v) = \pi_U\varphi\varepsilon_U\psi\pi_U\varphi\varepsilon_U(v) = \pi_U\varphi\varepsilon_U(v) = \varphi(v)$ for any $v \in U$.

To prove that $J(V_F)$ is strictly simple, it suffices to show that for any $0 \neq \varphi, \psi \in J(V_F)$, ψ belongs to the ideal generated by φ. Again, choose for φ and ψ a subspace $U \in \mathbb{O}(V_F)$ according to $(*)$. Applying Proposition 3.1(i) to $\pi_U\varphi\varepsilon_U$, $\pi_U\psi\varepsilon_U \in \mathsf{End}(U_F)$, we get that there are $m < \omega$ and $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_m \in \mathsf{End}(U_F)$ such that $\pi_U\psi\varepsilon_U = \sum_{i=1}^m \tau_i\pi_U\varphi\varepsilon_U\sigma_i$. Then according to (c), $\psi = \sum_{i=1}^m \varepsilon_U\tau_i\pi_U\varphi\varepsilon_U\sigma_i\pi_U$ and $\varepsilon_U\sigma_i\pi_U$, $\varepsilon_U\tau_i\pi_U \in J(V_F)$ for all $i \in \{1, \ldots, m\}$ by (e).

(ii) We prove first that $\varphi_0 J(V_F) \subseteq \varphi_1 J(V_F)$ is equivalent to $\mathsf{im}\,\varphi_0 \subseteq \mathsf{im}\,\varphi_1$ for any $\varphi_0, \varphi_1 \in \mathsf{End}^*(V_F)$. Suppose first that $\mathsf{im}\,\varphi_0 \subseteq \mathsf{im}\,\varphi_1$ and take an arbitrary $\psi \in J(V_F)$; then $\varphi_0\psi, \varphi_1\psi \in J(V_F)$. Choose for $\varphi_0\psi$ and $\varphi_1\psi$ a subspace $U \in \mathbb{O}(V_F)$ according to $(*)$. Then $\xi_i = \pi_U\varphi_i\psi\varepsilon_U \in \mathsf{End}(U_F)$ for any $i < 2$ and $\mathsf{im}\,\xi_0 \subseteq \mathsf{im}\,\xi_1$. As $\dim U_F < \omega$, $\xi_0 = \xi_1\chi$ for some $\chi \in \mathsf{End}(U_F)$. According to (c),

$$\varphi_0\psi = \pi_U\varphi_0\psi\varepsilon_U\pi_U = \xi_0\pi_U = \xi_1\chi\pi_U = \pi_U\varphi_1\psi\varepsilon_U\chi\pi_U = \varphi_1\psi\varepsilon_U\chi\pi_U \in \varphi_1 J(V_F),$$

as $\psi\varepsilon_U\chi\pi_U \in J(V_F)$ by (e). The reverse implication is trivial by Proposition 2.2.

Besides that, for any finite-dimensional subspace W of V_F, there is $\varphi \in J(V_F)$ such that $W = \mathsf{im}\,\varphi$. Indeed by Proposition 2.2, there is $U \in \mathbb{O}(V_F)$ such that $W \subseteq U$, whence $W = \mathsf{im}\,\psi$ for some $\psi \in \mathsf{End}(U_F)$. Then $W = \mathsf{im}\,\varphi$ with $\varphi = \varepsilon_U\psi\pi_U \in J(V_F)$ by Proposition 4.2 and (e). This establishes the claimed 1-1-correspondence.

(iii) It follows from (ii) that for an arbitrary element $v \neq 0$ in V, there is an idempotent $\pi_v \in J(V_F)$ such that $\mathsf{im}\,\pi_v = vF$. Moreover, $\pi_v R = \pi_v J(V_F)$ is obviously a minimal right ideal of both $J(V_F)$ and R. Consider any element $\varphi \in R$, such that $\varphi \neq 0$. There is $v \in V$ such that $\varphi(v) \neq 0$; in particular, $\mathsf{im}(\varphi\circ\pi_v) = \mathsf{im}\,\pi_v$, whence $\pi_v R$ is a minimal right ideal contained in φR according to (ii) and the above. If $\dim\varphi = n$, then $\varphi R = \sum_{i=1}^n \pi_{v_i} R$, where $v_1, \ldots, v_n$ is a basis of $\mathsf{im}\,\varphi$; in particular, $\varphi R = \pi_{v_1} R$ if φR is minimal. Thus R is atomic with $\mathsf{Soc}(R) = J(V_F)$ contained in any non-zero ideal.

(iv) Given $\varphi_1, \ldots, \varphi_n \in J(V_F)$, choose a subspace $U \in \mathbb{O}(V_F)$ according to $(*)$. Then $\mathsf{im}\,\varphi_i + \mathsf{im}\,\varphi_i^* \subseteq U$, whence $\pi_U\varphi_i = \varphi_i$ and $\pi_U\varphi_i^* = \varphi_i^*$.

(v) If π is a projection in the $*$-ring $J(V_F)$ then by Proposition 4.2, it is an orthogonal projection of V_F and $\langle v \mid v\rangle \neq 0$ for any $0 \neq v \in \mathsf{im}\,\pi$. Thus, V_F is not alternate. Conversely, assume that V_F is not alternate. If $\dim V_F = 1$, then id_V is obviously a projection generating a minimal right ideal. If $\dim V_F \geqslant 2$ then, in view of Proposition 2.1, we may assume that V_F is hermitian. By Proposition 2.2, there is a non-alternate space $0 \neq U \in \mathbb{O}(V_F)$. By [20, Chapter II §2, Corollary 1], U has an orthogonal basis. Thus $U = W \oplus W'$, where $W' \subseteq W^\perp$ and $\dim W = 1$. It follows that $W \in \mathbb{O}(V_F)$ and that π_W is a projection generating a minimal right ideal of $J(V_F)$.

Now, let V_F be alternate and let π be an idempotent with $\mathsf{im}\,\pi = wF \neq 0$. Then for all $v \in V$ there is $\lambda \in F$ such that $\langle v \mid \pi^*(w)\rangle = \langle \pi(v) \mid w\rangle = \langle w\lambda \mid w\rangle = 0$, whence $\pi^*(\pi(w)) = \pi^*(w) = 0$ and thus $\pi^* \circ \pi = 0$. The claim for π^* follows since $\dim \mathsf{im}\,\pi^* = 1$; indeed, with U for π according to $(*)$, one has $\mathsf{im}\,\pi = \mathsf{im}\,\pi_U\pi\varepsilon_U$, $\mathsf{im}\,\pi^* = \mathsf{im}\,\pi_U\pi^*\varepsilon_U = (\mathsf{ker}\,\pi_U\pi\varepsilon_U)^\perp$, and $\dim U/(\mathsf{ker}\,\pi_U\pi\varepsilon_U) = 1$.

(vi) If $\varphi^*\varphi = 0$ for given $\varphi \in \mathsf{End}^*(V_F)$ then $\mathsf{im}\,\varphi \subseteq (\mathsf{im}\,\varphi)^\perp$, whence $\mathsf{im}\,\varphi = 0$ provided that V_F is anisotropic. Conversely, assume $\langle v \mid v\rangle = 0$ for some $v \neq 0$ and choose φ such that $vF = \mathsf{im}\,\varphi$, in particular $0 \neq \varphi \in J(V_F)$. Then $\mathsf{im}\,\varphi \subseteq (\mathsf{im}\,\varphi)^\perp$ and $\varphi^*\varphi = 0$. ∎

Proposition 4.5. *Any $*$-Λ-subalgebra R of $\mathsf{End}^*(V_F)$ extends to a $*$-Λ-subalgebra $\hat{R}$ of $\mathsf{End}^*(V_F)$ such that $J(V_F)$ is the unique minimal ideal of $\hat{R}$. In particular, $\hat{R}$ is strictly subdirectly irreducible and atomic with the minimal right ideals being those of $J(V_F)$. Moreover, if R is regular then $\hat{R}$ is also regular.*

Proof. The $*$-regular case is due to [44, Proposition 3.12]. Let $\hat{R} = R + J(V_F)$. Clearly, $\hat{R}$ is a subalgebra of $\mathsf{End}^*(V_F)$; thus, Proposition 4.4(iii), applies to $\hat{R}$. Finally, Propositions 3.1(ii) and 4.4(i) imply the regularity of $\hat{R}$ when R is regular. ∎

In particular, Proposition 4.5 applies to the subalgebra R of $\mathsf{End}^*(V_F)$ consisting of the endomorphisms $v \mapsto v\lambda$ (also denoted as $\lambda\,\mathsf{id}_V$), where λ is in the center $C(F)$ of F; in this case, we denote the corresponding subalgebra $\hat{R}$ by $\hat{J}(V_F)$.

Corollary 4.6. *Let $\dim V_F \geq \omega$. $\hat{J}(V_F)$ is the directed union of its $*$-Λ- subalgebras $B_U = \{\varepsilon_U\varphi\pi_U + \lambda\,\mathsf{id}_V \mid \varphi \in \mathsf{End}^*(V_F),\ \lambda \in C(F)\}$ where $U \in \mathbb{O}(V_F)$. Moreover, each B_U is regular and embeds into $\mathsf{End}^*(W_F)$, where $U \subset W \in \mathbb{O}(V_F)$.*

Proof. The first claim is obvious. Now, $B_U \cong \mathsf{End}^*(U_F) \times C(F)$, and the latter embeds into $\mathsf{End}^*(U_F) \times \mathsf{End}^*\big((U^\perp \cap W)_F\big)$, which embeds in turn into $\mathsf{End}^*(W_F)$, where $U \subset W \in \mathbb{O}(V_F)$ cf. Propositions 2.2 and 4.3. ∎

A *representation* of a $*$-Λ-algebra R within a pre-hermitian space V_F is a homomorphism $\varepsilon\colon R \to \mathsf{End}^*(V_F)$ of $*$-Λ-algebras; it is *faithful* if ε is an injective map.

In the following Theorem 4.7, existence is due to Jacobson [33, Chapter IV, §12, Theorem 2] and Kaplansky [31, Theorem 1.2.2]. Uniqueness is based on an approach via the Jacobson Density Theorem, cf. [2, Theorem 4.6.8].

Theorem 4.7. *Let R be a primitive $*$-Λ-algebra having a minimal right ideal. Then*

(i) *R is atomic with $\mathsf{Soc}(R)$ as smallest non-zero ideal;*

(ii) *R admits a faithful representation ε within some pre-hermitian space V_F such that $\varepsilon(\mathsf{Soc}(R)) = J(V_F)$. Up to similitude, the space V_F is uniquely determined by $\mathsf{Soc}(R)$.*

Proof. The proof of (i) will be a by-product of the proof of (ii). If the underlying ring of R is a division ring then a representation is given via the scalar product $\langle \lambda \mid \mu \rangle = \lambda^* \mu$. Conversely, given a representation in V_F, we have $J(V_F) \cong R$ and may assume that $R = \mathsf{End}^*(F_F)$. Up to scaling, we have F with involution ν and scalar product $\langle \lambda \mid \mu \rangle = \lambda^\nu \langle 1 \mid 1 \rangle \mu = \lambda^\nu \mu$. For the endomorphism φ_λ given by $\varphi_\lambda(\mu) = \lambda\mu$, one obtains $\lambda^\nu = \langle \varphi_\lambda(1) \mid 1 \rangle = \langle 1 \mid \varphi_\lambda^*(1) \rangle = \varphi_\lambda^*(1)$; that is, ν is determined by the involution on R.

Assume that R is not a division ring. First, we ignore the action of Λ. By [33, Chapter IV §12, Theorem 1], there are a non-degenerate sesquilinear space V_F and an embedding $\varepsilon\colon R \to \mathsf{End}^*(V_F)$ such that $\varepsilon(R) \supseteq J(V_F)$. Since $\dim V_F \geqslant 2$, Proposition 2.1 applies, whence V_F is pre-hermitian, cf. [33, Chapter IV §12, Theorem 2]. The remaining claims about R follow from Proposition 4.4.

In order to discuss the uniqueness of V_F as well as the action of Λ, we have a closer look on how R relates to the pre-hermitian space V_F, given a $*$-ring embedding $\varepsilon\colon R \to \mathsf{End}^*(V_F)$ such that ε maps $J = \mathsf{Soc}(R)$ onto $J(V_F)$.

Let e be an idempotent such that eR is a minimal right ideal, $\pi = \varepsilon(e)$, $U = \mathsf{im}\,\pi$, and $W = \mathsf{im}(\mathsf{id}_V - \pi)$; then $V = U \oplus W$ and $\dim U = 1$. Choose $0 \neq u_0 \in U$. For $\lambda \in F$, there is unique $\varphi_\lambda \in \mathsf{End}(V_F)$ such that $\varphi_\lambda(u_0) = u_0\lambda$ and $\varphi_\lambda|_W = 0$. Then $\alpha(\lambda) = \varepsilon^{-1}(\varphi_\lambda)$ defines a ring isomorphism from F onto the subring eRe of R. Moreover, one has an α-semilinear bijection ω from V_F onto the right eRe-vector space Re; it is given by $\omega(v) = \varepsilon^{-1}(\varphi_v)$, where $\varphi_v(u_0) = v$ and $\varphi_v|_W = 0$. Now, the given ring embedding $\varepsilon\colon R \to \mathsf{End}^*(V_F)$ can be described by the formula $\varepsilon(r)(v) = \omega^{-1}\big(r\omega(v)\big)$. Compare the proof of [2, Proposition 4.6.4].

If the action of Λ on F is still to be defined, put $\zeta\lambda = \alpha^{-1}\big(\zeta\alpha(\lambda)\big)$ for any $\lambda \in F$ and $\zeta \in \Lambda$. Then ε is a Λ-algebra homomorphism from R into the Λ-algebra $\mathsf{End}(V_F)$; indeed for any $\zeta \in \Lambda$, $r \in R$, and $v \in V$, one has $\omega(v) = se$ for some $s \in R$. Hence

$$\begin{aligned}\varepsilon(\zeta r)(v) = \omega^{-1}\big((\zeta r)\omega(v)\big) = \omega^{-1}(\zeta rse) = \omega^{-1}(rsee\zeta e) = \omega^{-1}\big((r\omega(v))(e\zeta e)\big)\\ = \omega^{-1}\big(r\omega(v)\big)\alpha^{-1}(e\zeta e) = \big(\varepsilon(r)(v)\big)\alpha^{-1}(e\zeta e).\end{aligned}$$

Assume that e is a projection; then π is also a projection, whence $u_0 \not\perp u_0$ and $W = U^\perp$. In view of scaling, we may assume that $\langle u_0 \mid u_0 \rangle = 1$. Thus $\varphi_\lambda^* = \varphi_{\lambda^*}$

and α is an isomorphism of $*$-rings. Also, one obtains for all v, $w \in V$:

$$\langle v \mid w\rangle = \langle \varphi_v(u_0) \mid \varphi_w(u_0)\rangle = \langle u_0 \mid \varphi_v^*(\varphi_w(u_0))\rangle = \langle u_0 \mid u_0\lambda\rangle = \lambda,$$

where $\varphi_v^*\big(\varphi_w(u_0)\big) = u_0\lambda$ for some $\lambda \in F$, as $\mathsf{im}\,\varphi_v^* = W^\perp = \mathsf{im}\,\pi$. From $\varphi_\lambda|_W = 0 = \varphi_w|_W$, it follows that $\varphi_\lambda = \varphi_v^* \circ \varphi_w$. Summarizing, the space V_F is determined, up to scaling, by the $*$-ring $J(V_F)$. Given another pre-hermitian space $V'_{F'}$ and a faithful representation $\varepsilon'\colon R \to \mathsf{End}^*(V'_{F'})$ as in the statement of the theorem, and a vector $u'_0 \in \mathsf{im}\,\varepsilon'(e)$ chosen accordingly, we have $\alpha'\colon F' \to eRe$ and $\omega'\colon V' \to Re$ providing an isomorphism $\beta = (\alpha')^{-1} \circ \alpha\colon F \to F'$ of division rings and a β-semilinear bijection $\omega' \circ \omega\colon V_F \to V'_{F'}$ which combine into an isomorphism of the sesquilinear spaces obtained from V_F and $V'_{F'}$ by scaling, thus establishing the claimed similitude.

Now, assume that R (and thus $J(V_F)$) does not have a projection generating a minimal right ideal. By Proposition 4.4(v), V_F is an alternate space; in particular $\lambda = \lambda^*$ for all $\lambda \in F$, F is commutative, and $\pi \circ \pi^* = 0 = \pi^* \circ \pi$ for any idempotent π generating a minimal right ideal in $J(V_F)$. Choose $\pi = \varepsilon(e)$. It follows that

$$\mathsf{im}\,\pi \cap \mathsf{im}\,\pi^* = \mathsf{im}\,\pi \cap (\mathsf{im}\,\pi^*)^\perp = \mathsf{im}\,\pi^* \cap (\mathsf{im}\,\pi)^\perp = 0.$$

Also $U' = \mathsf{im}\,\pi \oplus \mathsf{im}\,\pi^* \in \mathbb{O}(V_F)$ and $W = \mathsf{im}\,\pi^* + U'$. Thus for any $v \in V$, $\mathsf{im}\,\varphi_v^* = W^\perp = \mathsf{im}\,\pi^*$. For $\psi \in \mathsf{End}(V_F)$, we have $\psi(\mathsf{im}\,\pi^*) = \mathsf{im}\,\pi$ and $\mathsf{ker}\,\psi = U' + \mathsf{im}\,\pi$ if and only if $\pi \circ \psi = \psi = \psi \circ \pi^*$. For any such endomorphism ψ there is unique $0 \neq \mu \in F$ such that $\psi(u_1) = u_0\mu$, and vice versa. Besides that, $\mathsf{im}\,\psi^* = (U' + \mathsf{ker}\,\pi)^\perp = \mathsf{im}\,\pi$. Choose u_1 such that $u_1F = \mathsf{im}\,\pi^*$ and $\langle u_1 \mid u_0\rangle = 1$. Choosing μ (and ψ), one has for any v, $w \in V$

$$\begin{aligned}\langle v \mid w\rangle &= \langle \varphi_v(u_0) \mid \varphi_w(u_0)\rangle = \langle \varphi_v(\psi(u_1\mu)) \mid \varphi_w(u_0)\rangle \\ &= \langle u_1\mu \mid \psi^*(\varphi_v^*(\varphi_w(u_0)))\rangle = \langle u_1 \mid u_0\rangle\mu\sigma = \mu\sigma,\end{aligned}$$

where $\psi^*(\varphi_v^*(\varphi_w(u_0))) = u_0\sigma$. Therefore, $\langle v \mid w\rangle = \mu\sigma$ if and only if $\psi^* \circ \varphi_v^* \circ \varphi_w = \varphi_\sigma$. Uniqueness of V_F up to similitude follows as above, with the additional choice of u'_1 and $\mu' = \mu$, cf. [2, Proposition 4.6.6]. ■

Remark 4.8. For a primitive ring R with a minimal right ideal, according to Kaplansky (cf. Corollary 4.3.4 and Theorem 4.6.2 in [2]), there is an idempotent $e \in R$ such that eR and e^*R are minimal right ideals and either $e = e^*$ or $ee^* = 0 = e^*e$. Given such an idempotent $e \in R$, the representation of Theorem 7.6 can be directly obtained from the Jacobson Density Theorem in the context of non-empty socle, cf. [2, Theorem 4.6.2]. In the first case, eRe is a $*$-subring of R. In the second case,

$$S = \{\lambda + \lambda^* \mid \lambda \in eRe\}$$

is a $*$-subring of R and $r^* = r$ for any $r \in S$. Moreover, the map $\lambda \mapsto \lambda + \lambda^*$ is a ring isomorphism from eRe onto S with inverse $\mu \mapsto e\mu$. One has $v\lambda = v(\lambda + \lambda^*)$ for all $v \in Re$ and all $\lambda \in eRe$. Thus in both cases, $\langle v \mid w \rangle = ev^*w$ provides the required scalar product on Re.

Proposition 4.9. *A $*$-Λ-algebra R is strictly simple artinian and regular if and only if $R \cong \mathsf{End}^*(V_F)$, where V_F is a pre-hermitian space and $\dim V_F < \omega$. Moreover, V_F is uniquely determined by R up to similitude.*

Proof. Having a unit, R is noetherian and $J(V_F) = \mathsf{End}^*(V_F)$, cf. [38, §3.3.5, Proposition 3]. Thus, this follows from Theorem 4.7. ■

5. Lattices with a Galois operator

We focus on lattices with Galois operator arising in Orthogonal Geometry, cf. [20, Chapter I §9], [21], and [24, §2]. For basics on modular lattices, we refer to [11, §3-4, §10, §13], alternatively [19, Chapter V §1, §5]. We consider *lattices* as algebraic structures with two binary operations, $\cdot$ (*meet*) and $+$ (*join*); that is, for a suitable (unique) partial order $\leq$, $ab = a \cdot b = \inf\{a, b\}$, $a + b = \sup\{a, b\}$. A lattice L is *modular* if for all a, b, $c \in L$,

$$a \geq c \text{ implies } a(b + c) = ab + c.$$

It is well known that the class of [modular] lattices can be defined by equations. If L has a smallest element 0 and if $ab = 0$, then we write $a \oplus b$ instead of $a + b$.

A *sublattice* of L is a subset of L closed under meets and joins and a lattice (modular if so is L) endowed with the restrictions of these operations; for example the *intervals* $[u, v] = \{x \in L \mid u \leq x \leq v\}$. A *homomorphism* $\varphi\colon L \to M$ between lattices is a map such that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all a, $b \in L$. A *congruence* (*relation*) on a lattice L is an equivalence relation θ which is compatible with meet and join, that is $a\,\theta\,b$ and $c\,\theta\,d$ jointly imply $ac\,\theta\,bd$ and $(a + c)\,\theta\,(b + d)$. If $\varphi\colon L \to M$ is a homomorphism then $a\,\theta\,b \Leftrightarrow \varphi(a) = \varphi(b)$ defines a congruence on L; any congruence arises this way with surjective φ and if L is modular so is M. A lattice L is *subdirectly irreducible* if it has a smallest non-trivial congruence μ, which is called the *monolith* of L.

A modular lattice L has *dimension* $n < \omega$, (which is denoted by $\dim L$), if L has $(n + 1)$-element maximal chains. If $a \in L$ and L has a smallest element 0, we put $\dim a = \dim[0, a]$ if that exists; we call a an *atom* if $\dim a = 1$. If $\dim a$ and $\dim b$ exist, then the well-known dimension formula applies:

$$\dim a + \dim b = \dim(ab) + \dim(a + b).$$

A *bounded lattice* has a smallest element $0 = \inf L$ and a greatest element $1 = \sup L$ which are considered as constants. A *dual atom* or *coatom* of L is an element c such that $\dim[c, 1] = 1$. A bounded lattice L is *complemented* if for any $a \in L$, there is $b \in L$ such that $a \oplus b = 1$. In a complemented modular lattice (a CML for short) L, any interval $[u, v]$ is complemented, too; L is *atomic* if for any $a > 0$ there is an atom $p \leq a$. It follows that for elements a and b of an atomic CML L, $a \nleq b$ in L if and only if there is an atom $p \in L$ such that $p \leq a$ and $p \nleq b$. Primary examples of atomic CMLs are the lattices of all subspaces of vector spaces, see Proposition 6.1 below.

Lattices relevant in orthogonal geometry also support an operation $X \mapsto X^\perp$, the Galois correspondence induced by the orthogonality relation, see [20]. This is captured by the following concept (cf. [21,24]). A *Galois lattice* is a bounded lattice L endowed with an additional operation $x \mapsto x'$ such that $x \leq y'$ implies $y \leq x'$ for any x, $y \in L$ and such that $1' = 0$. It is well known and easy to prove that

$x \leq x''$;
$x \leq y$ implies $y' \leq x'$;
$x''' = x'$;
$0' = 1$;
$(x + y)' = x'y'$

for any x, $y \in L$. For an equational definition, see [24, IV.2.5]. A *Galois sublattice* S of a Galois lattice L is a sublattice of L such that $0, 1 \in S$ and $x' \in S$ for any $x \in S$; thus, it is a Galois lattice with the inherited operations. Similarly, a homomorphism $\varphi\colon L \to M$ between Galois lattices L and M is a lattice homomorphism from L to M preserving 0, 1 and such that $\varphi(x') = \big(\varphi(x)\big)'$ for all $x \in L$. Also, a lattice congruence θ on L is a *Galois lattice congruence* if $a \,\theta\, b$ implies $a' \,\theta\, b'$ for all a, $b \in L$.

A *polarity lattice* is a Galois lattice L such that p' is a dual atom of L for any atom $p \in L$.[1] Note that the class of modular polarity lattices is not closed under substructures. Indeed, consider a subspace X of a Hilbert space H such that $X \neq X^c$, where X^c denotes the closure of X. Then 0, X, X^c, $X^\perp$, $X + X^\perp$, H form a subalgebra of the polarity lattice associated with H, cf. Proposition 7.1 below. It remains open whether the class of CML polarity lattices is closed under complemented subalgebras.

A Galois lattice L is a *lattice with involution* if, in addition, $x'' = x$ for all $x \in L$; equivalently, if $x \mapsto x'$ is a dual automorphism of order 2 of the lattice L; in particular, a lattice with involution is a polarity lattice. Furthermore, L

[1] The referee pointed out the need for such a concept and provided the smallest example of a Galois CML which is not a polarity lattice: the 4-element CML with $x' = 0$ for $x \neq 0$ and $0' = 1$.

is an *ortholattice* if, in addition, the involution satisfies the identity $xx' = 0$ (or equivalently, $x+x' = 1$). We write MIL [CMIL] for a [complemented] modular lattice with involution and MOL for a modular ortholattice. We use each abbreviation also to denote the class of *all* Galois lattices with the corresponding property. Observe that $\dim u = \dim[u', 1]$ in any MIL. Also, observe that in a CMIL, in general, x' fails to be a complement of x, cf. the 4-element CML with $p = p'$ for each atom. The following statement is straightforward to prove.

Lemma 5.1. *Let L_0, L_1 be lattices with involution.*

(i) *A map $\varphi\colon L_0 \to L_1$ is a homomorphism, if $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x') = \varphi(x)'$ for all x, $y \in L_0$, and $\varphi(0) = 0$.*

(ii) *A subset $X \subseteq L_0$ is a Galois sublattice of L_0, if $0 \in X$ and X is closed under the operations $+$ and $'$.*

For a modular polarity lattice L, let $L_{\mathsf{f}} = F \cup \{u' \mid u \in F\}$, where $F = \{u \in L \mid \dim u < \omega\}$.

Proposition 5.2. *If L is a polarity CML then L_{f} is an atomic Galois sublattice of L; moreover, L_{f} is a CMIL which is the directed union of its subalgebras $[0, u] \cup [u', 1]$, where $\dim u < \omega$ and $u \oplus u' = 1$ (which are all CMILs). If L is a CMIL, then $L_{\mathsf{f}} = \{a \in L \mid \dim a < \omega$ or $\dim[a, 1] < \omega\}$.*

Proof. We claim that $\dim[v', u'] \leqslant \dim[u, v]$ if the latter is finite. Indeed, if $\dim[u, v] = 1$ then $v = u + p$, where p is a complement of u in $[0, v]$, whence an atom and so $v' = u'p'$ is a lower cover of u' unless $v' = u'$. The claim now follows by induction, Now, assume $\dim u < \omega$. Then $\dim u'' \leq \dim[u', 1] \leq \dim u$, whence $u'' = u$. Consequently, $x \mapsto x'$ provides a pair of mutually inverse lattice anti-isomorphisms between the intervals $[0, u]$ and $[u', 1]$ of L; in particular, $[u', 1] \subseteq L_{\mathsf{f}}$. Since $\{u \in L \mid \dim u < \omega\}$ is closed under joins and $0 \in L_{\mathsf{f}}$, L_{f} is a Galois sublattice of L by Lemma 5.1(ii) and, in particular, L_{f} is a MIL and atomic since it contains all atoms of L.

If $X \subseteq L_{\mathsf{f}}$ is finite, then there is $u \in L$ such that $\dim u < \omega$ and $X = Y \cup Z$, where y, $z' \in [0, u]$ for all $y \in Y$, $z \in Z$. Choose v as a complement of $u + u'$ in $[u, 1]$. Then $\dim[u, v] = \dim[u + u', 1] \leqslant \dim[u', 1] = \dim u < \omega$ whence $\dim v < \omega$ We have u, u', $v \in L_{\mathsf{f}}$. Therefore, $u = v(u + u')$ implies $u' = v' + uu'$. It follows that $v + v' = v + u + uu' + v' = v + u + u' = 1$ and $vv' = (v + v')' = 1' = 0$ whence $X \subseteq [0, v] \cup [v', 1]$. This proves the second statement.

If L is a CMIL and $\dim[a, 1] < \omega$ then $\dim a' = \dim[a, 1] < \omega$, thus $a' \in L_{\mathsf{f}}$ and $a = a'' \in L_{\mathsf{f}}$.

■

For a lattice congruence θ on a MIL L, we put $a\,\theta'\,b$ if and only if $a'\,\theta\,b'$. Then θ' is also a lattice congruence on L and the Galois lattice congruences on L are exactly the lattice congruences θ on L such that $\theta = \theta'$. We call an MIL *strictly subdirectly irreducible* if the underlying lattice is subdirectly irreducible; in that case, one has $\mu = \mu'$ for the lattice monolith μ. Similarly, the MIL is *strictly simple* if the underlying lattice is simple. In the case of MOLs, one has $\theta = \theta'$ for all lattice congruences θ; thus subdirectly irreducible MOLs [simple MOLs] are strictly subdirectly irreducible [strictly simple, respectively]. The following is well known.

Proposition 5.3. *A subdirectly irreducible CML L is atomic provided it contains an atom. If L is, in addition, a CMIL with lattice monolith μ, then one has $a \in L_{\mathsf{f}}$ iff $a\,\mu\,0$ or $a\,\mu\,1$. In particular, L_{f} is strictly subdirectly irreducible and atomic, too.*

Proof. Let p be an atom in L. By modularity, the smallest lattice congruence μ such that $0\,\mu\,p$ is a minimal lattice congruence. Thus given $a > 0$, one has $0\,\theta\,p$ in the smallest lattice congruence such that $0\,\theta\,a$, whence by modularity, the quotient $p/0$ is projective to some subquotient c/d of $a/0$. Then any complement q of d in $[0, c]$ is an atom. Thus L is atomic and it follows that $x\,\mu\,y$ iff $\dim[xy, x + y] < \omega$. In view of Proposition 5.2, we are done. ∎

The next proposition associates a CMIL $\mathbb{L}(R)$ with a regular $*$-Λ-algebra R.

Proposition 5.4.

(i) *The principal right ideals of a regular ring R, possibly without unit, form a sublattice $L(R)$, containing 0, of the lattice of all right ideals of R; $L(R)$ is sectionally complemented and modular. In the case with unit, $L(R)$ is a CML with top element R.*

(ii) *For any regular [$*$-regular] $*$-Λ-algebra R, the CML $L(R)$ becomes a CMIL [MOL, respectively] endowed with the involution $eR \mapsto (eR)' := (1 - e^*)R$, where e is an idempotent [a projection, respectively]; we denote it by $\mathbb{L}(R)$.*

(iii) *For any regular [$*$-regular] $*$-Λ-algebras R_i, $i \in I$, and $R = \prod_{i \in I} R_i$ one has $\mathbb{L}(R) \cong \prod_{i \in I} \mathbb{L}(R_i)$.*

(iv) *If $\varepsilon\colon R \to S$ is a homomorphism and R, S are regular rings, then $\bar{\varepsilon}\colon L(R) \to L(S)$, $\bar{\varepsilon}\colon aR \mapsto \varepsilon(a)S$ is a lattice homomorphism preserving 0 and 1. If ε is injective, then so is $\bar{\varepsilon}$; if ε is surjective, then so is $\bar{\varepsilon}$. If R and S are regular $*$-Λ-algebras and ε a homomorphism of $*$-Λ-algebras, then $\bar{\varepsilon}\colon \mathbb{L}(R) \to \mathbb{L}(S)$ is a homomorphism of MILs.*

In Proposition 5.4(ii), one can consider the preorder $e \le f$ iff $fe = e$ on the set of idempotents of R and obtain the lattice $\mathbb{L}(R)$ factoring by the equivalence relation $e \sim f$ iff $e \leq f \leq e$; the involution is given by $e \mapsto 1 - e^*$. In the $*$-regular case, any of the equivalence classes contains a unique projection so that $\mathbb{L}(R)$ is also called the *projection* [*ortho*]*lattice* of R.

Recalling Proposition 4.4 for a pre-hermitian space V_F, the principal right ideals of $J(V_F)$ form an atomic sectionally complemented sublattice of the lattice of all right ideals of $J(V_F)$, which is isomorphic to the lattice of finite-dimensional subspaces of V_F via the map $\varphi\colon J(V_F) \mapsto \mathsf{im}\,\varphi$.

Proof. These results originate from [13]. For the proof of (i)–(ii), see [51, §8-3.3.13]. For R a regular $*$-Λ-algebra, the map $eR \mapsto R(1-e) \mapsto (1-e^*)R$ combines a dual isomorphism of $L(R)$ onto the lattice of principal left ideals with an isomorphism of the latter onto $L(R)$. For the proof of (iii)–(iv), see [51, §8-3.3.14–15]. ∎

6. Projective spaces and orthogeometries

Synthetic geometries are a convenient link between lattice and vector space structures. We follow [12, Chapter 2]. A *projective space* P is a set, whose elements are called *points*, endowed with a ternary relation $\Delta \subseteq P^3$ of *collinearity* satisfying the following conditions:

(i) if $\Delta(p_0, p_1, p_2)$, then $\Delta(p_{\sigma(0)}, p_{\sigma(1)}, p_{\sigma(2)})$ and $p_{\sigma(0)} \neq p_{\sigma(1)}$ for any permutation σ on the set $\{0,1,2\}$;

(ii) if $\Delta(p_0, p_1, a)$, $\Delta(p_0, p_1, b)$, and $a \neq b$, then $\Delta(p_0, a, b)$;

(iii) if $\Delta(p, a, b)$ and $\Delta(p, c, d)$, then $\Delta(q, a, c)$ and $\Delta(q, b, d)$ for some $q \in P$.

The space P is *irreducible* if for any $p \neq q$ in P there is $r \in P$ such that $\Delta(p, q, r)$. A set $X \subseteq P$ is a *subspace* of P if p, $q \in X$ and $\Delta(p, q, r)$ together imply that $r \in X$.

Any projective space P is the disjoint union of its irreducible subspaces P_i, $i \in I$, which are called its *components*. The set $L(P)$ of all subspaces of an [irreducible] projective space P is a complete [subdirectly irreducible] atomic CML, in which the atoms are the subspaces $\{p\}$, $p \in P$, all of which are compact. Moreover, $L(P) \cong \prod_{i \in I} L(P_i)$ via the map $X \mapsto (X \cap P_i \mid i \in I)$. Conversely, any complete atomic CML L with compact atoms is isomorphic to the lattice $L(P_L)$ via the map $a \mapsto \{p \in P_L \mid p \leq a\}$, where P_L is the set of atoms of L and distinct points p, q, $r \in P_L$ are collinear if and only if $r < p + q$. Recall that Jónsson's *Arguesian* lattice identity [34] holds in $L(P)$ if and only if P is desarguean. For a vector space V_F, let $L(V_F)$ denote the lattice of all linear subspaces of V_F.

Proposition 6.1.
(i) *For any vector space V_F, $L(V_F)$ is a CML. Moreover, there exists an irreducible desarguean projective space P such that $L(V_F) \cong L(P)$.*
(ii) *For any irreducible desarguean projective space P with $\dim L(P) > 2$, there is a vector space V_F such that $L(P) \cong L(V_F)$; F is unique up to isomorphism, V_F up to semilinear bijection.*
(iii) *If P is irreducible and $\dim L(P) > 3$, then P is desarguean.*
(iv) *Any subdirectly irreducible CML of dimension at least 4 is Arguesian.*

Proof. Claim (i) is the content of [12, Proposition 2.4.15]. For (ii), see [12, Proposition 2.5.6] and [12, Chapter 9]. For (iii), see [11, Chapter 13]. As to claim (iv), according to Frink [14], any CML L embeds into $L(P)$ for some projective space P. Since L is subdirectly irreducible as a lattice, it embeds into $L(P_i)$ for some irreducible component P_i of P, which is desarguean since $\dim L(P_i) > 3$, whence statement (iv) follows. ■

Proposition 6.2. *Let P be a projective space. There is a one-to-one correspondence between maps $X \mapsto X^\perp$ turning $L(P)$ into a Galois CML $\mathbb{L}(P, \perp)$ on one side and, on the other side, symmetric binary relations $\perp$ on P such that*
(a) *for any p, q, r, $s \in P$, if $p \perp q$, $p \perp r$, and $\Delta(q, r, s)$ then $p \perp s$;*
(b) *for any $p \in P$ there is $q \in P$ with $p \not\perp q$.*
The correspondence is given by

$$X^\perp = \{q \in P \mid q \perp p \textit{ for all } p \in X\} \quad nd \quad p \perp q \quad \textit{if and only if} \quad p \in \{q\}^\perp.$$

Given such a relation $\perp$, the following statements are equivalent.
(i) *The Galois CML $\mathbb{L}(P, \perp)$ is a polarity CML.*
(ii) *For all p, q, $r \in P$, if $p \neq q$, then $r \perp t$ for some $t \in P$ such that $\Delta(p, q, t)$.*
(iii) *Same as* (ii) *with additional hypotheses $r \not\perp p$ and $r \not\perp q$.*

A pair $(P, \perp)$, where P is a projective space and $\perp$ is a symmetric relation, satisfying (a)–(b) and (i)–(iii) of Proposition 6.2, is called an *orthogeometry*. Compare [12, Definition 14.1.1] and [22, §4]. $\mathbb{L}(P, \perp)$ is defined, accordingly.

Proof. Let $\perp$ be a symmetric relation on P satisfying (a)–(b). It follows that $X^\perp \in L(P)$ by (a), while (b) and symmetry of $\perp$ yield that $\mathbb{L}(P, \perp)$ is a Galois lattice. The converse is obvious.

Let (i) hold. If $\dim X = 2$ and $r \in P$, then $X \cap \{r\}^\perp \neq \varnothing$ by modularity, which proves (ii), cf. [12, Remark 14.1.2]. Statement (iii) is a special case of (ii); it implies that $\{r\}^\perp$ is a coatom. Indeed, according to (b), there is $p \in P$ with $p \not\perp r$. Now, if $q \in P$, $q \neq p$, and $q \notin \{r\}^\perp$, then one has $\Delta(p,q,t)$ for some $t \in \{r\}^\perp$, whence $q \in \{p\} + \{r\}^\perp$. This proves that $\{p\} + \{r\}^\perp = P \neq \{r\}^\perp$ and, by modularity, that $\{r\}^\perp$ is a coatom of $L(P)$. ■

Let P_L denote the set of atoms of a MIL L. We define a collinearity on P_L by putting $\Delta(p,q,r)$ for distinct atoms p, q, $r \in P_L$ such that $p \leq q + r$ in L. Furthermore, we put $p \perp q$ if $p \leq q'$.

Proposition 6.3. ([22, Lemma 4.2]) *For any MIL L, $\mathbb{G}(L) = (P_L, \perp)$ is an orthogeometry.*

Proposition 6.4. *For any orthogeometry $(P, \perp)$, the Galois lattice $L = \mathbb{L}(P,\perp)_{\mathsf{f}}$, consisting of all $X, X^\perp$ with $X \in L(P)$ and $\dim X < \omega$, is a CMIL with $L = L_{\mathsf{f}}$. Conversely, for any CMIL L with $L = L_{\mathsf{f}}$, one has $L \cong \mathbb{L}(\mathbb{G}(L))_{\mathsf{f}}$.*

Proof. See [22, Theorem 1.1] and Proposition 5.2. ■

7. Subspace lattices

For a pre-hermitian space V_F, let $\mathbb{G}(V_F) = (P, \perp)$, where $P = \{vF \mid 0 \neq v \in V\}$ and $vF \perp wF$ if and only if $v \perp w$. The following proposition relates any pre-hermitian space V_F with $\mathbb{G}(V_F)$ and $\mathbb{L}(V_F)$.

Proposition 7.1. *Let V_F be a pre-hermitian space.*

(i) *The CML of all linear subspaces endowed with the unary operation $U \mapsto U^\perp$ becomes a polarity lattice $\mathbb{L}(V_F)$.*
(ii) *$\mathbb{G}(V_F)$ is an orthogeometry.*
(iii) *The map $U \mapsto \{vF \mid 0 \neq v \in U\}$ defines an isomorphism from $\mathbb{L}(V_F)$ onto $\mathbb{L}\big(\mathbb{G}(V_F)\big)$.*

Proof. Since $X^\perp$ is a subspace for any $X \subseteq V$, (a) of Proposition 6.2 is satisfied, while (b) follows from the fact that V_F is non-degenerate. Thus $\mathbb{L}(V_F)$ is a Galois CML. Observe that for any $u \in V$, $f_u(v = \langle u \mid v\rangle$ is a linear map from V_F to F_F. Therefore, $uF^\perp = \ker f_u$ is a coatom and $\mathbb{L}(V_F)$ is a polarity lattice. The map $\mathbb{L}(V_F) \to \mathbb{L}\big(\mathbb{G}(V_F)\big)$ given in (iii) is obviously an isomorphism. Thus $\mathbb{G}(V_F)$ is an orthogeometry. See also [12, Proposition 14.1.6]. ■

Proposition 7.2. *Let V_F be a pre-hermitian space.*

(i) *The polarity lattice $\mathbb{L}(V_F)_{\mathsf{f}}$ is the directed union of its Galois sublattices $[0,U] \cup [U^\perp, V]$, $U \in \mathbb{O}(V_F)$. Moreover, for any $U \in \mathbb{O}(V_F)$, there is a Galois lattice embedding from $[0,U] \cup [U^\perp, V]$ into $\mathbb{L}(W_F)$ for some $W \in \mathbb{O}(V_F)$.*

(ii) *The Galois lattices* $\mathbb{L}\big(\mathbb{G}(V_F)\big)_{\mathsf{f}}$ *and* $\mathbb{L}(V_F)_{\mathsf{f}}$ *are isomorphic and strictly subdirectly irreducible Arguesian CMILs; if* V_F *is anisotropic, then* $\mathbb{L}(V_F)_{\mathsf{f}}$ *is a MOL.*

(iii) *For any strictly subdirectly irreducible Arguesian CMIL* L *of dimension at least* 3 *such that* $L = L_{\mathsf{f}}$*, there is a (unique up to similitude) pre-hermitian space* V_F *such that* $L \cong \mathbb{L}(V_F)_{\mathsf{f}}$*; if* L *is a MOL, then* V_F *is anisotropic.*

(iv) *If* $\dim V_F < \omega$ *then* $\mathbb{L}(V_F)$ *is a MIL.*

For cardinality reasons, the requirement $\dim V_F < \omega$ is also necessary in (iv); see also [37].

Proof. The first claim in (i) and the fact that $[0,U] \cup [U^\perp, V] \cong \mathbb{L}(U_F) \times \mathbf{2}$ whenever $U \neq V$ follow from Propositions 5.2 and 7.1. Moreover, this Galois lattice is isomorphic to the Galois sublattice $[0,U] \cup [U^\perp \cap W, W]$ of $\mathbb{L}(W_F)$, where the subspace W is such that $U \subset W \in \mathbb{O}(V_F)$, according to Proposition 2.2.

To prove (ii), we notice first that $\mathbb{L}\big(\mathbb{G}(V_F)\big)_{\mathsf{f}}$ is a CMIL by Propositions 7.1 and 6.4. Moreover, as a sublattice of $\mathbb{L}(V_F)$, $\mathbb{L}(V_F)_{\mathsf{f}}$ is an Arguesian lattice. Strict subdirect irreducibility of $\mathbb{L}\big(\mathbb{G}(V_F)\big)_{\mathsf{f}}$ follows from Fact 7.1, [12, Example 2.7.2], and [22, Corollary 1.5]. Furthermore, if V_F is anisotropic, then $X^\perp$ is an orthocomplement of X for any $X \in \mathbb{L}(V_F)$ with $\dim X < \omega$.

We prove now (iii). By [22, Corollary 1.5], there is an irreducible orthogeometry $(P, \perp)$ such that $L \cong \mathbb{L}(P,\perp)_{\mathsf{f}}$. Combining Proposition 6.1(ii) and [12, Theorem 14.1.8], one gets a pre-hermitian space V_F such that $\mathbb{L}(P,\perp)_{\mathsf{f}} \cong \mathbb{L}(V_F)_{\mathsf{f}}$. For uniqueness, see [12, Theorem 14.3.4] or [20, p. 33]. If L is a MOL, then V_F is obviously anisotropic.

If $\dim V_F < \omega$, then $\mathbb{L}(V_F) = \mathbb{L}(V_F)_{\mathsf{f}}$ is a MIL by Propositions 7.1 and 5.2. ■

Proposition 7.3. *Any Galois sublattice* L *of* $\mathbb{L}(V_F)$ *which is a MIL extends to a Galois sublattice* $\hat{L}$ *of* $\mathbb{L}(V_F)$ *which is a MIL and such that* $\hat{L}_{\mathsf{f}} = \mathbb{L}(V_F)_{\mathsf{f}}$*. In particular,* $\hat{L}$ *is a strictly subdirectly irreducible atomic MIL. Moreover, if* L *is a CMIL then* $\hat{L}$ *is a CMIL.*

Proof. Existence of $\hat{L}$ with the required properties follows from the proof of [22, Theorem 2.1]. In particular, $\hat{L}$ is atomic. Strict subdirect irreducibility of $\hat{L}$ follows from [22, Corollary 1.5], see also Proposition 7.2(ii). For a first such construction see [8]. ■

A *representation* of a MIL (or CMIL) L in V_F is a homomorphism $\varepsilon\colon L \to \mathbb{L}(V_F)$ of Galois lattices. It is *faithful* if it is injective, i.e. an embedding; in this case, we usually identify L with its image in $\mathbb{L}(V_F)$. A map $\varepsilon\colon L \to \mathbb{L}(V_F)$ is a representation if and only if it preserves joins, involution, and the least element.

Lemma 7.4. *Let ε be a representation of a MIL L in a pre-hermitian space V_F.*

(i) *Any element in the image of ε is closed.*

(ii) *If ε is faithful and V_F is anisotropic, then L is a MOL.*

Proof. Let $x \in L$ be arbitrary.

(i) We have $\varepsilon(x) = \varepsilon(x'') = \varepsilon(x')^\perp = \varepsilon(x)^{\perp\perp}$.

(ii) If V_F is anisotropic, then we have $\varepsilon(xx') = \varepsilon(x) \cap \varepsilon(x)^\perp = 0$. As ε is faithful, we conclude that $xx' = 0$. Hence $'$ is an orthocomplementation. ∎

A *representation* of a MIL L within an orthogeometry $(P, \perp)$ is a homomorphism $\eta\colon L \to \mathbb{L}(P, \perp)$. The following obvious fact relates the two concepts of a representation.

Proposition 7.5. *For a MIL L, ε is a [faithful] representation in V_F if and only if the mapping $\eta\colon a \mapsto \{p \in P_V \mid p \subseteq \varepsilon(a)\}$ is a [faithful] representation of L in the orthogeometry $\mathbb{G}(V_F)$.*

Theorem 7.6. *Let L be an Arguesian strictly subdirectly irreducible CMIL [MOL] such that $\dim L > 2$ and L has an atom. Then L admits a faithful representation ε within some [anisotropic] pre-hermitian space V_F such that ε induces a bijection between the sets of atoms of L and of $\mathbb{L}(V_F)$. In particular, ε restricts to an isomorphism from L_f onto $\mathbb{L}(V_F)_\mathsf{f}$. The space V_F is unique up to similitude.*

Proof. By Proposition 5.3, L is atomic and L_f is strictly subdirectly irreducible and atomic. Moreover by Proposition 7.2(iii), $L_\mathsf{f} \cong \mathbb{L}(V_F)_\mathsf{f}$ for some [anisotropic] pre-hermitian space V_F which is unique up to isomorphism and scaling. By definition and Proposition 6.4, $\mathbb{G}(L) = \mathbb{G}(L_\mathsf{f}) \cong \mathbb{G}(V_F)$. By [22, Lemma 10.4], L has a faithful representation within the orthogeometry $\mathbb{G}(L)$, whence in the orthogeometry $\mathbb{G}(V_F)$. The proof is done in view of Proposition 7.5. ∎

The next fact is a corollary of Theorem 7.6 which is in principle already in [7].

Proposition 7.7. *A lattice L is a strictly simple Arguesian CMIL of finite dimension $n > 2$ if and only if $L \cong \mathbb{L}(V_F)$, where V_F is a pre-hermitian space with $\dim V_F = n$. The space V_F is determined by L up to similitude; V_F is anisotropic, iff L is a MOL.*

Proposition 7.8.

(i) *If* ε *is a faithful representation of the regular* $*$-Λ*-algebra* R *in a pre-hermitian space* V_F*, then the map* $\eta\colon aR \mapsto \mathsf{im}\,\varepsilon(a)$ *defines a faithful representation of* $\mathbb{L}(R)$ *in* V_F*.*

(ii) *If* $\dim V_F < \omega$ *then* $\mathbb{L}(V_F) \cong \mathbb{L}(\mathsf{End}^*(V_F))$*.*

Proof. (i) We refer to [15]. We may assume that $R \subseteq \mathsf{End}^*(V_F)$; that is, ε is the inclusion map. By Propositions 3.1(i) and 5.4(iv), η is a 0 and 1 preserving lattice embedding of $L(R)$ into $L(V_F)$. Moreover, for any $v \in V$ and an idempotent $\varphi \in R$, one has $v \in \big(\eta(\varphi R)\big)^\perp = (\mathsf{im}\,\varphi)^\perp$ iff $\langle \varphi^*(v) \mid w\rangle = \langle v \mid \varphi(w)\rangle = 0$ for all $w \in V$, iff $\varphi^*(v) = 0$, iff $v = (\mathsf{id}_V - \varphi^*)(v)$, iff $v \in \mathsf{im}(id_V - \varphi^*) = \eta\big((\varphi R)'\big)$, whence η preserves the involution.

(ii) By (i) and Proposition 4.3(ii), the identical map ε on $\mathsf{End}^*(V_F)$ defines a faithful representation of $\mathbb{L}(V_F)$. It is surjective since any subspace is the image of some endomorphism $\varphi \in \mathsf{End}^*(V_F)$, cf. also Proposition 5.4(iv). ∎

8. Representations as multi-sorted structures

Given a commutative $*$-ring Λ, let $\mathcal{F}_\Lambda$ denote the class of all division rings with involution which are $*$-Λ-algebras. Unless stated otherwise, any pre-hermitian space V_F, where $F \in \mathcal{F}_\Lambda$, is dealt with as a 2-sorted structure with sorts V and F. That is, V carries the structure of an abelian group and F the structure of a ring with involution ν and with a unary operation $\lambda \mapsto \zeta\lambda$ associated to each $\zeta \in \Lambda$. Moreover, one has the maps $V \times F \to V$ with $(v, \lambda) \mapsto v\lambda$ and $V \times V \to F$ with $(v, w) \mapsto \langle v \mid w\rangle$.

In general, a *similarity type* for an n-sorted algebraic structure has a list $S_1, \ldots,$ S_n of names for sorts, a list of typed operation symbols $f\colon S_{j_1} \times \cdots \times S_{j_{k_f}} \to S_{j_{k_f+1}}$, and a list of typed relation symbols $R \subseteq S_{j_1} \times \cdots \times S_{j_{k_R}}$. A *structure* A of this type is a family $S_1^A, \ldots, S_n^A$ of sets together with a map $f^A\colon S_{j_1}^A \times \cdots \times S_{j_{k_f}}^A \to S_{j_{k_f+1}}^A$ for each operation symbol f and with a set $R^A \subseteq S_{j_1}^A \times \cdots \times S_{j_{k_R}}^A$ for each relation symbol R.

Recall the notion of an *ultrafilter* over a set I. A set $\mathcal{U}$ of subsets of I which is maximal with the following properties:

(i) $\varnothing \notin \mathcal{U}$;

(ii) $U \cap V \in \mathcal{U}$ for any $U, V \in \mathcal{U}$;

(iii) if $U \subseteq V \subseteq I$ and $U \in \mathcal{U}$, then $V \in \mathcal{U}$;

in particular for any $U \subseteq I$, either $U \in \mathcal{U}$ or $I \backslash U \in \mathcal{U}$. Given n-sorted structures A_i, $i \in I$, of a fixed sorted similarity type and any ultrafilter $\mathcal{U}$ over I, for each sort S_j, one has an equivalence relation $\equiv_{S_j}$ on the direct product $\prod_{i \in I} S_j^{A_i}$ of sets such that

$$(a_i \mid i \in I) \equiv_{S_j} (b_i \mid i \in I) \text{ if and only if} \\ \text{there is } U \in \mathcal{U} \text{ such that } a_i = b_i \text{ for all } i \in U.$$

The equivalence classes $[a_i \mid i \in I]_{S_j}$ are the elements of the ultraproduct $S_j^A = \prod_{i \in I} S_j^{A_i}/\mathcal{U}$ of the sort S_j. One defines the relations and operations on the *ultraproduct* $A = \prod_{i \in I} A_i/\mathcal{U}$ as follows:

$$\big([a_i^{j_1} \mid i \in I]_{S_{j_1}}, \ldots, [a_i^{j_{k_f}} \mid i \in I]_{S_{j_{k_R}}}\big) \in R^A \text{ if and only if} \\ \text{there is } U \in \mathcal{U} \text{ such that } (a_i^{j_1}, \ldots, a_i^{j_{k_R}}) \in R^{A_i} \text{ for all } i \in U$$

for each relation symbol R (of the type as above) and

$$f^A\big([a_i^{j_1} \mid i \in I]_{S_{j_1}}, \ldots, [a_i^{j_{k_f}} \mid i \in I]_{S_{j_{k_f}}}\big) = \big[f^{A_i}(a_i^{j_1}, \ldots, a_i^{j_{k_f}}) \mid i \in I\big]_{S_{j_{k_f+1}}}$$

for each operation symbol f (of the type as above). As one easily sees, the operations and relations are well defined.

Proposition 8.1. *Let $\Phi(x_1, \ldots, x_m)$ be a formula in the first order language associated to the given similarity type with free variables $x_1, \ldots, x_m$ of sorts $S_{j_1}, \ldots, S_{j_m}$ respectively. For an ultraproduct $A = \prod_{i \in I} A_i/\mathcal{U}$, one has the substituted formula*

$$\Phi\big([a_i^{j_1} \mid i \in I]_{S_{j_1}}, \ldots, [a_i^{j_m} \mid i \in I]_{S_{j_{k_m}}}\big)$$

valid in A if and only if there is $U \in \mathcal{U}$ such that $\Phi(a_i^{j_1}, \ldots, a_i^{j_{k_m}})$ is valid in A_i for all $i \in U$.

Proposition 8.1 is a variant of the well-known Theorem of Łoś [32, Theorem 9.5.1]. To derive it from the 1-sorted case, multi-sorted structures may be conceived as 1-sorted relational structures, assuming sorts to be pairwise disjoint and captured by unary predicates. The following is an immediate consequence of Proposition 8.1.

Lemma 8.2. *Let $\mathcal{U}$ be an ultrafilter over a set I. Let also $(V_i)_{F_i}$ be a pre-hermitian space over $F_i \in \mathcal{F}_\Lambda$ for all $i \in I$. Then $F = \prod_{i \in I} F_i/\mathcal{U} \in \mathcal{F}_\Lambda$ and $V = \prod_{i \in I} V_i/\mathcal{U}$ is a pre-hermitian space over F. Here, for $v = [v_i \mid i \in I]$ and $w = [w_i \mid i \in I]$ in the abelian group V and $\lambda = [\lambda_i \mid i \in I]$ in F, one has*

$$v\lambda = [v_i \lambda_i \mid i \in I], \quad \langle v \mid w \rangle = [\langle v_i \mid w_i \rangle_i \mid i \in I],$$

where $\langle v_i \mid w_i \rangle_i \in F_i$ is the value under the scalar product on $(V_i)_{F_i}$.

Recall that a representation of a $*$-Λ-algebra R within a pre-hermitian space V_F is a $*$-Λ-algebra homomorphism $\varepsilon\colon R \to \mathsf{End}^*(V_F)$. It is convenient to consider representations as unitary R-F-bimodules. More precisely, one has an action $(r, v) \mapsto rv = \varepsilon(r)(v)$ of R on the left and an action $(v, \lambda) \mapsto v\lambda$ of F on the right satisfying the laws of unitary left and right modules and such that

$$(\lambda r)v = (rv)\lambda = r(v\lambda) \quad \text{for all } v \in V,\ r \in R,\ \lambda \in \Lambda,$$

where $v\lambda = v(\lambda 1_F)$. Moreover,

$$\langle rx \mid y\rangle = \langle x \mid r^* y\rangle \quad \text{for all } r \in R,\ x, y \in V;$$
$$(\lambda r)^* v = (\lambda^* r^*)v = (r^* v)\lambda^* \quad \text{for all } v \in V,\ r \in R,\ \lambda \in \Lambda.$$

We denote a representation of R in V_F by ${}_RV_F$. The R-F-bimodule ${}_RV_F$ with scalar product will be considered as a 3-sorted structure with sorts V, R, and F; the $*$-Λ-algebras R and F are considered as 1-sorted structures, where $\lambda \in \Lambda$ serves to denote the unary operation $x \mapsto \lambda x$. Our main concern will be faithful representations; that is, representations ${}_RV_F$ such that $rv = 0$ for all $v \in V$ if only if $r = 0$. Observe that a regular algebra R is $*$-regular, if it admits a faithful representation in an anisotropic space.

The following is as obvious as crucial: A representation of a MIL $\varepsilon\colon L \to \mathbb{L}(V_F)$ can be viewed as a 3-sorted structure with sorts L, V, and F and with the map ε being captured by the binary relation (cf. [41, 42, 48] for this method)

$$\{(a, v) \mid v \in \varepsilon(a)\} \subseteq L \times V,$$

which we denote by ε again.

Lemma 8.3. *Under the hypotheses of Lemma 8.2, one has the following.*

(i) *If L_i is a MIL and $(L_i, V_i, F_i; \varepsilon_i)$ is a faithful representation for all $i \in I$, then the associated ultraproduct $(L, V_F, F; \varepsilon)$ is a faithful representation of $L = \prod_{i\in I} L_i/\mathcal{U}$.*

(ii) *If R_i is a $*$-Λ-algebra and ${}_{R_i}(V_i)_{F_i}$ a faithful representation, $i \in I$, then the associated ultraproduct ${}_RV_F$ is a faithful representation of $R = \prod_{i\in I} R_i/\mathcal{U}$.*

(iii) *Let U be an n-dimensional subspace of V_F, $n < \omega$. Then there are $J \in \mathcal{U}$ and n-dimensional subspaces U_i of $(V_i)_{F_i}$, $i \in J$, such that $U \cong \prod_{i\in J} U_i/\mathcal{U}_J$, where $\mathcal{U}_J = \{X \in \mathcal{U} \mid X \subseteq J\}$, and*

$$\mathbb{L}(U_F) \cong \prod_{i\in J} \mathbb{L}\big((U_i)_{F_i}\big)/\mathcal{U}_J, \quad \overset{*}{\mathsf{End}}(U_F) \cong \prod_{i\in J} \overset{*}{\mathsf{End}}\big((U_i)_{F_i}\big)/\mathcal{U}_J.$$

Proof. Statements (i) and (ii) follow from Proposition 8.1 and the observation that both types of 3-sorted structures can be characterized by first order axioms. In (iii), observe that for a fixed positive integer n, there is a first order formula in the two sorted language for vector spaces expressing that a family $(v_1, \dots, v_n)$ of vectors is independent [is a basis], as well as a first order formula expressing that a vector v is in the span of the $v_1, \dots, v_n$. Thus by the Łoś Theorem (cf. Proposition 8.1), a basis of U determines J and bases of spaces U_i, $i \in J$. Now, apply (i) to the lattices $L_i = \mathbb{L}\big((U_i)_{F_i}\big)$, $i \in J$, to get an embedding of $\prod_{i \in J} L_i/\mathcal{U}_J$ into $\mathbb{L}(U_F)$. Surjectivity of this embedding is granted by the sentence stating that for any $v_1, \dots, v_n$, there is a such that $v \in \varepsilon(a)$ if and only if v is in the span of $v_1, \dots, v_n$. Similarly, we apply (ii) in the ring case and use the sentence stating that for any basis $v_1, \dots, v_n$ and any $w_1, \dots, w_n$, there is r such that $rv_i = w_i$ for all $i \in \{1, \dots, n\}$. ∎

Inheritance of existence of representations under homomorphic images has been dealt with, in different contexts, in [22, 25] for CMILs and by Micol in [44] for $*$-rings. Apparently, this needs saturation properties of ultrapowers. Considering a fixed 1-sorted algebraic structure A, add a new constant symbol $\underline{a}$, called a *parameter*, for each $a \in A$. In what follows, $\Sigma(x_1, \dots, x_n)$ is a set of formulas with free variables $x_1, \dots, x_n$ in this extended language. Given an embedding $h\colon A \to B$, we call B *modestly saturated* [*ω-saturated*] *over A via h*, if for any $n < \omega$ and for any set of formulas $\Sigma(x_1, \dots, x_n)$, with parameters from A [and finitely many parameters from B, respectively], which is finitely realized in A [in B, respectively] is realized in B (where $\underline{a}$ is interpreted as $\underline{a}^B = h(a)$). The following is a particular case of [10, Corollary 4.3.14].

Proposition 8.4. *Every 1-sorted algebraic structure A admits an elementary embedding h into some structure B which is modestly saturated [ω-saturated] over A via h. One can choose B to be an ultrapower of A and h to be the canonical embedding. Identifying a with $h(a)$, one may assume B to be an elementary extension of A. An analogous result holds for multi-sorted algebraic structures.*

Theorem 8.5. *Let a CMIL L [a $*$-Λ-algebra R] have a faithful representation within a pre-hermitian space V_F. There is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L [such that for any regular ideal $I = I^*$, the algebra R/I] admits a faithful representation within $(U/\operatorname{rad} U)_{\hat{F}}$ for some closed subspace U of $\hat{V}_{\hat{F}}$.*

Proof. For a $*$-Λ-algebra R we use the same idea as in the proof of [30, Proposition 25]. Though here, the scalar product induced on U, as defined below, might be degenerate. According to Proposition 8.4, there is an ultrapower ${}_{\hat{R}}\hat{V}_{\hat{F}}$ of the faithful representation ${}_RV_F$ which is modestly saturated over ${}_RV_F$ via the canonical embedding. Then $\hat{V}$ is an R-module via the canonical embedding of R into $\hat{R}$ and the set

$$U = \{v \in \hat{V} \mid av = 0 \text{ for all } a \in I\} = \bigcap_{a \in I} (a^* \hat{V})^\perp$$

is a closed subspace of $\hat{V}_{\hat{F}}$ and a left (R/I)-module. Moreover as $I = I^*$, one has

$$\langle (r + I)v \mid w\rangle = \langle v \mid (r^* + I)w\rangle \text{ for all } v, w \in U \text{ and all } r \in R.$$

We observe that $U^\perp$ is also an (R/I)-module. Indeed, if $v \in U^\perp$ then

$$\langle (r + I)v \mid u\rangle = \langle v \mid (r^* + I)u\rangle = 0 \text{ for all } u \in U.$$

Thus with $W = \mathsf{rad}\, U$, one obtains an (R/I)-$\hat{F}$-bimodule U/W, where

$$(r + I)(v + W) = rv + W \text{ for all } r \in R \text{ and all } v \in U,$$

which is also a subquotient of V_F.

We show that ${}_{R/I}(U/W)_{\hat{F}}$ is a faithful representation of R/I; that is, for any $a \in R\backslash I$, there has to be $u \in U$ such that $au \notin W$. It suffices to show that for any $a \in R\backslash I$, there are u, $v \in U$ such that $\langle au \mid v\rangle \neq 0$. Since $u \in U$ means $bu = 0$ for all $b \in I$, we have to show that the set

$$\Sigma(x, y) = \{\langle \underline{a}x \mid y\rangle \neq 0\} \cup \{\underline{b}x = 0 = \underline{b}y \mid b \in I\}$$

of formulas with parameters from $\{a\} \cup I$ and variables x, y of type V is satisfiable in ${}_{\hat{R}}\hat{V}_{\hat{F}}$. Due to modest saturation, it suffices to show that for any $b_1, \ldots, b_n \in I$, there are u, $v \in V$ such that $\langle au \mid v\rangle \neq 0$ and $b_i u = b_i v = 0$ for all $i \in \{1, \ldots, n\}$. In view of Proposition 3.1(iv) and regularity of I, there is an idempotent $e \in I$ such that $b_i e = b_i$ for all $i \in \{1, \ldots, n\}$; in particular $b_i u = b_i v = 0$ whenever $eu = ev = 0$. Thus it suffices to show that there are u, $v \in V$ such that $eu = ev = 0$ but $\langle au \mid v\rangle \neq 0$.

Assume the contrary; namely, let $eu = ev = 0$ imply $\langle au \mid v\rangle = 0$ for all u, $v \in V$. For arbitrary u', $v' \in V$, let $u = (1 - e)u'$ and $v = (1 - e)v'$. As $eu = ev = 0$, we get by our assumption that $\langle (1 - e^*)au \mid v'\rangle = \langle au \mid v\rangle = 0$. This holds for all $v' \in V$, whence $(1-e^*)au = 0$ since V_F is non-degenerate. Thus $(1-e^*)a(1-e)u' = 0$ for all $u' \in V$, whence $(1 - e^*)a(1 - e) = 0$, as ${}_RV_F$ is a faithful representation. But then $a = e^*a + ae - e^*ae \in I$, a contradiction.

In the case of CMILs, given a representation $\varepsilon\colon L \to \mathbb{L}(V_F)$, let $G = \mathbb{G}(V_F)$ and let $\pi(v) = vF$ for $v \in V$. We consider the 4-sorted structure $(L, V, F, G; \varepsilon, \pi)$. According to Proposition 8.4, there is an ultrapower $(\hat{L}, \hat{V}, \hat{F}, \hat{G}; \hat{\varepsilon}, \hat{\pi})$ of $(L, V, F, G; \varepsilon, \pi)$

which is modestly saturated over $(L, V, F, G; \varepsilon)$ via the canonical embedding. By Lemma 8.3(i), $(\hat{L}, \hat{V}, \hat{F}; \hat{\varepsilon})$ is a faithful representation. In view of Proposition 7.2(ii), $\hat{G} \cong \mathbb{G}(\hat{V}_{\hat{F}})$ via $\hat{\pi}$; and $\hat{\rho}\colon W \mapsto \{v \in \hat{V} \mid \hat{\pi}(v) \in W\}$ defines an isomorphism from $\mathbb{L}(\hat{G})$ onto $\mathbb{L}(\hat{V}_{\hat{F}})$ by Proposition 7.2(iii).

Now, let θ be a congruence of the Galois lattice L. According to the proof of [22, Theorem 13.1], there is a faithful representation $\eta\colon L/\theta \to \mathbb{L}(W/W')$ in a subquotient W/W' of $\hat{G}$, where the subspace W is closed and $W' = W \cap W^{\perp}$. Then $\hat{\rho}(W)/\hat{\rho}(W')$ is a subquotient of $\hat{V}_{\hat{F}}$, $\hat{\rho}(W)$ is a closed subspace of $\hat{V}$, and $\hat{\rho}\eta$ is a faithful representation of L/θ in $\hat{\rho}(W)/\hat{\rho}(W')$ by Proposition 7.5. The proof is complete. ∎

Corollary 8.6. *Let a MOL L have a faithful representation within a pre-hermitian space V_F. There is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L admits a faithful representation within a pre-hermitian closed subspace $U_{\hat{F}}$ of $\hat{V}_{\hat{F}}$.*

Proof. According to the proof of [22, Theorem 13.1] and the proof of Theorem 8.5, there is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L admits a faithful representation within a subquotient W/W' of the orthogonal geometry $\mathbb{G}(\hat{V}_{\hat{F}})$. As L is a MOL, according to the definition of W' (given in [22, page 355] and denoted by U there), one has $W' = \varnothing$. Hence in the proof of Theorem 8.5, $\mathsf{rad}\, U = \hat{\rho}(W') = 0$. ∎

9. Classes of structures

We consider classes $\mathcal{C}$ of $*$-Λ-algebras on one side, of Galois lattices on the other. With the familiar concepts, by $\mathsf{H}(\mathcal{C})$, $\mathsf{S}(\mathcal{C})$, $\mathsf{P}(\mathcal{C})$, $\mathsf{P_s}(\mathcal{C})$, $\mathsf{P}_\omega(\mathcal{C})$, and $\mathsf{P_u}(\mathcal{C})$, we denote the class of all homomorphic images, subalgebras, direct products, subdirect products, direct products of finitely many factors, and ultraproducts of members of $\mathcal{C}$, respectively, allowing isomorphic copies in all cases. Of course, all fundamental operations have to be taken care of. In particular, in the case of $*$-Λ-algebras, this applies also to the unit 1, the additive inverse, and the "scalars" $\lambda \in \Lambda$; that is, "subalgebra" means $*$-subring and Λ-subalgebra with unit. In the case of Galois lattices, also the bounds 0, 1 and the operation $x \mapsto x'$ are to be preserved; that is, "subalgebra" means Galois sublattice. In terms of Universal Algebra, we have classes of algebraic structures of a given "similarity type" or "signature" and the associated class operators, cf. [9, Chapter II] and [18, Chapter I], also [43].

A class $\mathcal{C}$ of algebraic structures of the same type is a *universal class* if it is closed under S and $\mathsf{P_u}$; a *positive universal class* (shortly a *semivariety*), if it is closed also under H; a *variety* if, in addition, it is closed under P. Let $\mathsf{W}(\mathcal{C})$ and $\mathsf{V}(\mathcal{C})$ denote the smallest semivariety and the smallest variety containing the class $\mathcal{C}$. The following statement is well known and easily verified, cf. Theorem A.5 in Appendix A.

Proposition 9.1. *A class $\mathcal{K}$ is universal [a semivariety, a variety] if and only if it can be defined by universal sentences [positive universal sentences, identities, respectively].*

Dealing with a class $\mathcal{C}$ of $*$-Λ-algebras or MILs, let $\mathsf{S}_\exists(\mathcal{C})$ $[\mathsf{P}_{\mathsf{s}\exists}(\mathcal{C})]$ consist of all regular or complemented members of the class $\mathsf{S}(\mathcal{C})$ [of the class $\mathsf{P}_\mathsf{s}(\mathcal{C})$, respectively]. Call $\mathcal{C}$ an $\exists$-*semivariety* if it is closed under the operators H, $\mathsf{S}_\exists$, P_u and an $\exists$-*variety* if it is also closed under P, cf. [28], also [35] for an analogue within semigroup theory. Let $\mathsf{W}_\exists(\mathcal{C})$ $[\mathsf{V}_\exists(\mathcal{C})]$ denote the least $\exists$-semivariety [$\exists$-variety, respectively] which contains the class $\mathcal{C}$.

Recall that MIL also denotes the class of all MILs, similarly for CMIL and MOL. Let $\mathcal{A}_\Lambda$ denote the class of all $*$-Λ-algebras, with the subclasses $\mathcal{R}_\Lambda$, $\mathcal{R}^*_\Lambda$, and $\mathcal{F}_\Lambda$ consisting of its members which are regular, $*$-regular, and divisions rings, respectively.

Proposition 9.2. *Let $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ or $\mathcal{C} \subseteq$ CMIL.*

(i) $\mathsf{O}\mathsf{S}_\exists(\mathcal{C}) \subseteq \mathsf{S}_\exists\mathsf{O}(\mathcal{C})$ *for any class operator* $\mathsf{O} \in \{\mathsf{P}_\mathsf{u}, \mathsf{P}, \mathsf{P}_\omega\}$.

(ii) $\mathsf{S}_\exists\mathsf{H}(\mathcal{C}) \subseteq \mathsf{H}\mathsf{S}_\exists(\mathcal{C})$.

(iii) $\mathsf{W}_\exists(\mathcal{C}) = \mathsf{H}\mathsf{S}_\exists\mathsf{P}_\mathsf{u}(\mathcal{C})$.

(iv) $\mathsf{V}_\exists(\mathcal{C}) = \mathsf{H}\mathsf{S}_\exists\mathsf{P}(\mathcal{C}) = \mathsf{H}\mathsf{S}_\exists\mathsf{P}\mathsf{P}_\mathsf{u}(\mathcal{C}) = \mathsf{H}\mathsf{S}_\exists\mathsf{P}_\mathsf{u}\mathsf{P}_\omega(\mathcal{C}) = \mathsf{P}_{\mathsf{s}\exists}\mathsf{W}_\exists(\mathcal{C})$.

(v) $\mathsf{W}_\exists(\mathcal{C})$ *and* $\mathsf{V}_\exists(\mathcal{C})$ *are axiomatic classes.*

These statements are well known for arbitrary algebraic structures if the suffix $\exists$ is omitted.

Proof. In view of Proposition A.4 in Appendix A, Proposition A.6(i)–(iv) and Theorem A.5(iii)–(iv) apply to yield (i)–(v). The last equation in (iv) follows from Proposition A.6(v) and the distributivity of congruence lattices of lattices (cf. [19, Theorem 149]) and regular rings (cf. [51, Lemma 8-3.5]). ∎

Dealing with pre-hermitian spaces, we primarily adhere to the 2-sorted point of view, as explained in Section 8. A [2-sorted] *embedding* $V'_{F'}$ into V_F is given by a $*$-Λ-algebra embedding $\alpha\colon F' \to F$ and an injective α-semilinear map ω such that $\langle \omega(v) \mid \omega(w)\rangle = \alpha(\langle v \mid w\rangle')$ for all v, $w \in V'$. An embedding is an *isomorphism* if both α and ω are bijections. $V'_{F'}$ is a [2-sorted] substructure of V_F if it embeds into V_F with α and ω being inclusion maps. In contrast to that, a *subspace* of V_F will always mean an F-linear subspace with the induced scalar product; that is, here we follow the 1-sorted view on the vector space V_F.

Let $\mathcal{S}$ be a class of pre-hermitian spaces V_F, where $F \in \mathcal{F}_\Lambda$ and Λ is a fixed commutative $*$-ring. In such a case, we also speak of a class of spaces *over* Λ. Introducing operators for classes of spaces, let $\mathsf{S}(\mathcal{S})$ and $\mathsf{P_u}(\mathcal{S})$ denote the classes of all spaces isomorphic to non-degenerate 2-sorted substructures and to all ultraproducts of members of $\mathcal{S}$ respectively. In contrast to that, following the one-sorted view, let $\mathsf{S_{1f}}(\mathcal{S})$ [$\mathsf{S_{1q}}(\mathcal{S})$] denote the class of (isomorphic copies of) non-degenerate finite-dimensional subspaces [of all subquotients $U/\operatorname{\mathsf{rad}} U$ such that $V_F \in \mathcal{S}$, U_F is a subspace of V_F, and $U = U^{\perp\perp}$, respectively] of members of $\mathcal{S}$. The next statement follows from Propositions 2.2 and 2.3.

Lemma 9.3. *For any class $\mathcal{S}$ of spaces over Λ, $\mathsf{S_{1f}}(\mathcal{S}) \subseteq \mathsf{S_{1q}}(\mathcal{S})$ and $\mathsf{S_{1f}}\mathsf{S_{1q}}(\mathcal{S}) = \mathsf{S_{1f}}(\mathcal{S})$.*

Let also $\mathsf{I_s}(\mathcal{S})$ denote the class of spaces which arise from $\mathcal{S}$ by scaling and observe that $\mathsf{I_s}\mathsf{O}(\mathcal{S}) \subseteq \mathsf{O}\mathsf{I_s}(\mathcal{S})$ for any of the class operators introduced above. Call $\mathcal{S}$ a *universal class*, if it is closed under $\mathsf{P_u}$, S, and $\mathsf{I_s}$. Observe that $\mathsf{S}\mathsf{P_u}\mathsf{I_s}(\mathcal{S})$ is the smallest universal class containing a class $\mathcal{S}$. Call $\mathcal{S}$ a *semivariety* if it is closed under $\mathsf{P_u}$ and $\mathsf{S_{1f}}$. Of course, any universal class is a semivariety, and the smallest semivariety containing a class $\mathcal{S}$ is contained in $\mathsf{S}\mathsf{P_u}(\mathcal{S})$.

10. Reduction to finite dimension

The importance of representations for the universal algebraic theory of CMILs and regular $*$-rings derives from the following

Theorem 10.1. *Let V_F be a pre-hermitian space and let $L \in$ MIL [$R \in \mathcal{A}_\Lambda$] have a faithful representation within V_F. Then $L \in \mathsf{W}\big(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)\big)$ [$R \in \mathsf{W}\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$, respectively]. If $L \in$ CMIL [$R \in \mathcal{R}_\Lambda$], then $L \in \mathsf{W}_\exists\big(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)\big)$ [$R \in \mathsf{W}_\exists\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$, respectively].*

Proof. We may assume that $\dim V_F \geqslant \omega$. In view of Proposition 7.3, L embeds into an atomic MIL M which is a subalgebra of $\mathbb{L}(V_F)$ such that $M_\mathsf{f} = \mathbb{L}(V_F)_\mathsf{f}$ and M may be chosen a CMIL if L is a CMIL. Proposition 7.2(i)–(ii) yields that M_f is a CMIL and the directed union of its subalgebras $[0,U] \cup [U^\perp, V]$, $U \in \mathbb{O}(V_F)$, each of which is in $\mathsf{S}_\exists(W_F)$ for some $W \in \mathbb{O}(V_F)$. Since any directed union of algebraic structures A_i, $i \in I$, embeds into an ultraproduct of structures A_i, $i \in I$, (cf. [18, Theorem 1.2.12(1)]), one gets

$$M_\mathsf{f} \in \mathsf{S}_\exists\mathsf{P_u}\big(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)\big).$$

Finally, the proof of [22, Theorem 16.3] yields $M \in \mathsf{W}(M_{\mathsf{f}})$ and $M \in \mathsf{W}_{\exists}(M_{\mathsf{f}})$ in case M is complemented. The claim about L follows immediately.

Dealing with an algebra $R \in \mathcal{A}_\Lambda$, first observe that, in view of Corollary 4.6,

$$\hat{J}(V_F) \in \mathsf{W}_{\exists}\big(\overset{*}{\mathsf{End}}(U_F) \mid U \in \mathbb{O}(V_F)\big).$$

In view of Proposition 4.5, we may assume that R is a subalgebra of $\mathsf{End}^*(V_F)$ containing $A = \hat{J}(V_F)$. Let $J = J(V_F)$ and let J_0 denote the set of projections in J. By Proposition 8.4, there is an ultrapower $({}_{\hat{R}}\hat{V}_{\hat{F}}; \hat{A})$ of $({}_RV_F; A)$ which is ω-saturated over $({}_RV_F; A)$. We may assume that R is a subalgebra of $\hat{R}$ and $\hat{A}$ is an ultrapower of A; in particular, $\hat{A} \in \mathsf{W}_{\exists}\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$. For $a \in \hat{A}$ and $r \in R$, we put

$$a \sim r, \quad \text{if} \quad ae = re \text{ and } a^*e = r^*e \text{ for all } e \in J_0.$$

Claim 1. *For any $a \in \hat{A}$ and any r, $s \in R$, $a \sim r$ and $a \sim s$ imply $r = s$.*

Proof of Claim. For any $U \in \mathbb{O}(V_F)$, we have $\pi_U \in J_0$, whence $r\pi_U = a\pi_U = s\pi_U$. Considering r and s as endomorphisms of V_F, we get that they coincide on any $U \in \mathbb{O}(V_F)$, whence they coincide on V_F by Proposition 2.2. ∎

Claim 2. *$S = \{a \in \hat{A} \mid a \sim r$ for some $r \in R\}$ is a subalgebra of $\hat{A}$ and the map $g\colon \hat{A} \to R$, $\quad g\colon a \mapsto r$, where $a \sim r$ is a homomorphism.*

Proof of Claim. It follows from Claim 1 that g is well defined. Let a, $b \in \hat{A}$ and r, $s \in R$ be such that $a \sim r$ and $b \sim s$. Then, obviously, $a + b \sim r + s$, $\lambda a \sim \lambda r$ for any $\lambda \in \Lambda$, and $a^* \sim r^*$. Let $e \in J_0$, then $be \in J$. By Proposition 4.4(iv), there is $f \in J_0$ such that $fbe = be$. Therefore, we get $abe = afbe = rfbe = rbe = rse$, whence $ab \sim rs$.

Obviously, $0_{\hat{V}}$, $\mathsf{id}_{\hat{V}} \in \hat{A}$. For any $U \in \mathbb{O}(V_F)$ we have $\pi_U \in J_0$. Therefore, $0_{\hat{V}}\pi_U = 0_U$ and $\mathsf{id}_{\hat{V}}\,\pi_U = \pi_U$ imply in view of Proposition 2.2 that $0_{\hat{V}} \sim 0_R$ and $\mathsf{id}_{\hat{V}} \sim 1_R$. ∎

Claim 3. *The homomorphism g is surjective.*

Proof of Claim. Surjectivity of g is shown via the saturation property. Given $r \in R$, consider a finite set $E \subseteq J_0$. According to Proposition 4.4(iv), there is $e \in J_0$ such that $ef = f$ for all $f \in E$ and $er^*f = r^*f$ for all $f \in E$. Take $a = re$ and observe that $af = ref = rf$ and $a^*f = er^*f = r^*f$ for all $f \in E$. Thus the set of formulas

$$\Sigma(x) = \big\{[xe = re] \;\&\; [x^*e = r^*e] \mid e \in J_0\big\}$$

with a free variable x of type A is finitely realized in $({}_RV_F;A)$. As $({}_{\hat{R}}\hat{V}_{\hat{F}};\hat{A})$ is ω-saturated over $({}_RV_F;A)$, we get that there is $a \in \hat{A}$ with $a \sim r$. ∎

Claim 4. *If R is regular, then S is also regular.*

Proof of Claim. In view of Proposition 3.1(ii), it suffices to prove that $\mathsf{ker}\, g = \{a \in S \mid a \sim 0\}$ is regular. Observe that $a \sim 0$ means that $ae = 0 = a^*e$ for any $e \in J_0$, equivalently $(1-e)a = a = a(1-e)$. Again, let $E \subseteq J_0$ be finite. By Proposition 4.4(iv), there is $e \in J_0$ such that $ef = f$ for any $f \in E$. The ring A is regular by Propositions 4.4(i) and 4.5, whence $\hat{A}$ is also regular. Therefore, the ring $(1-e)\hat{A}(1-e)$ is regular by [5, 2.4]. Thus there is $b \in \hat{A}$ such that $aba = a$ and $(1-e)b = b = b(1-e)$; in particular, $be = 0 = eb$ whence $b^*e = 0$. This implies that $bf = bef = 0$ and $b^*f = b^*ef = 0$ for all $f \in E$. Therefore, the set of formulas

$$\Sigma(x) = \{axa = a\} \cup \big\{[xe = 0] \;\&\; [x^*e = 0] \mid e \in J_0\big\}$$

with a variable x of type A is finitely realized in $({}_{\hat{R}}\hat{V}_{\hat{F}};\hat{A})$. Thus $\Sigma(x)$ is realized in $({}_{\hat{R}}\hat{V}_{\hat{F}};\hat{A})$, and we obtain $b \in \hat{A}$ such that $aba = a$ and $b \sim 0$; that is, $b \in \mathsf{ker}\, g$. ∎

The proof for $*$-Λ-algebras is done in view of Claims 2–4. ∎

Remark 10.2. The statements of Theorem 10.1 concerning $*$-Λ-algebras were established in case of representability in inner product spaces in [30, Theorem 16]. Considering the operator W only, a more direct approach is possible. For $R \in \mathcal{A}_\Lambda$, one chooses in the proof of [30, Theorem 16] $I = \mathbb{O}(V_F)$. By Proposition 2.2, any finite-dimensional subspace of V_F is contained in some $U \in I$. Moreover, with the induced scalar product, U_F is a pre-hermitian space. A similar approach works for MILs.

11. (∃-)semivarieties of representable structures

We denote by $\mathcal{L}(\mathcal{S})$ [by $\mathcal{R}(\mathcal{S})$] the class of all CMILs [of all $R \in \mathcal{R}_\Lambda$ respectively] having a faithful representation within some member of $\mathcal{S}$ (we also say that these structures are *representable* within $\mathcal{S}$). Here, we consider conditions on $\mathcal{S}$ which ensure that the classes $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are ∃-(semi)varieties. Observe that

$$\mathcal{L}\big(\mathsf{I_s}(\mathcal{S})\big) = \mathcal{L}(\mathcal{S}) \quad \text{and} \quad \mathcal{R}\big(\mathsf{I_s}(\mathcal{S})\big) = \mathcal{R}(\mathcal{S}).$$

Proposition 11.1. *Let $\mathcal{S}$ be a [recursively] axiomatized class of pre-hermitian spaces over a [recursive] commutative $*$-ring Λ. Then $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are [recursively] axiomatizable.*

Proof. Let Γ_r denote the set of first order axioms defining representations ${}_RV_F$ with $R \in \mathcal{R}_\Lambda$ and $V_F \in \mathcal{S}$ and let Σ_r denote the set of all universal sentences in the first order language of $*$-Λ-algebras which are consequences of Γ_r. Then Σ_r defines the class of all $*$-Λ-algebras representable in $\mathcal{S}$ (we refer to the Fundamental Theorem in [48] for this kind of reasoning, see also [42]). Adding to Σ_r the $\forall\exists$-axiom of regularity one defines the subclass $\mathcal{R}(\mathcal{S})$. If Λ is recursive and $\mathcal{S}$ is recursively axiomatizable, then Γ_r is recursive; moreover, by Gödel's Completeness Theorem, Σ_r is recursively enumerable and, by Craig's trick [32, Exercise 6.1.3], Σ_r is also recursive.

Similarly, taking Γ_l to be the set of first order axioms defining representations of CMILs within spaces from $\mathcal{S}$, and denoting by Σ_l the set of all universal sentences in the signature of CMILs which are consequences of Γ_l, we get that Σ_l defines the class $\mathcal{L}(\mathcal{S})$ of all CMILs representable in $\mathcal{S}$. Moreover, if Γ_l is recursive, then Σ_l is also recursive. ∎

A *tensorial embedding* of a pre-hermitian space V_F into another one, W_K, is given by a $*$-Λ-algebra embedding $\alpha\colon F \to K$ and an injective α-semilinear map $\varepsilon\colon V_F \to W_K$ such that W_K is spanned by $\mathsf{im}\,\varepsilon$ as a K-vector space and $\langle\varepsilon(v) \mid \varepsilon(w)\rangle = \alpha\big(\langle v \mid w\rangle\big)$ for all $v, w \in V$; in particular, ε is an isomorphism of V_F onto a two-sorted substructure of W_K. A *joint tensorial extension* of spaces V_{iF_i}, $i \in \{0,1\}$, is given by a pre-hermitian space $W_F = U_0 \oplus^\perp U_1$ and a tensorial embedding of V_{iF_i} into U_{iF} for $i \in \{0,1\}$.

Lemma 11.2. *Let F, F_0, $F_1 \in \mathcal{A}_\Lambda$, let V_F be a pre-hermitian space, and let V_{0F_0} and V_{1F_1} be finite-dimensional pre-hermitian spaces.*

(i) *If α_0 and ε_0 defines a tensorial embedding of V_{0F_0} into V_F then $\mathsf{End}^*(V_{0F_0})$ embeds into $\mathsf{End}^*(V_F)$ and $\mathbb{L}(V_{0F_0})$ embeds into $\mathbb{L}(V_F)$.*

(ii) *If V_F is a joint tensorial extension of V_{0F_0} and V_{1F_1}, then $\mathsf{End}^*(V_{0F_0}) \times \mathsf{End}^*(V_{1F_1})$ embeds into $\mathsf{End}^*(V_F)$ and $\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1})$ embeds into $\mathbb{L}(V_F)$.*

Proof. (i) In view of Proposition 4.3(i), V_{0F_0} has a dual pair $(v_1,\dots,v_n)$, $(w_1,\dots,w_n)$ of bases; applying ε_0, one obtains such a pair for V_F. Indeed, V_F is obviously spanned by both, $(\varepsilon_0(v_1),\dots,\varepsilon_0(v_n))$ and $(\varepsilon_0(w_1),\dots,\varepsilon_0(w_n))$. Suppose that $\Sigma_{j=1}^n\varepsilon_0(v_j)\lambda_j = 0$ for some $\lambda_1, \dots, \lambda_n \in F$. Then for any $k \in \{1,\dots,n\}$, one gets

$$0 = \langle 0 \mid w_k\rangle = \big\langle \Sigma_{j=1}^n\varepsilon_0(v_j)\lambda_j \mid \varepsilon_0(w_k)\big\rangle = \Sigma_{j=1}^n\alpha_0\big(\langle v_j \mid w_k\rangle\big)\lambda_j = \lambda_k,$$

whence $(\varepsilon_0(v_1),\dots,\varepsilon_0(v_n))$ is a basis of V_F. Similarly, $(\varepsilon_0(w_1),\dots,\varepsilon_0(w_n))$ is a basis of V_F.

For $\varphi \in \mathsf{End}^*(V_{0F_0})$, let $\xi_0(\varphi)$ be the F-linear map on V defined by $\xi_0(\varphi)\colon \varepsilon_0(v_j) \mapsto \varepsilon_0\bigl(\varphi(v_j)\bigr)$ for all $j \in \{1,\dots,n\}$. Clearly, ξ_0 is a Λ-algebra embedding of $\mathsf{End}^*(V_{0F_0})$ into $\mathsf{End}^*(V_F)$. Moreover, for any j, $k \in \{1,\dots,n\}$, one has

$$\bigl\langle \varepsilon_0(v_j) \mid \xi_0(\varphi^*)\varepsilon_0(v_k)\bigr\rangle = \bigl\langle \varepsilon_0(v_j) \mid \varepsilon_0\varphi^*(v_k)\bigr\rangle = \alpha_0\bigl(\langle v_j \mid \varphi^*(v_k)\rangle\bigr) = \alpha_0\bigl(\langle \varphi(v_j) \mid v_k\rangle\bigr)$$
$$= \bigl\langle \varepsilon_0\varphi(v_j) \mid \varepsilon_0(v_k)\bigr\rangle = \bigl\langle \xi_0(\varphi)\varepsilon_0(v_j) \mid \varepsilon_0(v_k)\bigr\rangle,$$

whence $\xi_0(\varphi^*) = \xi_0(\varphi)^*$ by Proposition 4.3(iii). For the claim about polarity lattices, now apply Proposition 7.8.

(ii) As $V_F = U_0 \oplus^\perp U_1$, by (i), there are $*$-Λ-algebra embeddings

$$\xi_i\colon \overset{*}{\mathsf{End}}(V_{iF_i}) \to \overset{*}{\mathsf{End}}(U_{iF}), \quad i \in \{0,1\}.$$

Thus there is a unique embedding

$$\xi\colon \overset{*}{\mathsf{End}}(V_{0F_0}) \times \overset{*}{\mathsf{End}}(V_{1F_1}) \to \overset{*}{\mathsf{End}}(V_F)$$

such that $\xi(\varphi_0,\varphi_1)|_{U_i} = \xi_i(\varphi_i)$ for $i \in \{0,1\}$. By Propositions 5.4(iii), 4.3(i), and 7.8(ii), one has

$$\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1}) \cong \mathbb{L}\bigl(\overset{*}{\mathsf{End}}(V_{0F0})\bigr) \times \mathbb{L}\bigl(\overset{*}{\mathsf{End}}(V_{1F_1})\bigr) \cong \mathbb{L}\bigl(\overset{*}{\mathsf{End}}(V_{0F_0}) \times \overset{*}{\mathsf{End}}(V_{1F_1})\bigr).$$

By (ii) for $*$-Λ-algebras and Proposition 7.8(i), the latter admits a faithful representation in V_F. ■

Theorem 11.3. *Let $\mathcal{S}$ be a class of pre-hermitian spaces over Λ. Then*

(i) $\mathcal{L}\bigl(\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr) = \mathcal{L}\bigl(\mathsf{SP}_\mathsf{u}\mathsf{I}_\mathsf{s}(\mathcal{S})\bigr) = \mathsf{W}_\exists\bigl(\mathcal{L}(\mathcal{S})\bigr) = \mathsf{W}_\exists\bigl(\mathbb{L}(V_F) \mid V_F \in \mathsf{S}_{1\mathsf{f}}(\mathcal{S})\bigr)$*;*

(ii) $\mathcal{R}\bigl(\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr) = \mathcal{R}\bigl(\mathsf{SP}_\mathsf{u}\mathsf{I}_\mathsf{s}(\mathcal{S})\bigr) = \mathsf{W}_\exists\bigl(\mathcal{R}(\mathcal{S})\bigr) = \mathsf{W}_\exists\bigl(\mathsf{End}^*(V_F) \mid V_F \in \mathsf{S}_{1\mathsf{f}}(\mathcal{S})\bigr)$.

In particular, if the class $\mathcal{S}$ is a semivariety then the classes $\mathcal{L}(\mathcal{S}) = \mathcal{L}\bigl(\mathsf{SP}_\mathsf{u}\mathsf{I}_\mathsf{s}(\mathcal{S})\bigr)$ and $\mathcal{R}(\mathcal{S}) = \mathcal{R}\bigl(\mathsf{S}P_\mathsf{u}\mathsf{I}_\mathsf{s}(\mathcal{S})\bigr)$ are $\exists$-semivarieties generated by their strictly simple finite-dimensional or artinian members, respectively.

Proof. The proofs of (i) and (ii) follow the same lines. We prove (ii).

The fact that $\mathsf{S}_\exists\mathsf{P}_\mathsf{u}\bigl(\mathcal{R}(\mathcal{S})\bigr) \subseteq \mathcal{R}\bigl(\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr)$ follows immediately from Lemma 8.3(iii). Then $\mathsf{W}_\exists\bigl(\mathcal{R}(\mathcal{S})\bigr) \subseteq \mathcal{R}\bigl(\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr)$ by Theorem 8.5. By Theorem 10.1, $\mathcal{R}\bigl(\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr) \subseteq \mathsf{W}_\exists\bigl(\mathsf{End}^*(V_F) \mid V_F \in \mathsf{S}_{1\mathsf{f}}\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr)$. By Lemmas 8.3(iii) and 9.3, for any $V_F \in \mathsf{S}_{1\mathsf{f}}\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S}) = \mathsf{S}_{1\mathsf{f}}\mathsf{P}_\mathsf{u}(\mathcal{S})$, we have $V_F \in \mathsf{P}_\mathsf{u}\mathsf{S}_{1\mathsf{f}}(\mathcal{S})$ and $\mathsf{End}^*(V_F) \in \mathsf{P}_\mathsf{u}\bigl(\mathsf{End}^*(W_K) \mid W_K \in \mathsf{S}_{1\mathsf{f}}(\mathcal{S})\bigr)$. It follows that

$$\mathsf{W}_\exists\bigl(\mathcal{R}(\mathcal{S})\bigr) \subseteq \mathcal{R}\bigl(\mathsf{S}_{1\mathsf{q}}\mathsf{P}_\mathsf{u}(\mathcal{S})\bigr) \subseteq \mathsf{W}_\exists\bigl(\overset{*}{\mathsf{End}}(W_K) \mid W_K \in \mathsf{S}_{1\mathsf{f}}(\mathcal{S})\bigr) \subseteq \mathsf{W}_\exists\bigl(\mathcal{R}(\mathcal{S})\bigr).$$

Now, consider $R \in \mathcal{R}(\mathsf{SP_u}(\mathcal{S}))$; that is, R is represented in a 2-sorted substructure W_K of some $V_F \in \mathsf{P_u}(\mathcal{S})$. By Theorem 10.1, we have $R \in \mathsf{W}_\exists(\mathsf{End}^*(U_K) \mid U_K \in \mathsf{S_{1f}}(W_K))$. Let U'_F denote the F-subspace of V_F spanned by U. By Lemma 11.2(i), $\mathsf{End}^*(U_K) \in \mathsf{S}_\exists(\mathsf{End}^*(U'_F))$. Thus, $R \in \mathsf{W}_\exists(\mathcal{R}(\mathcal{S}))$. Hence

$$\mathcal{R}(\mathsf{SP_u I_s}(\mathcal{S})) \subseteq \mathcal{R}(\mathsf{I_s SP_u}(\mathcal{S})) = \mathcal{R}(\mathsf{SP_u}(\mathcal{S})) \subseteq \mathsf{W}_\exists(\mathcal{R}(\mathcal{S})) = \mathcal{R}(\mathsf{S_{1q}P_u}(\mathcal{S})).$$

The containment $\mathcal{R}(\mathsf{S_{1q}P_u}(\mathcal{S})) \subseteq \mathcal{R}(\mathsf{SP_u I_s}(\mathcal{S}))$ is trivial by Lemma 9.3. ∎

More closure properties on $\mathcal{S}$ are needed if one intends to get a one-to-one correspondence between classes of spaces and classes of structures as in Theorem 11.3.

Definition 11.4. Let V_F, W_K be pre-hermitian spaces over Λ, $\dim V_F < \omega$, and let $\mathcal{S}$ be a class of pre-hermitian spaces over Λ.

- (i) The sesquilinear space V_F is an *L-spread* of W_K if $\dim V_F > 2$ and $\mathbb{L}(V_F) \in \mathcal{L}(W_K)$. The class $\mathcal{S}$ is *L-spread closed*, if it contains all L-spreads of its members.
- (ii) The sesquilinear space V_F is an *R-spread* of W_K if $\mathsf{End}^*(V_F) \in \mathcal{R}(W_K)$. The class $\mathcal{S}$ is *R-spread closed*, if it contains all R-spreads of its members.
- (iii) An R-[L-]spread closed universal class or a semivariety $\mathcal{S}$ is *small*, if $\mathcal{S}$ coincides with the smallest R-[L-]spread closed universal class or a semivariety which contains all members of $\mathcal{S}$ of dimension $n < \omega$ [of dimension $2 < n < \omega$, respectively].

Example 11.5. Consider the class $\mathcal{S}$ of all anisotropic hermitian spaces, where $F \in \mathsf{SP_u}(\mathbb{Q})$; in particular, $F \models \forall x\,[x^2 \neq 2]$ and $\mathcal{S}$ is a universal class which does not contain K^3_K with the canonical scalar product, where $K = \mathbb{Q}(\sqrt{2})$. Nonetheless, $K^{3\times 3}$ (whence also $\mathbb{L}(K^{3\times 3})$) is representable within $\mathbb{Q}^6_{\mathbb{Q}} \in \mathcal{S}$:

$$a + b\sqrt{2} \mapsto a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + b\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad \text{where } a, b \in \mathbb{Q},$$

yields a $*$-ring embedding of K into $\mathbb{Q}^{2\times 2}$ thus giving rise to an embedding of $K^{3\times 3}$ into $(\mathbb{Q}^{2\times 2})^{3\times 3}$. In the sense of Definition 11.4, K^3_K is an L-spread and an R-spread of $\mathbb{Q}^6_{\mathbb{Q}}$.

Theorem 11.6.

- (i) *For any $\exists$-semivariety $\mathcal{V}$ of Arguesian CMILs generated by its strictly simple members of finite dimension at least* 3, *there is a small L-spread closed semivariety* [*universal class*] $\mathcal{S}$ *of pre-hermitian spaces over $\mathbb{Z}$ such that $\mathcal{V} = \mathcal{L}(\mathcal{S})$. Moreover, the class of members of $\mathcal{S}$ of dimension at least* 3 *is unique.*

(ii) *For any $\exists$-semivariety $\mathcal{V} \subseteq \mathcal{R}_\Lambda$ generated by its strictly simple artinian members, there is a small R-spread closed semivariety [universal class] $\mathcal{S}$ of pre-hermitian spaces over Λ such that $\mathcal{V} = \mathcal{R}(\mathcal{S})$. Moreover, such a class $\mathcal{S}$ is unique.*

The class $\mathcal{S}$ above is anisotropic, if $\mathcal{V}$ consists of MOLs or $\mathcal{V} \subseteq \mathcal{R}^*_\Lambda$.

Remark 11.7. If $\mathcal{V}$ consists of MOLs (in the context of (i)) or $\mathcal{V} \subseteq \mathcal{R}^*_\Lambda$ (in the context of (ii)), it suffices to require that $\mathcal{V}$ is generated by its simple members which are of finite dimension respectively artinian and that, in the context of (i), $\mathcal{V}$ is not 2-distributive. Then, in the context of (i), $\mathcal{V}$ contains all MOLs of dimension 2.

Proof of Theorem 11.6. (ii) Given an $\exists$-semivariety $\mathcal{V} \subseteq \mathcal{R}_\Lambda$ with all required properties, let $\mathcal{K}_\mathcal{V}$ denote the class of strictly artinian members of $\mathcal{V}$. By Proposition 4.9, for any $R \in \mathcal{K}_\mathcal{V}$, there is a pre-hermitian space V_F over Λ such that $R \cong \mathsf{End}^*(V_F)$. By $\mathcal{S}_\mathcal{V}$, we denote the class of spaces V_F over Λ such that $\mathsf{End}^*(V_F) \in \mathcal{K}_\mathcal{V}$.

We put $\mathcal{G}_0 = \mathsf{S}_{1\mathsf{f}}(\mathcal{S}_\mathcal{V})$. For any ordinal α, let $\mathcal{G}_{\alpha+1}$ be the union of two classes: $\mathsf{P_u}(\mathcal{G}_\alpha)$ and the class of all $V_F \in \mathsf{S}_{1\mathsf{f}}(V'_F)$, where V'_F is an R-spread of some $W_K \in \mathcal{G}_\alpha$. Let also $\mathcal{G}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\beta$, if α is a limit ordinal.

Claim 1. $\mathsf{S}_{1\mathsf{f}}(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ *and* $\mathsf{End}^*(V_F) \in \mathcal{V}$ *for any α and $V_F \in \mathcal{G}_\alpha$ with* $\dim V_F < \omega$.

Proof of Claim. We argue by induction on α. For $\alpha = 0$, the first claim follows from the definition of $\mathcal{G}_0$. Moreover, if $U_F \in \mathsf{S}_{1\mathsf{f}}(V_F)$ and $\mathsf{End}^*(V_F) \in \mathcal{V}$ then $\mathsf{End}^*(U_F) \in \mathsf{HS}_\exists(\mathsf{End}^*(V_F)) \subseteq \mathcal{V}$ by Proposition 4.3(iii). The limit step is trivial. In the step from α to $\alpha+1$, we assume first that V_F is isomorphic to an ultraproduct of spaces $V_{iF_i} \in \mathcal{G}_\alpha$, $i \in I$. If $U_F \in \mathsf{S}_{1\mathsf{f}}(V_F)$ and $n = \dim U_F$ then, by Lemma 8.3(iii), for some $J \subseteq I$, U_F is isomorphic to an ultraproduct of $U_{iF_i} \in \mathsf{S}_{1\mathsf{f}}(V_{iF_i})$ with $\dim U_{iF_i} = n$, $i \in J$. By the inductive hypothesis, $U_{iF_i} \in \mathcal{G}_\alpha$ and $\mathsf{End}^*(U_{iF_i}) \in \mathcal{V}$. Thus $U_F \in \mathcal{G}_{\alpha+1}$ and $\mathsf{End}^*(U_F) \in \mathcal{V}$ by Lemma 8.3(iii).

Now, let V'_F be an R-spread of $W_K \in \mathcal{G}_\alpha$ and let $V_F \in \mathsf{S}_{1\mathsf{f}}(W_K)$. If $U_F \in \mathsf{S}_{1\mathsf{f}}(V_F)$ then $U_F \in \mathsf{S}_{1\mathsf{f}}(V'_F)$, whence $U_F \in \mathcal{G}_{\alpha+1}$ by definition. By Theorem 10.1 and the inductive hypothesis,

$$\overset{*}{\mathsf{End}}(V'_F) \in \mathsf{W}_\exists\big(\overset{*}{\mathsf{End}}(W'_K) \mid W'_K \in \mathbb{O}(W_K)\big) \subseteq \mathcal{V}.$$

By Proposition 7.2(i), $\mathsf{End}^*(U_F) \in \mathsf{HS}_\exists(\mathsf{End}^*(V'_F)) \subseteq \mathcal{V}$. ∎

It follows that the R-spread closed semivariety $\mathbb{K}(\mathcal{V})$ of pre-hermitian spaces over Λ generated by $\mathcal{S}_\mathcal{V}$ is the union of the classes $\mathcal{G}_\alpha$, where α ranges over all ordinals. Thus in view of the assumption $\mathcal{V} = \mathsf{W}_\exists(\mathcal{K}_\mathcal{V})$ and Claim 1, one gets by Theorem 11.3(ii)

$$\mathcal{V} \subseteq \mathcal{R}\big(\mathbb{K}(\mathcal{V})\big) = \mathsf{W}_\exists\big(\overset{*}{\mathsf{End}}(V_F) \mid V_F \in \mathbb{K}(\mathcal{V}), \dim V_F < \omega\big) \subseteq \mathcal{V}.$$

To prove uniqueness, let $\mathcal{S}$ and $\mathcal{S}'$ be small R-spread closed semivarieties of pre-hermitian spaces over Λ such that $\mathcal{R}(\mathcal{S}) = \mathcal{V} = \mathcal{R}(\mathcal{S}')$. For any $V_F \in \mathcal{S}$ with $\dim V_F < \omega$, we have $\mathsf{End}^*(V_F) \in \mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{S}')$, whence V_F is an R-spread of $\mathcal{S}'$ and $V_F \in \mathcal{S}'$. Similarly, interchanging the roles of $\mathcal{S}$ and $\mathcal{S}'$, we get that $\mathcal{S}$ and $\mathcal{S}'$ have the same artinian members.

To deal with the case of universal classes, one includes into the union $\mathcal{G}_\alpha$ a third class, namely $\mathsf{S}(\mathcal{G}_\alpha)$. Claim 1 and its proof remain valid, only the case of the third class remains to be considered. Indeed, assume that $V_F \in \mathcal{G}_{\alpha+1}$ is a 2-sorted substructure of $W_K \in \mathcal{G}_\alpha$ and let $U_F \in \mathsf{S}_{1\mathsf{f}}(V_F)$. Then $U_F \in \mathsf{S}(W_K)$ and $U_F \in \mathcal{G}_{\alpha+1}$ by definition. Moreover, U_F is a 2-sorted substructure of the K-subspace U'_K of W_K spanned by U; that is, U_F is tensorially embedded into U'_K. In particular, $U'_K \in \mathsf{S}_{1\mathsf{f}}(W_K)$ and the inductive hypothesis yields $U'_K \in \mathcal{G}_\alpha$ and $\mathsf{End}^*(U'_K) \in \mathcal{V}$. As $\mathsf{End}^*(U_F)$ embeds into $\mathsf{End}^*(U'_K)$ by Lemma 11.2(i), it follows that $\mathsf{End}^*(U_F) \in \mathcal{V}$.

(i) The proof follows the same lines as the one of (ii) replacing Λ by $\mathbb{Z}$, Proposition 4.9 by Proposition 7.7, Proposition 4.3(iii) by Proposition 7.2(i), and Theorem 11.3(ii) by Theorem 11.3(i). For $L \in \mathcal{K}_\mathcal{V}$, one has to require that $3 \leqslant \dim L < \omega$.

■

For results of the same type as Theorem 11.6, see also [29, Theorems 4.4–5.4].

12. ∃-varieties and representations

We first consider a condition on $\mathcal{S}$ under which the class of representables is an ∃-variety. Then we review the approach of Micol [44] to capture ∃-varieties via the concept of generalized representation.

A semivariety $\mathcal{S}$ of pre-hermitian spaces over Λ is a *variety* if for any finite-dimensional V_{0F_0}, $V_{1F_1} \in \mathcal{S}$, there is a joint tensorial extension $V_F \in \mathcal{S}$.

Proposition 12.1. *If $\mathcal{S}$ is a variety of pre-hermitian spaces over Λ, then $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are ∃-varieties.*

Proof. In view of Proposition 9.2(iv) and Theorem 11.3, it suffices to notice that for any finite-dimensional spaces $V_{0F_0}, V_{1F_1} \in \mathcal{S}$, the structures $\mathsf{End}^*(V_{0F_0}) \times \mathsf{End}^*(V_{1F_1})$ and $\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1})$ have a faithful representation within some member of $\mathcal{S}$ by Lemma 11.2(ii).

■

Classes $\mathcal{L}(\mathcal{S})$ of CMILs having a faithful representation within some member of a class $\mathcal{S}$ of orthogeometries have been considered in [22]. The closure properties of Theorem 11.3(i) hold also in this case. with $\mathsf{S}(\mathcal{S})$ denoting formation of non-degenerate subgeometries of members of $\mathcal{S}$, $\mathsf{S}_{1\mathsf{f}}(\mathcal{S})$ and $\mathsf{S}_{1\mathsf{q}}(\mathcal{S})$ — formation of non-degenerate finite-dimensional subspaces and of subquotients $U/\operatorname{\mathsf{rad}} U$, where $U_F \in \mathcal{S}$ and $U = U^{\perp\perp}$. In addition, one has the class $\mathsf{U}(\mathcal{S})$ of all disjoint orthogonal unions of members of $\mathcal{S}$ and thus $\mathsf{P}\big(\mathcal{L}(\mathcal{S})\big) \subseteq \mathcal{L}\big(\mathsf{U}(\mathcal{S})\big)$, cf. [22, Theorem 2.2]. Moreover, mimicking the concept of an L-spread and the proof of Theorem 11.6, one obtains

Theorem 12.2. *For any $\exists$-variety $\mathcal{V}$ of CMILs generated by its finite-dimensional members, there is a small L-spread and U-closed semivariety [universal class] $\mathcal{S}$ of orthogeometries such that $\mathcal{V} = \mathcal{L}(\mathcal{S})$. Moreover, such a class $\mathcal{S}$ is unique.*

The objective of Micol [44] was to derive results for $*$-regular rings, analogous to those above. Of course, representation requires some structure of the type of sesquilinear spaces. Apparently, in general there is no axiomatic class of such spaces which would serve for representing direct products of representable structures. Micol solved this problem by introducing the concept of a *generalized representation*. This concept was transferred to MOLs by Niemann [46].

A *g-representation* of $A \in$ CMIL $[A \in \mathcal{R}_\Lambda]$ within a class $\mathcal{S}$ of pre-hermitian spaces is a family $\{\varepsilon_i \mid i \in I\}$ of representations ε_i of A in $V_{iF_i} \in \mathcal{S}$, $i \in I$. It is *faithful* if $\bigcap_{i\in I} \operatorname{\mathsf{ker}} \varepsilon_i$ is trivial. Let $\mathcal{L}_{\mathsf{g}}(\mathcal{S})$ $[\mathcal{R}_{\mathsf{g}}(\mathcal{S})]$ denote the class of all $A \in$ CMIL $[A \in \mathcal{R}_\Lambda]$ having a faithful g-representation within $\mathcal{S}$; equivalently, the class of structures A having a subdirect decomposition into factors $\varepsilon_i(A)$, $i \in I$, which have a faithful representation within $\mathcal{S}$.

Call an artinian algebra $R \in \mathcal{R}_\Lambda$ *strictly artinian* if $I = I^*$ for any ideal I of R. By the Wedderburn–Artin Theorem, this is equivalent to the fact that R is isomorphic to a direct product of strictly simple factors (cf. [38, §3.4]). Similarly, call a finite-dimensional CMIL L *strictly finite-dimensional* if $\theta = \theta'$ for any lattice congruence θ of L. By [6, Theorem IV.7.10]), this is equivalent to the fact that L is a direct product of strictly simple factors.

Proposition 12.3. *The following statements are true.*

(i) *For any semivariety $\mathcal{S}$ of pre-hermitian spaces, the class $\mathcal{L}_{\mathsf{g}}(\mathcal{S}) = \mathsf{P}_{\mathsf{s}\exists}\big(\mathcal{L}(\mathcal{S})\big)$ $[\mathcal{R}_{\mathsf{g}}(\mathcal{S}) = \mathsf{P}_{\mathsf{s}\exists}\big(\mathcal{R}(\mathcal{S})\big)]$ is an $\exists$-variety generated by its strictly simple finite-dimensional [artinian] members, which are of the form $\mathbb{L}(V_F)$ $[\operatorname{\mathsf{End}}^*(V_F)]$ with $V_F \in \mathcal{S}$, $\dim V_F < \omega$.*

(ii) *For any $\exists$-variety $\mathcal{V} \subseteq$ CMIL $[\mathcal{V} \subseteq \mathcal{R}_\Lambda]$ which is generated by its strictly finite-dimensional, of dimension at least 3, [artinian] members, there is a semivariety $\mathcal{S}$ of pre-hermitian spaces such that $\mathcal{V} = \mathcal{L}_{\mathsf{g}}(\mathcal{S})$ $[\mathcal{V} = \mathcal{R}_{\mathsf{g}}(\mathcal{S})]$.*

(iii) *$A \in \mathcal{L}_{\mathsf{g}}(\mathcal{S})$ $[A \in \mathcal{R}_{\mathsf{g}}(\mathcal{S})]$ if and only if A has an atomic extension $\hat{A}$ which is a subdirect product of atomic strictly subdirectly irreducible structures A_i such that $(A_i)_{\mathsf{f}} \cong \mathbb{L}(V_{iF_i})_{\mathsf{f}}$ [the minimal ideal of A_i is isomorphic to $J(V_{iF_i})$] with $V_{iF_i} \in \mathcal{S}$.*

Proof. Statement (i) follows from Propositions 9.2(iii)–(iv), 7.7, 4.9, and Theorem 11.3. Statement (ii) follows from Propositions 9.2(iv), 7.7, 4.9, and Theorem 11.6. Finally, statement (iii) follows from Propositions 7.3, 4.5 and Theorems 7.6, 4.7. ■

For $*$-regular rings, the result of Proposition 12.3 is in essence due to Micol [44]. To prove that g-representability is preserved under homomorphic images, she axiomatized families of inner product spaces as 3-sorted structures, where the third sort mimics the index set I. Again, a saturation property is needed for the proof and regularity is crucial. The fact that the $\exists$-variety of g-representable structures is generated by its artinian members was shown by her reducing to countable subdirectly irreducible structures R, deriving countably based representation spaces (and forming 2-sorted subspaces), and using the approach of Tyukavkin [50] with respect to a countable orthogonal basis. Conversely, a substantial part of Theorem 11.3 follows from Proposition 12.3.

A. Appendix: Existence semivarieties

We characterize $\exists$-(semi)varieties contained in CMIL or in $\mathcal{R}_\Lambda$ as model classes, proving at the same time the operator identities of Proposition 9.2. With no additional effort, this can be done to include other classes of algebraic structures.

Given a set Σ of first order axioms, by $\operatorname{Mod}\Sigma$ we denote the model class $\{A \mid A \models \Sigma\}$ of Σ. By $\operatorname{Th}\mathcal{C}$ $[\operatorname{Th}_L \mathcal{C}]$, we denote the set of sentences [from the fragment L] of first order language which are valid in $\mathcal{C}$. As usual, let $\overline{x}$ denote a sequence of variables of length being given by context.

Definition A.1. A class $\mathcal{C}_0$ of algebraic structures of the same similarity type is *regular* if there is a (possibly empty) set Ψ_0 of conjunctions $\alpha(\overline{x}, y)$ of atomic formulas (i.e. formulas of the form $\bigwedge_{i=1}^k s_i(\overline{x}, y) = t_i(\overline{x}, y)$) and a class $\mathcal{S}$ such that

(i) $\mathcal{C}_0 = \mathcal{S} \cap \operatorname{Mod}\{\forall\overline{x}\exists y\ \alpha(\overline{x}, y) \mid \alpha(\overline{x}, y) \in \Psi_0\}$;
(ii) $\mathcal{S}$ is closed under S and $\mathcal{C}_0$ is closed under H and P;
(iii) for any structures A, $B \in \mathcal{C}_0$, for any surjective homomorphism $\varphi\colon A \to B$, for any formula $\alpha(\overline{x}, y) \in \Psi_0$, and for any $\overline{a}$, $b \in B$ such that $B \models \alpha(\overline{a}, b)$, there are $\overline{c}$, $d \in A$ such that $\varphi(\overline{c}) = \overline{a}$, $\varphi(d) = b$, and $A \models \alpha(\overline{c}, d)$.

More generally, one may admit $\alpha(\overline{x}, y)$ of the form $\bigwedge_{i=1}^{k} p_i\big(t_1(\overline{x}, y), \ldots, t_m(\overline{x}, y)\big)$, where p_i is a relation symbol of arity m.

From Definition A.1(ii) it follows immediately that any regular class is closed under $\mathsf{P_u}$. In the sequel, we shall fix a regular class $\mathcal{C}_0$ and write for any $\mathcal{C} \subseteq \mathcal{C}_0$:

$$\mathsf{S}_{\exists}(\mathcal{C}) = \mathcal{C}_0 \cap \mathsf{S}(\mathcal{C}) \quad \text{and} \quad \mathsf{P}_{\mathsf{s}\exists}(\mathcal{C}) = \mathcal{C}_0 \cap \mathsf{P_s}(\mathcal{C}).$$

Let $\mathcal{C}_0$ be a regular class. A *Skolem expansion* A^* of $A \in \mathcal{C}_0$ adds for each $\alpha(\overline{x}, y) \in \Psi_0$ an operation f_α on A such that $A \models \alpha(\overline{a}, f_\alpha(\overline{a}))$ for all $\overline{a} \in A$.

Definition A.2. A class $\mathcal{C}_0$ is *strongly regular* if it is regular and

(iii′) for any structures A, $B \in \mathcal{C}_0$, for any surjective homomorphism $\varphi\colon A \to B$, for any formula $\alpha(\overline{x}, y) \in \Psi_0$, for any $\overline{a}$, $b \in B$ such that $B \models \alpha(\overline{a}, b)$, and for any $\overline{c} \in A$ such that $\varphi(\overline{c}) = \overline{a}$ there is $d \in A$ such that $\varphi(d) = b$ and $A \models \alpha(\overline{c}, d)$.

Remark A.3. It is obvious that if a class $\mathcal{C}_0$ satisfies (iii′) of Definition A.2, then $\mathcal{C}_0$ satisfies (iii) of Definition A.1. For any strongly regular class $\mathcal{C}_0$, for any A, $B \in \mathcal{C}_0$, and for any surjective homomorphism $\varphi\colon A \to B$, if B^* is a Skolem expansion of B, then there is a Skolem expansion A^* of A such that $\varphi\colon A^* \to B^*$ is a homomorphism. Clearly, $\mathcal{C}_0$ is strongly regular if it satisfies (i)–(ii) of Definition A.1 and for any $\alpha \in \Psi_0$ and for any $\overline{a} \in A \in \mathcal{C}_0$, there is unique b such that $\alpha(\overline{a}, b)$. This applies, in particular, to completely regular [inverse] semigroups.

In what follows, when we speak of a [strongly] regular class $\mathcal{C}$, we always assume that the set of formulas Ψ_0 and the classes $\mathcal{C}_0$ and $\mathcal{S}$ are given according to Definition A.1 [Definition A.2, respectively].

Proposition A.4. *For any variety $\mathcal{V}$ having ring* [*bounded modular lattice*] *reducts the class of all regular* [*complemented, respectively*] *members is a strongly regular class. For any variety $\mathcal{V}$ having ∗-ring reducts, the class of all ∗-regular members is a strongly regular class; in particular, $\mathcal{R}^*_\Lambda$ is strongly regular.*

Proof. See [28, Lemma 9]. For convenience, we give a proof here. In the ring case, let Ψ_0 consist of the formula $xyx = y$ and let $\mathcal{S} = \mathcal{V}$. Then $\mathcal{C}_0$ defined as in Definition A.1(i) consists of the regular members of $\mathcal{V}$. Closure of $\mathcal{C}_0$ under H and P is obvious. The proof of (iii′) essentially goes as in [17, Lemma 1.4]. Indeed, the two-sided ideal $I = \ker\varphi$ is regular (Proposition 3.1(i)). Let $c \in A$ be such that $a = \varphi(c)$, and let $aba = a$ in B. There is $y \in A$ such that $\varphi(y) = b$. Then $c - cyc \in I$. Since I is regular, there is $u \in I$ such that $(c - cyc)u(c - cyc) = c - cyc$. It follows from the latter that $cuc - cycuc - cucyc + cycucyc + cyc = c$. Taking $d = u - ucy - ycu + ycucy + y$, we get

$cdc = cuc - cucyc - cycuc + cycucyc + cyc = c$ and $d - y = u - ucy - ycu + ycucy \in I$, whence $\varphi(d) = b$. In the $*$-regular variant let $\mathcal{S} = \mathcal{V} \cap \mathrm{Mod}(\forall x\ xx^* = 0 \to x = 0)$ and use Proposition 3.1(v).

In the lattice case, let $a \oplus b = 1$ in B and $c \in A$ such that $a = \varphi(c)$. Choose y with $\varphi(y) = b$ and then d_1, d_2, d which are complements of cy in $[0, y]$, $c + y$ in $[y, 1]$, and y in $[d_1, d_2]$, respectively, to obtain $c \oplus d = 1$ and $\varphi(d) = b$. ■

We consider fragments of the first order language associated with a given regular class $\mathcal{C}_0$. Let L_u consist of all quantifier free formulas; up to equivalence, we may assume that L_u consists of conjunctions of formulas $\bigwedge_{i=1}^{n} \beta_i \ \to \ \bigvee_{j=1}^{m} \gamma_j$, where β_i, γ_j are atomic formulas and n, $m \geqslant 0$. The set $L_q \subseteq L_u$ of all *quasi-identities* is defined by $m = 1$. The set L_p consists of all formulas of the form

$$\bigwedge_{i=1}^{n} \alpha_i(\overline{x}_i, y_i) \ \to \ \bigvee_{j=1}^{m} \gamma_j,$$

where $n \geqslant 0$, $m \geqslant 1$, and $\alpha_i(\overline{x}_i, y_i) \in \Psi_0$. Then $L_e \subseteq L_p$ is defined by $m = 1$; its members are called *conditional identities*, while those of L_p are *conditional disjunctions of equations*. As usual, validity of a formula means validity of its universal closure. We write Th_x instead of Th_{L_x}. Define the concepts of $\exists$-*semivariety* and $\exists$-*variety* and the operators $\mathsf{W}_\exists$ and $\mathsf{V}_\exists$ in analogy to Section 9.

Theorem A.5. *Let $\mathcal{C}_0$ be a regular class and let $\mathcal{C} \subseteq \mathcal{C}_0$. Then the following statements hold.*

(i) $\mathcal{C}_0 \cap \mathrm{Mod}\,\mathrm{Th}_u\,\mathcal{C} = \mathsf{S}_\exists\mathsf{P}_\mathsf{u}(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by universal sentences relatively to $\mathcal{C}_0$ if and only if it is closed under $\mathsf{S}_\exists$ and P_u.*

(ii) $\mathcal{C}_0 \cap \mathrm{Mod}\,\mathrm{Th}_q\,\mathcal{C} = \mathsf{S}_\exists\mathsf{P}_\mathsf{u}\mathsf{P}_\omega(\mathcal{C}) = \mathsf{S}_\exists\mathsf{P}\mathsf{P}_\mathsf{u}(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by quasi-identities relatively to $\mathcal{C}_0$ if and only if it is closed under $\mathsf{S}_\exists$, P_u, and P_ω [under $\mathsf{S}_\exists$, P_u, and P, respectively].*

(iii) $\mathcal{C}_0 \cap \mathrm{Mod}\,\mathrm{Th}_p\,\mathcal{C} = \mathsf{H}\mathsf{S}_\exists\mathsf{P}_\mathsf{u}(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by conditional disjunctions of equations relatively to $\mathcal{C}_0$ if and only if it is an $\exists$-semivariety.*

(iv) $\mathcal{C}_0 \cap \mathrm{Mod}\,\mathrm{Th}_e\,\mathcal{C} = \mathsf{H}\mathsf{S}_\exists\mathsf{P}_\mathsf{u}\mathsf{P}_\omega(\mathcal{C}) = \mathsf{H}\mathsf{S}_\exists\mathsf{P}\mathsf{P}_\mathsf{u}(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by conditional identities relatively to $\mathcal{C}_0$ if and only if it is an $\exists$-variety [closed under H, $\mathsf{S}_\exists$, P_u, and P_ω, respectively].*

Of course, the statements of Theorem A.5 are well-known results in the case of empty Ψ_0. Proofs of (i) and (ii) are included since they can be seen as a preparation for proofs of (iii)–(iv); the latter are our primary interest.

Proof. For the model classes, relatively to $\mathcal{C}_0$, closure under the operators is granted by the properties of $\mathcal{C}_0$ and well-known preservation properties for the relevant kinds of first order sentences. In particular in cases (iii)–(iv), the inclusion $\mathsf{H}(\mathcal{C}) \subseteq \operatorname{Mod}\operatorname{Th}_x \mathcal{C}$ follows directly from Definition A.1(iii).

Conversely, we have to show that any member of the relative model class can be obtained from $\mathcal{C}$ via the operators. This relies on adapting the method of diagrams. Given a structure A, let $a \mapsto x_a$ be a bijection onto a set of variables and let $\bar{x} = (x_a \mid a \in A)$. We consider quantifier free formulas $\chi(\bar{x})$ in these variables; evaluations $\bar{x}$ in a structure B are given as $\bar{b} = (b_a \mid a \in A) \in B^A$, and we write $B \models \chi(\bar{b})$ if $\chi(\bar{x})$ is valid under evaluation $\bar{b}$. For a set $\Phi = \Phi(\bar{x})$ of formulas, $B \models \Phi(\bar{b})$ if $B \models \chi(\bar{b})$ for all $\chi(\bar{x}) \in \Phi$. Let At denote the set of atomic formulas and let

$$\Delta^+(A) = \{\chi(\bar{x}) \in At \mid A \models \chi(\bar{a})\};$$
$$\Delta^-(A) = \{\neg\chi(\bar{x}) \mid \chi(\bar{x}) \in At,\ A \not\models \chi(\bar{a})\};$$
$$\Delta^0(A) = \Big\{\alpha\big(t_1(\bar{x}), \dots, t_n(\bar{x}), x_a\big) \,\Big|\, t_1, \dots, t_n \text{ are terms, } \alpha(x_1, \dots, x_n, y) \in \Psi_0, \\ A \models \alpha\big(t_1(\bar{a}), \dots, t_n(\bar{a}), a\big)\Big\};$$
$$\Delta_u(A) = \Delta_q(A) = \Delta^+(A) \cup \Delta^-(A);$$
$$\Delta_p(A) = \Delta_e(A) = \Delta^0(A) \cup \Delta^-(A).$$

For $x \in \{u, q, p, e\}$ and a finite subset Φ of $\Delta_x(A)$, let $\Phi^- = \Phi \cap \Delta^-(A)$, $\Phi^+ = \Phi \setminus \Phi^-$, and let $\Phi^\dagger$ denote the formula

$$\bigwedge_{\varphi \in \Phi^+} \varphi \to \bigvee_{\neg\chi \in \Phi^-} \chi;$$

while for $\neg\chi \in \Phi^-$, let $\Phi^\dagger_\chi$ denote the quasi-identity

$$\bigwedge_{\varphi \in \Phi^+} \varphi \to \chi.$$

Thus for any finite $\Phi \subseteq \Delta_u(A)$ and for $\chi \in \Phi^-$, we have $\Phi^\dagger \in L_u$ and $\Phi^\dagger_\chi \in L_q$, while for any finite $\Phi \subseteq \Delta_p(A)$ and for $\chi \in \Phi^-$, we have $\Phi^\dagger \in L_p$ and $\Phi^\dagger_\chi \in L_e$. Observe that $A \not\models \Phi^\dagger$ and $A \not\models \Phi^\dagger_\chi$ in any case (verified by substituting x_a with a). Let $A \in \mathcal{C}_0 \cap \operatorname{Mod}\operatorname{Th}_x \mathcal{C}$. We have to obtain A from $\mathcal{C}$ by means of operators.

First, we consider the case $x \in \{u, p\}$. Let $\Phi \subseteq \Delta_x(A)$ be finite. As $A \not\models \Phi^\dagger$, we have that $\Phi^\dagger \notin \operatorname{Th}_x \mathcal{C}$. Thus there are a structure $B_\Phi \in \mathcal{C}$ and $\bar{b}_\Phi = (b_{\Phi a} \mid a \in A) \in B_\Phi^A$ such that $B_\Phi \not\models \Phi^\dagger(\bar{b}_\Phi)$, i.e. $B_\Phi \models \Phi(\bar{b}_\Phi)$.

As in the proof of the Compactness Theorem, let I be the set of all finite subsets of $\Delta_x(A)$ and let $\mathcal{U}$ be one of the ultrafilters containing all sets $\{\Psi \in I \mid \Psi \supseteq \Phi\}$, where $\Phi \in I$. Let $B = \prod_{\Phi \in I} B_\Phi/\mathcal{U}$, $b_a = (b_{\Phi a} \mid \Phi \in I)/\mathcal{U}$ and $\bar{b} = (b_a \mid a \in A)$. By (the quantifier free part of) the Łoś Theorem, we have $B \models \Delta_x(A)(\bar{b})$. Moreover, $B \in \mathsf{P_u}(\mathcal{C}) \subseteq \mathcal{C}_0$.

Let C be the subalgebra of B generated by the set $\{b_a \mid a \in A\}$. We claim that $C \in \mathcal{C}_0$, i.e. $C \in \mathsf{S}_\exists(B)$. Indeed, let $\alpha(x_1, \dots, x_n, y) \in \Psi_0$ and let $c_1, \dots, c_n \in C$. As C is generated by the set $\{b_a \mid a \in A\}$, there are terms $t_1(\bar{x}), \dots, t_n(\bar{x})$ such that $c_i = t_i(\bar{b})$ for all $i \in \{1, \dots, n\}$. Since $A \in \mathcal{C}_0$, by Definition A.1(i) there is $a \in A$ such that

$$A \models \alpha\big(t_1(\bar{a}), \dots, t_n(\bar{a}), a\big).$$

Therefore,

$$\alpha\big(t_1(\bar{x}), \dots, t_n(\bar{x}), x_a\big) \in \Delta^+(A) \cap \Delta^0(A).$$

Since $B \models \Delta_x(A)(\bar{b})$, we conclude that $B \models \alpha\big(t_1(\bar{b}), \dots, t_n(\bar{b}), b_a\big)$. This implies that $C \models \alpha(c_1, \dots, c_n, b_a)$. On the other hand, $B \in \mathsf{P_u}(\mathcal{C}) \subseteq \mathcal{C}_0 \subseteq \mathcal{S}$, as $\mathcal{C}_0$ is closed under $\mathsf{P_u}$ by Definition A.1(ii). Therefore, $C \in \mathsf{S}(B) \subseteq \mathsf{S}(\mathcal{S}) \subseteq \mathcal{S}$ again by Definition A.1(ii). This implies by Definition A.1(i) that $C \in \mathcal{C}_0$ which is our desired conclusion. Furthermore, the map

$$\varphi\colon C \to A; \quad t(\bar{b}) \mapsto t(\bar{a})$$

is well defined (since $B \models \Delta^-(A)(\bar{b})$), a homomorphism (in view of term composition), and surjective (since $\varphi(b_a) = a$). Moreover, in case $x = u$, φ is an isomorphism, as $B \models \Delta^+(A)(\bar{b})$. This proves (i) and (iii).

Let $x \in \{q, e\}$. Given a finite subset $\Phi \subseteq \Delta_x(A)$ and $\neg\chi \in \Phi^-$, one has $A \not\models \Phi^\dagger_\chi$, whence $\Phi^\dagger_\chi \notin \mathrm{Th}_x\,\mathcal{C}$. Thus there are a structure $B_{\Phi,\chi} \in \mathcal{C}$ and $\bar{b}_{\Phi\chi} = (b_{\Phi\chi a} \mid a \in A) \in B^A_{\Phi,\chi}$ such that

$$B_{\Phi,\chi} \models \Phi^+(\bar{b}_{\Phi\chi}) \quad \text{and} \quad B_{\Phi,\chi} \models \neg\chi(\bar{b}_{\Phi\chi}).$$

Taking $B_\Phi = \prod_{\neg\chi \in \Phi^-} B_{\Phi,\chi} \in \mathsf{P}_\omega(\mathcal{C})$ and $b_{\Phi a} = (b_{\Phi\chi a} \mid \neg\chi \in \Phi^-)$, we get that $B_\Phi \models \Phi(\bar{b}_\Phi)$. As above, let $B = \prod_{\Phi \in I} B_\Phi/\mathcal{U}$, $b_a = (b_{\Phi a} \mid \Phi \in I)/\mathcal{U}$, so that $B \models \Delta_x(A)(\bar{b})$. Let C be again the subalgebra of B generated by the set $\{b_a \mid a \in A\}$. We get as above that $C \in \mathsf{S}_\exists\mathsf{P_u}\mathsf{P}_\omega(\mathcal{C})$. Thus $A \in \mathsf{H}(C)$ for $x = e$ and $A \cong C$ for $x = q$ follow exactly as above.

It remains to show that $A \in \mathsf{HS}_\exists\mathsf{PP_u}(\mathcal{C})$ if $x = e$ and $A \in \mathsf{S}_\exists\mathsf{PP_u}(\mathcal{C})$ if $x = q$. Here, we fix $\neg\chi \in \Delta(A)^-$ and consider the set $I_\chi = \{\Phi \in I \mid \neg\chi \in \Phi^-\}$. Then there is a non-principal ultrafilter $\mathcal{U}_\chi$ on I which contains all sets $\{\Psi \in I_\chi \mid \Psi \supseteq \Phi\}$ with $\Phi \in I_\chi$. Take

$$B_\chi = \prod_{\Phi \in I_\chi} B_{\Phi,\chi}/\mathcal{U}_\chi; \quad b_{\chi a} = (b_{\Phi\chi a} \mid \Phi \in I_\chi)/\mathcal{U}_\chi; \quad \bar{b}_\chi = (b_{\chi a} \mid a \in A),$$

so that $B_\chi \models \neg\chi(\bar{b}_\chi)$ and $B_\chi \models \Delta^+(A)(\bar{b}_\chi)$ if $x = q$, $B_\chi \models \Delta^0(A)(\bar{b}_\chi)$ if $x = e$. Then

$$B' = \prod_{\neg\chi \in \Delta^-(A)} B_\chi \in \mathsf{PP_u}(\mathcal{C}); \quad B' \models \Delta_x(A)(\bar{b}'), \text{ where } b'_a = (b_{\chi a} \mid \chi \in \Delta^-(A)).$$

Let C' be the subalgebra of B' generated by the set $\{b'_a \mid a \in A\}$. As above, $C' \in \mathsf{S}_\exists(B')$ and $A \in \mathsf{H}(C')$ (if $x = e$) or $A \cong C'$ (if $x = q$) via the map $\varphi'\big(t(\bar{b}')\big) = t(\bar{a})$. The proof is now complete. ■

The following recaptures part of [28, Proposition 10]. For convenience, we include proofs.

Proposition A.6. *Let $\mathcal{C}_0$ be a strongly regular class and let $\mathcal{C} \subseteq \mathcal{C}_0$. Then statements* (i)–(iv) *of Proposition 9.2 hold, analogously. Moreover*

(v) *If all members of $\mathcal{C}_0$ have a distributive congruence lattice, then $A \in \mathsf{W}_\exists(\mathcal{C})$ for any subdirectly irreducible structure $A \in \mathsf{V}_\exists(\mathcal{C})$.*

Proof. (i) This follows immediately from the well-known rules for S and the fact that $\mathcal{C}_0$ is closed under P and $\mathsf{P_u}$.

(ii) Consider structures A, B and C such that $A \in \mathcal{C}$, $C \in \mathsf{S}_\exists(B)$, and let $\varphi\colon A \to B$ be a surjective homomorphism. Then $B, C \in \mathcal{C}_0$ by Definition A.1(ii). Choose a Skolem expansion C^* of C and extend it to a Skolem expansion B^* of B. According to Remark A.3, there is a Skolem expansion A^* of A such that $\varphi\colon A^* \to B^*$ is a homomorphism. Then $C^* \in \mathsf{S}(B^*) \subseteq \mathsf{SH}(A^*) \subseteq \mathsf{HS}(A^*)$, whence $C^* \in \mathsf{H}(D^*)$ for some $D^* \in \mathsf{S}(A^*)$ and $C \in \mathsf{H}(D)$ with $D \in \mathsf{S}_\exists(A)$.

(iii) According to Theorem A.5(iii), $\mathsf{HS}_\exists\mathsf{P_u}(\mathcal{C})$ is definable, relatively to $\mathcal{C}_0$, by conditional conjunctions of equations. By the same token, it is a $\exists$-semivariety, that is the smallest one containing $\mathcal{C}$.

(iv) By the same kind of reasoning, based on Theorem A.5(iv), $\mathsf{V}_\exists(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{PP_u}(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{P_u}\mathsf{P}_\omega(\mathcal{C})$. Straightforward inclusions $\mathsf{P_u}(\mathcal{C}) \subseteq \mathsf{HP}(\mathcal{C})$ and $\mathsf{PH}(\mathcal{C}) \subseteq \mathsf{HP}(\mathcal{C})$ together with (i) imply

$$\mathsf{V}_\exists(\mathcal{C}) \subseteq \mathsf{HS}_\exists\mathsf{PHP}(\mathcal{C}) \subseteq \mathsf{HS}_\exists\mathsf{HP}(\mathcal{C}) \subseteq \mathsf{HS}_\exists\mathsf{P}(\mathcal{C}).$$

The reverse inclusion is obvious.

(v) Let $A \in \mathsf{V}_\exists(\mathcal{C})$ be subdirectly irreducible. Then by (ii), there is $B \in \mathsf{S}_\exists\mathsf{P}(\mathcal{C})$ such that $A \in \mathsf{H}(B)$. By Jónsson's Lemma, there is $C \in \mathsf{SP_u}(\mathcal{C})$ such that $A \in \mathsf{H}(C)$ and $C \in \mathsf{H}(B)$. The latter inclusion implies by Definition A.1(ii) that $C \in \mathcal{C}_0$, whence $C \in \mathsf{S}_\exists\mathsf{P_u}(\mathcal{C})$. ■

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