JOURNAL OF ALGEBRA 38, 85–92 (1976)

I-Rings with Involution

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Communicated by I. N. Herstein

Received September 18, 1974

The conditions of von Neumann regularity, π-regularity, and being an I-ring are placed on symmetric subrings of a ring with involution, and we determine when the whole ring must satisfy the same property. It is shown that any symmetric subring must satisfy any one of these properties if the whole ring does.

In this paper, we continue our investigations begun in [6], in which we determined conditions on the symmetric elements of a ring with involution that force the ring to be an I-ring. In the spirit of [5], the conditions of von Neumann regularity, π-regularity, and being an I-ring are assumed for certain subrings generated by symmetric elements, to see if these conditions are then implied for the whole ring. Any of these conditions on the ring will restrict to the subrings under consideration, but we can only show that the I-ring condition extends up. In addition, we prove that π-regularity extends to the whole ring in the presence of a polynomial identity, and that the regularity condition cannot in general, be extended to the whole ring.

1. Symmetric Subrings

Throughout this work, R will denote an associative ring with involution $*$. Let $S = \{x \in R \mid x^* = x\}$ be the symmetric elements of R, $T = \{x + x^* \mid x \in R\}$, the set of traces, and $N = \{xx^* \mid x \in R\}$, the set of norms. In [5], we considered various conditions placed on the subring generated by S. Here, we move to a more general setting, that of symmetric subrings, introduced by Lee in [7].

* This work was supported by NSF Grant GP-38601.

DEFINITION. A subring U of R is a symmetric subring if:

(1) U is generated as a ring by elements of S;

(2) $T \cup N \subset U$; and

(3) $xUx^* \subset U$ for all $x \in R$.

Condition (1) implies that $U^* = U$, and condition (2) implies that U is a Lie ideal of R (See [5, Lemma 1.1]), that is, $ux - xu \in U$ for all $u \in U$ and $x \in R$. Consequently, as is well known, U is either commutative, or contains the nonzero ideal of R generated by $[u, v] = uv - vu$ for all $u, v \in U$. (See [3, Lemma 1.3] or [5, Lemma 1.2]), Henceforth, this fact will be used without further elaboration. That the subring generated by S, or by $T \cup N$, is a symmetric subring follows, as in [7, Lemma 2] by induction and the identity, $xu_1 \cdots u_{n-1}u_nx^* = [x, u_1 \cdots u_{n-1}][u_n, x^*] + u_1 \cdots u_{n-1}(xu_nx^*) + (xu_1 \cdots u_{n-1}x^*)\, u_n - u_1 \cdots u_{n-1}xx^*u_n$.

The question of generality aside, the advantage of considering symmetric subrings, rather than the subring generated by S, and the importance of this new idea, stems from the fact that any $*$-homomorphic image of a symmetric subring is again a symmetric subring. Specifically, if $A^* = A$ is an ideal of R, then R/A inherits the involution $(r + A)^* = r^* + A$. It is easy to show that the image in R/A of a symmetric subring U is a symmetric subring of R/A. However, since R/A may contain symmetric elements that are not images of symmetric elements of R, properties of the image in R/A of the subring generated by S may not extend to the ring generated by all the symmetric elements of R/A.

2. Von Neumann Regularity and π-Regularity

We recall that a ring R is called von Neumann regular if for each $x \in R$ there exists $y \in R$ with $xyx = x$. The ring R is called π-regular if for each $x \in R$ there is a $y \in R$ and a positive integer n with $x^nyx^n = x^n$.

Note that a von Neumann regular ring must be semisimple, since xy is a nonzero idempotent for $x \neq 0$. Similarly, when R is π-regular, $J(R)$, the Jacobson radical of R, must be nil.

Our first result, which is quite easy, was discovered independently by Herstein, and the proof is essentially his.

PROPOSITION 1. *If R is von Neumann (π-) regular, so is any symmetric subring U of R.*

Proof. If R is von Neumann regular and $t \in U$, then $tyt = t$ for some $y \in R$. Consequently, $ty^*t = (tyt)y^*t$, so $t(y + y^* - yty^*)t = t$ and

$y + y^* - yty^* \in U$, by definition. If R is π-regular, then $t \in U$ satisfies $t^n y t^n = t^n$ for some $y \in R$, and we proceed as above.

The converse to Proposition 1, for von Neumann regularity, is false, as one can see by considering the direct sum of an $n \times n$ matrix ring over a field with a 2-torsionfree ring with trivial multiplication. Using transpose as the involution on the matrix ring and the map $x^* = -x$ on the second factor, one obtains a ring that is not von Neumann regular, but in which the subring generated by S is von Neumann regular.

Even assuming that R is semiprime, one cannot prove the converse to Proposition 1, as [6, Example 2] shows. The defect with this example is that S is in the center of R, which implies that R is a subdirect product of orders in simple algebras, each four-dimensional or less over its center. Since it frequently happens that such "small" algebras must be excluded from consideration, one would like a better example. We present such an example when R is primitive. Note that one cannot assume much more and obtain a counterexample, since, when R is simple, either $R = U$ or R is finite-dimensional over its center; in either case, R is von Neumann regular if U is.

EXAMPLE. Let F be a field with char $F \neq 2$, and V a countable dimensional vector space over F with basis $\{v_0, v_1, \ldots\}$. Define a bilinear form (,) on V by setting $(v_{2i}, v_{2i-1}) = 1$, $(v_{2i+1}, v_{2i}) = -1$, and $(v_k, v_j) = 0$ for $j = k$ or $|j - k| > 1$, and extending to V bilinearly. Let H be the ring of all finite-rank continuous linear transformations from V to V. Then, H is a simple ring with involution, where $*$ is the adjoint with respect to (,) [4, Chap. 4]. Furthermore, H is von Neumann regular, which follows in our situation from the fact that H is isomorphic to the ring of countable by countable matrices over F having only finitely many nonzero entries [4, p. 89].

Define T on V by $(v_{2i})T = v_{2i+1}$ and $(v_{2j+1})T = 0$. Then, $T^2 = 0$, $T \notin H$, and $T^* = -T$. The last fact follows from the relations

$$\begin{aligned}(v_{2i+1}T, v_{2j}) &= 0 = (v_{2i+1}, (v_{2j})(-T)),\\ (v_{2i}T, v_{2j}) &= 0 \quad \text{if } i \neq j\\ &= -1 \quad \text{if } i = j,\end{aligned}$$

and

$$\begin{aligned}(v_{2i}, (v_{2j})(-T)) &= 0 \quad \text{if } i \neq j\\ &= -1 \quad \text{if } i = j.\end{aligned}$$

Let R be the subalgebra of $\mathrm{Hom}_F(V, V)$ generated by H and T. R is primitive since V is irreducible as an H module, and the subring generated by S is H, so it is von Neumann regular. Given that $r \in R$, write $r = h + fT$ for $h \in H$

and $f \in F$. If T is a regular element of R, then $TrT = T$ for some $r \in R$. But $T(h + fT)T = ThT \in H$, since H is an ideal of R and $T^2 = 0$. Thus, $T \notin TRT$ and R is not von Neumann regular.

Note that in our example, $R^2 \subset H$, so R is π-regular. In fact, this must be the case for any such example, as we shall show in Theorem 4. Our next result considers the situation when a symmetric subring is π-regular.

THEOREM 2. *If R satisfies a polynomial identity and U is a π-regular symmetric subring of R, then R is π-regular.*

Proof. Using [2, Theorem 2.3, proof of Corollary 2.4], it suffices to show that each prime factor ring of R is von Neumann regular. Begin by considering a prime ideal P of R with $P^* \neq P$. The image of P^* in R/P is a nonzero ideal, each element of which is the image of a trace from R. Consequently, $(P^* + P)/P \subset (U + P)/P$ and is π-regular. Also, $(P^* + P)/P$ satisfies a polynomial identity and is a prime ring, since it is an ideal of R/P. Hence, $Z((P^* + P)/P)$, the center of $(P^* + P)/P$, is not zero [8], and for $\bar{c} \in Z((P^* + P)/P)$, $(\bar{c})^n \bar{a} (\bar{c})^n = (\bar{c})^n$, where $\bar{a} \in (P^* + P)/P$. But $\bar{c}$ is not a zero divisor in R/P, so by standard arguments, R/P has an identity and $\bar{c}$ is invertible. Hence, $(P^* + P)/P = R/P$, the center is a field, so R/P is a simple ring [8] satisfying a polynomial identity, forcing R/P to be von Neumann regular by Kaplansky's theorem [4, p. 226].

Now, assume that $P^* = P$. Then, R/P is a prime ring with involution and satisfies a polynomial identity. As above, $Z(R/P) \neq 0$, and for $\bar{c} \in Z(R/P)$, we have $\bar{c}\bar{c}^* \in ((U + P)/P) \cap Z(R/P)$. We obtain that $\bar{c}\bar{c}^*$ is invertible when $\bar{c} \neq 0$, so $\bar{c}$ is invertible, and the same argument as before yields that R/P is von Neumann regular. Having shown that each prime factor ring of R is von Neumann regular, the proof is complete.

COROLLARY 3. *If U is a commutative symmetric subring of R and is π-regular, then R is π-regular.*

Proof. Since the set of traces satisfies the identity $xy - yx$, R itself satisfies a polynomial identity by a result of Amitsur [1]. The corollary is now immediate from Theorem 2.

Of course, the proof of Corollary 3 works just as well under the assumption that U satisfies any polynomial identity, but for our next theorem, we need the corollary only for U commutative.

THEOREM 4. *If U is a von Neumann regular, symmetric subring of R, then R is π-regular.*

Proof. Should U be commutative, the result follows from Corollary 3.

Assuming that U is not commutative, it contains the ideal K of R generated by all $xy - yx$, for $x, y \in U$. Since K is an ideal of U, it is a von Neumann regular ring. The image of U in R/K is a symmetric subring of R/K and it is also von Neumann regular. Furthermore, the definition of K implies that this image of U is a commutative ring. Applying Corollary 3 yields that R/K is π-regular. Hence, for any $x \in R$, $x^n y x^n - x^n \in K$ for some $y \in K$ and for some integer n. Using the fact that K is von Neumann regular, one easily obtains that $x^m t x^m = x^m$ for some $t \in R$, proving that R is π-regular.

3. I-Rings

A ring is called an I-ring if each nonnil left (right) ideal contains a nonzero idempotent. Equivalently, the Jacobson radical $\mathscr{J}(R)$, of R, is nil and each $x \in R - \mathscr{J}(R)$ has a multiple that is a nonzero idempotent. In this section, we show that R is an I-ring if and only if U is, for U a symmetric subring of R. The first lemma appears in [7] and is important for reducing to the semisimple case.

Lemma 5. *If U is a symmetric subring of R, then $\mathscr{J}(U) = U \cap \mathscr{J}(R)$.*

Proof. [7, Theorem 17].

To show that U an I-ring forces R to be an I-ring we must know that $\mathscr{J}(R)$ is nil. Our next lemmas demonstrate this fact.

Lemma 6. *If U is a nil symmetric subring of R, then R is nil.*

Proof. The result follows from [7, Theorem 11, Lemma 3]. Also, [5, proof of Theorem 2.2], which is a statement of the lemma for the subring generated by S, can be followed exactly for an arbitrary symmetric subring.

Lemma 7. *Let U be a symmetric subring of R. Then, $\mathscr{J}(R)$ is nil if and only if $\mathscr{J}(U)$ is nil.*

Proof. That $\mathscr{J}(R)$ is nil implies that $\mathscr{J}(U)$ is nil follows immediately from Lemma 5. Suppose that $\mathscr{J}(U) = U \cap \mathscr{J}(R)$ is nil. Since $\mathscr{J}(R)^* = \mathscr{J}(R)$, all norms and traces of $\mathscr{J}(R)$, considered as a ring with involution, are contined in $U \cap \mathscr{J}(R) = \mathscr{J}(U)$. Hence, the subring generated by these norms and traces is nil. But the subring generated by the norms and traces in any ring with involution is a symmetric subring, as we indicated in Section 1. Thus, from Lemma 6 we may conclude that $\mathscr{J}(R)$ is nil.

The main step in showing that R is an I-ring when U is comes from a modification of [6, Theorem 3].

LEMMA 8. *Let U be a symmetric subring of R. If U is a semisimple I-ring, then R is an I-ring with $\mathscr{J}(R)$ nil of index* 2.

Proof. First, for $x \in \mathscr{J}(R)$, xx^* and $x + x^*$ are in $\mathscr{J}(R) \cap U = \mathscr{J}(U) = 0$. Thus, $x^2 = x(x + x^*) = 0$ and $\mathscr{J}(R)$ is nil of index 2. To complete the proof, we show that if $x \in R$ has no nonzero idempotent as a multiple, then $x \in \mathscr{J}(R)$.

Since no multiple of x is a nonzero idempotent, and every nonzero element of U has such a multiple, we conclude that $xx^* = x^*x = xUx^* = x^*Ux = 0$. If $x + x^* \neq 0$, then for some $t \in U$, $(x + x^*)t(x + x^*)t = (x + x^*)t \neq 0$. Using $xUx^* = x^*Ux = 0$, this equation reduces to $xtxt + x^*tx^*t = xt + x^*t$. Right multiplication by xt yields $(xt)^3 = (xt)^2$, forcing $(xt)^2$ to be an idempotent, and so, $(xt)^2 = 0$. Similarly, $(x^*t)^2 = 0$, which implies that $(x + x^*)t = 0$. Consequently, $x + x^* = 0$ and $x^2 = xx^* = 0$. But is no multiple of x is a nonzero idempotent, the same must hold for xr, for any $r \in R$. Thus, $(xr)^2 = 0$, xR is a nil right ideal of R, and as desired, $x \in \mathscr{J}(R)$.

Finally we can prove

THEOREM 9. *If U is an I-ring and a symmetric subring of R, then R is an I-ring.*

Proof. If U is semisimple, just apply Lemma 8. Assuming that $\mathscr{J}(U) \neq 0$, then $\mathscr{J}(R)$ is nil by Lemma 7, so it suffices to show that $R/\mathscr{J}(R)$ is an I-ring. The image of U in $R/\mathscr{J}(R)$ is $(U + \mathscr{J}(R)/\mathscr{J}(R) \cong U/(U \cap \mathscr{J}(R)) \cong U/\mathscr{J}(U)$, by Lemma 5. Thus, the image of U is a semisimple I-ring, and a symmetric subring of $R/\mathscr{J}(R)$. Once again, using Lemma 8 will establish the theorem.

The converse to Theorem 9 is our most difficult result, and we begin by considering the case when U is commutative. The argument for this special case is an adaptation of the proof of [6, Theorem 5].

LEMMA 10. *Let U be a commutative symmetric subring of R. If R is an I-ring, then U is an I-ring.*

Proof. Since U is commutative, $U \subset S$. Also, because $\mathscr{J}(U) = U \cap \mathscr{J}(R)$ from Lemma 5, we have that $x \in U - \mathscr{J}(U)$ implies that $x \notin \mathscr{J}(R)$, and of course, that $\mathscr{J}(U)$ is nil. Thus, for any $x \in U - \mathscr{J}(U)$, there is some $y \in R$ with $xyxy = xy \neq 0$. We may assume that $yxy = y$, for if not, with $y_1 = yxy$ one easily shows that $(xy_1)^2 = xy_1 \neq 0$ and $y_1xy_1 = y_1$. Consequently, $y^*xy^* = y^*$, so $xy^* = f$ and $xy = e$ are nonzero idempotents.

Considering that $ef = xyxy^* = x(yxy^*) \in U$ and $fe = xy^*xy = x(y^*xy) \in U$, we must have $ef^2e = fe^2f$, or equivalently, $efe = fef$. Thus, $(ef)^2 = fef$ and $(ef)^3 = (efe)(fef) = (efe)(efe) = e(fef)e = efe = fef = (ef)^2$. As a result,

$(ef)^2 = x(yxy^*)x(yxy^*) = xt$ is an idempotent with $t \in U$. Should $(ef)^2 = 0$, then $fef = efe = 0$. But now, $g = e + f - ef - fe$ is an idempotent and $g = x(y + y^* - yxy^* - y^*xy) = xt$ for $t \in U$. Finally, should $g = 0$, then $e + f = ef - fe$ and multiplication by e on both sides gives $e = 0$, a contradiction. Hence, either $(ef)^2 \neq 0$ or $g \neq 0$, so xt is a nonzero idempotent for some $t \in U$, and U is an I-ring, completing the proof.

Before our final result, we recall an important definition and some trivial, but useful, facts pertaining to it.

DEFINITION. For J an ideal of R, $\operatorname{Ann} J = \{x \in R \mid Jx = 0\}$.

When R is semiprime one has that $\operatorname{Ann} J = \{x \in R \mid xJ = 0\}$, $J \cap \operatorname{Ann} J = 0$, and if $J^* = J$, then $(\operatorname{Ann} J)^* = \operatorname{Ann} J$.

THEOREM 11. *If R is an I-ring, then any symmetric subring U of R is also an I-ring.*

Proof. That $\mathscr{J}(U)$ is nil follows at once from Lemma 7. Also, using $\mathscr{J}(U) = U \cap \mathscr{J}(R)$ from Lemma 5, the image of U in $R/\mathscr{J}(R)$ is isomorphic to $U/\mathscr{J}(U)$, a symmetric subring of $R/\mathscr{J}(R)$. Thus, it suffices to prove that $U/\mathscr{J}(U)$ is an I-ring, so we have reduced to the case when R is semisimple.

Should U be commutative, we may conclude that it is an I-ring by Lemma 10. If U is not commutative, it contains the ideal K of R, generated by all $xy - yx$ for $x, y \in U$. Since K is an ideal of R, it is itself an I-ring. Consequently, if $x \in U$ and $xK \neq 0$, then $xkt \neq 0$ is an idempotent, where $k, t \in K$. Hence, $x(kt)$ is an idempotent for $kt \in K \subset U$. So if some $x \in U$ fails to have a multiple in U that is an idempotent, $xk = 0$, or equivalently, $x \in \operatorname{Ann} K$. We wish to show that $U \cap \operatorname{Ann} K$ is an I-ring.

First, note that as a consequence of the definition, $K^* = K$, so $(\operatorname{Ann} K)^* = \operatorname{Ann} K$, and also, as an ideal of R, $\operatorname{Ann} K$ is an I-ring. For $x, y \in U \cap \operatorname{Ann} K$, $xy - yx \in K \cap \operatorname{Ann} K = 0$, since R has no nilpotent ideals. Thus, $U \cap \operatorname{Ann} K$ is a commutative subring of R. Since $U \cap \operatorname{Ann} K$ is invariant under the involution, its set of symmetric elements is $S \cap U \cap \operatorname{Ann} K$, which is a subring because of commutativity.

For any $k \in \operatorname{Ann} K$, $k + k^*$ and kk^* are in U, by definition of symmetric subring, so $k + k^*$, $kk^* \in S \cap U \cap \operatorname{Ann} K$. Also, for $t \in S \cap U \cap \operatorname{Ann} K$ and $k \in \operatorname{Ann} K$, $ktk^* \in S \cap U \cap \operatorname{Ann} K$. Thus, $S \cap U \cap \operatorname{Ann} K$ is a symmetric subring of $\operatorname{Ann} K$, so applying Lemma 10 with $\operatorname{Ann} K$ replacing R and with $S \cap U \cap \operatorname{Ann} K$ as the symmetric subring, we may conclude that $S \cap U \cap \operatorname{Ann} K$ is an I-ring. Clearly, $S \cap U \cap \operatorname{Ann} K$ is also a symmetric subring of $U \cap \operatorname{Ann} K$. Therefore, by Theorem 9, $U \cap \operatorname{Ann} K$ is an I-ring.

Recall that if $x \in U$ and has no multiple in U that is an idempotent, then $x \in U \cap \operatorname{Ann} K$. Since $U \cap \operatorname{Ann} K$ is an I-ring, it follows that

$x \in \mathscr{J}(U \cap \text{Ann } K)$, which, of course, is nil. For $y \in U$, xy must not have any multiple in U that is an idempotent, so $xU \subset \mathscr{J}(U \cap \text{Ann } K)$ is a nil right ideal of U. Thus, $x \in \mathscr{J}(U) = U \cap \mathscr{J}(R) = 0$, proving the theorem.

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