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A parameter for subdirectly irreducible modular lattices with four generators

CHRISTIAN HERRMANN

BIRKHOFF [1; Problem 43] suggested to study modular lattices with four generators by imposing relations, first — e.g. — the relations expressing that the generators split into two complemented pairs. Basing on more special results of DAY, HERRMANN, and WILLE [2] and SAUER, SEIBERT, and WILLE [9] Birkhoff's problem has been solved in [6]. Remarkably enough, the subdirectly irreducible factors can be given by diagrams (including infinite ones) — these factors are the lattices M_4, $S(n, 4)$, R_∞ and its dual defined in § 1. In [7] there have been constructed lattice polynomials s_n (and their duals s_n^* — see § 2) such that a subdirectly irreducible modular lattice M (with more than 5 elements) is one of the above if and only if $s_n=1$ and $s_n^*=0$ holds in M for all n. In the present note we want to provide a basis for the study of subdirectly irreducible four generated modular lattices not being one of the above. In particular, we show that an inductive approach is possible using the polynomials s_n.

Theorem. *Let M be a subdirectly irreducible modular lattice with four generators a, b, c, d not being isomorphic to any of the lattices M_4, $S(n, 4)$ $(n<\infty)$, R_∞ or its dual. Then there is an n such that either*

(i) $s_n(a, b, c, d) = 0 = ab = ac = ad = bc = bd = cd$

or

(ii) $s_n^*(a, b, c, d) = 1 = a+b = a+c = a+d = b+c = b+d = c+d.$

Examples of such lattices are the rational projective geometries of finite dimension (GELFAND and PONOMAREV [4; § 8]) and, more generally, all subdirectly irreducible modular lattices generated by a frame ([5] and [7]). The use of the s_n in the analysis these examples has been pointed out in [7]. Clearly, such lattices can be visualized by diagrams in the most trivial cases, only.

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Corollary. *The* M_4, $S(n, 4)$ $(n<\infty)$, R_∞ *and its dual are the only subdirectly irreducible modular lattices generated by* a, b, c, d *such that* $a+b=c+d=1$ *and* $ab=cd=0$ [6]. M_4 *and* R_∞ *are the only ones for which, in addition,* $ac=ad=bc=bd=0$ (Sauer, Seibert, and Wille [9]). R_∞ *is the modular lattice freely generated by the partial lattice* J_1^4 (Day, Herrmann, and Wille [2]).

Also, it follows that the lattices listed in the Corollary are the only four generated subdirectly irreducible modular lattices of breadth $\leqq 2$ (Freese [3]) or, more generally, satisfying the 2-distributive law ([6]).

The proofs do not depend on [2] nor [9]. From [6] we need only § 2 and 3 and from [7] § 1 and 5. The basic tool is the neutral element method from [6] — see § 3.

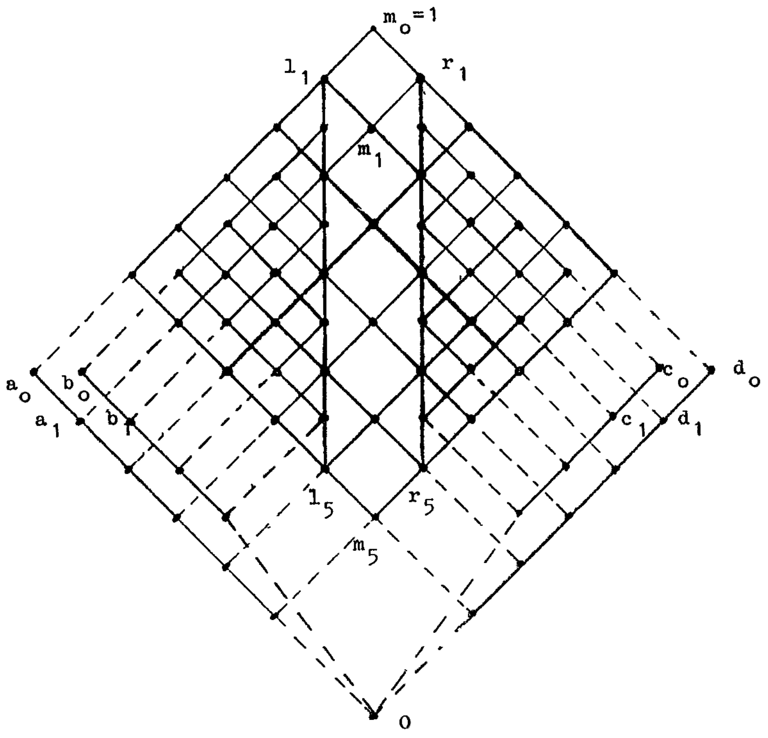

Figure 1

Replace a_i, b_i, c_i, d_i, m_i, l_i, r_i, 0, 1 respectively

a) by $\hat{a}_i$, $\hat{b}_i$, $\hat{c}_i$, $\hat{d}_i$, $\hat{m}_i$, $\hat{l}_i$, $\hat{r}_i$, $\hat{0}$, $\hat{1}$, b) by $\bar{a}_i$, $\bar{c}_i$, $\bar{b}_i$, $\bar{d}_i$, $\bar{m}_i$, $\bar{l}_i$, $\bar{r}_i$, $\bar{0}$ $\bar{1}$,

c) by $\tilde{a}_i$, $\tilde{d}_i$, $\tilde{b}_i$, $\tilde{c}_i$, $\tilde{m}_i$, $\tilde{l}_i$, $\tilde{r}_i$, $\tilde{0}$, $\tilde{1}$.

§ 1. The breadth two models

First, let us introduce the lattices referred to in the main theorem. M_n is the length two lattice with n atoms. Let A_∞ (cf. Fig. 1) consist of the elements $x(i, j)$ $0 \leqq i \leqq j \leqq \infty$, $x \in E = \{a, b, c, d\}$ with the equalities $a(i, i) = b(i, i) = c(i, i) = d(i, i) =: m_i$ $(0 \leqq i \leqq \infty)$, $a(i-1, i) = b(i-1, i) =: l_i$ and $c(i-1, i) = d(i-1, i) =: r_i$ $(1 \leqq i < \infty)$ and no others. The relation $\leqq$ on A_∞ is defined in the following way (with $x \neq y$ in E, $0 \leqq i \leqq j \leqq \infty$, and $0 \leqq k \leqq l \leqq \infty$)

$$x(i, j) \leqq x(k, l) \quad \text{if and only if} \quad k \geqq i \text{ and } l \geqq j,$$

$$x(i, j) \leqq y(k, l) \quad \text{if and only if} \quad \begin{cases} l \leqq i & \text{for} \quad \{x, y\} \neq \{a, b\}, \{c, d\} \\ l \leqq i+1 & \text{and} \quad k \leqq i \text{ else.} \end{cases}$$

This yields a modular lattice order on A_∞ such that

$$x(i, j) + x(k, l) = x(s, t) \quad \text{with} \quad s = \min(i, k), \quad t = \min(j, l)$$

$$x(i, j) \cdot x(k, l) = x(s, t) \quad \text{with} \quad s = \max(i, k), \quad t = \max(j, l)$$

$$\left.\begin{aligned} x(i, j) + y(k, l) &= x(i, s) \text{ for } i \leqq k \text{ and } s = \min(j, k) \\ x(i, j) \cdot y(k, l) &= x(s, j) \text{ for } j \geqq l \text{ and } s = \max(i, l) \end{aligned}\right\} \text{ if } \{x, y\} \neq \{a, b\}, \{c, d\}$$

$$\left.\begin{aligned} x(i, j) + y(k, l) &= x(i, s) \text{ for } i \leqq k \text{ and } s = \min(k+1, j, l) \\ x(i, j) \cdot y(k, l) &= x(s, j) \text{ for } j \geqq l \text{ and } s = \max(i, l) \end{aligned}\right\} \text{ else.}$$

Put $x_i = x(i, \infty)$. Then every element of A_∞ has a unique representation $m_i (0 \leqq i \leqq \infty)$, l_i, r_i $(1 \leqq i < \infty)$, x_i $(0 \leqq i < \infty)$, or $x_i + m_n$ $(0 \leqq i \leqq n-2)$ with x in E. A_∞ is generated by the x_0 $(x \in E)$ as one derives from the relations $m_0 = 1$, $m_\infty = 0$, $l_{n+1} = a_n + b_n$, $r_{n+1} = c_n + d_n$, $m_{n+1} = r_{n+1} l_{n+1}$, and $x_{n+1} = x_0 m_{n+1}$.

Observe that every proper quotient of A_∞ contains a prime quotient $x(i, j)/x(k, l)$ with $l = j$ and $k = i+1$ or $k = i$ and $l = j+1$. Moreover, $x(i, j)/x(i+1, j)$ is transposed upward to $y(k, l)/y(s, t)$ if and only if $x = y$, $i+1 = s = k+1$, and $j \geqq l = t$ or $x \neq y$, $\{x, y\} \neq \{a, b\}$, $\{c, d\}$, $k = s$, and $i+1 \geqq t = l+1$ or, finally, $\{x, y\} \in \{\{a, b\}, \{c, d\}\}$ and $l = s = t = i+1 = k+1$ or $k = s \leqq i$, $t \leqq i+2$, and $t = l+1$. On the other hand $x(i, j)/x(i, j+1)$ is transposed upward to $y(k, l)/y(s, t)$ if and only if $x = y$, $k = s \leqq i$, $l = j$, and $t = j+1$ respectively $\{x, y\} \in \{\{a, b\}, \{c, d\}\}$ and $i = j = l$, $k = i-1 = s$, $t = i+1$ or $i = j$, $k = s \leqq i-2$, $l = i = t-1$. Thus, every prime quotient is projective to one of $1/l_1$ and $1/r_1$. Let Q consist of all quotients $x(i, n)/x(i+1, n)$ with i even and $x = c, d$ or i odd and $x = a, b$ as well as the quotients $x(i, n)/x(i, n+1)$ with n even and $x = a, b$ or n odd and $x = c, d$ and, finally, the r_i/r_{i+1} with i odd and l_i/l_{i+1} with i even. Then $1/l_1$ is in Q and Q describes a minimal congruence θ. Let R_∞ be the homomorphic image A_∞/θ. Its operation table can be derived easily from that of A_∞. (Actually, R_∞ is the lattice

$FM(J_1^4)$ from [2] where its diagram is given.) Let φ be defined as 0 interchanging "odd" with "even". By symmetry, A_∞/φ is isomorphic to R_∞. The intersection $0\cap\varphi$ is the identity and every proper congruence of A_∞ contains 0 or φ. Thus, R_∞ is subdirectly irreducible. Since $A_\infty/0\vee\varphi$ is the simple lattice M_4 there are no other homomorphic images of A_∞.

The section $[m_n, 1]$ of A_∞ is called A_n. It is generated by the $x(0, n)$ (x in E). The restrictions of the congruences 0 and φ to A_n yield a subdirect decomposition into two isomorphic simple factors called $S(n, 4)$ — use the same arguments as above! Clearly, $S(n, 4)$ is isomorphic to the section $[[m_n]0, 1]$ of R_∞.

§ 2. Some lattice polynomials

We have to recall some definitions and results from [7]. Let F be the modular lattice with 0 and 1 freely generated by four elements $a=e_1$, $b=e_2$, $c=e_3$, $d=e_4$. Write $E=\{a, b, c, d\}$ and $\mathbf{n}=\{1, \dots, n\}$. Put $q_1=(a+b)(c+d)$, $q_2=(a+c)(b+d)$, $q_3=(a+d)(b+c)$. Let $x\mapsto x^i=x(aq_i, bq_i, cq_i, dq_i)$ denote the endomorphism of F with $1\mapsto q_i$, $0\mapsto 0$, and $e\mapsto eq_i$ for $e\in E$. Define by induction

$$s_0 = 1, \quad s_1 = a+b+c+d, \qquad s_{n+1} = \sum (s_n^i | i\in\mathbf{3})$$

$$t_0 = 1, \quad t_1 = (a+b+c)(a+b+d)(a+c+d)(b+c+d), \quad t_{n+1} = \sum (t_n^i | i\in\mathbf{3}).$$

Let x^* be the dual of x. Then 1.1, 1.3, 1.2, and 5.1 of [7] yield

Lemma 2.1. *For $n\geqq 0$ and $i\neq j$ in $\mathbf{3}$ one has*

(1) $q_iq_j = q_j^i$ *and* $(x^i)^j = (x^j)^i$ *for all x in F.*

(2) $s_{n+1} = s_n^i + s_n^j$ *and* $t_{n+1} = t_n^i + t_n^j$ *for* $n \geqq 1$.

(3) $q_is_{n+1} = s_n^i$ *and* $q_it_{n+1} = t_n^i$.

(4) $s_m^* \leqq s_{n+1} \leqq t_n \leqq s_n$ *and* $ef \leqq s_n$ *for all m and $e \neq f$ in E.*

(5) $q_i(e_l+e_k) = q_ie_l+q_ie_k$ *for* $k \neq l$ *in* $\mathbf{4}$ *with* $|\{i, i+1, k, l\}| = 3$.

Lemma 2.2. s_1, t_1, s_2, *and* t_2 *are neutral elements of F. For $i\neq j$ in $\mathbf{3}$ and e in E one has* $s_2q_i+s_2q_j=s_2$ *and* $et_2=et_2q_i+et_2q_j$.

Lemma 2.3. *Let u be s_n or t_n ($n\geqq 1$), i in $\mathbf{3}$, and e, f, g distinct elements of E. Then the sublattices generated by $e, f+g, u$ and e, q_i, u and e, f, u, respectively, are distributive. Moreover*

$$q_i(a+u, b+u, c+u, d+u) = q_i+u \quad \text{and} \quad u(a+u, b+u, c+u, d+u) = u,$$

$$q_i(au, bu, cu, du) = q_iu \qquad \text{and} \quad u(au, bu, cu, du) = u.$$

Proof. For $n \leqq 2$ anything follows by neutrality (Lemma 2.2). The distributivity of $\langle e, f+g, u\rangle$ and $\langle e, q_i, u\rangle$ and $u=u(a+u, b+u, c+u, d+u)$ have been shown in [7; 5.3]. Thus, $e+h, f+g, u$ is distributive, too. Assuming $t_1=1$ we have $(e+h)u+(f+g)u=eu+(f+g)u=u$. We prove the remaining claims by induction. For $n \geqq 2$ we get by 2.1 and the inductive hypothesis $as_{n+1}+bs_{n+1} \geqq a^2 s_n^2+b^2 s_n^2+ +a^3 s_n^3+b^3 s_n^3=(a^2+b^2)s_n^2+(a^3+b^3)s_n^3=(a+b)q_2 s_{n+1}+(a^3+b^3)s_n^3=(a+b)s_{n+1}(q_2+ +(a^3+b^3)s_n^3)$. Now $q_2+(a^3+b^3)s_n^3 \geqq q_2+q_2^3+q_1^3 s_n^3 \geqq q_2+s_n^3 \geqq s_{n+1}$ by 2.1 (2) whence $as_{n+1}+bs_{n+1}=(a+b)s_{n+1}$. By symmetry, $es_{n+1}+fs_{n+1}=(e+f)s_{n+1}$ for all $e \neq f$ in E. Thus, $(es_{n+1}+hs_{n+1})(fs_{n+1}+gs_{n+1})=(e+h)(f+g)s_{n+1}$.

By the inductive hypothesis we have $(q_2 s_n)^1=(q_2(as_n, bs_n, cs_n, ds_n))^1=((as_n+ +cs_n)(bs_n+ds_n))^1=(a^1 s_n^1+c^1 s_n^1)(b^1 s_n^1+d^1 s_n^1)=(q_1 as_{n+1}+q_1 cs_{n+1})(q_1 bs_{n+1}+q_1 ds_{n+1})= =q_2^1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1}) \leqq q_1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ using 2.1 (3) and (1). Similarly, $(q_3 s_n)^1 \leqq q_1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ whence $q_1 s_{n+1}=s_n^1=(q_2 s_n+q_3 s_n)^1= =(q_2 s_n)^1+(q_3 s_n)^1 \leqq q_1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ by 2.1 (2) and (3). The converse inclusion holds due to monotony. By symmetry we get $q_i s_{n+1}=q_i(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ for all $i \in \mathbf{3}$. Finally, with the inductive hypothesis and 2.1 (3) it follows

$$\begin{aligned} s_{n+1}(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1}) &= \sum s_n^i(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1}) = \\ = \sum s_n(q_i as_{n+1}, q_i bs_{n+1}, q_i cs_{n+1}, q_i ds_{n+1}) &= \sum s_n(a^i s_n^i, b^i s_n^i, c^i s_n^i, d^i s_n^i) = \\ = \sum s_n(as_n, bs_n, cs_n, ds_n)^i &= \sum s_n^i = s_{n+1}. \end{aligned}$$

For t_n the proof is quite analogous.

Corollary 2.4. *Let u and v be any of the s_n, t_n $(n \geqq 0)$ such that $u \geqq v$. Then $u(au+v, bu+v, cu+v, du+v)=u$, $v(au+v, bu+v, cu+v, du+v)=v$, and $q_j(au+ +v, bu+v, cu+v, du+v)=q_j u+v$ for j in $\mathbf{3}$.*

Define by induction $q_{0i}=1$ and $q_{n+1,i}=q_i(aq_{ni}, bq_{ni}, cq_{ni}, dq_{ni})$. Write $\varrho_i x=x^i$ and $\varrho_i^0 x=x$.

Lemma 2.5. *$\varrho_i^n 1=q_{ni}$, and $\varrho_i^n e=eq_{ni}$ for i in $\mathbf{3}$ and e in E.*

Proof. The first claim is 1.5 in [7]. The other follows by induction on n: $\varrho_i^{n+1} e= =\varrho_i^n q_{1i} e=\varrho_i^n q_{1i} \varrho_i^n e=\varrho_i^{n+1} 1\, eq_{ni}=eq_{n+1,i}$.

§ 3. The neutral element method revisited

An element of a modular lattice M is *neutral*, if for all a and b in M the sublattice generated by u, a, and b is distributive. Then the map $x \mapsto (ux, u+x)$ yields a subdirect representation of M. In [6] we proved

Proposition 3.1. *Let u be an element of a modular lattice M. Let S be a lattice and α an order preserving map of S in M such that $x \mapsto u+\alpha x$ preserves meets and*

$x \mapsto u\alpha x$ preserves joins. Moreover, let M be generated by the union of all intervals $[u\alpha x, \alpha x]$ and $[u\alpha x, u]$ with x in S. Then u is a neutral element of M.

Here, we need a more sophisticated version.

Proposition 3.2. *Let M be a finitely generated subdirectly irreducible modular lattice and u_n $(n \geqq 0)$ a descending chain of elements of M. Let S be a lattice and γ a meet homomorphism of S into M such that M is generated by the image of γ. Assume that for all x and y in S and $n \geqq 0$ there is an $m \geqq n$ with $u_m\gamma x + u_m\gamma y = u_m\gamma(x+y)$. Then either M is a homomorphic image of S or there is an n such that u_n is the smallest element of M.*

Proof. Let $\mathscr{F}(M)$ denote the lattice of all filters on M with partial order dual to set inclusion. Then $\mathscr{F}(M)$ is a dually algebraic lattice having M as a sublattice. Write $\prod$ for the meets in $\mathscr{F}(M)$. In particular, let $u = \prod u_n$ be the filter generated by the u_n $(n \geqq 0)$. Let M' be the sublattice generated by M and u. By lower continuity and the hypothesis we have for any x, y in S: $u\gamma x + u\gamma y = \prod u_n\gamma x + + \prod u_n\gamma y = \prod (u_n\gamma x + u_n\gamma y) = \prod u_n\gamma(x+y) = u\gamma(x+y) \geqq u(\gamma x + \gamma y)$. Thus, $x \mapsto u\gamma x$ is a join homomorphism of S into M' and the sublattice generated by u, γx, and γy is distributive for all x, y in S. Consequently, $(u+\gamma x)(u+\gamma y) = u + \gamma x\gamma y = u + \gamma xy$ and Prop. 3.1 applies to conclude that u is neutral in M'.

Therefore, the map $x \mapsto (ux, u+x)$ yields a subdirect representation of M'. M being subdirectly irreducible the induced subdirect representation of M has to be trivial, i.e. one of the maps $x \mapsto ux$ $(x \in M)$ and $x \mapsto u+x$ $(x \in M)$ has to be an embedding. In the first case we get $x = ux$ i.e. $x \leqq u$ for all x in M. Then, $x \mapsto u\gamma x = = \gamma x$ is a homomorphism of S onto M.

In the second case we have $x = u+x$ i.e. $x \geqq u$ for all x in M. Then, $u \leqq 0_M$, the smallest element of M. Since 0_M is the smallest element of $\mathscr{F}(M)$, too, it follows $u = 0_M$. The filter u being generated by the descending chain u_n $(n \geqq 0)$ there has to be an n such that $u_n = 0_M$.

§ 4. Proof of the Theorem

Let M be as in the Theorem. The Lemma in [6] states that either

$$\text{(i}'\text{)} \quad ab = ac = ad = bc = bd = cd = \prod (q_{n1}q_{n2}q_{n3} \mid n < \infty)$$

or the dual of (i′) takes place. Thus, let us assume (i′). For any map ε of $\{a_0, b_0, c_0, d_0\}$ onto $\{a, b, c, d\}$ we define a map $\gamma = \gamma^\varepsilon$ of A_∞ into M recursively:

$$\gamma m_0 = 1$$

$$\gamma l_{n+1} = \varepsilon a_0 \gamma m_n + \varepsilon b_0 \gamma m_n, \quad \gamma r_{n+1} = \varepsilon c_0 \gamma m_n + \varepsilon d_0 \gamma m_n$$

$$\gamma(m_{n+1} + x_0) = \varepsilon x_0 + \gamma l_{n+1} \quad \text{for} \quad x = a, b, \quad \gamma(m_{n+1} + x_0) = \varepsilon x_0 + \gamma r_{n+1} \quad \text{for} \quad x = c, d,$$

and for $1 \leqq i \leqq n-1$

$$\gamma(m_{n+1}+x_i) = \gamma(m_{n+1}+x_0)\gamma(m_n+x_i); \quad \gamma m_{n+1} = \gamma l_{n+1}\gamma r_{n+1};$$
$$\gamma x_k = \varepsilon x_0 \gamma m_n \quad \text{for} \quad x = a, b, c, d; \quad \gamma m_\infty = 0.$$

Claim 1. γ^ε is a meet homomorphism of A_∞ into M.

Proof. In section 2 of [6] it has been shown that γ^ε restricted to A_n is a meet homomorphism for every n. Due to (i′) and the definition of γ^ε the claim follows, immediately.

Proposition 3.2 will be applied with L being a subdirect product of three copies of A_∞. We use the notation $\hat{x}=x$ for elements in the first, $\bar{a}_i=a_i$, $\bar{b}_i=c_i$, $\bar{d}_i=d_i$, $\bar{m}_i=m_i$ for elements in the second, and $\tilde{a}_i=a_i$, $\tilde{b}_i=c_i$, $\tilde{c}_i=d_i$, $\tilde{d}_i=b_i$, $\tilde{m}_i=m_i$ for elements in the third copy — see Fig. 1. In analogy, we write $\hat{\gamma}=\gamma^\varepsilon$ with $\varepsilon\hat{e}_0=e$, $\bar{\gamma}=\gamma^\varepsilon$ with $\varepsilon\bar{e}_0=e$, and $\tilde{\gamma}=\gamma^\varepsilon$ with $\varepsilon\tilde{e}_0=e$ for $e\in E$. Observe (by induction) that $\hat{\gamma}\hat{m}_n=q_{n1}=:\hat{q}_n$, $\bar{\gamma}\bar{m}_n=q_{n2}=:\bar{q}_n$, and $\tilde{\gamma}\tilde{m}_n=q_{n3}=:\tilde{q}_n$. Define $L=\{(0,0,0)\}\cup\bigcup$ $\bigcup([(m_i, m_j, m_k), (1,1,1)]\cup\{(e_i, e_j, e_k)|e\in E\})|i,j,k<\infty)$.

Claim 2. L is the sublattice of $A_\infty\times A_\infty\times A_\infty$ generated by the elements $\check{e}=(\hat{e}_0, \bar{e}_0, \tilde{e}_0)$ with $e\in E$.

Proof. Component wise calculation yields the sublattice property, easily. We show by induction on i that the union of the intervals $[(m_i, 1, 1), (1, 1, 1)]$ and $[\hat{e}_i, \hat{e}_0]$ $(e\in E)$ belongs to the sublattice S generated by the $\check{e}$. Namely, with $g=(\hat{m}_i, 1, 1)$ we have $(\hat{m}_{i+1}, 1, 1)=(\check{a}g+\check{b}g)(\check{c}g+\check{d}g)$ in S whence $(\hat{e}_j+\hat{m}_{i+1}, 1, 1)$ for $j\leqq i$ and $(\hat{e}_{i+1}, 1, 1)=(\hat{e}_0, 1, 1)(\hat{m}_{i+1}, 1, 1)$ are in S, too. Using symmetry and forming meets we get that S contains L. Trivially one obtains

Claim 3. $\gamma(\hat{x}, \bar{y}, \tilde{z})=\hat{\gamma}\hat{x}\bar{\gamma}\bar{y}\tilde{\gamma}\tilde{z}$ defines a meet homomorphism of L into M with $\gamma\check{e}=e$, $\gamma(\hat{m}_i, \bar{m}_j, \tilde{m}_k)=\hat{q}_i\bar{q}_j\tilde{q}_k$, and $\gamma(\hat{e}_i, \bar{e}_j, \tilde{e}_k)=e\hat{q}_i\bar{q}_j\tilde{q}_k$.

For $m\geqq 0$ define the map σ_m: $L\to M$ by $\sigma_m x=s_m\gamma x$. For $n\geqq 0$ define

$$S_n = [(\hat{m}_n, \bar{m}_n, \tilde{m}_n), (1, 1, 1)]\cup\{(\hat{e}_i, \bar{e}_j, \tilde{e}_k)|e\in E,\ i, j, k < n\}.$$

Claim 4. S_n is a join subsemilattice of L and $\sigma_m|S_n$ a join homomorphism if $m>3n$.

Proof. Let us write $1=(1, 1, 1)$. Observe that for $i=n-1$ and $e\neq f$ in E $(\hat{e}_i, \bar{e}_i, \tilde{e}_i)+(\hat{f}_i, \bar{f}_i, \tilde{f}_i)\geqq(\hat{m}_n, \bar{m}_n, \tilde{m}_n)$. Since $\{(\hat{e}_i, \bar{e}_j, \tilde{e}_k)|i,j,k<n\}=[(\hat{e}_{n-1}, \bar{e}_{n-1}, \tilde{e}_{n-1}),$ $(\hat{e}_0, \bar{e}_0, \tilde{e}_0)]$ and $[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ are intervals this suffices to prove that S_n is closed under joins.

The second claim will be shown by induction on n. The modular lattice identities (a)—(f) we refer to shall be proved at the end of the section. The case $n=0$ is trivial. Let be $n\geqq 1$, $m>3n$, and assume that $\sigma_m|S_{n-1}$ is a join homomorphism.

Step 1. $\sigma_m|[(\hat{l}_n, 1, 1), 1]$ and $\sigma_m|[(\hat{r}_n, 1, 1), 1]$ preserve joins. Since $[(\hat{l}_n, 1, 1), 1]$ is the union of $[(\hat{m}_{n-1}, 1, 1), 1]$, $\{(\hat{l}_n, 1, 1)\}$, and the chains $[(\hat{e}_{n-2}+\hat{m}_n, 1, 1), (\hat{e}_0+\hat{m}_n, 1, 1)]$ $(e=a, b)$ it suffices to show $\sigma_m(\hat{a}_{n-2}, 1, 1)+\sigma_m(\hat{b}_{n-2}, 1, 1)\geqq\sigma_m(\hat{m}_{n-1}, 1, 1)]$ i.e.

(a) $s_m a\hat{q}_{n-2}+s_m b\hat{q}_{n-2} \geqq s_m\hat{q}_{n-1}$

and $\sigma_m(\hat{e}_i, 1, 1)+\sigma_m(\hat{m}_{n-1}, 1, 1)=\sigma_m(\hat{e}_i+\hat{m}_{n-1}, 1, 1)$, i.e.

(b) $s_m e\hat{q}_i+s_m\hat{q}_{n-1} = s_m(e+f\hat{q}_{n-2})\hat{q}_i$ for $\{e,f\} = \{a, b\}$ and $i \leqq n-2$.

(We have $\hat{\gamma}(\hat{e}_i+\hat{m}_{n-1})=\hat{\gamma}(\hat{e}_0+\hat{m}_{n-1})\hat{\gamma}\hat{m}_i$ since $\hat{\gamma}$ is a meet homomorphism.) The second claim follows by symmetry.

Step 2. $\sigma_m|[(\hat{m}_n, 1, 1), 1]$ is a join homomorphism. Since $[(\hat{m}_n, 1, 1), 1]$ is the union of $[(\hat{l}_n, 1, 1), 1]$, $[(\hat{r}_n, 1, 1), 1]$ and $\{(\hat{m}_n, 1, 1)\}$ and because of $(\hat{l}_n, 1, 1)+(\hat{r}_n, 1, 1)=(\hat{m}_{n-1}, 1, 1)$ it suffices to show $\sigma_m(\hat{l}_n, 1, 1)+\sigma_m(\hat{r}_n, 1, 1)=\sigma_m\{\hat{m}_{n-1}, 1, 1)$, i.e.

(c) $s_m(a\hat{q}_{n-1}+b\hat{q}_{n-1})+s_m(c\hat{q}_{n-1}+d\hat{q}_{n-1}) = s_m\hat{q}_{n-1}$.

Step 3. $\sigma_m|[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ is a join homomorphism. By symmetry, the restriction of σ_m to any of $[(\hat{m}_n, 1, 1), 1]$, $[(1, \bar{m}_n, 1), 1]$, and $[(1, 1, \tilde{m}_n), 1]$ is a join homomorphism. In view of

(i) $s_m\bar{q}_n+s_m\tilde{q}_n = s_m$ and $s_m\hat{q}_n+s_m\bar{q}_n\tilde{q}_n = s_m$

the $\sigma_m(\hat{m}_n, 1, 1)$, $\sigma_m(1, \bar{m}_n, 1)$, and $\sigma_m(1, 1, \tilde{m}_n)$ are dually independent in $[0, s_m]$. $\sigma_m|[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ being the product of the above three restrictions it is a join homomorphism, too.

Step 4. $\sigma_m|\{(\hat{e}_i, \bar{e}_j, \tilde{e}_k)|i, j, k<n\}$ is a join homomosphism for $e\in E$. This means for $i, j, k, r, s, t<n$, $u=\min(i, r)$, $v=\min(j, s)$, $w=\min(k, t)$

(d) $s_m e\hat{q}_i\bar{q}_j\tilde{q}_k+s_m e\hat{q}_r\bar{q}_s\tilde{q}_t = s_m e\hat{q}_u\bar{q}_v\tilde{q}_w$.

Step 5. $\sigma_m|S_n$ is a join homomorphism. Since S_n is the union of the intervals $[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ and $[(\hat{e}_i, \bar{e}_i, \tilde{e}_i), (\hat{e}_0, \bar{e}_0, \tilde{e}_0)]$ $(i=n-1, e\in E)$ it suffices to check $\sigma_m(\hat{e}_i, \bar{e}_i, \tilde{e}_i)+\sigma_m(\hat{f}_i, \bar{f}_i, \tilde{f}_i)\geqq\sigma_m\{\hat{m}_n, \bar{m}_n, \tilde{m}_n)$, i.e.

(e) $s_m e\hat{q}_i\bar{q}_i\tilde{q}_i+s_m f\hat{q}_i\bar{q}_i\tilde{q}_i \geqq s_m\hat{q}_n\bar{q}_n\tilde{q}_n$ for $i = n-1$, $e \neq f$ in E

and $\sigma_m(\hat{e}_i, \bar{e}_j, \tilde{e}_k)+\sigma_m(\hat{m}_n, \overline{m}_n, \tilde{m}_n)=\sigma_m(\hat{e}_i+\hat{m}_n, \bar{e}_j+\overline{m}_n, \tilde{e}_k+\tilde{m}_n)$ for $i, j, k<n$ and e in E. Due to symmetry and Step 3 the latter is satisfied if $\sigma_m(\hat{e}_i, \bar{e}_n, \tilde{e}_n)+$ $+\sigma_m(\hat{m}_n, \overline{m}_n, \tilde{m}_n)=\sigma_m(\hat{e}_i+\hat{m}_n, \overline{m}_n, \tilde{m}_n)$, i.e.

(f) $s_m e\hat{q}_i\bar{q}_n\tilde{q}_n+s_m\hat{q}_n\bar{q}_n\tilde{q}_n = s_m(e+f\hat{q}_{n-1})\hat{q}_i\bar{q}_n\tilde{q}_n$ for $i<n$ and $\{e,f\} = \{a, b\}$.

Now, we are ready to prove the Theorem. Observe that M_4 and R_∞ are the only subdirectly irreducible homomorphic images of L. Namely, L is a subdirect product of six copies of R_∞ having M_4 as its only proper homomorphic image. Thus, the subdirectly irreducible lattice M cannot be a homomorphic image of L. Due to Claims 3 and 4 we may apply Proposition 3.2 and conclude that there is an n such that $s_n=\sigma_n 1=0$.

To prove the Corollary observe that induction yields $s_n=1$ and $s_n^*=0$ for all n and all lattices listed there. Namely, $q_1=1$ whence by Lemma 2.1 $s_{n+1}\geqq$ $\geqq q_1 s_n=s_n=1$. For the additional results recall that according to A. HUHN [8] in a 2-distributive lattice frames may have order at most 2. In view of Corollary 1.4 and 2.1, 3.2, and 3.3 from [7] this implies that $t_n=s_{n+1}$ for $n\geqq 1$ and $t_n=s_n$ for $n\geqq 3$. Thus, by Lemma 2.2 the only subdirectly irreducibles with $s_n=0$ for an n may be D_2 and M_3.

Before we come to the proof of the formulas (a)—(f) we need a Lemma.

Lemma 4.1. *For all* $m\geqq n$ *and* $i\in\mathbf{3}$ *one has* $s_m q_{ni}=\varrho_i^n s_{m-n}$. *Also,* e, q_{ni}, *and* s_m *generate a distributive sublattice for all* e *in* E.

Proof. By induction on n. For $n=1$ this is Lemma 2.1 (3) and 2.3. For $n>1$ one has by 2.5 $s_m q_{ni}=s_m q_i q_{ni}=\varrho_i s_{m-1}\varrho_i q_{n-1,i}=\varrho_i\varrho_i^{n-1}s_{m-n}=\varrho_i^n s_{m-n}$. Show $\varrho_i^n(e+s_k)=q_{ni}(e+s_{k+n})$ for all k. Indeed $\varrho_i^{n+1}(e+s_k)=\varrho_i^n\varrho_i(e+s_k)=\varrho_i^n(q_i e+$ $+q_i s_{k+1})=\varrho_i^n q_i(e+s_{k+1})=\varrho_i^n q_i\varrho_i^n(e+s_{k+1})=q_{n+1,i}q_{ni}(e+s_{k+1+n})=q_{n+1,i}(e+s_{k+n+1})$ by the hypothesis, and 2.5. Thus, $eq_{ni}+s_m q_{ni}=\varrho_i^n e+\varrho_i^n s_{m-n}=\varrho_i^n(e+s_{m-n})=q_{ni}(e+s_m)$ and the distributivity follows.

Proof of (a). $s_m a\hat{q}_{l-1}+s_m b\hat{q}_{l-1}=\varrho_1^{l-1}(as_{m-l+1}+bs_{m-l+1})=\varrho_1^{l-1}(a+b)s_{m-l+1}\geqq$ $\geqq\varrho_1^{l-1}q_1 s_{m-l+1}\geqq\hat{q}_l s_m$ for $l\leqq m+1$ by 2.5 and 4.1, 2.3 and 2.5, and 4.1 again.

Proof of (c). By 2.3 one has $s_k(a+b)+s_k(c+d)=s_k$ for $k\geqq 1$. (c) follows immediately applying the homomorphism ϱ_1^{n-1} in the case $k=m-n+1$ and appealing to 2.5 and 4.1.

Proof of (b). By 4.1 one has $s_k a+s_k\hat{q}_j=s_k(a+\hat{q}_j)$ for $k\geqq j$. Apply the homomorphism ϱ_1^i in the case $j=l-i$ and $k=m-i$ (for $i\leqq l<m$) to obtain $s_m a\hat{q}_i+s_m\hat{q}_l=s_m\hat{q}_i(a\hat{q}_i+\hat{q}_l)=s_m\hat{q}_i(a+\hat{q}_l)$. Now $a+\hat{q}_l=a+(a+b\hat{q}_{l-1})(c\hat{q}_{l-1}+d\hat{q}_{l-1})=$ $=(a+b\hat{q}_{l-1})(a+c\hat{q}_{l-1}+d\hat{q}_{l-1})$ by modularity and $a+c+d\geqq t_1\geqq s_{m-e+1}$ whence $a+c\hat{q}_{l-1}+d\hat{q}_{l-1}\geqq s_m$ (applying ϱ_1^{l-1}) and $s_m aq_i+s_m\hat{q}_l\geqq s_m\hat{q}_i(a+q_{l-1})$. Due to

$s_m\hat{q}_n\bar{q}_l\tilde{q}_l \geqq s_m\hat{q}_n\bar{q}_n\tilde{q}_n$ and the following Lemma (e) may be obtained from the formula proved under (a) (with $l-1=i=n-1$ and $m>3n-2i>l$) by application of the homomorphism $\varrho_2^l\varrho_3^l$.

Lemma 4.2. *$\varrho_j^m q_{ni}=q_{mj}q_{ni}$ for all $i\neq j$ in* 3 *and* $m, n\geqq 0$.

Proof. We show $\varrho_j q_{ni}=q_j q_{ni}$ by induction over n: $\varrho_j q_{n+1,i}=\varrho_j\varrho_i q_{ni}= =\varrho_i\varrho_j q_{ni}=\varrho_i(q_j q_{ni})=\varrho_i q_j \varrho_i q_{ni}=q_i q_j q_{n+1,i}=q_j q_{n+1,i}$ by 2.1 (1) and 2.5. Now we induce over m: $\varrho_j^{m+1}q_{ni}=\varrho_j\varrho_j^m q_{ni}=\varrho_j(q_{jm}q_{ni})=\varrho_j q_{mj}\varrho_j q_{ni}=q_{m+1,j}q_{mj}q_{ni}=q_{m+1,j}q_{ni}$.

Next, observe that (f) and (d) are consequences of the following formula

(g) $\bar{q}_j\tilde{q}_k s_m e+\hat{q}_l s_m \geqq \bar{q}_j\tilde{q}_k s_m(e+\hat{q}_l)$ for $j+k+l<m$ and e in E.

Namely, for (f) put $j=k=l=n$, multiply both sides with $\hat{q}_i\bar{q}_n\tilde{q}_n$ and observe $a+\hat{q}_l\geqq \geqq s_m(a+b\hat{q}_{l-1})$ as proved under (b).

For (d) assume w.l.o.g. $j\geqq s$, $k\geqq t$, and $i\leqq r=l$ and multiply both sides of (g) with $e\hat{q}_i\bar{q}_s\tilde{q}_t$.

In the proof of (g) assume w.l.o.g. $e=a$. Fisst, we show that q_1, as_h, and q_3s_h distribute for $h\geqq 3$: By 2.1 and 2.3 we have $q_1s_ha+q_1q_3s_h=(s_{h-1}a+q_3s_{h-1})^1= =(s_{h-1}(a+q_3))^1=(s_{h-1}(a+b+c)(a+d))^1=(s_{h-1}(a+d))^1=s_h(q_1a+q_1d)=s_hq_1(a(c+ +d)+d)=s_hq_1(a+d)\geqq q_1(s_ha+s_hq_3)$.

Now, $\varrho_1^l(s_ha+s_hq_3)=q_{l1}(s_{h+l}a+s_{h+l}q_3)$ for $h\geqq 2$ follows by induction: $\varrho_1^{l+1}(s_ha+s_hq_3)=\varrho_1^l\varrho_1(s_ha+s_hq_3)=\varrho_1^l(q_1s_{h+1}a+q_1s_{h+1}q_3)=\varrho_1^lq_1(s_{h+1}a+s_{h+1}q_3)= =\varrho_1^lq_1\varrho_1^l(s_{h+1}a+s_{h+1}q_3)=q_{l+1,1}q_{l1}(s_{h+1+l}a+s_{h+1+l}q_3)=q_{l+1,1}(s_{h+l+1}a+s_{h+l+1}q_3)$ using 2.1 and 2.5. Thus, for $h-l\geqq 2$ q_{l1}, s_ha, and s_hq_3 distribute: $q_{l1}s_ha+q_{l1}s_hq_3= =\varrho_1^ls_{h-l}a+\varrho_1^ls_{h-l}q_3=\varrho_1^l(s_{h-l}a+s_{h-l}q_3)=q_{l1}(s_ha+s_hq_3)$ by 4.1 and 4.2.

Induction on $j+k$ yields $\varrho_2^j\varrho_3^k(s_ha+s_hq_{l1})=q_{j2}q_{k3}(as_{h+j+k}+q_{l1}s_{h+j+k})$ for $h>l$: $\varrho_2^j\varrho_3^k(s_ha+s_hq_{l1})=\varrho_2^j\varrho_3^{k-1}\varrho_3(s_ha+s_hq_{l1})=\varrho_2^j\varrho_3^{k-1}(aq_3s_{h+1}+q_3q_{l1}s_{h+1})= =\varrho_2^j\varrho_3^{k-1}q_3(as_{h+1}+q_{l1}s_{h+1})=\varrho_2^j\varrho_3^{k-1}q_3\varrho_2^j\varrho_3^{k-1}(as_{h+1}+q_{l1}s_{h+1})=q_{j2}q_{k3}(as_{h+j+k}+q_{l1}s_{h+j+k})$ assuming $k>0$ w.l.o.g. (since $\varrho_2^j\varrho_3^k=\varrho_3^k\varrho_2^j$ by 2.1 (1)), and using 2.3 and 4.2. Finally, we get $\bar{q}_j\tilde{q}_ks_ma+\hat{q}_ls_m\geqq\bar{q}_j\tilde{q}_ks_ma+\bar{q}_j\tilde{q}_k\hat{q}_ls_m=\varrho_2^j\varrho_3^ks_{m-j-k}a+\varrho_2^j\varrho_3^k\hat{q}_ls_{m-j-k}= =\varrho_2^j\varrho_3^k(as_{m-j-k}+\hat{q}_ls_{m-j-k})=\bar{q}_j\tilde{q}_k(as_m+\hat{q}_ls_m)=\bar{q}_j\tilde{q}_ks_m(a+\hat{q}_l)$ applying the above, 4.2 and 4.1.

Finally, to prove (i) we show by induction on m:

(j) $s_m\hat{q}_j+s_m\bar{q}_k\tilde{q}_l=s_m$ for $j+k+l\leqq m$.

The cases $m\leqq 1$, $j=0$, or $k=l=0$ being trivial, let $m\geqq 2$, $j\geqq 1$, $k\geqq 1$. Then

$$s_m\hat{q}_j+s_m\bar{q}_k\tilde{q}_l=s_m\hat{q}_j+s_m\hat{q}_1\bar{q}_k\tilde{q}_l+s_m\hat{q}_j\bar{q}_1+s_m\bar{q}_k\tilde{q}_l=\hat{\varrho}(s_{m-1}\hat{q}_{j-1}+s_{m-1}\bar{q}_k\tilde{q}_l)+$$

$$+\bar{\varrho}(s_{m-1}\hat{q}_j+s_{m-1}\bar{q}_{k-1}\tilde{q}_l)=\hat{\varrho}s_{m-1}+\bar{\varrho}s_{m-1}=s_m.$$

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TECHNISCHE HOCHSCHULE
FACHBEREICH MATHEMATIK
SCHLOSSGARTENSTR. 7
D 61 DARMSTADT, BRD