ON THE EQUATIONAL THEORY OF FINITE MODULAR LATTICES

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ABSTRACT. It is shown that there is N such that there is no algorithm to decide for identities in at most N variables validity in the class of finite modular lattices. This is based on Slobodskoi's result that the Restricted Word Problem is unsolvable for the class of finite groups and relies on Freese's technique of capturing group presentations within free modular lattices.

1. INTRODUCTION

Since Dedekind's early result on modular lattices with 3 generators, calculations in modular lattices have served to reveal structure, particularly in algebraic and geometric contexts. Though, as shown by Hutchinson [15] and Lipshitz [21], the Restricted Word Problem for modular lattices is unsolvable (5 generators suffice) and so is the Triviality Problem. These results remain valid for any class of modular lattices containing the subspace lattices of some infinite dimensional vector space. Applying von Neumann's rings associated with frames, i.e. coordinate systems, in modular lattices, the proof relies on interpreting a finitely presented group with unsolvable word problem cf. Section 9, below.

On the other hand, for many rings R, including all division rings and homomorphic images of $\mathbb{Z}$, the equational theory of the class of all lattices $L({}_RM)$ of submodules of R-modules is decidable [11]; a thorough analysis has been given by Gábor Czédli and George Hutchinson [16].

Again based on frames and the fact, shown by András Huhn [14], that frames freely generate projective modular lattices, Ralph Freese [8] proved unsolvability of the Word Problem for the modular lattice $FM(5)$ with 5 free generators. On the model side, he used a construction, due to Dilworth and Hall, obtaining a modular lattice matching an upper section of one with a lower section of the other - here applied to height 2 intervals in subspace lattices of 4-dimensional vector spaces. The matching was such that results of Cohn and McIntyre could be

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used to capture group presentations within one of the associated skew fields. On the syntactic side, this structure is reflected in a sublattice of $FM(5)$, by a method to be called *Freese's technique* (definitions are given below):

- Start with terms providing, in a free modular lattice, a configuration composed of frames.
- Use "reduction" of frames (mimicking subquotients of modules) to force additional relations, e.g. characteristic p of frames for some prime p.
- Based on integers in coordinate rings, construct elements behaving well under reduction (called "stable" in [12]), a property which is inherited under reduction and change of the reference frame via glueing.
- Use stable elements obtained via glueing as group generators and force group relations on these via reduction.

Actually, Freese's proof associates a projective modular lattice with any finitely presented group and his result remains valid for all varieties of modular lattices containing all infinite modular lattices of height 6.

The case of the free modular lattice with 4 generators has been done in [12] interpreting finitely presented 2-generator groups G via a concept of "skew-frames of characteristic $p \times p$", providing 2 stable elements. Here, models are obtained by a glueing construction involving a lattice ordered system of components given as lattices of submodules of free $(\mathbb{Z}/p\mathbb{Z})G$-modules.

The result crucial for the present note is the following.

Theorem 1.1. Slobodskoi [22]. *The Restricted Word Problem for the class of finite groups is unsolvable. That is, there is a list $\bar{g} = (g_1, \ldots, g_n)$ of generator symbols and a finite set of relations $\rho_i(\bar{g})$ in the language of groups such that there is no algorithm to decide, for any word $w(\bar{g})$, whether $w(\bar{a}) = e$ for all finite groups G and all $\bar{a}$ in G satisfying the relations $\rho_i(\bar{a})$ for all i.*

Kharlampovich [17] proved the analogue for finite nilpotent groups, even more restricted classes of finite groups have been dealt with in [19]. A concise review of Slobodkoi's result has been given in [2, Section 2]. For a detailed analysis see [18, Section 7.4].

A rather immediate consequence of Thm. 1.1 is that the Restricted Word Problem is unsolvable for any class of finite modular lattices containing all subspace lattices of finite vector spaces cf. Section 9. The same applies to the Triviality Problem [13], based on the unsolvability for the class of finite groups, proved by Bridson and Wilton [2].

Theorem 1.2. *With n from Slobodkoi's result, the set of identities in $n+6$ variables, valid in all finite modular lattices, is non-recursive.*

Thm 1.2 adapts to all classes of finite modular lattices containing the particular ones constructed in Thm. 7.4 from groups in a class with unsolvable restricted word problem.

In a recent related result, Kühne and Yashfe [20] show that there is no algorithm to decide, for any finite geometric lattice L with dimension function δ, whether there is a join embedding ε of L into the subspace lattice of some vector space (over fields from any specified class) such that, for some $c \in \mathbb{N}$, $\dim \varepsilon(x) = c \cdot \delta(x)$ for all $x \in L$.

Concerning the proof of Thm. 1.2, on the model side, the construction given in [12, Section 4] is extended combining n skew frames (into a "tower") to deal with n group generators (Section 7). On the formal side, for convenience, we first capture towers by presentations projective within modular lattices (Section 3). Reductions of towers and stable elements are discussed in Section 4, coordinates and Freese's method of forcing group relations in Section 5. Finally, in Section 8 reduction according to [12, Lemma 9] is used to turn each skew frame of a tower, one at a time, into one of characteristic $p \times p$ and to obtain the stable element provided by [12, Cor. 13]. Stable elements associated with other skew frames will be transformed (thanks to Ralph Freese) into stable elements, again, and it does not matter that characteristic $p \times p$ is (supposedly) lost. For easy reference, a description of the (well known) general method is given in Section 2.

2. Presentations and reduction to identities

2.1. **Presentations.** Given a similarity type of algebraic structures, we fix a variety $\mathcal{T}$ with solvable word problem for free algebras $F\mathcal{T}(\bar{x})$ in free generators $\bar{x} = (x_1, \ldots, x_n)$, called *variables*. Elements $t(\bar{x})$ will be called *terms*. For binary operator symbols, say $+$, we write $s + \bar{t}$ to denote the list $(s + t_1, \ldots, s + t_n)$.

Due to the solvability of the word problem there is an algorithm to decide, for any terms $t(\bar{x}), s(\bar{x})$ in the absolutely free algebra, whether the *identity* $t(\bar{x}) = s(\bar{x})$ is valid in $\mathcal{T}$, i.e. whether the terms denote the same element of $F\mathcal{T}(\bar{x})$. This applies also to expansions by new constants.

To simplify notation, a list $(t_1(\bar{x}), \ldots, t_n(\bar{x}))$ is also written as $\bar{t}(\bar{x})$ and $\bar{t}(\bar{x})|_m$ stands for $(t_1(\bar{x}), \ldots t_m(\bar{x}))$ where $m \leq n$. Also, we write $\bar{t}(\bar{u}(\bar{y})) = \bar{t}(u_1(\bar{y}), , \ldots, u_n(\bar{y}))$ and the like.

Given a list $\bar{c}$ of (pairwise distinct) new constants, called *generator symbols*, a *relation* $\rho(\bar{c})$ is an expression $t(\bar{c}) = s(\bar{c})$ where $t(\bar{x})$ and $s(\bar{x})$ are terms. A (finite) *presentation* Π (also written as $(\Pi, \bar{c})$) is then given by $\bar{c}$ and a finite set of relations $\rho(\bar{c})$. Constant (i.e. variable free) terms in the language expanded by $\bar{c}$ are also referred to as "terms over Π".

A relation $\rho(\bar{c})$, as above, is *satisfied* by $\bar{a} = (a_1, \ldots, a_n)$ in $A \in \mathcal{T}$ if $t(\bar{a}) = s(\bar{a})$, we write $A \models \rho(\bar{a})$. $(A, \bar{a})$ is a *model* of Π, written as $A \models \Pi$, if all relations of Π are satisfied in A. Par abuse de language we also say that $\bar{a}$ is a Π in A and we use $\bar{c}$ to denote $\bar{a}$.

In the sequel, let $\mathcal{A} \subseteq \mathcal{T}$ denote a class of algebraic structures closed under subalgebras. We say that $(A, \bar{a})$ *is in* $\mathcal{A}$ if $A \in \mathcal{A}$. A relation $\rho(\bar{c})$ is a *consequence* of Π in $\mathcal{A}$, also *implied* by Π in $\mathcal{A}$, if $A \models \rho(\bar{a})$ for all models $(A, \bar{a})$ of Π in $\mathcal{A}$. The *Restricted Word Problem* for $\mathcal{A}$ is *unsolvable* if there is a presentation $(\Pi, \bar{c})$ such there is no algorithm to decide, for any relation $\rho(\bar{c})$, whether $\rho(\bar{c})$ is a consequence of $(\Pi, \bar{c})$ within $\mathcal{A}$.

2.2. **Transformations and strengthening.** A *transformation* within $\mathcal{A}$ of Π to a presentation Ψ in generator symbols $\bar{d} = (d_1, \ldots, d_m)$ is given by a list of terms $u_j(\bar{x})$, $j = 1, \ldots, m$, such that one has $(A, (u_1(\bar{a}), \ldots, u_m(\bar{a}))$ a model of Ψ for each model $(A, \bar{a})$ of Π in $\mathcal{A}$. The *composition* with a further transformation Ψ to Φ, given by the $v_k(\bar{y})$, is the transformation obtained by the terms $v_k(u_1(\bar{x}), \ldots, u_m(\bar{x}))$. Thus, one obtains transformations by iterated composition. The presentations Π and Ψ are *equivalent* within $\mathcal{A}$ if, in the above, one has $\Phi = \Pi$ and $A \models \bar{v}(\bar{u}(\bar{a})) = \bar{a}$, and $B \models \bar{u}(\bar{v}(\bar{b})) = \bar{b}$ for all models $(A, \bar{a})$ of Π and $(B, \bar{b})$ of Ψ. In particular, if Ψ is obtained from Φ adding generators (that is, $m > n$ and $c_i = d_i$ for $i \leq n$) and relations then Π and Ψ are equivalent within $\mathcal{A}$ if and only if there is a transformation of Φ to Ψ within $\mathcal{A}$ such that $u_j = x_j$ for $j \leq n$.

Consider presentations Π and Π^+ in the same generator symbols $\bar{c} = (c_1, \ldots, c_n)$. A transformation from Π to Π^+ within $\mathcal{A}$ given by terms $u_i(\bar{x})$, $i = 1, \ldots, n$ (also written as $u_{c_i}(\bar{x})$ with $x_i = x_{c_i}$) *strengthens* Π to Π^+ within $\mathcal{A}$ if the following hold.

(1) The relations of Π are consequences of Π^+ within $\mathcal{A}$.
(2) $u_i(\bar{a}) = a_i$ for $i = 1, \ldots, n$ and all models $(A, \bar{a})$ of Π^+ in $\mathcal{A}$.

That is, from any model $(A, \bar{a})$ of Π in $\mathcal{A}$ one obtains the model $(A, \bar{u}(\bar{a}))$ of Π^+ while models of Π^+ remain unchanged.

Considering a model $(A, \bar{a})$ of Π, it is common use to write also c_i in place of a_i, that is, the generator symbol c_i denotes the element a_i of A. In view of this, we use the notation $c_i := u_i(\bar{c})$ to indicate the terms $u_i(\bar{x})$ defining the strengthening of Π to Π^+ - without mentioning $c_i := c_i$ if $u_i(\bar{x}) = x_i$. In particular this is done if we construct a sequence of strengthenings - which, of course, provides a strengthening of the original presentation.

2.3. **Projective presentations.** A presentation Π is *projective* within $\mathcal{A}$ if there are (*witnessing*) terms $t_1^\Pi(\bar{x}), \ldots, t_n^\Pi(\bar{x})$ such that the following hold for all $A \in \mathcal{A}$ and $\bar{a}$ in A.

(1) $(A, (t_1^\Pi(\bar{a}), \ldots, t_n^\Pi(\bar{a})) \models \Pi$.
(2) If $(A, \bar{a}) \models \Pi$ then $t_i^\Pi(\bar{a}) = a_i$ for all i.

Then, of course, Π is projective within any $\mathcal{B} \subseteq \mathcal{A}$.

Fact 2.1. *If Π_1 and Π_2 are projective within $\mathcal{A}$ then so is their disjoint union, e.g. if Π_2 introduces additional generators, but no relations.*

Fact 2.2. *If Π is strengthened to Π^+ within $\mathcal{A}$ then Π^+ is projective in $\mathcal{A}$ if so is Π.*

Fact 2.3. *If Π is projective in $\mathcal{A}$, with witnessing terms $t_i^\Pi(\bar{x})$, then the identity*

$$t(t_1^\Pi(\bar{x}), \ldots, t_n^\Pi(\bar{x})) = s(t_1^\Pi(\bar{x}), \ldots, t_n^\Pi(\bar{x}))$$

is valid in $\mathcal{A}$ if and only if $t(\bar{a}) = s(\bar{a})$ for all models $(A, \bar{a})$ of Π in $\mathcal{A}$.

Now, assume that $\mathcal{A}$ is a variety, i.e. an equationally definable class. Then for each presentation $(\Pi, \bar{c})$ one has "the" algebra $F\mathcal{A}(\Pi, \bar{c})$ in $\mathcal{A}$ freely generated by $\bar{c}$ under the relations Π; here, $\bar{c}$ also denotes its image under the canonical homomorphism. This algebra is projective within $\mathcal{A}$ if and only if so is the presentation $(\Pi, \bar{c})$.

Strengthening the presentation $(\Pi, \bar{c})$ to Π^+ with additional relation $s(\bar{c}) = t(\bar{c})$ then means to provide $\bar{b}$ in $F\mathcal{A}(\Pi, \bar{c})$ such that $s(\bar{b}) = t(\bar{b})$ and $\phi(\bar{b}) = \phi(\bar{c})$ for all $A \in \mathcal{A}$ and homomorphisms $\phi : F\mathcal{A}(\Pi, \bar{c}) \to A$ such that $s(\phi(\bar{c})) = t(\phi(\bar{c}))$.

2.4. **Reducing quasi-identities to identities.** Given a signature, an *identity* or *equation* is a sentence of the form $\forall \bar{x}.\ t(\bar{x}) = s(\bar{x})$, a *quasi-identity* a sentence of the form $\forall \bar{y}.\ \alpha(\bar{y}) \Rightarrow t(\bar{y}) = s(\bar{y})$ with *antecedent* $\alpha(\bar{y}) \equiv \bigwedge_i t_i(\bar{y}) = s_i(\bar{y})$; here $t(\bar{x})$, $s(\bar{x})$, $t_i(\bar{y})$, and $s_i(\bar{y})$ are terms. Observe that, replacing variables by new constants, α is turned into a presentation.

Consider classes $\mathcal{M}_0$ and $\mathcal{G}_0$ of algebraic structures in not necessarily distinct signatures, both closed under subalgebras. The task is to *reduce* quasi-identities for $\mathcal{G}_0$ to equations for $\mathcal{M}_0$; that is, given a set Λ of quasi-identities in the language of $\mathcal{G}_0$ to construct an algorithm associating with each $\beta \in \Lambda$ an equation β^* in the language of $\mathcal{M}_0$ such that β holds in $\mathcal{G}_0$ if and only if β^* holds in $\mathcal{M}_0$.

In the sequel we describe the general structure of such algorithm to be applied to the case where $\mathcal{G}_0$ is the class of all finite groups, $\mathcal{M}_0$ the class of all finite modular lattices. Fix a set Λ_0 of formulas $\alpha(\bar{y}) \equiv \bigwedge_{i=1}^{h} w_i(\bar{y}) = v_i(\bar{y})$, $\bar{y} = (y_1, \ldots, y_{n_\alpha})$, in the language of $\mathcal{G}_0$.

Hypothesis: There is an algorithm which constructs the following in the language of $\mathcal{M}_0$.

(a) For any given $\alpha \in \Lambda_0$ a presentation $(\Pi, \bar{c})$ with $\bar{c} = (c_1, \ldots, c_N)$ and terms $\bar{u} = (u_1, \ldots, u_N)$ with $N := N_\alpha \geq n := n_\alpha$

(b) For each r-ary operation symbol f of $\mathcal{G}_0$, a term $f^{\#}(\bar{z}, \bar{x})$, $\bar{z} = (z_1, \ldots, z_r)$ where $\bar{x} = (x_1, \ldots, x_N)$.

Now, for a formula $\gamma(\bar{y})$ in the language of $\mathcal{G}_0$, the translation according to (b) into a formula in the language of $\mathcal{M}_0$ is denoted by $\gamma^{\#}(\bar{y}, \bar{x})$ and the following are required.

(i) $(\Pi, \bar{c})$ is projective for $\mathcal{M}_0$ with witnessing terms $\bar{t}$.

(ii) If $(L, \bar{a})$ is a model of $(\Pi, \bar{c})$ in $\mathcal{M}_0$ then so is $(L, \bar{u}(\bar{a}))$.

(iii) For any $\alpha \in \Lambda_0$, $(\Pi, \bar{c})$ implies $\alpha^{\#}(\bar{u}(\bar{c})|_n), \bar{u}(\bar{c}))$ in $\mathcal{M}_0$.

(iv) For any model $(L, \bar{a})$ of $(\Pi, \bar{c})$ with $L \in \mathcal{M}_0$, the algebra $G = G(L, \bar{u}|_n(\bar{a}))$ generated by $\bar{u}|_n(\bar{a})$ under the operations $\bar{b} \mapsto f^{\#}(\bar{b}, \bar{u}|_n(\bar{a}))$, f an operation symbol of $\mathcal{G}_0$, is a member of $\mathcal{G}_0$ and $(G, \bar{u}|_n(\bar{a})) \models \alpha(\bar{u}|_n(\bar{a}))$.

(v) For any $\alpha \in \Lambda_0$ and $G \in \mathcal{G}_0$, with generators $\bar{g} = (g_1, \ldots, g_n)$ such that $G \models \alpha(\bar{g})$, there is a model $(L(G, \bar{g}), \bar{a})$ of $(\Pi, \bar{c})$ with $L(G, \bar{g}) \in \mathcal{M}_0$ and $\bar{u}(\bar{a}) = \bar{a}$ and, moreover, such that there is an embedding $\omega : G \to G(L(G, \bar{g}), \bar{a})$ with $\omega(g_i) = a_i$ for $i = 1, \ldots, N$.

Lemma 2.4. *Given an algorithm satisfying the above hypothesis, there is an algorithm associating with any quasi-identity β, with antecedent $\alpha \in \Lambda_0$ in the language of $\mathcal{G}_0$, an equation β^* in the language of $\mathcal{M}_0$ such that β holds in $\mathcal{G}_0$ if and only if β^* holds in $\mathcal{M}_0$.*

In particular, if a presentation Ψ in n generators in the language of $\mathcal{G}_0$ is given, then the restricted word problem for Ψ within $\mathcal{G}_0$ reduces to the decision problem for N-variable identities within $\mathcal{M}_0$ where N is the number of generators in the presentation $(\Pi, \bar{c})$, required in (a) above.

Proof. By (b), there is an algorithm associating, uniformly for all n, N ($n \leq N$), with any term $w(\bar{y})$ in the language of $\mathcal{G}_0$ a term $w^{\#}(\bar{y}, \bar{x})$ in the language of $\mathcal{M}_0$ such that $y_i^{\#}(\bar{y}, \bar{x}) = y_i$ and

$$(f(w_1(\bar{y}), \ldots, w_n(\bar{y})))^{\#}(\bar{y}, \bar{x}) = f^{\#}(w_1^{\#}(\bar{y}, \bar{x}), \ldots, w_n^{\#}(\bar{y}, \bar{x})).$$

Now, given $\beta \equiv \forall y.(\alpha(\bar{y}) \Rightarrow w(\bar{y}) = v(\bar{y}))$ where $\alpha \in \Lambda_0$, let β^* denote the identity $\forall \bar{x}.\ \gamma(\bar{x})$ where $\gamma(\bar{x})$ denotes

$$w^{\#}(u_1(\bar{t}(\bar{x})), \ldots, u_n(\bar{t}(\bar{x})), \bar{u}(\bar{t}(\bar{x}))) =$$
$$= v^{\#}(u_1(\bar{t}(\bar{x})), \ldots, u_n(\bar{t}(\bar{x})), \bar{u}(\bar{t}(\bar{x})))$$

with $u_i(\bar{x})$ according to (a). Assume that β^* holds in $\mathcal{M}_0$ and consider $G \in \mathcal{G}_0$ and $\bar{g}$ in G such that $G \models \alpha(\bar{g})$. Given $(L(G, \bar{g}), \bar{a})$ according to (v), one has $u_i(\bar{a}) = a_i$ for all i whence, due to validity of β^*,

$$\omega(w(\bar{g})) = w^{\#}(\bar{a}|_n, \bar{a}) = w^{\#}(\bar{u}(\bar{a})|_n), \bar{u}(\bar{a})) =$$
$$= v^{\#}(\bar{u}(\bar{a})|_n, \bar{u}(\bar{a})) = v^{\#}(\bar{a}|_n, \bar{a})) = \omega(v(\bar{g}))$$

and $w(\bar{g}) = v(\bar{g})$ follows, verifying β for $\mathcal{G}_0$.

Conversely, assume that β holds for all G in $\mathcal{G}_0$ and consider any $L \in \mathcal{M}_0$ and $\bar{a} = (a_1, \ldots, a_N)$ in L. That $(L, \bar{u}(\bar{t}(\bar{a})))$ is a model of $(\Pi, \bar{c})$ is obtained combining (i) and (ii) and by (iii) it follows that $L \models \alpha^{\#}(\bar{u}(\bar{t}(\bar{a}))|_n, \bar{u}(\bar{t}(\bar{a})))$. Thus, by (iv) $G := G(L, \bar{u}(\bar{t}(\bar{a}))|_n)$ is in $\mathcal{G}_0$ and $\alpha(\bar{u}(\bar{t}(\bar{a}))|_n)$ holds in G. Now, the quasi-identity β being valid in $\mathcal{G}_0$, it follows $G \models \gamma(\bar{u}(\bar{t}(\bar{a}))|_n, \bar{u}(\bar{t}(\bar{a})))$; that is, the identity β^* holds in L for the substitution $\bar{a}$.. ☐

3. Some projective modular lattice presentations

3.1. **Terms and lattices.** For concepts of lattice theory we refer to Birkhoff [1], for modular lattices also von Neumann [23]. For better readability, joins and meets will be written as $x + y$ and $x \cdot y = xy$, assuming associativity, commutativity. and idempotency for both operations. That is, the term algebra $F\mathcal{T}(\bar{x})$ is the free algebra in the variety $\mathcal{T}$ of algebras $(A, +, \cdot)$ where $(A, +)$ and $(A, \cdot)$ are commutative idempotent monoids. Thus, the word problem for free algebras in $\mathcal{T}$ has a (simple) solution and we may use expressions $\sum_i a_i$ and $\prod_i a_i$ - to be read as $(\sum_i a_i)$ and $(\prod_i a_i)$, respectively. For convenience, we also use the rule that $s \cdot t + u = st + u$ reads as $(st) + u$.

A *lattice* L is a member of $\mathcal{T}$ which satisfies the *absorption laws*

$$x(x + y) = y \text{ and } x + xy = x.$$

For lattices, $x \leq y \Leftrightarrow x = xy$ (we also write $y \geq x$) defines a partial order $\leq$ and one has $a \leq b$ if and only if $a = a + b$. With respect to this partial order, $a + b$ is the supremum, ab the infimum of a, b. If L has a smallest resp. greatest element these will be denoted by $\perp^L$ and $\top^L$, respectively. A set $\{a_1, \ldots, a_n\}$ in L such that $\sum_i a_i = \top^L$ and $\prod_i a_i = \perp^L$ will be called *spanning* in L. In particular, this applies if L is generated by $a_1, \ldots, a_n$. For $a \leq b$ in L, the *interval* $[a, b] = \{c \in L \mid a \leq c \leq b]$ is a sublattice of L; an *ideal* of L is a sublattice I such that $b \in I$ for any $b \leq a \in I$. The word problem for free lattices is well known to be solvable, but for simplicity we prefer to consider terms in $\mathcal{T}$.

A *chain* C_n of length n is a presentation with generators d_i, $i = 0, \ldots, n$, and relations $d_i \leq d_{i+1}$, $i < n$. Obviously, chains are projective within the class of all lattices.

3.2. **Modular lattices.** A lattice is *modular* if it satisfies the identity $x(y + xz) = xy + xz$, equivalently, if $a(b + c) = ab + c$ for all $c \leq a$. The class of all modular lattices is denoted by $\mathcal{M}$. *Projectivity* of presentations will always refer to $\mathcal{M}$. Examples of modular lattices are the lattices $L({}_RM)$ of all submodules of R-modules, with operations $+$ and $\cap$.

Fact 3.1. *In a modular lattice, $x \mapsto x + b$ is an isomorphism of $[ab, a]$ onto $[b, a + b]$ with inverse $y \mapsto ya$.*

Accordingly, we define $\bar{x} \nearrow \bar{y}$ to stand for the formula

$$\bigwedge_{i=1}^{n} \Big(y_i = x_i + \prod_{j=1}^{n} y_j \quad \wedge \quad x_i = y_i \cdot \sum_{j=1}^{n} x_j\Big).$$

Observe that for $x_1 \geq x_2$ and $y_1 \geq y_2$ one has $x_1, x_2 \nearrow y_1, y_2$ if and only if $x_1 \leq y_1$, $x_2 \leq y_2$, and $y_2, x_2 \nearrow y_1, x_1$. Also, $\bar{x} \nearrow \bar{y}$ and $\bar{y} \nearrow \bar{z}$ jointly imply $\bar{x} \nearrow \bar{z}$. Moreover,, $\bar{x} \nearrow \bar{y}$, $\bar{x} \nearrow \bar{z}$ and $\bar{y} \leq \bar{z}$ jointly imply $\bar{y} \nearrow \bar{z}$. Writing $\bar{x}^1 \nearrow \ldots \nearrow \bar{x}^m$ we require $\bar{x}^i \nearrow \bar{x}^j$ for all $i < j \leq m$.

Call elements $a_1, \ldots, a_n$ of a modular lattices *relatively independent* (*over* b) if $b = a_k \cdot \sum_{i<k} a_i$ for all $1 < k \leq n$. This implies that any permutation of $a_1, \ldots, a_n$ is independent over b, too, and that the a_i generate a boolean sublattice B with smallest element b and each a_i is either b or an atoms of B.

Fact 3.2. *In a modular lattice, if u, v, w are relatively independent over t, then $(x, y, z) \mapsto x + y + z$ defines an embedding of $[t, u] \times [t, v] \times [t, w]$ into $[t, u + v + w]$. In particular, for $t \leq u' \leq u'' \leq u$, $t \leq v' \leq v'' \leq v$, and $t \leq w' \leq w'' \leq w$ the sublattice generated by these 3 chains is isomorphic to the direct product of these chains. and the above embedding restricts to isomorphisms $x \mapsto x + v'$ of $[u' + w', u'' + w'']$ onto $[u' + v' + w', u'' + v' + w'']$ and $y \mapsto y + u'$ of $[v' + w', v'' + w'']$ onto $[u' + v' + w', u' + v'' + w'']$, respectively.*

See Fig. 1. For the proof observe, that one can assume $t = 0$ and that the case $w = 0$ is well known. The analogous result holds for any number of relatively independent elements.

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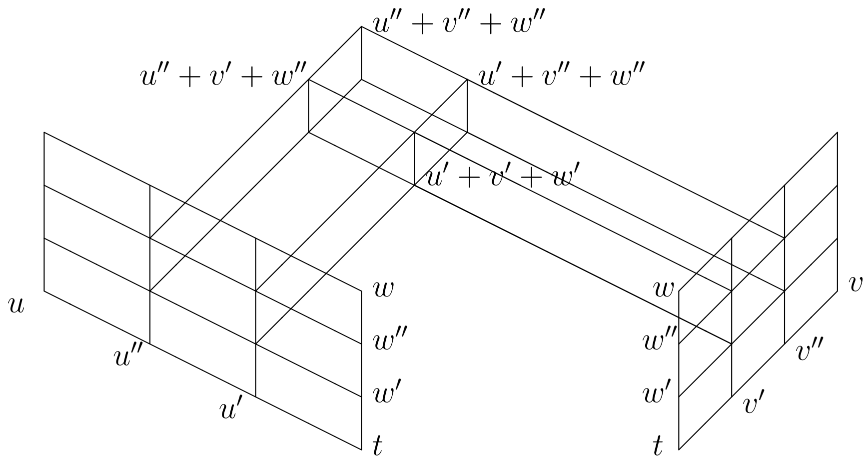

FIGURE 1. Direct product of chains

3.3. **Products of presentations.** Given presentations $(\Pi^j, (\bot, \bar{c}^j))$, where $\bar{c}^j = (c_1^j, \dots, c_{n_j}^j)$ for $j = 1,2$ with pairwise distinct c_i^j and, for $j = 1,2$, relations in Π^j implying $\bot \leq c_i^j$ for all i, the *product* $(\Pi^1, (\bot, \bar{c}^1)) \times (\Pi^2, (\bot, \bar{c}^2))$ is the presentation $(\Pi, (\bot, \bar{c}))$ with generator symbols

$$\bot \text{ and } \bar{c} = (c_1^1, \dots, c_{n_1}^1, c_1^2, \dots c_{n_2}^2)$$

and, in addition to the relations of the $(\Pi^j, (\bot, \bar{c}^j))$, the relations

$$\sum_{i=1}^{n_1} c_i^1 \cdot \sum_{k=1}^{n_2} c_k^2 = \bot.$$

Fact 3.2 implies the following well known fact.

Fact 3.3. *Within $\mathcal{M}$, products of projective presentations are projective. Moreover, given models $(L_j, (a_0^j, \bar{a}^j))$ of $(\Pi^j, (\bot, \bar{c}^j))$ in $\mathcal{M}$ one has $L_1 \times L_2$ a model of the product with generators mapped to (a_i^1, a_0^2) and (a_0^1, a_k^2), respectively; moreover, any model of the product of the presentations is isomorphic to such.*

3.4. **Frames.** Frames have been introduced by von Neumann [23] for coordinatizing complemented modular lattices. Given n independent generators e_i of an R-module ${}_RM$, the *canonical* n-frame in $L({}_RM)$ consists of $a_i = Re_i$, $c_{1j} = R(e_1 - e_j)$, and $a_\bot = \{0\}$. This is mimicked by the following presentation.

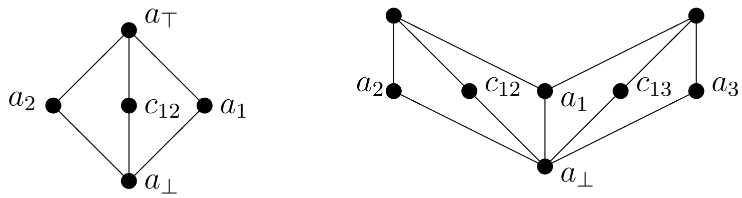

FIGURE 2. 2-frame and 3-frame

An *n-frame* Φ is a lattice presentation with generators $a_\perp, a_1, \ldots, a_n$, $c_{1j} = c_{j1}$ $(2 \leq j \leq n)$ and relations

$$\begin{array}{llllll}
(1) & a_\perp & = & a_j(\sum_{i=1}^{j-1} a_i) & & \\
(2) & a_\perp & = & a_1 c_{1j} & = & a_j c_{1j} \\
(3) & a_1 + a_j & = & a_1 + c_{1j} & = & a_j + c_{1j}
\end{array}$$

where $2 \leq j \leq n$. See Fig. 2

An equivalent presentation is obtained by replacing $a_\perp$ by $a_1 a_2$. A model (in $\mathcal{M}$) of (an) n-frame is referred to as "an n-frame in a modular lattice"; otherwise, speaking of "an n-frame" we mean a presentation as above, possibly with renamed generators.

We define $a_\top = \sum_{i=1}^n a_i$ and write also $a^\top = a_\top^\Phi = \top^\Phi$, $a_i = a_i^\Phi$, $c_{1j} = c_{1j}^\Phi$, and $a_\perp = a_\perp^\Phi = \perp^\Phi$. The list of generators with indices not involving fixed $k >$ is written as $\Phi_{\neq k}$. Observe that Φ implies, within $\mathcal{M}$, the relations of $n-1$-frames for $\Phi_{\neq k}$ and $a_k + \Phi_{\neq k}$ and that

$$\Phi_{\neq k} \nearrow a_k + \Phi_{\neq k}.$$

Also observe, that the concept of n-frame can be defined, recursively: start with that of 2-frame, as defined above; now, given the concept of n-frame Φ, obtain the $n+1$-frame Φ^+ adding to the generators and relations of Φ the generators a_{n+1} and $c_{1,n+1}$ and relations (2) and (3) for $j = n+1$ - renaming $a_\perp^\Phi$ into $a_\perp^{\Phi^+}$. The following is a special case of Dedekind's description of 3-generated modular lattices.

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Fact 3.4. *The modular lattice freely generated by a, a', c such that $aa' \leq c \leq a + a'$ has diagram given in Fig. 3. In particular, with $b = ac$ and $d = a(a'+c)$ one has a 2-frame $b + a'c$, $d + a'c$, $b + a'(a+c)$, c and $b, d \nearrow b + a'c, d + a'c$.*

3.5. **Reduction.** Given an n-frame Φ and variables x, y, put

$$a_\perp(x,y) \equiv x + \sum_{j>1} a_j(x + c_{1j}) \text{ and } a_\top(x,y) \equiv y + \sum_{j>1} a_j(y + c_{1j})$$

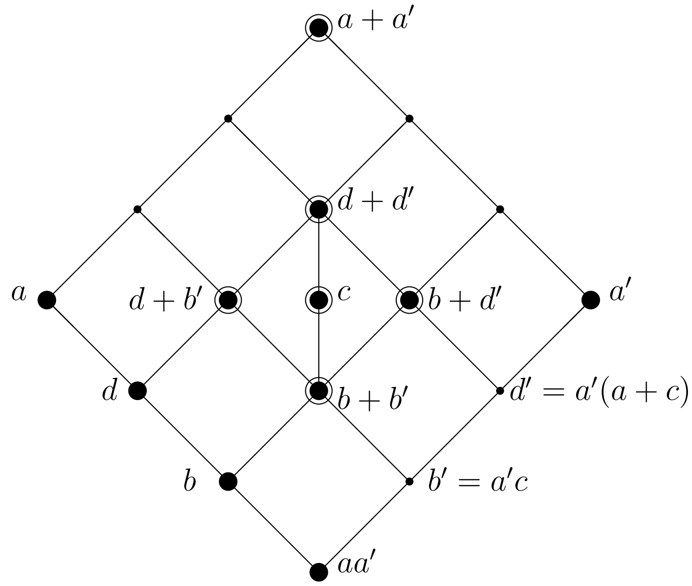

FIGURE 3. Modular lattice generated by a, a', c with $aa' \leq c \leq a+a'$

and introduce for each remaining generator symbol c in Φ the term

$$\hat{c}(x,y) \equiv c \cdot a_\top(x,y) + a_\perp(x,y).$$

Observe that for all models of Φ in a modular lattice L and b, d in L with $a_\perp \leq b \leq d \leq a_1$ one has the identity $\hat{c}(b,d) = (c + a_\perp(b,d)) \cdot a_\top(b,d)$. Let $\Phi(x,y) = \Phi_x^y$ denote the list of terms $\hat{c}(x,y)$, c a generator symbol of Φ; this is called the *reduction setup* for n-frames.

If Φ is part of a presentation Π and b, d are terms over Π then Φ_d^b is obtained substituting b, d for x, y; Φ_d^b is called the *reduction* of Φ via b, d. We put $\Phi^b = \Phi_{a_\perp}^b$ and $\Phi_d = \Phi_d^{a_1}$.

If $B \subseteq D$ are left-ideals of the ring R, then the reduction of the canonical n-frame of $L({}_R R^n)$ by $b = Be_1 \leq d = De_1$ is given by

$$a'_\perp = \sum_{i=1}^n Be_i,\ a'_i = a'_\perp + De_i,\ c'_{1j} = a'_\perp + D(e_1 - e_j),\ a'_\top = \sum_{i=1}^n De_i.$$

See [7, Lemma 1.1] and Fig. 4 for the following.

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Lemma 3.5. *For any n-frame Φ and $a_\perp \leq b \leq d \leq a_1$ in a modular lattice, L, one has the following*

(1) Φ_d^b *is an n-frame in L.*

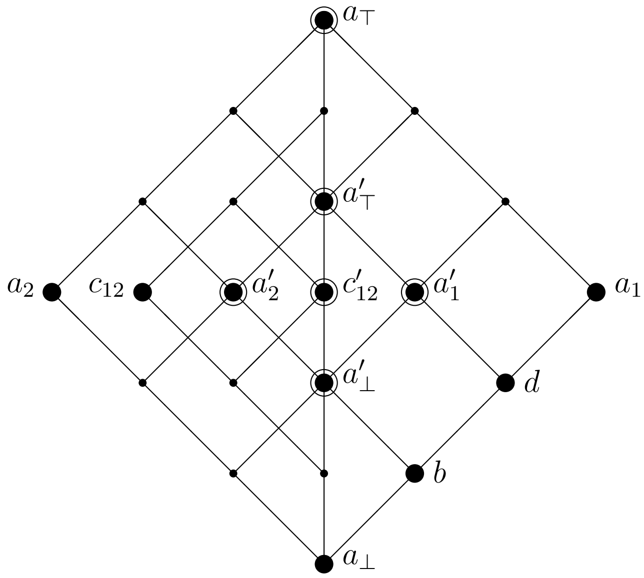

FIGURE 4. Reduction 2-frame (Φ, a_i, c_{12}) to $(\Phi^b_d, a'_i, c'_{12})$

(2) $d, b \nearrow (a_1)^{\Phi^b_d}, (a_\perp)^{\Phi^b_d} \nearrow \sum \Phi^b_d, \sum_{i\neq 1} a_i^{\Phi^b_d} \nearrow$
$\nearrow d + \sum_{i\neq 1} a_i, b + \sum_{i\neq 1} a_i$.

(3) $a_\top^{\Phi^b_d} \cdot \sum \Phi_{\neq 1},\ a_\perp^{\Phi^b_d} \cdot \sum \Phi_{\neq 1} \nearrow (\Phi^b_d)_{\neq 1}, a_\perp^{\Phi^b_d}$.

(4) $(\Phi_{\neq k})^b_d \nearrow (\Phi^b_d)_{\neq k} \nearrow (a_k)^{\Phi^b_d} + (\Phi^b_d)_{\neq k} \nearrow (a_k + \Phi_{\neq k})^{a_k+b}_{a_k+d}$ *for* $k > 1$..

(5) *If* $b = a_\perp$ *and* $d = a_1$ *then* $\Phi^b_d = \Phi$.

3.6. **Towers of 2-frames.** An *n-tower of 2-frames* is a presentation $\Delta(n) = \Delta(2, n)$ which is the disjoint union of 2-frames $(\Phi^k, a^k_\perp, a^k_i, c^k_{1j})$, $k = 1, \ldots, n$, with the additional relations

$$a^k_\top, a^k_2 \nearrow a^{k+1}_1, a^{k+1}_\perp \text{ for } 1 \leq k < n.$$

It follows that $a^n_2 a^1_1 = a^1_\perp$ and $a^n_2 + a^1_1 = a^n_\top$. Referring to the reduction setups $\Phi^k(x, y)$ of the 2-frames Φ^k, define the *reduction setup* $\Delta(n)(x, y)$ as the union of the $\Phi^k(x + a^k_\perp, y + a^k_\perp)$ and the *reduction* $\Delta(n)^b_d = \Delta(n)(b, d)$. The following is due to Alan Day [3, Thm.5.1]

Lemma 3.6. (i) *Within $\mathcal{M}$, n-towers of 2-frames are projective.*

(ii) *$F\mathcal{M}(\Delta(n))$ is the disjoint union of 5-element interval sublattices $\Phi^k \cup \{a^k_\top\}$.*

(iii) *$F\mathcal{M}(\Delta(n))$ is generated by the $n + 2$-elements $a^1_1, a^n_2, c^k_{12}(1 \leq k \leq n)$.*

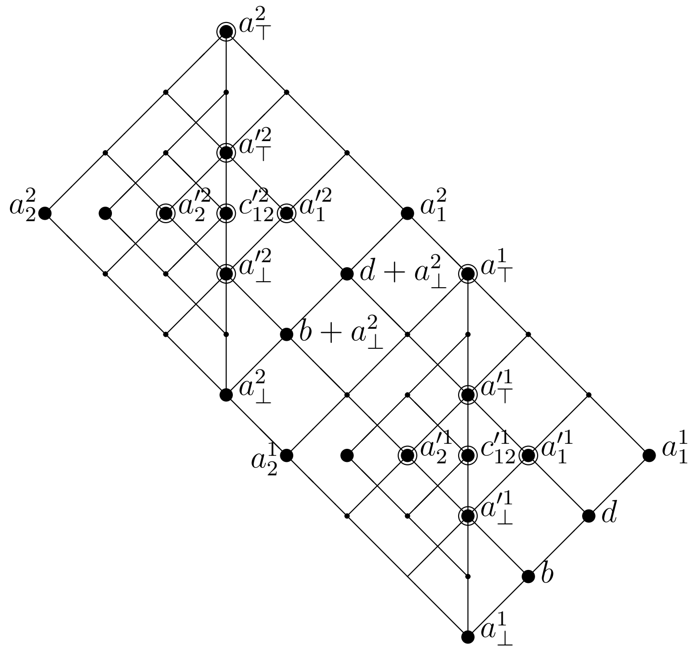

FIGURE 5. Reduction of $\Delta(2)$

(iv) *If $\Delta(n)$ is an n-tower of 2-frames in a modular lattice L and if $a^1_\perp \leq b \leq d \leq a^1_1$ then $\Delta(n)^b_d$ is an n-tower of 2-frames $a'^k_\perp, a'^k_1, a'^k_2, c'^k_{12}$ in L and with $u = a^n_2 \geq u'' = ua'^n_2 \geq u' = ua'^1_\perp$ and $w = a^1_1 \geq w'' = d \geq w' = b \geq a^1_\perp$ one has $uw = a^1_\perp$, $a'^1_\perp = u' + w'$, and $a'^n_\top = u'' + w''$. Moreover, if $d = a^1_1$ then $w'' = w$ and $u'' = u$. If, in addition, $b = a^1_\perp$ then $\Delta(n)^b_d = \Delta$.*

Proof. (i)-(iii) are in [3]. (iv) follows from (1) and (2) of Lemma 3.5, readily. See Fig. 5. ☐

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3.7. **Towers of 3-frames.** An *n-tower* of 3-frames is a presentation $\Delta(3,n)$ consisting of the product of an n-tower $\Delta(n)$ of 2-frames with the chain $a^1_\perp \leq a^1_3$ and an additional generator c^1_{13} such that the $a^1_\perp, a^1_1, a^1_3, c^1_{13}$ form a 2-frame Φ. In particular, by Fact 3.2 $a^n_2(a^1_1+a^1_3) = a^1_\perp = (a^1_1 + a^1_2)a^1_3$ whence Φ^1 together with a^1_3, c^1_{13} forms a 3-frame. See Fig. 6

Lemma 3.7. *n-towers of 3-frames are projective within $\mathcal{M}$.*

Proof. By Facts 3.3, 2.1, and Lemma 3.6, the product of $\Delta(n)$ with the chain $a^1_\perp \leq a^1_3$ is projective within $\mathcal{M}$ and strengthening with

$$c^1_{13} := (c^1_{13} + a^1_\perp)(a^1_1 + a^1_3)$$

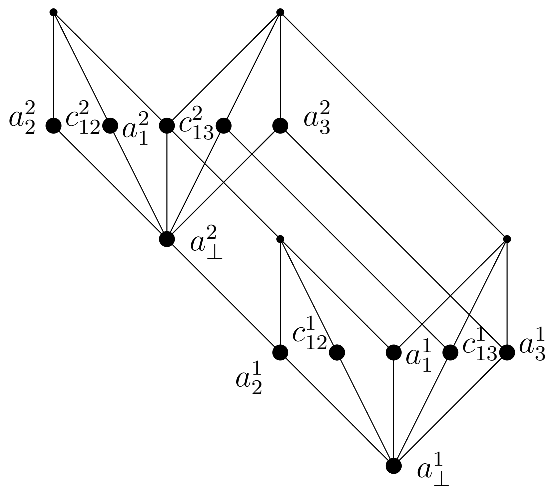

FIGURE 6. 2-tower of 3-frames

yields the additional relations $a_\perp^1 \leq c_{13}^1 \leq a_1^1 + a_3^1$. Now, in view of Fact 3.4 put

$$b = a_1^1 c_{13}^1 \text{ and } d = a_1^1(c_{13}^1 + a_3^1)$$

$$(*) \quad v = a_3^1 \geq v'' = a_3^1(a_1^1 + c_{13}^1) \geq v' = a_3^1 c_{13}^1 \geq a_\perp^1$$

to obtain the 2-frame $\Phi' = (b+v', d+b+v', v''+b, c_{13}^1+b+v', d+v'')$. Now, together with the chains defined in (iv) of Lemma 3.6, apply Fact 3.2 to obtain the n-tower $v' + \Delta(n)_d^b$ spanning $[u' + w' + v', u'' + w'']$ and the 2-frame $u' + \Phi'$ spanning $[u' + w' + v', w'' + v'']$. This verifies the strengthening

$$\Delta(n) := v' + \Delta(n)_d^b,\ a_3^1 := u' + w' + v' + a_3^1,\ c_{13}^1 := u' + w' + v' + c_{13}^1$$

and proves the lemma. □

The *reduction setup* $\Delta(3, n)(x, y)$ is the union of $\Delta(n)(x, y)$ and $\Phi(x, y)$.

Corollary 3.8. (i) *If $\Delta(3, n)$ is an n-tower of 3-frames in a modular lattice L and if $b, d \in L$ such that $a_\perp^1 \leq b \leq d \leq a_1^1$ then $\Delta(3, n)_d^b := \Delta(3, n)(b, d)$ is also an n-tower of 3-frames in L and spans the interval $[u' + w' + v', u'' + w'' + v'']$ with the chains from $(*)$ and (iv) of Lemma 3.6.*

(ii) *Redefining two of the chains into one, namely $u := u + v \geq u'' := u'' + v'' \geq u' := u'' + v''$, so one has that $\Delta(3, n)_d^b$ spans the interval $[u' + w', u'' + w'']$. Moreover, if $d = a_1^1$ then $w = w''$ and $u = u''$.*

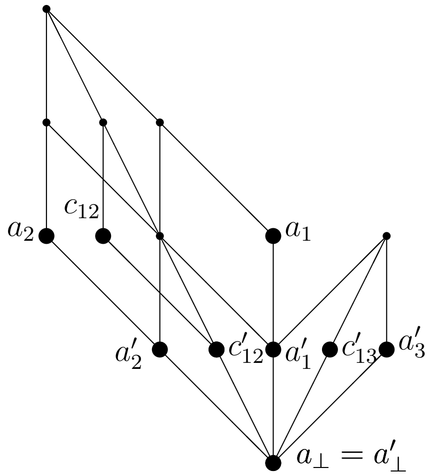

FIGURE 7. Skew $(3,2)$-frame $(\Phi', a'_i, c'_{1j}; \Phi, a_i, c_{12})$

3.8. **Towers of skew frames.** A *skew $(n+1,n)$-frame* is the lattice presentation Ψ given by the product of an n-frame Φ with the chain $a_\perp \leq a'_{n+1}$ and an additional generator $c'_{1,n+1}$ such that the list of terms $a_\perp, b = a_1(a'_{n+1}+c'_{1,n+1}), a'_{n+1}, c'_{1,n+1}$ satisfies the relations of a 2-frame. See Fig. 7.

An *n-tower $\Omega(m,n)$ of skew $(m,m-1)$-frames* is the presentation given as the product of an n-tower $\Delta(m-1,n)$ of $m-1$-frames with the chain $a^1_\perp \leq a'_m$ and an additional generator c'_{1m} subject to relations stating that $a^1_\perp, a^1_1(a'_m + c'_{1m}), a'_m, c'_{1m}$ is a 2-frame. Compare Fig. 8 for the case of towers of skew $(3,2)$-frames. Dealing with the case $m=4$, we put $\Omega(n) = \Omega(4,n)$ and speak of *n-towers of skew frames.*

Theorem 3.9. *n-towers of skew frames can be defined in terms of $n+6$ generators and are projective within $\mathcal{M}$.*

Proof. $n+2$ generators provide the n-tower of 2-frames, 2 more the n-tower of 3-frames, and another 2 are used to turn this into an n-tower of skew frames.

Omitting the relations concerning c'_{14}, projectivity within $\mathcal{M}$ follows from Lemma 3.7 and Facts 3.3, 2.1. A first strengthening

$$c'_{14} := (c'_{14} + a^1_\perp)(a^1_1 + a'_4)$$

adds the relations $a^1_\perp \leq c'_{14} \leq a^1_1 + a'_4$. In view of Fact 3.4 put $b = a^1_1 c'_{14}$ and $v = v'' = a'^1_4(a^1_1 + c'^1_{14}) \geq v' = a'^1_4 c'^1_4$ and let M denote the interval $[b+v', a^1_1 + v']$ of the sublattice generated by a^1_1, a'^1_4, c'^1_{14}. Consider the

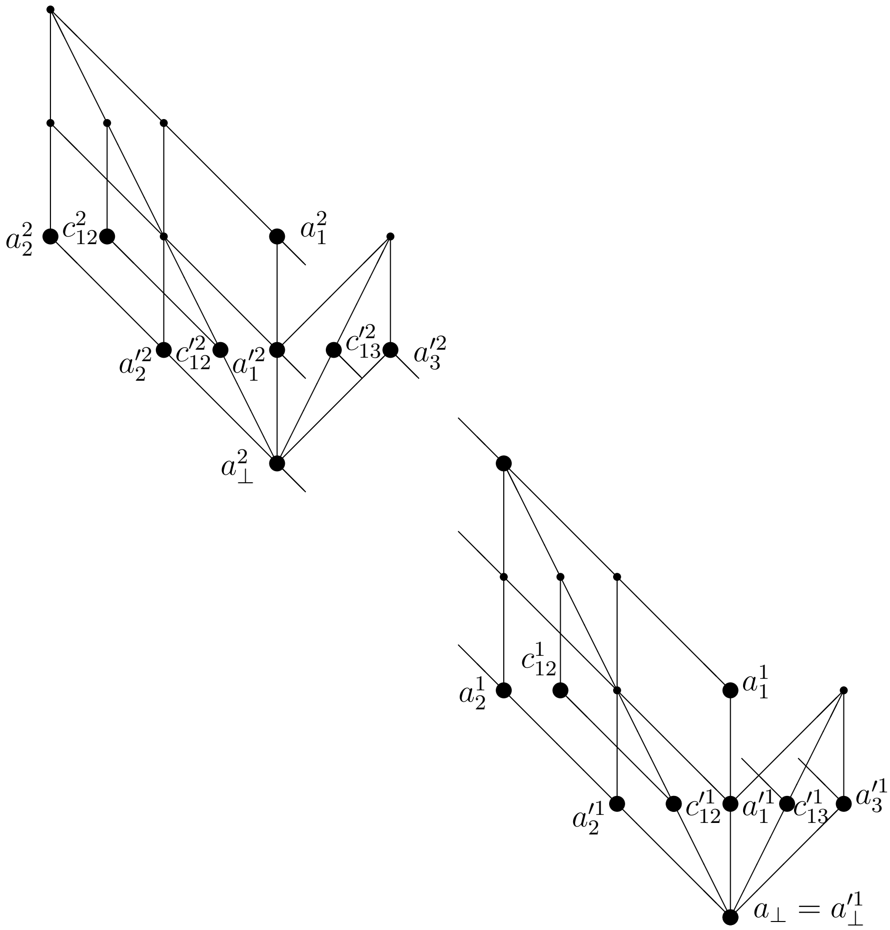

FIGURE 8. 2-tower of skew $(3,2)$-frames

reduction $\Delta(3,n)^b_{a_1^1}$ and apply Fact 3.2 to the chains $u' = u'' \geq u'$ and $w = w'' \geq w'$ from (ii) of Cor.3.8 and $v = v'' \geq v'$. This yields the n-tower $v' + \Delta(3,n)^b_{a_1^1}$ spanning $[u' + v' + w', u + v' + w]$ and $u' + M$ spanning $[u' + v' + w', u' + v + w]$ and verifies the final strengthening

$$\Delta(3,n) := \Delta(3,n)^b_{a_1^1},\ c_{14}'^1 := c_{14}'^1 + u',\ a_4'^1 := (a_1^1 + u' + v')(c_{14}'^1 + v + u').$$

□

The presentations n-tower of 2- resp. 3-frames and n-tower of skew $(4,3)$-frames have been constructed explicitly and uniformly for all n. This results in the following.

Corollary 3.10. *There is an algorithm constructing for each n the presentation $\Omega(n)$, projective within the class of modular lattices.*

4. Structure of towers of skew frames

While projectivity within $\mathcal{M}$ of the presentation "n-tower (of skew frames)" has been established in Theorem 3.9, in this section we collect the needed concepts and results about models of this and related presentations - considered as lists of elements or *configurations*. Maps are supposed to match such lists. If such list is the concatenation of parts we say that it is obtained by *combining* these parts. We will use $\bar{a}$ to denote lists of elements, in general, not necessarily the $a_1, \ldots, a_n$ of an n-frame.

First, we recall some more details about particular elements in sublattices generated by (skew) frames - used in equivalent presentations e.g. in [12].

4.1. **Frames.** A well known equivalent definition of frames is obtained es follows cf. [6]. Consider an n-frame Φ in a modular lattice L and define $c_{ij} = c_{ji} = (a_i + a_j)(c_{1i} + c_{1j})$ for $1 \neq i \neq j \neq 1$, Then it follows

$$(\sum_{i\in I} a_i)\cdot(\sum_{j\in J} a_j) = \left\{ \begin{array}{ll} a_\perp & \text{if } I\cap J=\emptyset \\ \sum_{k\in I\cap J} a_k & \text{else} \end{array} \right. \quad \text{for } I, J \subseteq \{1,\ldots,n\},$$

and for pairwise distinct i, j, k

$$a_i + a_j = a_i + c_{ij},\ \ a_i \cdot c_{ij} = a_\perp,$$
$$c_{ik} = (a_i + a_k)\cdot(c_{ij} + c_{jk}).$$

We also write $\perp^\Phi = a^\Phi_\perp, a^\Phi_i, c^\Phi_{ij}$ for these elements and $\top^\Phi = \sum_{i=1}^n a_i$. Observe that, for any k, the a_i, c_{ij} with $i, j \neq k$ form an $n-1$-frame $\Phi_{\neq k}$ in L. An important property of frames Φ is the existence of the *perspectivities* $\pi_{kl} = \pi^\Phi_{kl}$, $k \neq l$, that is lattice isomorphisms between intervals of L matching $\Phi_{\neq k}$ with $\Phi_{\neq l}$

$$\pi_{kl} : [a_\perp, \sum_{i\neq k} a_i] \to [a_\perp, \sum_{i\neq l} a_i] \text{ where } \pi_{kl}(x) = (x + c_{kl})\sum_{i\neq l} a_i.$$

Thus, $\pi^\Phi_{kl}(a)$ is obtained from a lattice term $\hat{\pi}_{kl}(x, \bar{z})$ substituting a for x and Φ for $\bar{z}$ (actually, we use only the case where Φ is a 4-frame.)

4.2. **Reduction of frames.** Given a frame Φ and $b_1 \in L$ such that $a_\perp \leq b_1 \leq a_1$, define

$$b_j = a_j(b_1 + c_{1j}) \text{ for } j \neq 1, b = \sum_{j=1}^n b_i, \text{ and } b_{ij} = (b_i + b_j)c_{ij}$$

to obtain, by *upper reduction* with b_1, the frame

$$\Phi^{b_1} = \Phi^{b_1}_{a_\perp} = (b, b + a_{i(1\leq i\leq n)}, b + c_{ij(i\neq j)})$$

and by *lower reduction* the frame

$$\Phi_{b_1} = \Phi^{a_1}_{b_1} = (a_\perp, b_{i(1\leq i\leq n)}, b_{ij(i\neq j)}).$$

In view of the perspectivities, in both cases the resulting frame is the same if, for some $i \neq 1$, the construction is carried out based on $a_\perp \leq b_i \leq a_i$ such that $b_1 = a_1(b_i + c_{1i})$.

4.3. **Stable elements.** Given a modular lattice L, element s of L and an n-frame Φ in L, the element s is *j-stable* in L for Φ and $j \geq 2$ if

$$sa_1 = sa_j = a_\perp \text{ and } s + a_1 = s + a_j = a_1 + a_j$$

and for all $b_1 \in L$ with $a_\perp \leq b_1 \leq a_1$ one has

$$s + b_1 = s + b_j \text{ where } b_j = a_j(b_1 + c_{1j}).$$

Obviously, if Φ' is another n-frame in L such that $a_\perp, a_1, a_j, c_{1j} \nearrow a'_\perp, a'_1, a'_j, c'_{1j}$ in L then s is j-stable for Φ if and only if $s + a'_\perp$ is j-stable for Φ'. Also, in view of the perspectivities, if s is j-stable for Φ then $\pi_{jk}(s)$ is k-stable for Φ. This crucial concept is due to Ralph Freese [8]. See [12, Lemma 2,3] for the following

Fact 4.1. *If s is j-stable in L for Φ then for all $b \in L$ with $a_\perp \leq b \leq a_1$, one has $s + a^{\Phi^b}_\perp$ j-stable for Φ^b and $sa^{\Phi_b}_\top$ j-stable for Φ_b.*

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4.4. **Skew frames.** Using the above view (Subsection 4.1) on frames in modular lattices, an equivalent definition of a skew $(n+1, n)$-frame $\Psi = (\Phi', \Phi)$ in a modular lattice L is that of a configuration which is composed by the n-frame Φ and the $n+1$-frame Φ' such that $\Phi'_{\neq n}$ is the reduction $\Phi_{a'_1}$, in particular $a_\perp = a'_\perp$. Observe that the perspectivity π_{kl} of Φ induces the perspectivity π_{kl} of $\Phi'_{\neq n}$. We will deal with the cases $(3, 2)$ and $(4, 3)$, only.

Our basic example of a skew $(4, 3)$-frame is as follows: For a prime p consider the $\mathbb{Z}$-module A with generators e_i, $i = 1, 2, 3, 4$ and relations $p^2 e_i = 0$ for $i \leq 3$, $pe_4 = 0$. Then a skew $(4, 3)$-frame in the submodule lattice $L(A)$ is obtained as follows: Let $a_\perp = 0$, $a_i = \mathbb{Z}e_i$, $c_{ij} = \mathbb{Z}(e_i - e_j)$, $a'_i = \mathbb{Z}pe_i$, $c'_{ij} = \mathbb{Z}(pe_i - pe_j)$, for $i, j \leq 3$, $a'_4 = \mathbb{Z}e_4$, and $c'_{i4} = \mathbb{Z}(pe_i - e_4)$.

Dealing with a skew $(4, 3)$-frame Ψ we consider $\Psi_{(3,2)}$ given by generators not involving index 2 and $\Psi^{(3,2)} = a_2 + \Psi_{(3,2)}$. Observe that $\Psi_{(3,2)} \nearrow \Psi^{(3,2)}$ and that the relations of a skew $(3, 2)$-frame are implied by those of a skew $(4, 3)$-frame, in both cases.

For $a_\perp \leq b \leq a'_1 \leq d \leq a_1$ in L the *lower reduction* $\Psi_{b,d}$ is the skew $(n+1,n)$-frame which combines the lower reductions Φ'_b and Φ_d. For $a_\perp \leq b \leq a'_1$ in L the *upper reduction* Ψ^b combines the upper reductions Φ'^b and Φ^b.

4.5. **Towers of skew frames.**

Observation 4.2. *Within $\mathcal{M}$ a presentation equivalent to that of an n-tower $\Omega(n)$ is given by a list $\bar{a}$ combining the skew $(4,3)$-frames $\Psi^k = (\Phi'^k, \Phi^k)$ $(1 \leq k \leq n)$ each consisting of the 4-frame Φ'^k and the 3-frame Φ^k where*

$$\Phi'^k = \bar{a}'^k = (a_i'^k, c_{ij}'^k \mid i,j \leq 4, i \neq j),\ \Phi^k = \bar{a}^k = (a_i^k, c_{ij}^k \mid i,j \leq 3, i \neq j)$$

such that

$$(*)\ \Psi^k_{(3,2)} \nearrow (\Psi^k)^{(3,2)} \nearrow \Psi^l_{(3,2)} \nearrow (\Psi^l)^{(3,2)} \textit{ for } 1 \leq k < l \leq n.$$

Observe that $\Psi^k_{(3,2)}$ consists of the $a_i^k, c_{ij}^k, a_i'^k, c_{ij}'^k$ where $i,j \neq 2$ and that $(\Psi^k)^{(3,2)} = a_2^k + \Psi^k_{(3,2)}$. The exceptional role of index 2 (linked to the "upward direction") comes out of the construction of n-towers and fits to the application of [12].

Proof. Recall the definition of n-towers $\Omega(n)$ of skew frames in Subsection 3.8 and observe that the $a_i^1, c_{ij}^1, a_i'^1, c_{ij}'^1$ form a skew $(4,3)$ frame Ψ^1, that $a_2^n, a_1^1, a_3^1, a_4'^1$ are relatively independent over $a_\perp^1$, and that $a_1^k = a_\perp^k + a_1^1$. Thus, $\Psi^1_{(3,2)}$ is a skew $(3,2)$-frame whence so is (for each $k > 1$) $a_\perp^k + \Psi^1_{(3,2)}$ which combines with a_2^k, c_{12}^k to form a skew $(4,3)$-frame Ψ^k such that $\Psi^1_{(3,2)} \nearrow \Psi^k_{(3,2)}$. This proves $(*)$ in view of the remarks following Fact 3.1 and applies also to skew frame as considered in the preceding Subsection. $\square$

A model is obtained from the submodule lattice of the free $\mathbb{Z}$ module with generators e_1, e_2, e_3, e_4 and relations $p^2 e_1 = 0$, $p^{3n-1} e_2 = 0$, $p^2 e_3 = 0$ and $p e_4 = 0$. Indeed, put (where $i = 1,3$)

$$a_\perp^k = \mathbb{Z}p^{3(n-k)+2} e_2,\ a_2^k = \mathbb{Z}p^{3(n-k)} e_2,\ c_{2i}^k = \mathbb{Z}(p^{3(n-k)} e_2 - e_i)$$

$$a_i^k = a_\perp^k + \mathbb{Z}e_i,\ c_{13}^k = a_\perp^k + \mathbb{Z}(e_1 - e_3),\ c_{i2}^k = a_\perp^k + \mathbb{Z}(e_i - p^{3(n-k)} e_2)$$

$$a_2'^k = \mathbb{Z}p^{3(n-k)+1} e_2,\ c_{2i}'^k = \mathbb{Z}(p^{3(n-k)+1} e_2 - p e_i),\ c_{24}'^k = \mathbb{Z}(p^{3(n-k)+1} e_2 - e_4),$$

$$a_i'^k = a_\perp^k + \mathbb{Z}p e_i,\ c_{13}'^k = a_\perp^k + \mathbb{Z}(p e_1 - p e_3),\ c_{i4}'^k = a_\perp^k + \mathbb{Z}(p e_i - e_4).$$

4.6. **Reduction of towers.** Given an n-tower $\Omega(n) = \bar{a}$ in a modular lattice L, consider fixed $m > 0$ and $b, d \in L$ such that

$$a_\perp^m \leq b \leq a_1'^m \leq d \leq a_1^m.$$

The *lower reduction* $\Omega(n)_{b,d}$ of $\Omega(n)$ combines the following reductions of skew-frames $\Psi^k = (\Phi'^k, \Phi^k)$

$$\begin{array}{lll} (\Phi'^k_{a_1^k b}, & \Phi^k_{a_1^k d}) & \text{for } 1 \leq k < m \\ (\Phi'^m_b, & \Phi^m_d) & \\ (\Phi'^k_{b+a_\perp^k}, & \Phi^k_{d+a_\perp^k}) & \text{for } m < k \leq n \end{array}$$

Given $b \in L$ such that

$$a_\perp^m \leq b \leq a_1'^m$$

the *upper reduction* $\Omega(n)^b$ of $\Omega(n)$ combines the following reductions of skew-frames $\Psi^k = (\Phi'^k, \Phi^k)$

$$\begin{array}{lll} ((\Phi'^k)^{a_1^k b}, & (\Phi^k)^{a_1^k b}) & \text{for } 1 \leq k < m \\ ((\Phi'^m)^b, & (\Phi^m)^b) & \\ ((\Phi'^k)^{b+a_\perp^k}, & (\Phi^k)^{b+a_\perp^k}) & \text{for } m < k \leq n \end{array}$$

We speak of the lower resp. upper reduction of $\Omega(n)$ *induced* by the reduction of Ψ^k.

Observation 4.3. *If the configuration $\Omega(n)$ is an n-tower of skew frames $\Psi^k = (a'^k, c'^k_{ij}, a^k, c^k_{ij})$ in a modular lattice, L, then so are any of its lower reductions $\Omega(n)_{b,d}$ where $b, d \in L$ and any of its upper reductions $\Omega(n)^b$ where $b \in L$. Moreover, if $\phi : L \to L'$ is a homomorphism into a modular lattice L' such that $\phi(b) = \phi(a_\perp'^m)$ and $\phi(d) = \phi(a_1^m)$ resp. $\phi(b) = \phi(a_\perp^m)$ then $\phi(\Omega(n)) = \phi(\Omega(n)_{b,d})$ resp. $\phi(\Omega(n)) = \phi(\Omega(n)^b)$, as configurations in L'.*

5. Coordinates and characteristic

5.1. **Coordinate ring.** Following von Neumann [23] (cf. Freese [6, 7, 8] and [13, Lemma 6]) with any 4-frame Φ in a modular lattice L and choice of 3 different indices (here we use $1, 3, 4$) one obtains a (*coordinate*) ring $R(\Phi, L)$ with unit c_{13} and zero a_1, the elements of which are the $r \in L$ such that $ra_3 = a_\perp$ and $r + a_3 = a_1 + a_3$. More precisely, there are binary lattice terms $x \oplus_{\bar{z}} y$ and $x \otimes_{\bar{z}} y$ and a unary term $\ominus_{\bar{z}} x$ defining these coordinate rings. Here, one has $\bar{z} = (z_i, z_{ij} | i, j \neq 2)$ corresponding to the 3-frame $(a_i, c_{ij} | i, j \neq 2)$. For given L and Φ these rings are isomorphic for any choice of the triple of indices - via the perspectivities resp. compositions thereof.

If L embeds into the the subgroup lattice of an abelian group A and if $\Phi = (a_i, c_{ij} \mid i, j \neq 2)$ is a 3-frame in L then the above definitions apply to obtain the ring $R(\Phi, L)$, embedded into the endomorphism ring of the associated subquotient of A.

An element r of $R(\Phi, L)$ is invertible if and only if $ra_1 = a_\perp$ and $r + a_1 = a_1 + a_3$; these form the group $R^*(\Phi, L)$ of units in the ring $R(\Phi, L)$. Moreover, there is a lattice term $t(x, \bar{z})$ such that $t(r, \bar{a})$ is the inverse of r if r is invertible.

5.2. **Stable elements.** Obviously, 3-stable elements are invertible. Again, the following crucial tool is due to Ralph Freese [8].

Lemma 5.1. *For a modular lattice L containing a 4-frame Φ as above one has the following.*

- (i) *The elements of L which are 3-stable for Φ in L form a subgroup $R^\#(\Phi, L)$ of the group $R^*(\Phi, L)$ of units.*
- (ii) *For each $b \in L$ with $a_\perp \leq b \leq a_1$ the map $r \mapsto r + \perp^{\Phi^b}$ is a homomorphism $\beta_b : R^\#(\Phi, L) \to R^\#(\Phi^b, L)$.*
- (iii) *If r is 3-stable for Φ and $b = a_1(r + c_{13})$ then $\beta_b(r) = c_{13}^{\Phi^b}$ is the unit of $R^\#(\Phi^b, L)$.*

Proof. (i) and (ii) are Lemma 1.3-6 of [8]. For convenience, we prove (iii). With $b = \perp^{\Phi^b}$ one has $r+b = r+b+b = (a_1+r)(c_{13}+r)+b \geq c_{13}+b$ and equality follows since by (ii) both are complements of $a_3 + b$ in $[b, b + a_1 + a_3]$. $\square$

5.3. **Characteristic.** With the term $x \oplus_{\bar{z}} y$ of Subsection 5.1, define recursively, $1 \otimes_{\bar{z}} z_{14} = z_{14}$ and $(n+1) \otimes_{\bar{z}} z_{14} = z_{14} \oplus_{\bar{z}} (n \otimes_{\bar{z}} z_{14})$. In the sequel, p will be a fixed prime. The 4-frame $\Phi = \bar{a}$ has *characteristic p* if $p \otimes_{\bar{a}} c_{14} = a_1$. Ralph Freese [6] has shown that, for any frame $\Phi = \bar{a}$ in a modular lattice L, the frame $\Phi_{a_1(p \otimes_{\bar{a}} c_{14})}$ has characteristic p - and equals Φ if Φ has characteristic p, already.

Let $(\bar{z}', \bar{z})$ denote a list of variables to be used for substituting skew $(4, 3)$-frames. In [12, p. 516], a term $p_{32}(\bar{z})$ has been defined and a skew $(4, 3)$-frame $(\Phi', \Phi) = (\bar{a}', \bar{a})$ has been called of *characteristic $p \times p$* if Φ' is of characteristic p and $p_{32}(\bar{a}) \geq a_3'$ and $a_3 + p_{32}(\bar{a}) = a_2' + p_{32}(\bar{a}) = a_2' + a_3$. The following is [12, Lemma 9] (in the proof given, there, observe that $b_3 \geq a_3'$ since $p_{32} \geq a_3'$).

.

Lemma 5.2. *There are terms $b^*(\bar{z}', \bar{z})$ and $d^*(\bar{z}', \bar{z})$ such that for any skew (4.3)-frame $\Psi = (\Phi', \Phi)$ in a modular lattice L one has*

$$a_\perp \leq \mathbf{b} := b^*(\bar{a}', \bar{a}) \leq a_1' \leq \mathbf{d} := d^*(\bar{a}', \bar{a}) \leq a_1$$

and obtains $\Psi_{\mathbf{b},\mathbf{d}}$ of characteristic $p \times p$. Moreover, if Ψ has characteristic $p \times p$ then $\mathbf{b} = a_\perp$ and $\mathbf{d} = a_1$, that is $\Psi_{\mathbf{b},\mathbf{d}} = \Psi$.

As in Freese's result, one can derive projectivity but, in contrast, it appears unlikely that characteristic $p \times p$ is preserved under reductions. Though, the existence of stable elements is preserved (see Fact 4.1). Recall the term $g_1^*(\bar{z}', \bar{z})$ from [12, Cor.13] and, applying the "perspectivity term" $\hat{\pi}_{23}$ define

$$g^+(\bar{z}', \bar{z}) = \hat{\pi}_{23}(g_1^*(\bar{z}', \bar{z}), \bar{z}').$$

Lemma 5.3. *For any skew* $(4,3)$*-frame* $\Psi = (\Phi', \Phi) = (\bar{a}', \bar{a})$ *of characteristic* $p \times p$ *in a modular lattice* L*, one has* $g^+(\Psi) = g^+(\bar{a}', \bar{a})$ *an element of* L *which is* 2*-stable for* Φ'*.*

Proof. According to [12, Cor.13] one has $g_1^*(\bar{a}', \bar{a})$ a 2-stable element of $R(\Phi')$ and the claim follows via perspectivity. □

6. Glueing constructions

Extending early work of Dilworth and Hall, analysis and construction of lattices L as unions of interval sublattices have been studied by several authors, see [9, 5] and [12, Section 3]), also [4] for a survey. Here, we need only the special case where L and the "skeleton" S are finite modular lattices. Though, without additional effort, one can allow arbitrary bounded lattices L and skeletons S which are modular of finite height.

6.1. **Glueing construction of Dilworth and Hall.** Given intervals $L_i = [a_i, b_i]$, $i = 1, \ldots, n$, in a modular lattice, L, such that $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$ for $1 \leq i < n$, the union of these intervals is a sublattice of L and one has isomorphisms $\alpha_i : [c_i, b_i] \to [a_{i+1}, d_i]$, $i < n$, with $\alpha_i(x) = x + a_{i+1}$ (and inverse $\alpha_i^{-1}(y) = b_i y$) where $c_i = b_i a_{i+1}$ and $d_i = b_i + a_{i+1}$.

Conversely, given pairwise disjoint modular lattices $L_i = [a_i, b_i]$ $(i \leq n)$ and isomorphisms $\alpha_i : [c_i, b_i]_{L_i} \to [a_{i+1]}, d_{i+1}]_{L_{i+1}}$ $(i < n)$ where $c_i \in L_i$ and $d_{i+1} \in L_{i+1}$ there is a modular lattice L, the *Dilworth-Hall glueing*, which is the union of interval sublattices L_i, related as above.

Also, one obtains a homomorphic image of L in which the intervals $[c_i, b_i]$ and $[a_{i+1}, d_i]$ are identified via α_i.

6.2. **Decomposition of lattices as glued sums.** In the sequel let S a modular lattice of finite height with bottom $\mathbf{0}$ and top $\mathbf{1}$. We write $x \prec y$ if x is a lower cover of y in S.

Consider a lattice M, a join embedding $\sigma : S \to L$, and a meet embedding $\pi : S \to L$ such $\sigma y \leq \pi x$ for all $x \prec y$ in S.- Then the union L of interval sublattices $L_x = [\sigma x, \pi x]$, $x \in S$. of L is a sublattice of M. L is called an S*-glued sum* of the L_x, $(x \in S)$. L has greatest element $1 = \pi(1)$ and smallest element $0 = \sigma(0)$.

If L' is another S glued sum given by σ', π' then L is isomorphic to L' if and if there are isomorphisms $\chi_x : L_x \to L'_x$ such that χ_x and χ_y both induce, for $x \prec y$, the same isomorphism of $L_x \cap L_y$ onto $L'_x \cap L'_y$.

Clearly, if $T = [u, v]$ is an interval sublattice of S then $\bigcup_{x \in T} L_x$ is a T-glued sum and an interval sublattice of L. This allows to use induction over the height of S.

Observe that given $x < y$ in S, $a \in L_x$, and $b \in L_y$, one has $a \leq b$ if and only if for some/each chain $x = x_1 \prec x_2 \ldots \prec x_n = y$ in S one has $a_i \in L_{x_i} \cap L_{x_{i+1}}$

$$a \leq_{x_1} a_1 \leq_{x_2} a_2 \ldots \leq_{x_{n-2}} a_{n-1} \leq_{x_n} a_n = b.$$

Claim 6.1. *An S-glued sum L is simple of so are the L_x ($x \in S$).*

Proof. For any homomorphism ϕ of L onto L' one has $\phi(\sigma(x)) = \phi(\pi(x))$ for all $x \in S$ since the L_x are simple. If $x \prec y$ in S then $\sigma y \leq \pi x$ whence $\phi(\sigma(y)) \leq \phi(\pi(x)) = \phi(\sigma(x))$ and it follows $\phi(\sigma(x)) = \phi(\sigma(y))$ for all $x \leq y$, in particular $\phi(0) = \phi(\sigma(0)) = \phi(\sigma(1)) = \phi(\pi(1)) = \phi(1)$. □

6.3. **Modularity.**

Lemma 6.2. *Any S-glued sum L of modular lattices L_x is modular.*

Proof. Consider $b \leq a$ and c in L such that $ac \leq b \leq a \leq b + c$. Let $a \in L_u$, $b \in L_v$, and $c \in L_w$. From $\sigma v \leq b \leq a \leq \pi u$ one obtains $\sigma(uv) \leq \sigma v \leq b \leq (\pi u)(\pi v) = \pi(uv)$ which allows to assume $v \leq u$. Now $\pi(v + w) \geq b + c \geq a$ so that w.l.o.g. $v + w = \mathbf{1}$. Dually, one may assume $uw = \mathbf{0}$ whence $v = u$ by modularity of S. If $\mathbf{0} \prec x < u$ then $w \prec x + w < \mathbf{1}$ and, by induction, $[\sigma x, \mathbf{1}]$ and $[\mathbf{0}, \pi(w + x)]$ are modular. whence Dilworth-Hall applies. Similarly, if $\mathbf{0} \prec x < w$. This leaves the case that u, w are atoms or $\mathbf{0}$. If, say, $u = \mathbf{0}$ then one has the Dilworth-Hall glueing $[0, \pi u] \cup [\sigma w, 1]$ whence modularity of L. If both u, w are atoms then $[0, \pi u]$ and $[\sigma w, 1]$ are modular by Dilworth-Hall and then so is L.

To prove the second claim, choose $x \prec \mathbf{1}$; by inductive hypothesis, there is a maximal chain

□

6.4. **Calculations in glued sums.**

We write $\sigma x = 0_x$ and $a +_x b = a + b$ for $a, b \in L_x$. For maximal chains C in intervals $[x, z]$ of S and $a \in L_x$ we define, recursively, $a +_C 0_x = a$ if $C = \{x\}$ and

$$a +_C 0_z = (a +_D 0_u) +_u 0_z \text{ where } D = C \setminus \{z\} \text{ and } u \prec z, u \in C$$

$$a +_C c = (a +_C 0_z) +_z c \text{ for } c \in L_z.$$

Now, for $a \in L_x$, $b \in L_y$, $z = x + y$, and maximal chains C in $[x, z]$ and D in $[y, z]$ define

$$a +_{C,D} b = (a +_C 0_z) +_z (b +_D 0_z).$$

The following is obvious, as is its dual.

Claim 6.3. (i) *$a +_C c = a + c$ for all $x \leq z$ in S, $a \in L_x$, $c \in L_z$ and maximal chains C in $[x, z]$.*

(ii) *$a +_{C,D} b = a + b$ for all x, y in S, $a \in L_x$,, $b \in L_y$, and maximal chains C in $[x, x + y]$, D in $[y, x + y]$.*

The dual results hold for meets.

For our main result it will be crucial that certain calculations can be carried out in a partial sublattice of L. We introduce some notation which will be useful later. For $x \prec y$ in S we put $0_{y,x} = \sigma y$ and $1_{y,x} = \pi x$.

Now, given $P \subseteq S^2$ where $x \prec y$ for all $(x, y) \in P$ let $L_{|P} = \bigcup_{(x,y)\in P}(L_x \cup L_y)$ endowed with the partial operations $a +_P b = c$ if and only if $a, b \in L_x$ for some $(x, y) \in P$ and $c = a +_x b$ or if $a \in L_x$, $b = 0_y$, and $c = a +_x 0_{y,x}$ for some $(x, y) \in P$ or, similarly, interchanging a with b. Partial meets are defined, dually. In view of Claim 6.3 any calculation in L can be composed by calculations in $L_{|P}$ where $(x, y) \in P$ for all $x \prec y$.

6.5. **Glueing of sets.** Again, S is a finite height modular lattice. In particular, S is a directed graph with edges (x, y) where $x \prec y$. Thus, a chain $x_1 \prec x_2 \ldots \prec x_n$ is a (directed) path from x_1 to x_n.

A *glueing* of a family $L_x (x \in S)$ of pairwise disjoint sets is given by injective partial maps $\gamma_{yx} : L_x \to L_y$, $x \prec y$, such that

$$(*) \quad \gamma_{x+y,x}\gamma_{x,xy} = \gamma_{x+y,y}\gamma_{y,xy} \text{ for } xy \prec x, y \prec x + y.$$

We put $\gamma_{x,x} = \gamma_C$ the identity on L_x where $C = \{x\}$. Observe that these conditions are satisfied if one replaces the order by its dual and the $\gamma_{y,x}$ by their inverses. This provides the counterparts of concepts and results.

Now, for a chain $C = \{x_1 \prec x_2 \ldots \prec x_n\}$ define

$$\gamma_C = \gamma_{x_n,x_{n-1}} \circ \ldots \circ \gamma_{x_2,x_1} : L_{x_1} \to L_{x_n}$$

which is again an injective partial map, possibly empty.

Claim 6.4. (i) *If C, D are maximal chains in $[x, z]$ then $\gamma_D = \gamma_C$.*

(ii) *If C_i is a maximal chain in $[x_{11}x_{21}, x_{in_i}]$ for $i = 1, 2$ then there are maximal chains D_i in $[x_{n_i}, x_{1n_1} + x_{2n_2}]$ such that $\gamma_{D_1}\gamma_{C_1} = \gamma_{D_2}\gamma_{C_2}$.*

Proof. To prove (i) we proceed by induction on the height of $[x, z]$. Consider $x \prec u, v$, $u \in C$, $v \in D$ and the maximal chains $C' = C \setminus \{u\}$ in $[u, z]$ and $D' = D \setminus \{v\}$ in $[v, z]$. If $u = v$ then one has $\gamma_C = \gamma_{C'}\gamma_{ux} = \gamma_{D'}\gamma_{ux} = \gamma_D$ by inductive hypothesis. Now, assume $u \neq v$ and $w = u{+}v$. Choose a maximal chain E in $[w, z]$. Again by induction and by $(*)$, one has

$$\gamma_C = \gamma_{C'}\gamma_{ux} = \gamma_E\gamma_{wu}\gamma_{ux} = \gamma_E\gamma_{wv}\gamma_{vx} = \gamma_{D'}\gamma_{vx} = \gamma_D.$$

To prove (ii) let D_j the image of C_i under the isomorphism $x \mapsto x{+}x_{jn_j}$ of $[x_{11}x_{21}, x_{in_i}]$ onto $[x_{jn_j}, x_{1n_1} + x_{2n_2}]$ for $\{i, j\} = \{1, 2\}$. $\square$

For $a_i \in L_{x_i}$ $(i = 1, 2)$ define $a_1 \sim a_2$ if and only if there are maximal chains C_i in $[x_i, x_1 + x_2]$ such that $\gamma_{C_1}a_1 = \gamma_{C_2}a_2$. In view of the dual of (ii) in Claim 6.4 one has $a_1 \sim a_2$ if and only if there are maximal chains D_i in $[x_1x_2, x_i]$ such that $\gamma_{D_1}^{-1}a_1 = \gamma_{D_2}^{-1}a_2$.

Claim 6.5. *$\sim$ is an equivalence relation on $\bigcup_{x\in S} L_x$ which restricts to identity on each L_x.*

Proof. For $a_i \in L_{x_i}$ consider chains C_i, D_i witnessing $a_i \sim a_{i+1}$ for $i = 1, 2$; that is $\gamma_{C_i}(a_i) = \gamma_{D_i}(a_{i+1})$. By (ii) of Claim 6.4 there are chains E_1, E_2 such that $\gamma_{E_1}\gamma_{D_1}(a_2) = \gamma_{E_2}\gamma_{C_2}(a_2)$ whence $\gamma_{E_1}\gamma_{C_1}(a_1) = \gamma_{E_2}\gamma_{D_2}(a_3)$ and so $a_1 \sim a_3$. If $x_1 = x_2 = x$ then $C_1 = D_1 = \{x\}$ and $\gamma_{C_1} = \gamma_{D_1}$ is identity of L_x whence $a_1 = a_2$. $\square$

We write $M = \left(\bigcup_{x\in S} L_x\right)$ and $L = M/\sim$ and denote the equivalence class of a by $[a]$. The following is immediate by (ii) of Claim 6.4 and its dual.

Claim 6.6. *For each $a \in M$ there are largest u resp. smallest v in S such that $L_u \cap [a] \neq \emptyset$ resp. $L_v \cap [a] \neq \emptyset$.*

We write $u = \lambda(a)$, $v = \mu(a)$ and observe that for all y with $\mu(a) \leq y \leq \lambda(a)$ one has unique $b = \tau_y(a) \in L_y \cap [a]$. We also write $\mu^*(a)$ for the unique $b \in L_{\mu(a)}$ such that $a \sim b$ and $\lambda^*(a)$ for the unique $c \in L_{\lambda(a)}$ such that $a \sim c$.

6.6. **Glueing of posets.** Now, assume each $L_x(x \in S)$ to be endowed with a partial order $\leq_x$, each with smallest element 0_x and greatest element 1_x. Also, assume the γ_{yx}, $x \prec y$ in S, to be order isomorphisms $\gamma_{y.x} : [0_{y,x}, 1_x]_{L_x} \to [0_y, 1_{y,x}]_{L_y}$, mapping an interval of L_x onto an interval of L_y. Moreover, it is required that $0_x <_x 0_{y,x}$ and $1_{y,x} <_y 1_y$.

Again, observe that these conditions are satisfied if one replaces the order by its dual and the $\gamma_{y,x}$ by their inverses.

Observe that for $x \prec y \prec z$ the map $\gamma_{zx} = \gamma_{zy} \circ \gamma_{yx}$ is either empty or an order isomorphism $[0_{z,x}, 1_x]_{L_x} \to [0_z, 1_{z,x}]_{L_z}$ where $0_{z,x} = \gamma_{zx}^{-1}(0_z)$

and $1_{z,x} = \gamma_{zx}(1_x)$. Thus, for a maximal chain C in $[x,y]$, γ_C is either empty or an order isomorphism $[0_{y,x}, 1_x]_{L_x} \to [0_y, 1_{y,x}]_{L_y}$.

For $a, b \in M$ define $a \leq^0 b$ if and only $a \in L_x$ and $b \in L_y$ for some $x \leq y$ and if there are $x = x_0 \prec x_1 \ldots \prec x_n = y$ and $a_i, b_i \in L_{x_i}$ $(0 \leq i \leq n)$ such that $a = b_0 \leq_{x_0} a_1$, $b_i = \gamma_{x_i x_{i-1}} a_i$ $(1 \leq i \leq n)$, $b_i \leq_{x_i} a_{i+1}$ $(1 \leq i < n)$, and $a_{n+1} = b$. Here, put $a <^0 b$ if $b_i <_{x_i} a_{i+1}$ for some $0 \leq i < n$. Observe that $a \sim b$, otherwise. Clearly, $\leq^0$ is transitive.

Observe that applying the above scheme to $S' = [u,v]$ and $\bigcup_{x \in S'} L_x$ one obtains the restriction of $\leq^0$ for this subset of M. Thus, one may proceed by induction on the height of S. In particular, with $a \leq^0 b$ witnessed as above, one has $\gamma_{\mathbf{1}x_i} b_i \leq_{\mathbf{1}} \gamma_{\mathbf{1}x_i} a_{i+1}$. Now, for $a, b \in M$ define

$$[a] \leq [b] \text{ if and only if there are } a' \sim a,\ b' \sim b \text{ such that } a' \leq^0 b'.$$

Claim 6.7. *$(L, \leq)$ is a partially ordered set such that the following hold for all $x, y \in S$ and $a, b \in M$.*

(i) *For $a, b \in L_x$ one has $a \leq_x b$ if and only if $[a] \leq [b]$.*

(ii) *$x \mapsto [0_x]$ is an order embedding of S into L and $[0_x] \leq [b]$ if and only if $x \leq \lambda(b)$.*

(iii) *$[0_{x+y}] = \sup([0_x], [0_y])$ in $(L, \leq)$ for all $x, y \in S$.*

Proof. We claim that $a <^0 b$ implies $[a] \neq [b]$. To prove this, we derive a contradiction from assuming $a <^0 b$ and $b \leq \gamma_{yx} a$. Proceeding by induction, it suffices to consider $x = \mathbf{0}$ and $y = \mathbf{1}$ in the definition of $<^0$. Thus, we have $0_{\mathbf{1},\mathbf{0}} \leq a = b_0 \leq_{\mathbf{0}} a_1$ and $\gamma_{\mathbf{1},\mathbf{0}} a_1 \geq \gamma_{\mathbf{1},\mathbf{0}} a \geq b$. If $a = a_1$ then we apply induction for $S' = [x_1, \mathbf{1}]$. If there is $j > 0$ such that $b_j <_{x_j} a_{j+1}$ then $a_1 <^0 b$ and we are done by induction. Otherwise, $b_j = a_{j+1}$ for all $j > 0$, $a <_{\mathbf{0}} a_1$, and $\gamma_{\mathbf{1},\mathbf{0}} a_1 = b \leq_{\mathbf{1}} \gamma_{\mathbf{1},\mathbf{0}} a <_{\mathbf{1}} \gamma_{\mathbf{1},\mathbf{0}} a_1$ contradicting the requirement that $\gamma_{\mathbf{1},\mathbf{0}}$ is an order isomorphism. It follows that $[a] \leq [b] \leq [a]$ implies $[a] = [b]$ proving that $\leq$ is a partial order on L.

To prove (i), consider $a <^0 b' \sim b$; one may assume $b' = \gamma_{zx} b$ with $z \geq x$ and, in view of induction, $x = \mathbf{0}$ and $z = \mathbf{1}$. Thus, $\gamma_{z,\mathbf{0}} b$ is defined for all z. Now choose $u \leq v$ in S with (u,v) minimal in S^2 such that there are $a' \in L_u$ and $b'' \in L_v$ with $\gamma_{u,\mathbf{0}} a <_u a' \leq^0 b'' \sim b$. By minimality of v one has $u = v$ and so $a' \leq_u b''$. Since $\gamma_{u,\mathbf{0}}^{-1} b''$ is defined, so is $\gamma_{u,\mathbf{0}}^{-1} a'$ and it follows $a = \gamma_{u,\mathbf{0}}^{-1} \gamma_{u,\mathbf{0}} a <_{\mathbf{0}} \gamma_{u,\mathbf{0}} a' \leq_{\mathbf{0}} \gamma_{u,\mathbf{0}}^{-1} b'' = b$. The converse is obvious.

The first claim in (ii) follows, immediately, from $0_x <_x 0_{y,x} \sim 0_y$ for $x \prec y$. In $0_x \leq^0 b$ we may assume $b \in L_{\lambda(b)}$ and conclude $x \leq \lambda(b)$ from the definition of $\leq^0$. The converse is obvious by this definition.

Ad (iii). By (ii) we have $b \geq^0 0_x, 0_y$ if and only if $\lambda(b) \geq^0 x, y$, that is $\lambda(b) \geq x + y$, which in turn is equivalent to $0_{\lambda(b)} \geq^0 0_{x+y}$. Thus, $b \geq^0 0_x, 0_y$ implies $b \geq^0 0_{x+y}$. The converse is obvious. □

6.7. **Glueing of semilattices and lattices.** As an immediate consequence of Claims 6.3 and 6.7 and of duality one obtains the following.

Lemma 6.8. *Consider a finite height modular lattice S and a glueing L of bounded posets L_x $(x \in S)$.*

(i) *If the L_x are join semilattices then so is L. Moreover, with the join operation $+_x$ on L_x one has $\sigma(x) = [0_x]$ a join embedding $S \to L$ and joins in L are computed according to Claim 6.3, identifying $a \in L_x$ with $[a]$.*

(iii) *If the L_x are lattices then so is L. Moreover, L is an S-glued sum given by $\sigma(x) = [0_x]$, $\pi(x) = [1_x]$ for $x \in S$.*

(iii) *If L is an S-glued sum of lattice L_x given by σ, π then L is isomorphic to the glueing L' of lattices $L'_x = L_x \times \{x\}$ with $0_x = (\sigma(x), x)$, $1_x = (\pi(x), x)$, $0_{y,x} = (\sigma(y), x)$, $1_{y,x} = (\pi(x), y)$, and glueing maps $\gamma_{yx}((a, x)) = (a, y)$ for $\sigma(y) \leq a \leq \gamma(x)$.*

6.8. **Partial isomorphisms between glued sums.**

Lemma 6.9. *Consider a lattice L obtained by glueing L_x $(x \in S)$ via γ_{yx}. Let $U \subseteq S$ an antichain and assume that L' is obtained by a glueing ϕ_{yx} $(x \prec y$ in $S)$ of the lattices L_x (having image L'_x in L') such that the following hold.*

(1) *$\phi_{yx} = \gamma_{yx}$ if $x \prec y$ in S and $\{x, y\} \cap U = \emptyset$.*

(2) *If $x \prec u \prec y$ and $u \in U$ then $\phi_{yu}(a) = \gamma_{yx}(\phi^{-1}_{ux}(a))$ for all $a \in [\phi_{ux}(0_{y,x}), 1_{u,x}]$.*

(3) *If $x \prec u \prec y$ and $u \in U$ then $\phi_{yu}(b) = \gamma_{yu}(b)$ for all $b \in [1_{u,x}, 1_u]$.*

Then the following hold.

(i) *There is an isomorphism $\chi : L_{|P} \to L'_{|P}$ such that $\chi \circ \gamma_{yx} = \phi_{yx} \circ \chi$ for all $(x, y) \in P$ where P consists of the (x, y), $x \prec y$ in S with $\{x, y\} \cap U = \emptyset$.*

(ii) *If T is an ideal of S such that $U \cap T = \emptyset$ then χ restricts to an isomorphism of the ideal $L_T = \bigcup_{x \in T} L_x$ of L onto the ideal $L'_T = \bigcup_{x \in T} L'_x$ of L'.*

According to (ii) we may identify L_T with L'_T. Also, by (i) we may use computations in $L_{|P}$ to verify relations in $L'_{|P}$.

Proof. Referring to (iii) of Lemma 6.8 define $\chi(a, x) = (a, x)$ for $a \in L_x$ and $x \notin U$ to obtain a bijective map $\chi : L_{|P} \to L'_{|P}$ which restricts to an isomorphism $L_x \to L'_x$ for each $x \notin U$ and such that $\chi(0^L_{y,x}) = 0^{L'}_{y,x}$ and $\chi(1^L_{y,x}) = 1^{L'}_{y,x}$ if $(x,y) \in P$. □

7. Basic models

7.1. **The lattice directly associated with a group.** Adapting the construction in [12, Section 4], fix a prime p. Recall the abelian group A and its lattice $L(A)$ from Subsection 4.4. In particular, A is a $\mathbb{Z}_{p^2}$-module. The proper "skeleton" will be the ideal $S = L(pA)$ of $L(A)$.
.

Fact 7.1. *$L(A)$ is an S-glued sum with embedding of S into $L(A)$ given by $\sigma(X) = X$ and $\pi(X) = \{a \in A \mid pa \in X\}$. The interval sublattices $[\sigma(X), \pi(X)]$ of $L(A)$ are isomorphic to lattices $L(\mathbb{Z}_p^4)$ and $L(A)$ is a finite simple modular lattice.*

Proof. pA is a free $\mathbb{Z}$-module with generators $f_i = pe_i$ and relations $pf_i = 0$, $i = 1,2,3$- Now, with $V = pA + \mathbb{Z}e_4$ one obtains an isomorphism $f_i \mapsto e_i + V \in A/V$ and a lattice isomorphism $\pi : S \to [V, A] \subseteq L(A)$ such that $\sigma(X) \leq pA \leq \pi(Y)$ for all $X, Y \in S$. It follows that $L = \bigcup_{x\in S}[\sigma(X), \pi(X]$ is a sublattice of $L(A)$. Now, for any $C \in L(A)$ one has $X := pC \in S$ and $C \subseteq \pi(X)$ whence $L = L(A)$. Finally, $L(A)$ is simple in view of Lemma 6.1. □

Given a group G, let Q denote the group ring $\mathbb{Z}_{p^2}(G)$ with coefficients the integers modulo p^2. The free Q-module B with generators e_1, e_2, e_3, e_4 and relation $pe_4 = 0$ has a subgroup A generated by e_1, e_2, e_3, e_4

Fact 7.2. *$L(A)$ embeds into the Q-submodule lattice $L(B)$ of B via $X \mapsto \varepsilon(X) = QX$; moreover, $\sigma'(X) = QX$ and $\pi'(X) = Q\pi(X)$ are lattice embeddings of S into $L(B)$ establishing an S-glued sum $L(G)$ within $L(B)$. Finally, with the ideal $T = [0, \mathbb{Z}e_1 + \mathbb{Z}e_3]$ of S, $L_T(G) = \bigcup_{X\in T}[QX, Q\pi(X)]$ is an ideal of $L(G)$.*

Observe that B and $L(B)$ are finite if so is G and that $L(G)$ is a "rather small" sublattice of $L(B)$.

Proof. Since Q is a free $\mathbb{Z}_{p^2}$-module (with basis G), Q is a flat $\mathbb{Z}_{p^2}$-module and the map $X \mapsto QX = Q \otimes_{\mathbb{Z}_{p^2}} X$ is a lattice homomorphism $L(A) \to L(B)$ and injective since $L(A)$ is simple. Thus, σ' and π' are lattice embeddings, too, and $L(G) = \bigcup_{X\in S}[\sigma'(X), \pi'(X)]$ a sublattice of $L(B)$ which is an S-glued sum. □

7.2. **Capturing a group generator by a stable term.** In $L_T(G)$ there is a (canonical) skew $(3,2)$-frame $\Psi^0 = (\Phi'^0, \Phi^0)$ given by the submodules

$$\Phi'^0: \; Qpe_1, Qpe_3, Qe_4, Q(pe_1 - pe_3), Q(pe_1 - e_4), Q(pe_3 - e_4)$$

$$\Phi^0: \; Qe_1, Qe_3, Q(e_1 - e_3).$$

Choose $L_0(G)$ as the ideal $[0, \top^{\Psi^0_{(3,2)}}]$ of $L_T(G)$. The group G embeds into the group of units of the coordinate ring $R(\Phi'^0, L_0(G))$ via

$$g \mapsto Q(pe_1 - gpe_3).$$

Recall the definitions of the derived skew frames $\Psi_{(3,2)}$ in Subsection 4.4 and the terms $g_1^*(\bar{z}', \bar{z})$ and $g^+(\bar{z}', \bar{z})$ from Lemma 5.3.

Lemma 7.3. *For each group G and $g \in G$ there is a modular lattice $L(G,g)$ (finite if G is finite) with spanning skew $(4,3)$-frame $\Psi = (\Phi', \Phi)$ of characteristic $p \times p$ and an isomorphism ω from $L_0(G)$ onto the interval $L_0(G,g) = [0, \top^{\Psi_{(3,2)}}]$ of $L(G,g)$ matching the skew $(3,2)$-frame Ψ^0 of $L_0(G)$ with $\Psi_{(3,2)}$, inducing an isomorphism of coordinate rings $R(\Phi'^0, L_0(G)) \to R(\Phi', L_0(G,g))$, and such that*

$$(*) \quad \omega(Q(pe_1 - gpe_3) = g^+(\Psi)$$

which is a stable element w.r.t. the 3-frame $\Phi'_{\neq 2}$.

Proof. Leaving $(*)$ aside, the lattice $L(G,g)$ and the isomorphism ω have been constructed in [12, Section 4] relying on the method established in Lemma 6.9. Modularity follows from Lemma 6.2.

In particular, we may assume $L_T(G)$ an ideal of $L(G,g)$ and ω identity. Moreover, according to [12, Lemma 18] one has

$$Q(pe_1 - gpe_2) = g_1^*(\Psi) \in L_T(G)$$

with $g_1^*(\Psi)$ stable for Φ' according to [12, Cor.13] and $(*)$ follows applying the perspectivity π_{23} cf. Lemma 5.3.

Observe that in [12] Z_p has been used to denote both the ring $\mathbb{Z}/p\mathbb{Z}$ and the ideal pZ_{p^2} of $Z_{p^2} = \mathbb{Z}/p^2\mathbb{Z}$. Similarly, R denoted both the ring Q/pQ and the ideal pQ of Q. Referring to the ideal, given an element $a = \sum_{i=1}^3 r_i e_i \in A$ one has the subgroup $Z_p a = \mathbb{Z} \sum_{i=1}^3 r_i pe_i$ of pA and given $b = \sum_{i=1}^3 r_i e_i \in B$ one has the Q-submodule $Rb = Q \sum_{i=1}^3 r_i pe_i$ of pB. In each case, the proper meaning is obvious from the context. $\square$

7.3. **Basic model.**

Theorem 7.4. *For each group G with generators $\bar{g} = (g_1, \ldots, g_n)$ in G there is a modular lattice $L(G, \bar{g})$ such that the following hold*

(1) $L(G,\bar{g})$ *contains an* n*-tower* $\Omega(n)_{can}$ *(to be referred to as* canonical*) of skew* $(4,3)$*-frames* $\Psi^i = (\Phi'^i, \Phi^i)$ *of characteristic* $p \times p$, $i = 1, \ldots, n$
(2) *There is an embedding* $\gamma : G \to R(\Phi'^n, L(G,\bar{g}))$ *such that, for all* i, $g^+(\Psi^i) + \perp^{\Psi^n} = \gamma(g_i)$ *and* $\gamma(g_i) \cdot (a_1^{\Phi'^i} + a_3^{\Phi'^i}) = g^+(\Psi^i)$.
(3) $L(G,\bar{g})$ *is finite if* G *is finite.*

Proof. Given i, consider $L(G,g_i)$ from Lemma 7.3 with skew $(4,3)$-frame $\Psi^i = (\Phi'^i, \Phi^i)$ and isomorphism $\omega_i : L_0(G) \to L_0(G,g_i)$. We may assume that the $L(G,g_i)$ are pairwise disjoint lattices. Now,

$$\alpha_i(x) = \omega_{i+1}(\omega_i^{-1}(x \cdot \top^{\Psi^i_{(3,2)}})) \in L_0(G,g_i)$$

defines an isomorphism

$$\alpha_i : [a_2^{\Psi^i}, \top^{\Psi^i}]_{L(G,g_i)} \to [\perp^{\Psi^{i+1}}, \top^{\Psi^{i+1}_{(3,2)}}]_{L_{i+1}(G,g_{i+1})}.$$

Let $L(G,\bar{g})$ arise by Dilworth-Hall glueing (as described in Subsection 6.1) the $L(G,g_i)$ via the isomorphisms α_i. One has, due to the glueing via the α_i,

$$(**)\ \omega_i(x) + \perp^{\Psi^j} = \omega_j(x) \text{ for } x \in L_0(G)$$

for $j = i+1$. The case $i \leq j < n$ as well as the relations required for an n-tower follow by induction and transitivity of $\nearrow$ - recall $\Psi^i_{(3,2)} \nearrow \Psi^{i(3,2)}$. With the canonical embedding

$$\eta : G \to R(\Phi'^0, L(G,\bar{g})) \text{ where } \eta(x) = Q(pe_1 - xpe_3)$$

equation $(*)$ of Lemma 7.3 together with $(**)$ for $j = n$ yield

$$\omega_n(\eta(g_i)) = \omega_i(\eta(g_i)) + \perp^{\Psi^n} = g^+(\Psi^i) + \perp^{\Psi^n}.$$

Since G is generated by $\bar{g}$, $\omega_n \circ \eta$ restricts to an embedding $\gamma : G \to R(\Phi'^n, L(G,\bar{g}))$ as required in (2). □

8. Unsolvability

In order to prove Theorem 1.2 applying Slobodkoi's Theorem 1.1, we show that there is an algorithm reducing the Uniform Word Problem for the class $\mathcal{G}_0$ of all finite groups to the decision problem for the equational theory of the class $\mathcal{M}_0$ of finite modular lattices. Moreover, we observe that this reduction produces identities in $n+6$ lattice variables, if applied to group presentations in n generators. To prove the reduction, we verify the hypotheses of Lemma 2.4 which are given just before that lemma.

Proof. Consider a finite group presentation given by words $w_j(\bar{g})$, $1 \leq j \leq h$ in a list $\bar{g} = (g_1, \ldots, g_n)$ of generator symbols and relations $w_j(\bar{g}) = e$, $j = 1, \ldots, h$. We construct a series of n-towers Ω_m of skew $(4,3)$-frames

$$\Omega_m = (\Psi^k_m \mid_{k=1,\ldots,n}) = (\Phi'^k_m, \Phi^k_m \mid_{k=1,\ldots,n})$$

$$= (a'^k_{mi}, c'^k_{mij}; a^k_{mi}, c^k_{mij} \mid_{k=1,\ldots,n}), 0 \leq m \leq \mu = n + h.$$

The list of generators of Ω_0 is also denoted by $\bar{a}$, that of Ω_m by $\bar{a}_m$. Let $F = F_0$ denote the modular lattice freely generated by the n-tower $\Omega = \Omega_0$. The construction will be such that Ω_m generates a sublattice F_m of F_0 so that $F_{m+1} \subseteq F_m$ for all $m < \mu$. Moreover, the following will hold.

(A) For the 4-frame $\Phi'^n_\mu = (a'^n_{\mu,i}, c'^n_{\mu,ij})$ in F_μ one has a list of elements $\bar{s}_\mu = (s_{\mu 1}, \ldots, s_{\mu n})$ in the group $R^\#(\Phi'_\mu, F)$ such that $w_j(\bar{s}_\mu) = c'^n_{\mu,13}$ for $1 \leq j \leq h$.

(B) For any group G and $\bar{g}$ in G with $w_j(\bar{g}) = e$ for $1 \leq j \leq h$ one has $\phi(\bar{a}) = \phi(\bar{a}_\mu)$ and $\phi(s_{\mu i}) = \gamma(g_i)$ for $i = 1, \ldots, n$ where $\phi : F \to L(G, \bar{g})$ is the homomorphism mapping $\bar{a}$ onto the canonical n-tower Ω_{can} of the lattice $L(G, \bar{g})$ constructed in Thm. 7.4.

Observe that ϕ in (B) exists by Thms. 3.9 and 7.4 (1). This construction will be uniform for all group presentations, to be implemented by an algorithm as required in Lemma 2.4.

In the context of this lemma, we consider quasi-identities β in the language of groups with antecedent α the conjunction of identities $w_j(\bar{y}) = e$, $j = 1, \ldots, h$, where $\bar{y} = (y_1, \ldots, y_n)$. The presentation required in (a) of Lemma 2.4 is that of an n-tower Ω of skew frames - with generator symbols $\bar{a}$. Recall from Thm. 3.9 that Ω can be defined in terms of $n + 6$ generators.

The terms $u_i(\bar{x})$ are chosen such that $\bar{u}(\bar{a})$ is the n-tower generating F_μ within F. Hypothesis (i) is satisfied due to Cor 3.10. Concerning hypothesis (ii), consider a homomorphism $\phi : F \to L \in \mathcal{M}$ and observe that $\phi(\bar{u}(\bar{a})) = \phi(\bar{a}_\mu) = \phi(\bar{a})$ by (B).

The translation required in (b) is given by the constant c'^n_{13} defining the multiplicative identity and the terms defining multiplication and inversion in the group $R^\#(\Phi'^n, F)$ related to the 4-frame $\Phi'^n = (a'^n_i, c'^n_{ij})$ which is part of the n-tower $\bar{a}$. According to Subsections 5.1 and 5.2 this translation satisfies hypothesis (iii) within $\mathcal{M}$. Also by this, the algebra G in (iv) is indeed a group, finite if L is finite. Moreover, the generators $\bar{u}|_n(\phi(\bar{a}))$ satisfy α by (A).

Finally, hypothesis (v) is granted by Theorem 7.4 and (B).

Outline of construction: To obtain $\bar{a}_\mu$ we put $\bar{a}_0 = \bar{a}$ and construct, iteratively, n-towers $\Omega_m = \bar{a}_m$, $m \leq \mu$.

Case 1: $m \leq n$

(1) The n-tower $\Omega_m = \bar{a}_m$ is obtained from the n-tower $\Omega_{m-1} = \bar{a}_{m-1}$ by lower reduction, induced by a reduction of Ψ^{m-1}_{m-1} to Ψ^m_m, within the sublattice F_{m-1} of F generated by $\bar{a}_{m-1}$,

(2) One has m elements $s_{m1}, \ldots, s_{mm}$ 3-stable in F for the 4-frame Φ'^n_m in Ω_m.

(3) s_{mi} ($i \leq m-1$) is obtained as in Fact 4.1 by the lower reduction in (1) from $s_{m-1,i}$ stable in F for Φ'^n_{m-1} while $s_{mm} = s + \perp^{\Psi^n_m}$ where $s = g^+(\Psi^m_m)$ is 3-stable in F for the 4-frame Φ'^m_m.

(4) The reduction in (3) is chosen such that the skew frame Ψ^m_{m-1} is reduced as in Lemma 5.2 to the skew frame Ψ^m_m having characteristic $p \times p$.

Case 2: $n < m \leq \mu = n + h$.

(5) $\bar{a}_m$ is obtained from $\bar{a}_{m-1}$ by upper reduction within the sublattice F_{m-1} of F generated by $\bar{a}_{m-1}$.

(6) One has a list $\bar{s}_m$ of n elements stable in F for the 4-frame Φ'^n_m and satisfying $w_j(\bar{s}_m) = c'^n_{m,13}$ for $j \leq m - n$, within the group $R^\#(\Phi'^n_m, F)$.

(7) These are obtained from $\bar{s}_{m-1}$ by the upper reduction in (5).

Proof of (A) and (B). We show, by induction, that for all $m \leq \mu$

- $\phi(\bar{a}_m) = \phi(\bar{a})$, that is $\phi(\Omega_m) = \phi(\Omega)$.
- $\bar{s}_m$ is a list of stable elements for Φ'^n_m
- $\phi(s_{mi}) = \gamma(g_i)$ for $i \leq \min(m, n)$.
- $w_j(\bar{s}_m) = c'^n_{m,13}$ where $m \geq n$ and $j \leq h = m - n$.

The case $m = 0$ is just the definition of ϕ. For $m \leq n$, we apply Lemma 5.2 to Ψ^m_{m-1}, that is with $\mathbf{b} = b^*(\bar{a}'^m_{m-1}, \bar{a}^m_{m-1})$ and $\mathbf{d} = d^*(\bar{a}'^m_{m-1}, \bar{a}^m_{m-1})$. By inductive hypothesis one has $\phi(\Psi^m_{m-1}) = \phi(\Psi^m)$ which is of characteristic $p \times p$ as part of the canonical n-tower of $L(G, \bar{g})$, whence $\phi(\mathbf{b}) = \phi(\perp^{\Psi^m_m})$ and $\phi(\mathbf{d}) = \phi(a_1'^{\Psi^m_m})$ in view of (4). It follows

$$\phi(\Psi^m_m) = \phi((\Psi^m_{m-1})_{\mathbf{b},\mathbf{d}}) = (\phi(\Psi^m_{m-1}))_{\phi(\mathbf{b}),\phi(\mathbf{d})} = \phi(\Psi^m_{m-1}) = \phi(\Psi^m).$$

This in turn implies $\Phi(\Omega_m) = \Phi(\Omega_{m-1})$ in view of Observation 4.3. The element s in (3) being chosen as $s = g^+(\Psi^m_m)$ according to Lemma 5.3 we have s 3-stable for Φ'^m_m and s_{mm} 3-stable for Φ'^n_m due to the isomorphism induced by $(\Psi^m_m)_{3,2} \nearrow (\Psi^n_m)_{3,2}$ which matches Φ'^m_m with Φ'^n_m. Moreover, $\phi(s_{mm}) = \gamma(g_m)$ by (2) of Thm. 7.4. For $i < m$ the other hand, according to (3) s_{mi} is obtained from $s_{m-1,i}$ as in Fact 4.1 applying the reduction with $\mathbf{b}_1 + a^n_\perp$ to the 4-frame Φ'^n_{m-1}. In particular, s_{mi} is stable for Φ'^n_m. Since $\phi(\Omega_m) = \phi(\Omega_{m-1})$ it follows that $\phi(s_{mi}) = \phi(s_{m-1,i}) = \gamma(g_i)$.

For $m = n + j$ we proceed with the same kind of reasoning, now referring to Lemma 5.1, to add $w_j(\bar{s}_m) = c'^n_{m,13}$, while stable $\bar{s}_{m-1}$ leads to stable $\bar{s}_m$, and $\phi(s_{m-1,i}) = \gamma(g_i)$ to $\phi(s_{m,i}) = \gamma(g_i)$ and $w_i(\bar{s}_{m-1}) = c'^n_{m-1,13}$ to $w_i(\bar{s}_m) = c'^n_{m,13}$ for all $i < m - n$.

□

9. Remarks

Reducing (Restricted) Word Problems for groups to such for modular lattices follows the same scheme in the finite and in the infinite case. Recall from Subsection 5.1 that with any 4 frame in a modular lattice one has the associated von Neumann coordinate ring $R(\Phi)$ with subgroup $R^*(\Phi)$ of units, all defined in terms of the frame.

Now, with a group presentation $(\Pi, \bar{g})$ associate the lattice presentation $\lambda(\Pi, \bar{g})$ obtained as follows: To the 4-frame Φ add a generator symbol g_i for each g_i; also, to the relations add the relations $a_1 g_i = a_\perp$, $a_1 + g_i = a_1 + a_2$, and $w_i(\bar{g}) = c_{13}$. Here. $w_i(\bar{g}) = e$ is a relation of $(\Pi, \bar{g})$.

By this construction, if $w(\bar{g}) = e$ is a consequence of $(\Pi, \bar{g})$ for (finite) groups, then $w(\bar{g}) = c_{13}$ is a consequence of $\lambda(\Pi, \bar{g})$ for (finite) modular lattices L, since $R^*(\Phi, L)$ is a (finite) group for any L.

On the other hand, if $(G, \bar{h})$ is a model of $(\Pi, \bar{g})$ such that $w(\bar{h}) \neq e$ (and G finite), then a (finite) model $(L, \Phi, \bar{h})$ with $w(\bar{h}) \neq c_{13}$ is obtained choosing a (finite) vector space ${}_FV$ of $\dim_F V = 4|G|$; then the lattice L of R-submodules of R^4, R the group ring $F[G]$, embeds into the lattice $L({}_FV)$ of subspaces; moreover, the canonical 4-frame Φ of L together with the $R(e_1 - h_i e_3) \in R^*(\Phi)$ provide the required model of $\lambda(\Pi, \bar{g})$ such that $w(\bar{g}) \neq c_{13}$. In particular, the model embeds into the subspace lattice $L({}_FV)$. Consequently, the relevant class of models consists of sublattices of $L({}_FV)$ where $\dim_F V$ is infinite respectively of $L({}_{F_d}V_d)$ where $\dim_{F_d} V_d \to \infty$. In particular, from Thm. 1.1 one has the following.

Corollary 9.1. *Let $\mathcal{C}$ be any class of finite modular lattices such that for all $d \in \mathbb{N}$ there are lattices of subspaces $L({}_{F_d}V_d)$ in $\mathcal{C}$ with $\dim_{F_d} V_d \geq d$. Then the Restricted Word Problem for $\mathcal{C}$ is unsolvable.*

With a simple modification one can restrict the number of lattice generators to 5: For any $n \geq 3$ n-frames have an equivalent presentation in modular lattice theory to a presentation with 4-generators [10, Satz 4.1] and with $g = \sum_{j=1}^{n} \pi_{3,j+3}(g_j)$ one obtains $g_j = \pi_{j+3,3}(g \cdot (a_1 + a_{j+3}))$

for $j = 1, \ldots, n$ to replace $\bar{g}$ equivalently by g and to proceed with the 4-frame given by a_i, c_{ij} where $i, j \leq 4$.

For reduction to identities, the scheme is modified as described in Subsection 2.4. In order to associate with group generators lattice terms which allows one to force group relations within the lattice, several frames are combined via some kind of glueing. This leads to models which are non-Arguesian lattices and, in particular, do not embed into lattices of normal subgroups.

In all examples, discussed, one has a certain set Σ of quasi-identities in the language of groups and for each $\beta \in \Sigma$ an associated quasi-identity $\lambda(\beta)$ in the language of lattices and a class $\mathcal{S}$ of (finite) modular lattices, the class of "models", such that the following hold.

- If β holds for all (finite) groups then $\lambda(\beta)$ holds for all (finite) modular lattices.
- If β fails for some (finite) group then $\lambda(\beta)$ fails for some "model" lattice in $\mathcal{S}$.

Thus, if Σ is undecidable for the class of (finite) groups, then the set of $\lambda(\beta)$ valid in all (finite) modular lattices and the set of $\lambda(\beta)$ failing in some lattice in $\mathcal{S}$ are recursively inseparable. In other words, the undecidability results extend to all classes of (finite) modular lattices containing the relevant class of models.

Observe that the number of generators in Slobodkoi's Theorem is $3m+61$ where m is the minimum number of states of a two tape Minsky machine computing some partial recursive function with non-recursive domain.

Problem 9.2. *What is the minimal N such that the N-variable equational theory of finite modular lattices is undecidable.*

Since skew (n,m)-frames can be generated by 8 elements, the following could be of use.

Problem 9.3. *Can one find $n - m$ stable elements in the modular lattice freely generated by a skew (n,m)-frame of characteristic $p \times p$?*

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