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On perfect pairs for quadruples in complemented modular lattices and concepts of perfect elements

Christian Herrmann

Abstract. Gel'fand and Ponomarev [11] introduced the concept of perfect elements and constructed such in the free modular lattice on 4 generators. We present an alternative construction of such elements u (linearly equivalent to theirs) and for each u a direct decomposition $u, \overline{u}$ of the generating quadruple within the free complemented modular lattice on 4 generators; $u, \overline{u}$ are said to form a perfect pair. This builds on [17] and fills a gap left there. We also discuss various notions of perfect elements and relate them to preprojective and preinjective representations.

1. Introduction

According to Gel'fand and Ponomarev [11], a K-*perfect* element for a partially ordered set S is given by a lattice term u over S such that under any representation ρ of S in a finite dimensional K-vector space V_ρ, i.e., order preserving map $\rho\colon S \to L(V_\rho)$ into the lattice of subspaces, the subspace ρu given by the evaluation of u admits a complement $\overline{U}_\rho$ such that $\rho x = \rho u \cap \rho x + \overline{U}_\rho \cap \rho x$ for all $x \in S$, i.e., a direct decomposition of ρ. Equivalently, $\rho u \in \{0, V_\rho\}$ for any indecomposable ρ. u is *perfect* if it is K-perfect for all fields K. The stricter concept referring to all division rings has gained less attention in the literature.

We follow common usage and speak, instead of terms, of elements of lattices freely generated by S in suitable varieties of modular lattices. The proper framework is the variety generated by all lattices $L(V)$ over a fixed field K, resp. arbitrary, fields or, slightly more generally, all complemented modular lattices. In the first case, equality in the free lattice is K-linear equivalence (but only the characteristic of K matters), in the second, linear equivalence. Also, perfect elements are neutral in these lattices. See Section 9 for a discussion.

In the context of preprojective and preinjective representations, perfect elements have been established by Gel'fand and Ponomarev [11] for n-element antichains. These form a sublattice of the free modular lattice on n generators which is a linear sum of 2^n-element Boolean lattices.

For the union $2+2+2$ of three 2-element chains, Stekolshchik [31, 32] gave an explicit construction of perfect elements based on atomic elements

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and a detailed analysis of the related structure. Again, these are associated with preprojectives, resp. preinjectives. The preprojective part of the lattice is a union of a chain of sublattices isomorphic to C^3, C a 4-element chain, neighbouring ones related via an isomorphism between a boolean, upper resp., lower section.

All these constructions are independent of the base field. Cylke [6] suggested a notion of stable perfect elements reflecting this independence and studied the relations between perfect elements of posets arising by differentiation with respect to a maximal element. Cylke [6] also shows that in the case of tame posets with a chain of strongly coupled critical subsets each perfect element induces an oriented partition of the Auslander-Reiten quiver with one class finite (whence consisting of preprojectives, resp. preinjectives).

These results are based on the functorial methods of representation theory, Coxeter functors in particular. Cf. Simson [30] for representation theory of partially ordered sets in general.

The classification of indecomposable representations of quadruples in finite dimension was given by Nazarova [25], Gel'fand and Ponomarev [10], and Brenner [3]. The defect of ρ being defined as $\sum_{i=1}^{4} \dim \rho e_i - 2 \dim V$, the only values are $0, \pm 1, \pm 2$ with -1 and -2 corresponding to the preprojective, 1 and 2 to the preinjective indecomposables. In [17] elements $s_{n+1} \leq t_n \leq p_{ni} \leq s_n$ ($n \in \mathbb{N},\ i = 1, 2, 3, 4$) in the modular lattice on 4 free generators have been defined by a simple recursion (also used in [14, 15, 16] and Baur [1]) such that defect -2 is characterized by the relations $\rho t_n = V_\rho$, $\rho s_{n+1} = 0$ and defect -1 by $\rho t_n = 0$ and, for some i, $\rho p_{ni} = V_\rho$, $\rho p_{nj} = 0$ for $j \neq i$. The dual elements $s^*_{n+1} \geq t^*_n \geq p^*_{ni} \geq s^*_n$ are associated with preinjectives and $s_n \geq s^*_n$ for all n. In defect 0 one has $s_n = 1$ and $s^*_n = 0$ for all n. Thus, these elements are perfect and linearly equivalent to those established by Gel'fand and Ponomarev [11]. A detailed analysis of the relationship between both sets of elements has been given by Stekolshchik [32].

The idea of a perfect pair is to give a uniform description of the $\overline{U}_\rho$, too, and to generalize to representations $\rho\colon S \to L$ in arbitrary complemented modular lattices L. Thus, we consider a term $\overline{u}$ involving an additional unary operation symbol and require that ρu, $\rho \overline{u}$ is a direct decomposition for any modular lattice L endowed with a unary operation $x \mapsto x'$ such that $x \oplus x' = 1$. Then $u, \overline{u}$ is a *perfect pair.*

In [17] a recursion for completing elements $u \in \{s_n, t_n, p_{ni}\}$ to perfect pairs $u, \overline{u}$ was described. However, a faulty claim [17, 6.4] was used for the case $n = 2$. Here, we replace this by more specific arguments to make the construction work. Our methods are purely lattice theoretic, but guided by structural insight due to representation theory.

In Sections 10–12 we will discuss different notions of perfect elements, in particular one which relates perfect elements to the Auslander-Reiten quiver of preprojectives, resp. preinjectives. More precisely, a GP-*system of perfect* *elements* is an embedding $X \mapsto \gamma(X) = u$ of the quiver of preprojectives, resp. preinjectives, into the ordered set of perfect elements such that for any indecomposable representation, independently of the base field, $\rho u = 0$ if and only if there is path from ρ to X, resp. dually. The results quoted above provide GP-systems.

Extending work of Gel'fand and Ponomarev [11, 12, 13] on antichains it is shown in Section 12 that, given a GP-system, the only neutral elements u not K-linearly equivalent modulo some K to some element in the image of γ satisfy $\gamma(Y) < u < \gamma(X)$ for all preprojective X and preinjective Y. It follows that for fixed K there are at most two perfect elements not in the image of γ, provided that any two regular indecomposables are connected by a chain of non-zero morphisms — as established by Dlab and Ringel [8] for antichains with more than 4 elements.

In Section 13 we give a summary of results about modular lattices with four generators. We describe in more detail how the lattice structure is related to the perfect elements from the GP-system. In particular, we recall from [17] that for fixed K the neutral elements are exactly the members of perfect pairs as described above.

2. Basic concepts

We consider modular lattices with constants 0 and 1 for smallest and greatest element. We write meets as $a \cdot b = ab$ and joins as $a + b$, resp. $a \oplus b$, if $ab = 0$. We follow the usual bracket rules. a and b are *complements* of each other if $a \oplus b = 1$. Any section $[0, u] = \{x \in L \mid x \leq u\}$ of L is a *section sublattice* with the constants taking value 0 and u.

An element u of L is *neutral* if the sublattice generated by u, a, b is distributive for all $a, b \in L$. It follows that the maps $x \mapsto xu$ and $x \mapsto u + x$ are lattice homomorphisms onto the intervals $[0, u]$ and $[u, 1]$ of L, respectively, providing a subdirect decomposition of L. The neutral elements of L form a distributive sublattice; cf. [2, 5].

Let $\mathcal{C}$ be the class of all complemented modular lattices and $\mathsf{V}\mathcal{C}$ the variety generated by $\mathcal{C}$. Let $F(S)$ denote the lattice (with $0, 1$) freely generated in $\mathsf{V}\mathcal{C}$ by the finite partially ordered set S.

Lemma 2.1. *S embeds into $F(S)$, canonically. $F(S)$ embeds into some complemented modular lattice.*

Proof. The first claim follows from the fact that S embeds into a Boolean lattice. Since posets can be retracted under lattice homomorphisms, $F(S)$ can be constructed as sublattice of a direct product of complemented modular lattices, which is itself complemented modular. □

A *representation* of S in a complemented modular lattice, L, is an order preserving map $\rho\colon S \to L$. Equivalently, we have a homomorphism $\rho\colon F(S) \to L$, in particular, $\rho 0 = 0_L$ and $\rho 1 = 1_L$. Given elements $a_1, \dots, a_n$ in L we consider this as a representation of the n-element antichain, n, naturally. A *decomposition* of ρ is given by a pair u, v of mutual complements in L satisfying for each $x \in S$ some of the following

$$u + v\rho x \geq \rho x, \quad v + u\rho x \geq \rho x, \quad (u + \rho x)v \leq \rho x, \quad (v + \rho x)u \leq \rho x.$$

Lemma 2.2. *Let $\rho \colon S \to L$ be a representation with decomposition u, v. Then $u, v \in L$ are neutral in the sublattice of L they generate together with the $\rho x\,(x \in S)$. In particular, this lattice is a direct product of the sublattice of $[0, u]$ generated by the $u\rho x$ $(x \in S)$ and the sublattice of $[0, v]$ generated by the $v\rho x$ $(x \in S)$. Moreover, $\rho x = u\rho x + v\rho x = (u + \rho x)(v + \rho x)$ for all $x \in F(S)$.*

Proof. Modularity yields the identities for $x \in S$ which then are inherited by the sublattice of L generated by u, v and the $\rho x\,(x \in S)$: If $\rho x = u\rho x + v\rho x$ and $\rho y = u\rho y + v\rho y$, then $\rho(x+y) \geq u\rho(x+y) + v\rho(x+y) \geq u\rho x + u\rho y + v\rho x + v\rho y = \rho x + \rho y = \rho(x+y)$ and dually for meets. It follows that this lattice is isomorphic to the direct product of its interval sublattices $[0, u]$ and $[0, v]$ with $u \mapsto (u, 0)$ and $v \mapsto (0, v)$. Cf. Poguntke [26]. □

More generally, a *representation* of a lattice F is a 0-1-homomorphism $\rho \colon F \to L$ into a complemented modular lattice L. Then, the decomposition condition has to be required for all $x \in L$. The lemma shows that this is consistent with the definition in the special case where $F = F(S)$. Observe that ρ does not admit any nontrivial direct decomposition if ρ is surjective and L directly indecomposable.

3. Auxiliary results

Lemma 3.1. *Let L be a modular lattice generated by a, b, c, d such that $a \oplus b = c \oplus d = a + c = 1$ and the sublattices generated by a, b, c and a, c, d respectively are distributive. Then L is distributive and $ac \oplus (b + d) = 1$.*

Proof. By the supposed distributivity, $ad(b + c) = d(ab + ac) = dac = 0$ and $ad + b + c = (a + c)(d + c) + b = d + c + b = 1$. Now, each $x = a, b, c, d$ is comparable with ac or with $b + d$ and, due to modularity, the sublattice generated by x, ac, $b+d$ is distributive. Therefore, ad, $b+c$ is a decomposition of the quadruple a, b, c, d. By symmetry, we have the decomposition bc, $a + d$. From Lemma 2.2 it follows that L is isomorphic to a direct product of a factor with $ad = 1$ and a factor with $ad = 0$. If $ad = 1$, then we have complements $b = c = 0$ and a 2-element distributive factor. Applying the second decomposition to the second factor, the latter decomposes into a direct product of two factors with $bc = 1$, resp. $bc = 0$. $bc = 1$ yields a 2-element factor, again. It remains to consider the case $ad = bc = 0$ with complements $b+c = a+d = 1$. From $1 = a \oplus b = b \oplus c$ and the distributivity of the sublattice generated by a, b, c we conclude $a = c = a + c = 1$. Again, $b = d = 0$ and the factor is 2-element. Thus, $L \cong \{0,1\}^3$ is distributive, and it follows that $ac(b+d) = acb + acd = 0$ and $ac + b + d = (a+b+d)(c+b+d) = 1$. □

Corollary 3.2. *Let $\rho\colon S \to L$ be a representation. If u, v belong to the sublattice generated by $\rho(S)$, if $u + v = 1$, and if $u, \overline{u}$ and $v, \overline{v}$ are decompositions of ρ, then $uv, \overline{u} + \overline{v}$ is also a decomposition of ρ.*

Proof. By Lemma 2.2, the sublattices generated by $u, \overline{u}, v$, resp. $v, \overline{v}, u$, are distributive. Thus, by Lemma 3.1, $uv \oplus (\overline{u} + \overline{v}) = 1$. Again by Lemma 2.2, for $x \in S$ the sublattice generated by $u, \overline{u}$, and $v\rho x \in \rho(F(S))$ is distributive, whence $uv\rho x + (\overline{u} + \overline{v})\rho x \geq uv\rho x + \overline{u}v\rho x + \overline{v}\rho x = v\rho x + \overline{v}\rho x = \rho x$. □

Proposition 3.3. *Let L be a complemented modular lattice with sublattice M generated by a_1, a_2, a_3, a_4 where $u \geq a_4$ is a neutral element of M such that*

$$\text{(a)}\quad u + a_i + a_j = 1_L \quad \textit{and} \quad \text{(b)}\quad (u + a_i)(u + a_j) = u,$$

for $i \neq j$.

Define $b_i = a_i(a_j + a_k)$ for $\{i, j, k\} = \{1, 2, 3\}$ and consider $w_i \in L$ such that

$$\text{(c)}\quad b_1 = w_1 \oplus ub_1 \quad \textit{and} \quad \text{(d)}\quad b_2(w_1 + b_3) = w_2 \oplus b_2b_3.$$

Then $u, w_1 + w_2$ is a decomposition of the quadruple a_1, a_2, a_3, a_4.

Proof. This can be derived from the proof of Satz 1.2 in Poguntke [26]. We give a direct proof, here. We write (m) and (n) for application of modularity and of neutrality of u, respectively. Observe that

$$\text{(e)}\quad b_i \leq b_j + b_k, \qquad \text{(f)}\quad u + b_i = u + a_i.$$

The first is immediate by modularity, in the second we have

$$u + a_i(a_j + a_k) =_{(\mathrm{n})} (u + a_i)(u + a_j + a_k) =_{(\mathrm{a})} u + a_i.$$

Calculating meets we get

$$\begin{aligned} u(w_1 + b_3) &=_{(\mathrm{m}),(\mathrm{c})} u((u + b_3)b_1w_1 + b_3) \\ &=_{(\mathrm{n})} u(ub_1 + b_3b_1)w_1 + b_3) \leq_{(\mathrm{b})} u(uw_1 + b_3) =_{(\mathrm{c})} ub_3. \end{aligned}$$

It follows that $uw_2 =_{(\mathrm{d})} u(w_1 + b_3)b_2w_2 \leq b_3b_2w_2 =_{(\mathrm{d})} 0$. On the other hand, $(u + w_2)w_1 \leq_{(\mathrm{d})} (u + b_2)w_1 \leq_{(\mathrm{b})} uw_1 =_{(\mathrm{c})} 0$. Thus we obtain $u(w_1 + w_2) =_{(\mathrm{m})} u((u + w_2)w_1 + w_2) = uw_2 = 0$.

The decomposition condition for a_4 follows from $u \geq a_4$, that for a_2 and a_1 is obtained as follows:

$$\begin{aligned} u(a_2 + w_1 + w_2) &=_{(\mathrm{d})} u(a_2 + w_1) =_{(\mathrm{m})} u(a_2 + (u + a_2)w_1) \\ &=_{(\mathrm{b})} u(a_2 + uw_1) =_{(\mathrm{c})} ua_2 \end{aligned}$$

and

$$a_1u + w_1 + w_2 \geq_{(\mathrm{c})} a_1u + b_1 =_{(\mathrm{m})} a_1(u + b_1) =_{(\mathrm{f})} a_1.$$

Concerning a_3 we get

$$\begin{aligned} ua_3 + w_1 + w_2 &\geq_{(\mathrm{b})} ua_3 + w_1 + w_2 + b_2b_3 \geq_{(\mathrm{d})} ua_3 + w_1 + b_2(w_1 + b_3) \\ &=_{(\mathrm{m})} ua_3 + (w_1 + b_2)(w_1 + b_3) \\ &= ua_3 + (w_1 + a_2(a_1 + a_3))(w_1 + a_3(a_1 + a_2)) \\ &=_{(\mathrm{m})} ua_3 + (w_1 + a_2)(w_1 + a_3) \\ &=_{(\mathrm{m})} (ua_3 + w_1 + a_2)(w_1 + a_3) \geq a_3 \end{aligned}$$

since

$$\begin{aligned} ua_3 + w_1 + a_2 &=_{(\mathrm{n})} w_1 + u(a_2 + a_3) + a_2 + ua_3 \geq_{(\mathrm{a})} w_1 + ub_1 + a_2 + ua_3 \\ &=_{(\mathrm{c})} b_1 + a_2 + ua_3 \geq_{(\mathrm{n})} (b_1 + a_2 + a_3)(b_1 + a_2 + u) \geq_{(\mathrm{f}),(\mathrm{a})} a_3. \end{aligned}$$

Finally, it follows $u+w_1+w_2 \geq u+ua_1+ua_3+w_1+w_2 \geq u+a_1+a_3 =_{(\mathrm{a})} 1$. □

4. Perfect pairs

According to Gel'fand and Ponomarev [11] an element u of $F(S)$ is *perfect* if for all subspace lattices L of finite dimensional vector spaces and representations $\rho\colon F(S) \to L$ there is a decomposition $\rho u, v$ of ρ. We shall consider the case where any complemented modular lattices L are admitted and where v can be chosen "uniformly" for all ρ.

In order to do so, we consider modular lattices endowed with a unary operation, *complementation*, $x \mapsto x'$ such that $x \oplus x' = 1$. Observe that each section sublattice $[0, u]$ inherits the complementation $x \mapsto ux'$ and so becomes a *section subalgebra*.

Let $FU(S)$ be the free object generated by S in the class of all modular lattices with complementation operation. In view of Lemma 2.1 we may identify $F(S)$ with the sublattice of $FU(S)$ generated by S. Given a representation $\rho\colon F(S) \to L$ and a complementation operation on L there is a unique homomorphism $FU(S) \to L$ extending ρ and also denoted by ρ.

For $t \in F(S)$, let $F_t(S)$ denote the sublattice generated by the $xt\,(x \in S)$ in the section sublattice $[0, t]$ of $F(S)$ and $FU_t(S)$ the subalgebra generated by the $xt\,(x \in S)$ in the section subalgebra $[0, t]$ of $FU(S)$.

Lemma 4.1. *Let t be neutral in $F(S)$ and τ the representation of S in the section subalgebra $[0, t]$ of $FU(S)$ given by $\tau x = tx$ for $x \in S$. Then $\tau t = 1$ and τ is identity on $FU_t(S)$. For any representation $\rho\colon P \to L$ such that $\rho t = 1$ there is a homomorphism $\sigma\colon FU_t(S) \to L$ such that $\rho = \sigma \circ \tau$.*

In other words, $FU_t(S)$ is freely generated (in the class considered) by S subject to the relation $t = 1$.

Proof. By neutrality, $x \mapsto tx\,(x \in L(S))$ is a lattice homomorphism with $t \mapsto t$. On the generators, it coincides with τ, so it coincides with τ on $L(S)$. This proves $\tau t = t$ and it follows $\tau tx = \tau t\tau x = tx$ for $x \in S$. Thus τ is identity on $FU_t(S)$ since it is so on the generators. Now, given ρ with $\rho t = 1$, let σ be

ρ restricted to $FU_t(S)$. For $x \in S$ one has $\rho x = \rho t \rho x = \rho t x = \sigma \tau x$ whence $\rho = \sigma \circ \tau$. □

Let $t \in F(S)$, $u \in F_t(S)$, and $\overline{u} \in FU_t(S)$. Then $u, \overline{u}$ form a *perfect pair for S under the proviso* $t = 1$ if, for each representation $\rho\colon F(S) \to L$ such that $\rho t = 1$ and complementation on L, the pair $\rho u, \rho \overline{u}$ provides a decomposition of ρ. Here, if t is the element 1 of $F(S)$, then $u, \overline{u}$ are said to form a *perfect pair*. Equivalently, we have $u, \overline{u}$ a decomposition of the identical representation of S in $FU(S)$. With Lemmas 2.2 and 4.1 and Corollary 3.2 it follows:

Corollary 4.2. *If $u, \overline{u}$ is a perfect pair for S, then u is neutral in $F(S)$.*

Corollary 4.3. *Let t be neutral in $F(S)$, $u \in F_t(S)$, and $\overline{u} \in FU_t(S)$. Then $u, \overline{u}$ form a perfect pair under the proviso $t = 1$ if and only if they yield a decomposition of the representation τ of S in $FU_t(S)$ given by $\tau x = tx$ for $x \in S$.*

Corollary 4.4. *Let t be neutral in $F(S)$, $u, v \in F_t(S)$ with $u + v = t$, and $\overline{u}, \overline{v} \in FU_t(S)$. If each $u, \overline{u}$ and $v, \overline{v}$ is a perfect pair under the proviso $t = 1$, then so is $uv, \overline{u} + \overline{v}$.*

Lemma 4.5. *Let $u \leq t$ in $F(S)$ and $\overline{t}, \overline{u} \in FU(S)$ such that $t, \overline{t}$ is a perfect pair and $u, \overline{u}$ is a perfect pair under the proviso $t = 1$. Then u, $\overline{u} + \overline{t}$ is a perfect pair.*

Proof. Consider the identical representation of S in $FU(S)$. By Corollary 4.2, t is neutral in $F(S)$ and we have a representation τ as in Lemma 4.1. By hypothesis, $\tau u, \tau \overline{u}$ is a decomposition of τ. Put $v = \overline{u} + \overline{t} = \tau \overline{u} + \overline{t}$. Since $\tau u = u \leq t$ and $\tau \overline{u} \leq t$, it follows by modularity: $uv = \tau u \cdot t(\tau \overline{u} + \overline{t}) = \tau u(\tau \overline{u} + t\overline{t}) = \tau u \cdot \tau \overline{u} = 0$. Also,

$$u + v = \tau u + \tau \overline{u} + \overline{t} = \tau t + \overline{t} = t + \overline{t} = 1$$

and

$$xu + v = \tau x u + \tau \overline{u} + \overline{t} \geq \tau x t + \overline{t} = xt + \overline{t} \geq x$$

for $x \in S$. □

5. Relevant elements of $F(4)$

In the sequel, we deal with the 4-element antichain S consisting of $a = e_1$, $b = e_2$, $c = e_3$ and $d = e_4$, and $F(4) = F(S)$. Consider the following pairwise-commuting permutations of $\{1, 2, 3, 4\}$:

$$\alpha_1 = (12)(34), \quad \alpha_2 = (13)(24), \quad \alpha_3 = (14)(23).$$

Following [17] we define elements of $F(4)$ for $i = 1, 2, 3$:

$$q_i = (e_k + e_{\alpha_i k})(e_l + e_j), \text{ where } \{k, \alpha_i k, l, j\} = \{1, 2, 3, 4\}.$$

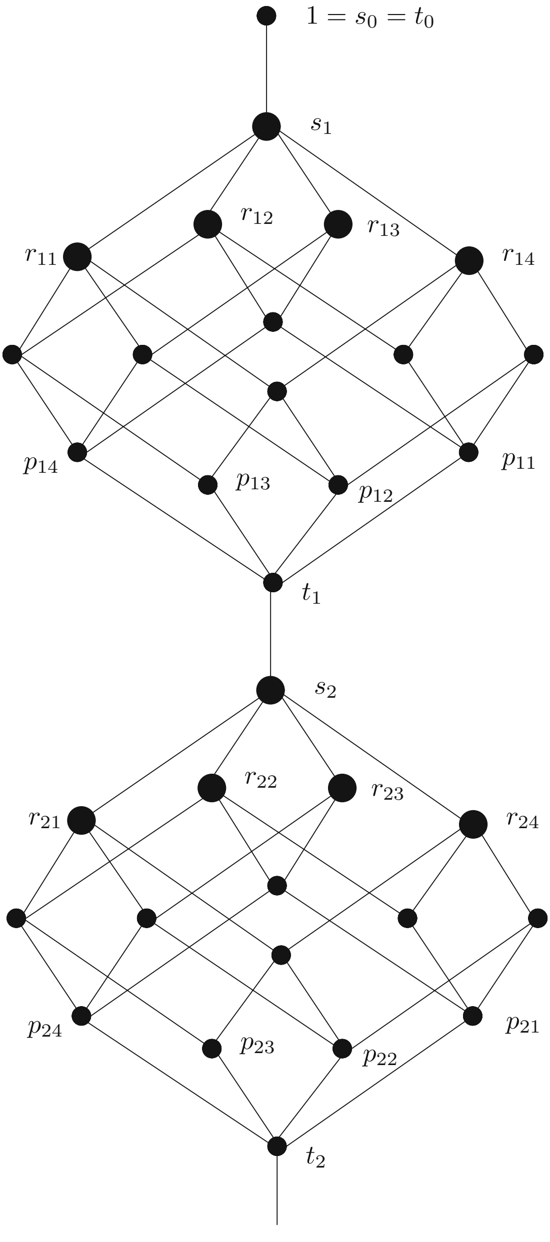

Figure 1. Perfect elements in $D^4 = FM(1+1+1+1)$

Let ϕ_i be the homomorphism of $F(4)$ into the section sublattice $[0, q_i]$ such that

$$\phi_i e = eq_i \ (e \in S).$$

Define by simultaneous recursion

$$\begin{aligned} s_0 &= 1, & s_1 &= \textstyle\sum_{i=1}^4 e_i, & s_{n+1} &= \textstyle\sum_{i=1}^3 \phi_i s_n, \\ r_{0k} &= 1, & r_{1k} &= \textstyle\sum_{i \neq k} e_i, & r_{n+1,k} &= \textstyle\sum_{i=1}^3 \phi_i r_{n\alpha_i k}, \\ t_0 &= 1, & t_1 &= \textstyle\prod_{k=1}^4 r_{1k}, & t_{n+1} &= \textstyle\sum_{i=1}^3 \phi_i t_n, \\ p_{0k} &= 1, & p_{1k} &= e_k + t_1, & p_{n+1,k} &= \textstyle\sum_{i=1}^3 \phi_i p_{n\alpha_i k}. \end{aligned}$$

Recall from §1 and Lemma 3.1 in [17] the following, which are rather immediate consequences of modularity:

$$\phi_i q_j = q_i q_j, \qquad \text{for } i \neq j, \tag{5.1}$$
$$\phi_i \circ \phi_j = \phi_j \circ \phi_i, \tag{5.2}$$
$$s_{n+1} = \phi_i s_n + \phi_j s_n, \qquad \text{for } i \neq j,\ n \geq 1, \tag{5.3}$$
$$t_{n+1} = \phi_i t_n + \phi_j t_n, \qquad \text{for } i \neq j,\ n \geq 1, \tag{5.4}$$
$$p_{n+1,k} = \phi_i p_{n\alpha_i k} + \phi_j p_{n\alpha_j k}, \qquad \text{for } i \neq j,\ n \geq 1, \tag{5.5}$$
$$q_i s_{n+1} = \phi_i s_n, \qquad \text{for } n \geq 0, \tag{5.6}$$
$$q_i t_{n+1} = \phi_i t_n, \qquad \text{for } n \geq 0, \tag{5.7}$$
$$q_i p_{n+1,k} = \phi_i p_{n\alpha_i k}, \qquad \text{for } n \geq 1, \tag{5.8}$$
$$t_{n+1} \leq s_{n+1} \leq t_n \leq p_{ni} \leq r_{nj} \leq s_n, \qquad \text{for } i \neq j, \tag{5.9}$$
$$s_2 \leq q_i + q_j, \qquad \text{for } i \neq j, \tag{5.10}$$
$$et_2 \leq eq_i + eq_j, \qquad \text{for } e \in S,\ i \neq j, \tag{5.11}$$
$$r_{nk} = \textstyle\sum_{i \neq k} p_{ni}, \tag{5.12}$$
$$s_n = p_{n1} + p_{n2} + p_{n3} + p_{n4}, \qquad \text{for } n \geq 1. \tag{5.13}$$

For $n = 1$, identity (5.12) follows immediately from $t_1 \leq r_{1k}$ and then (5.13) from $r_{11} + r_{12} = s_1$. Induction yields the general case.

Proposition 5.1. *The s_n, t_n, p_{ni}, r_{ni} $(n \leq 2)$ are neutral in $F(4)$. A modular lattice has generators e_1, e_2, e_3, e_4 such that $s_2 = 1 > r_{2k} = 0$ if and only if it is 5-element with atoms e_i $(i \neq k)$ and $e_k = 0$.*

Proof. The neutrality is Satz 5.1 in [17]. The second claim follows from Satz 3.2 and 3.3 or the proof of Satz 5.1. □

We need some more information about the r_{nk}.

$$r_{n+1,k} = \phi_i r_{n\alpha_i k} + \phi_j r_{n\alpha_j k}, \qquad \text{for } i \neq j,\ n \geq 1, \tag{5.14}$$
$$r_{n\alpha_i k} = q_i r_{n+1,k}, \qquad \text{for } n \geq 0. \tag{5.15}$$

Proof. In (5.14) let, e.g., $i = 1$, $j = 2$, $k = 1$. By (5.12) and (5.5),

$$\begin{aligned} r_{n+1,1} &= \textstyle\sum_{k\neq 1} p_{n+1,k} = \sum_{k\neq 1} \phi_1 p_{n\alpha_1 k} + \phi_2 p_{n\alpha_2 k} \\ &= \textstyle\sum_{k\neq 2} \phi_1 p_{nk} + \sum_{k\neq 3} \phi_2 p_{nk} = \phi_1 r_{n2} + \phi_2 r_{n3}. \end{aligned}$$

(5.15) follows with induction. For $n = 0$ we have $q_i r_{1k} = q_i = \phi_1 r_0$. Now by (5.14), for $n \geq 1$ and $i \neq j$, $q_i r_{n+1,k} = \phi_1 r_{n\alpha_i k} + q_i \phi_j r_{n\alpha_j k} = \phi_1 r_{n\alpha_i k}$ since $q_i \phi_j r_{n\alpha_j k} = q_i q_j \phi_j r_{n\alpha_j k} = \phi_j (q_i r_{n\alpha_j k}) = \phi_j \phi_i r_{n-1,\alpha_i \alpha_j k} = \phi_i \phi_j r_{n-1,\alpha_j \alpha_i k} \leq \phi_i r_{n\alpha_i k}$ by (5.1), induction, and definition. □

Lemma 5.2. *Let $\rho\colon F(4) \to L$ be a surjective homomorphism and $u \leq \rho s_2$ be neutral in L. Then $u = u\rho q_i + u\rho q_j$ for $i \neq j$.*

Proof. $u\rho q_i + u\rho q_j = u(\rho q_i + \rho q_j) = u\rho(q_i + q_j) \geq u\rho s_2 \geq u$ by neutrality and (5.10). □

6. Perfect pairs for quadruples

Theorem 6.1. *Each of the elements s_n, t_n, and r_{ni} of $F(4)$ can be completed to a perfect pair. In particular, each is neutral in $F(4)$.*

The proof is by induction on $n \geq 1$. In the next section, assuming that the perfect pair $t_n, \tilde{t}_n$ is already given, we define $\overline{s}_{n+1}$ such that $s_{n+1}, \overline{s}_{n+1}$ is perfect under the proviso $t_n = 1$. We then apply Lemma 4.5 to obtain the perfect pairs $s_{n+1}, \tilde{s}_{n+1}$ with $\tilde{s}_{n+1} = \tilde{t}_n + \overline{s}_{n+1}$. Continuing in the section to follow, we define $\overline{r}_{n+1,i}$ such that $r_{n+1,i}, \overline{r}_{n+1,i}$ is perfect under the proviso $s_{n+1} = 1$ and, in turn, the perfect pair $r_{n+1,i}$, $\tilde{r}_{n+1,i}$ with $\tilde{r}_{n+1,i} = \overline{r}_{n+1,i} + \tilde{s}_{n+1}$. In particular, the $r_{n+1,k}$ neutral in $F(4)$ by Corollary 4.2. And so is $u = \prod_{k=1}^{4} r_{n+1,k}$. Applying Lemma 5.2 and induction we get

$$t_n = \prod_{k=1}^{4} r_{nk}. \tag{6.1}$$

Indeed,

$$\begin{aligned} u = q_1 u + q_2 u &= \sum_{i=1}^{2} \prod_{k=1}^{4} q_i r_{n+1,k} = \sum_{i=1}^{2} \prod_{k=1}^{4} \phi_i r_{n\alpha_i k} \\ &= \sum_{i=1}^{2} \phi_i \prod_{k=1}^{4} r_{n\alpha_i k} = \phi_1 t_n + \phi_2 t_n = t_{n+1} \end{aligned}$$

by (5.15) and (5.4). Due to (5.13), under the proviso s_{n+1} we may apply Corollary 4.4 to the perfect pairs $r_{n+1,i}, \overline{r}_{n+1,i}$ with $i \neq j$ to obtain the perfect pairs $r_{n+1,1} r_{n+1,2}$, $\overline{r}_{n+1,1} + \overline{r}_{n+1,2}$ and $r_{n+1,3} r_{n+1,4}$, $\overline{r}_{n+1,3} + \overline{r}_{n+1,4}$ and then the perfect pair $t_{n+1}, \sum_{i=1}^{4} \overline{r}_{n+1,i}$ under the proviso $s_{n+1} = 1$. Finally, we get a perfect pair $t_{n+1}, \tilde{t}_{n+1}$ according to Lemma 4.5.

We denote the complementation on $FU(S)$ by $x \mapsto x'$ and observe that the homomorphisms ϕ_i defined in the preceding section extend to unique homomorphisms ϕ_i of $FU(4)$ into its section subalgebras $[0, q_i]$ and that as before

$$\phi_i \circ \phi_j = \phi_j \circ \phi_i. \tag{6.2}$$

Considering a representation $\rho\colon S \to L$ with $\rho t = 1$ we do the actual computations in L but use elements of $FU(S)$ to denote their images in L. We also use the ϕ_i in that notation, but we do not suggest that there are corresponding homomorphisms defined on (parts of) L. All these calculations could be done within suitable $FU_t(S)$ according to Corollary 4.3 if we would replace $e \in S$ by et, everywhere.

7. Perfect pairs with s_n

We define

$$\begin{aligned}\overline{s}_1 &= s_1',\\ \overline{s}_2 &= x_1 + x_a + x_c,\\ \overline{s}_{n+1} &= \phi_1 \overline{s}_n + \phi_2 \overline{s}_n, \quad \text{for } n \geq 2,\end{aligned}$$

where

$$x_1 = (q_1 s_2)' q_1, \quad x_a = (ab)'(x_1 + b) a t_1, \quad x_c = (cd)'(x_1 + d) c t_1.$$

We claim that, for $n \geq 1$, the pair $s_n, \overline{s}_n$ is perfect under the proviso $t_{n-1} = 1$. The proof is by induction on n, the case $n = 0$ being trivial. We add the following claim to be proved:

$$q_1 \overline{s}_n = \phi_1 \overline{s}_{n-1}, \quad \text{for } n \geq 2. \tag{7.1}$$

Let us consider the pair $s_2, \overline{s}_2$. Observe that the symmetry interchanging a and c as well as b and d leaves s_2 and $\overline{s}_2$ invariant. By neutrality of t_1 (Proposition 5.1) we have $q_1, s_2 \in FU_{t_1}(4)$ whence $x_1, x_a, x_c \in FU_{t_1}(4)$ using modularity and $ab \leq t_1$, $cd \leq t_1$. Thus, $\overline{s}_2 \in FU_{t_1}(4)$.

Now, consider any representation $\rho\colon S \to L$ satisfying $\rho t_1 = 1$ and calculate in L. By modularity and $t_1 = 1$ we have $a + q_1 = b + q_1 = a + b$, by neutrality and definition of s_2,

$$(a + s_2)(b + s_2) = (a + s_2)(q_1 + s_2) = (b + s_2)(q_1 + s_2) = s_2.$$

By modularity, $q_1 = q_1 s_2 \oplus x_1$ and $x_a + ab = ((ab)' + ab)(x_1 + b) a t_1 = (x_1 + b) a$ since $t_1 = 1$ whence $x_a \oplus ab = (x_1 + b) a$. Thus, we may apply Proposition 3.3 with $a_1 = q_1$, $a_2 = a$, $a_3 = b$, $a_4 = u = s_2$ and $w_1 = x_1$, $w_2 = x_a$ in the section sublattice $[0, a + b + s_2]$ of L. It follows that s_2, $x_1 + x_a$ is a decomposition for the quadruple q_1, a, b, s_2, in particular

$$a + b + s_2 = s_2 \oplus (x_1 + x_a), \;\; x_1 + x_a + s_2 a \geq a, \;\; x_1 + x_a + b s_2 \geq b.$$

By symmetry, we have

$$c + d + s_2 = s_2 \oplus (x_1 + x_c), \;\; x_1 + x_c + s_2 c \geq c, \;\; x_1 + x_c + d s_2 \geq d.$$

Now, by neutrality and definition of s_2,

$$(s_2 + c)(a + b) \leq (s_2 + c)(s_2 + a + b) = s_2 + c(a + b) = s_2,$$

whence

$$(s_2 + c)(x_1 + x_a) \leq (s_2 + c)(a + b)(x_1 + x_a) \leq s_2(x_1 + x_a) = 0$$

and, by modularity,

$$s_2(x_1 + x_a + x_c) = s_2(x_c + (s_2 + c)(x_1 + x_a)) = s_2 x_c = 0.$$

It follows that $s_2 \oplus \overline{s}_2 \geq s_1 = 1$ showing that one has a decomposition, indeed.

In the inductive step observe that, by hypothesis, $\overline{s}_n \in FU_{t_{n-1}}(4)$ which is generated by the et_{n-1} $(e \in S)$. It follows that $\phi_i \overline{s}_n$ is in the subalgebra of $[0, \phi_i t_{n-1}]$ generated by the $\phi_i(et_{n-1}) = eq_i t_n = et_n(at_n + bt_n)(ct_n + dt_n)$ (by (5.7) and the neutrality of t_n given by the inductive hypothesis and Lemma 4.2) whence in $FU_{t_n}(S)$. Thus, $\overline{s}_{n+1} \in FU_{t_n}(S)$.

Next, we verify $q_1 \overline{s}_{n+1} = \phi_1 \overline{s}_n$. Indeed, by definition and modularity, $q_1 \overline{s}_{n+1} = \phi_1 \overline{s}_n + q_1 \phi_2 \overline{s}_n = \phi_1 \overline{s}_n$ since $q_1 \phi_2 \overline{s}_n = q_1 q_2 \phi_2 \overline{s}_n = \phi_2(q_1) \phi_2 \overline{s}_n = \phi_2(q_1 \overline{s}_n) = \phi_2 \phi_1 \overline{s}_{n-1} = \phi_1 \phi_2 \overline{s}_{n-1} \leq \phi_1 \overline{s}_n$ by (5.1), induction, (6.2), and definition.

Now, consider any representation $\rho \colon S \to L$ such that $\rho t_n = 1$. Calculating in L, for $e \in S$ we have

$$\begin{aligned} es_{n+1} + \overline{s}_{n+1} &\geq e\phi_1 s_n + e\phi_2 s_n + \phi_1 \overline{s}_n + \phi_2 \overline{s}_n \\ &\geq \phi_1(es_n + \overline{s}_n) + \phi_2(es_n + \overline{s}_n) \geq \phi_1 e + \phi_2 e \geq et_2 \geq et_n = e \end{aligned}$$

by (5.11). It follows that $s_{n+1} + \overline{s}_{n+1} \geq s_1 = 1$.

On the other hand, the elements $\phi_1 s_n + q_1 q_2$, $\phi_1 \overline{s}_n + q_1 q_2$, $\phi_2 s_n + q_1 q_2$, and $\phi_2 \overline{s}_n + q_1 q_2$ are independent in the interval $[q_1 q_2, 1]$. Indeed, the join $(\leq q_1)$ of the first two meets the join $(\leq q_2)$ of the last two in $q_1 q_2$ and

$$(\phi_i s_n + q_1 q_2)(\phi_i \overline{s}_n + q_1 q_2) = q_1 q_2 + \phi_i(s_n(\overline{s}_n + q_j)) = q_1 q_2$$

for $\{i, j\} = \{1, 2\}$ since $\phi_i(s_n(\overline{s}_n + q_j)) \leq \phi_i(s_n(\tilde{s}_n + q_j)) \leq \phi_i q_j = q_i q_j$ from the fact that $s_n, \tilde{s}_n$ is a decomposition, Lemma 2.2, and (5.1). It follows that

$$s_{n+1} \overline{s}_{n+1} \leq (\phi_1 s_n + \phi_2 s_n + q_1 q_2)(\phi_1 \overline{s}_n + \phi_2 \overline{s}_n + q_1 q_2) \leq q_1 q_2,$$

whence $s_{n+1} \overline{s}_{n+1} \leq q_1 s_{n+1} \overline{s}_{n+1} \leq \phi_1 s_n \phi_1 \overline{s}_n = \phi_1(s_n \overline{s}_n) = 0$ using (5.6), (7.1), and induction.

8. Perfect pairs with r_{ni}

For $n = 1$, let

$$\overline{r}_{1k} = (e_k r_{1k})' e_k.$$

Consider $\rho\colon S \to L$ with $\rho s_1 = 1$. Then in L, $e_k = \overline{r}_{1k} \oplus e_k r_{1k}$ whence $\overline{r}_{1k} r_{1k} = \overline{r}_{1k} e_k r_{1k} = 0$, $\overline{r}_{1k} + r_{1k} = e_k + r_{ik} = s_1 = 1$. Also, $e_k \geq \overline{r}_{1k}$ and $e_i \leq r_{1k}$ for $i \neq k$, whence one has a decomposition of ρ.

Let $n = 2$, $\{k, \alpha_1 k, i, j\} = \{1, 2, 3, 4\}$, and $i < j$. We define

$$\overline{r}_{2k} = u_k + v_k, \text{ where } u_k = \phi_1 \overline{r}_{1\alpha_1 k},\ v_k = (e_i e_j)'(u_k + e_j) e_i,$$

$$\overline{r}_{n+1,k} = \phi_1 \overline{r}_{n\alpha_1 k} + \phi_2 \overline{r}_{n\alpha_2 k},$$

and add the following claim to be proven

$$q_1 \overline{r}_{n,k} = \phi_1 \overline{r}_{n-1,\alpha_1 k} \text{ for } n \geq 2. \tag{8.1}$$

Considering $n = 2$, observe that $e_i e_j \leq s_2$ and $e_i q_j \in F_{s_2}(4)$ by the neutrality of s_2 (Proposition 5.1). Thus, $u_k \in FU_{s_2}(4)$. Since $u_k \leq \phi e_{\alpha_1 k} \leq e_{\alpha_1 k}$, modularity yields

$$(u_k + e_j) e_i = (u_k + e_j (e_{\alpha_1 k} + e_i)) e_i (e_{\alpha_1 k} + e_j) \tag{8.2}$$

whence $v_k \in FU_{s_2}(4)$. By the case $n = 1$, definition, and (5.14), (5.15) we have

$$u_k \oplus \phi_1 (e_{\alpha_1 k} r_{1\alpha_1 k}) = u_k \oplus e_{\alpha_1 k} q_1 r_{2k} = \phi_1 e_{\alpha_1 k} = q_1 e_{\alpha_1 k}, \tag{8.3}$$

$$v_k \oplus e_i e_j = (u_k + e_j) e_i. \tag{8.4}$$

Now, consider any representation $\rho\colon S \to L$ with $\rho s_2 = 1$. We calculate in L, using notation from $FU(4)$. By Proposition 5.1 we have $u = r_{2k}$ neutral in $M = \rho(F(S))$, $u = u + e_k$, and pairwise complements $u + e_l$ ($l \neq k$) in $[u, 1]$. Consider

$$a_1 = e_{\alpha_1 k},\quad a_2 = e_i,\quad a_3 = e_j,\quad a_4 = e_k,$$

$$b_1 = a_1(a_2 + a_3),\quad b_2 = a_2(a_1 + a_3),\quad b_3 = a_3(a_1 + a_2).$$

By (8.3) we have $u_k \oplus u b_1 = b_1$. As $u_k \leq \phi_1 e_{\alpha_1 k} \leq e_{\alpha_1 k}$ it follows by (8.2) and (8.4) that

$$v_k \oplus b_2 b_3 = v_k \oplus e_i e_j = (u_k + e_j) e_i = (u_k + b_3) b_2.$$

Thus, we may conclude from Proposition 3.3 that $r_{2k}, \overline{r}_{2k}$ is a decomposition of ρ.

Finally,

$$\begin{aligned} q_1 v_k &\leq q_1 (u_k + e_j) e_i = (u_k + q_1 e_j) q_1 e_i = (\phi_1 \overline{r}_{1\alpha_1 k} + \phi_1 e_j) \phi_1 e_i \\ &\leq \phi_1((\overline{r}_{1\alpha_1 k} + e_j) e_i) = \phi_1((\overline{r}_{1\alpha_1 k} + e_j) r_{1\alpha_1 k} e_i) \leq \phi_1(e_j e_i) \leq e_i e_j \end{aligned}$$

since $e_i \leq r_{1\alpha_1 k}$. It follows by (8.4) that $q_1 v_k = 0$ and $q_1 \overline{r}_{2k} = q_1(u_k + v_k) = u_k + q_1 v_k = u_k = \phi_1 \overline{r}_{1\alpha_1 k}$, verifying (8.1) for $n = 2$.

The inductive step is similar to that in case of defect -2, one just has to pay heed to the second index. Observe that, by hypothesis, $\overline{r}_{n\alpha_i k} \in FU_{s_n}(4)$ which is generated by the es_n ($e \in S$). It follows that $\phi_i \overline{r}_{n\alpha_i k}$ is in the subalgebra of $[0, \phi_i s_n]$ generated by the $\phi_i(es_n) = eq_i s_n = es_{n+1}(as_{n+1} + bs_{n+1})(cs_{n+1} + ds_{n+1})$ (by (5.7) and the neutrality of s_{n+1} given by inductive hypothesis and Lemma 4.2) whence in $FU_{s_{n+1}}(S)$. Thus, $\overline{r}_{n+1,k} \in FU_{s_{n+1}}(S)$.

We verify $q_1 \overline{r}_{n+1,k} = \phi_1 \overline{r}_{n\alpha_1 k}$. Indeed, by definition and modularity, $q_1 \overline{r}_{n+1,k} = \phi_1 \overline{r}_{n\alpha_1 k} + q_1 \phi_2 \overline{r}_{n\alpha_2 k} = \phi_1 \overline{r}_{n\alpha_1 k}$ since by (5.1), induction, (6.2), and definition

$$\begin{aligned} q_1 \phi_2 \overline{r}_{n\alpha_2 k} &= q_1 q_2 \phi_2 \overline{r}_{n\alpha_2 k} = \phi_2 q_1 \phi_2 \overline{r}_{n\alpha_2 k} \\ &= \phi_2 (q_1 \overline{r}_{n\alpha_2 k}) = \phi_2 \phi_1 \overline{r}_{n-1,\alpha_1\alpha_2 k} = \phi_1 \phi_2 \overline{r}_{n-1,\alpha_2\alpha_1 k} \leq \phi_1 \overline{r}_{n\alpha_1 k}. \end{aligned}$$

Now, consider any representation $\rho\colon S \to L$ such that $\rho s_{n+1} = 1$. Calculating in L, for $e \in S$ we have

$$\begin{aligned} er_{n+1,k} + \overline{r}_{n+1,k} &\geq e\phi_1 r_{n\alpha_1 k} + e\phi_2 r_{n\alpha_2 k} + \phi_1 \overline{r}_{n\alpha_1 k} + \phi_2 \overline{r}_{n\alpha_2 k} \\ &= \phi_1 (er_{n\alpha_1 k} + \overline{r}_{n\alpha_1 k}) + \phi_2 (er_{n\alpha_2 k} + \overline{r}_{n\alpha_2 k}) \\ &\geq \phi_1 e + \phi_2 e \geq et_2 = e \end{aligned}$$

by induction and (5.11). It follows $r_{n+1,k} + \overline{r}_{n+1,k} \geq s_1 \geq s_{n+1} = 1$.

On the other hand, the elements $\phi_1 r_{n\alpha_1 k} + q_1 q_2$, $\phi_1 \overline{r}_{n\alpha_1 k} + q_1 q_2$, $\phi_2 r_{n\alpha_2 k} + q_1 q_2$, $\phi_2 \overline{r}_{n\alpha_2 k} + q_1 q_2$ are independent in the interval $[q_1 q_2, 1]$. Indeed, the join ($\leq q_1$) of the first two meets the join ($\leq q_2$) of the last two in $q_1 q_2$ and

$$(\phi_i r_{n\alpha_i k} + q_1 q_2)(\phi_i \overline{r}_{n\alpha_i k} + q_1 q_2) = q_1 q_2 + \phi_i (r_{n\alpha_i k}(\overline{r}_{n\alpha_i k} + q_j)) = q_1 q_2$$

for $\{i, j\} = \{1, 2\}$. Namely,

$$\phi_i (r_{n\alpha_i k}(\overline{r}_{n\alpha_i k} + q_j)) \leq \phi_i (r_{n\alpha_i k}(\overline{r}_{n\alpha_i k} + \overline{s}_{n+1} + q_j)) \leq \phi_i q_j = q_i q_j,$$

since by inductive hypothesis $r_{n\alpha_i k}, \overline{r}_{n\alpha_i k} + \overline{s}_{n+1}$ is a decomposition. Also use Lemma 2.2 and (5.1). It follows that

$$r_{n+1,k} \overline{r}_{n+1,k} \leq (\phi_1 r_{n\alpha_1 k} + \phi_2 r_{n\alpha_2 k} + q_1 q_2)(\phi_1 \overline{r}_{n\alpha_1 k} + \phi_2 \overline{r}_{n\alpha_2 k} + q_1 q_2) \leq q_1 q_2$$

whence, using (5.6), (8.1), and induction,

$$r_{n+1,k} \overline{r}_{n+1,k} \leq q_1 r_{n+1,k} \overline{r}_{n+1,k} \leq \phi_1 r_{n\alpha_1 k} \phi_1 \overline{r}_{n\alpha_1 k} = \phi_1 (r_{n\alpha_1 k} \overline{r}_{n\alpha_1 k}) = 0. \quad \square$$

9. Some theory of modular lattices

See [2, 5] for general reference. D_2 denotes the 2-element lattice and M_n the height 2 lattice with n atoms. In a distributive lattice L, quotients a/b (i.e., $a \geq b$) and c/d generate the same congruence if and only if they are *transposed*, i.e., $b = ac$ and $c = a + d$ or, dually, $d = bc$ and $c = a + d$; cf. [5]. This congruence is minimal, if and only if a/b is a *prime quotient*, i.e., if the interval $[b, a]$ is 2-element. Also, if L satisfies the ascending chain condition, then for each prime quotient a/b there is a unique meet irreducible element c such that $c \geq a$, $c \not\geq b$ and, consequently, a/b transposed to $(b + c)/c$.

Consider a poset S with largest element 1 such that all intervals $[x, 1]$ are finite. Then the finitely generated filters of S form a distributive lattice $L(S)$ in which all intervals $[u, 1]$ are finite and the mapping $x \mapsto [x, 1]$ is an isomorphism of S onto the poset of meet irreducible elements of $L(S)$.

Within the class of all lattices, posets are projective in the following sense: there are lattice terms t_x $(x \in S)$ such that for any order preserving $\phi\colon S \to L$, surjective homomorphism $\pi\colon M \to L$, and choice of $a_y \in M$ with $\pi a_y = \phi y$ for $y \in S$, it follows that the map $\psi\colon S \to M$ with $\psi x = t_x(a_y \mid y \in S)$ is order preserving and satisfies $\phi = \pi \circ \psi$.

Recall that 0 and 1 are considered as constants and that, for any class $\mathcal{L}$, the *variety* $\mathsf{V}\mathcal{L}$ *generated by* $\mathcal{L}$ is the model class of the equations valid in $\mathcal{L}$ and consists of the homomorphic images of sublattices of direct products of lattices in $\mathcal{L}$ (if $0, 1$ are not considered as constants one obtains all lattices which become members of this class after adding $0, 1$). For a variety $\mathcal{L}$ let $F\mathcal{L}(S)$ denote the lattice in $\mathcal{L}$ freely generated by the finite partially ordered set S. Write $F\mathcal{L}(n)$ if S is the n-element antichain. In particular, $F(4) = F\mathsf{V}\mathcal{C}(4)$. We write $\phi\colon S \to L$ if ϕ is order preserving and consider $\phi\colon S \to L \in \mathsf{V}\mathcal{L}$ also as a homomorphism of $F\mathsf{V}\mathcal{L}(S)$ into L and write $\mathsf{im}\phi$ for its image, i.e., the sublattice of L generated by the ϕa $(a \in S)$.

Lemma 9.1. *Given a set $\mathcal{L}$ of lattices, $F\mathsf{V}\mathcal{L}(S)$ is a subdirect product of the* $\mathsf{im}\phi$ *where $\phi\colon S \to L_\phi \in \mathcal{L}$ is order preserving.*

Proof. As in the construction of free algebras (cf. [2]), $F\mathsf{V}\mathcal{L}(S)$ can be constructed as the sublattice generated by the $(\phi a \mid \phi \in I)$ $(a \in S)$ in $\prod_{\phi\in I} L_\phi$ where I is the set of all $\phi\colon S \to L_\phi \in \mathcal{L}$. □

Let $\mathcal{M}$ and $\mathcal{A}$ denote the class of all modular, resp. all Arguesian, lattices. Let $L(V)$ denote the subspace lattice of the K-vector space V and $\mathcal{L}_K$ the variety generated by all such lattices. K_p denotes the prime field of characteristic p and $\mathcal{L}_p = \mathcal{L}_{K_p}$. See [20], [24], and [21] for the following.

Proposition 9.2. *$\mathcal{L}_K$ is generated by the $L(K^n)$, $n < \infty$. For any K-vector space V with* $\dim V \geq 3$*, $L(V) \in \mathcal{L}_p$ if and only if p is the characteristic of K and, in this case, $\mathcal{L}_K = \mathcal{L}_p$.*

Corollary 9.3. *Given $p < \infty$ and K of characteristic p, $F\mathcal{L}_p(S)$ is a subdirect product of* $\mathsf{im}\rho$ *where $\rho\colon S \to L(K^n)$, $n < \infty$.*

Proposition 9.4. *$L(K_0^n)$ belongs to the variety generated by the $L(K_p^n)$ with prime $p \to \infty$.*

Proof. Any ultraproduct of the $L(K_p^n)$ is isomorphic to some $L(K^n)$ where K is of characteristic 0 so that $L(K_0^n)$ is a sublattice. □

Corollary 9.5. *The variety $\mathcal{L}_\infty$ generated by all $L(V)$ is generated by the $L(K_p^n)$, p prime, $n < \infty$.*

$\mathcal{L}_\infty$ is generated by $\mathcal{A} \cap \mathcal{C}$. $\mathsf{V}\mathcal{C}$ is generated by $\mathcal{A} \cap \mathcal{C}$ together with the non-desarguean projective planes; cf. [5]. In the sequel, we consider p ranging over 0, ∞, and all primes, and say that p divides ∞. From [20, 24] and projectiveness of posets one obtains the following.

Proposition 9.6. *The equational theory of $\mathcal{L}_p$ is decidable. The word problem for $F\mathcal{L}_p(S)$ is solvable.*

In order to discuss the role of characteristics, we refer to the concept of an *n-frame* Φ_n, originally due to von Neumann (cf. [21]), a system of lattice generators and relations mimicking projective coordinate systems in lattices $L(V)$ with $\dim V = n$. In particular, M_3 can be viewed as a 2-frame. Within modular lattices, frames are projective systems of generators and relations (in the sense explained above for posets). In particular, $F\mathsf{V}\mathcal{L}(\Phi_n)$ is a subdirect product of sublattices of lattices in $\mathcal{L}$.

Proposition 9.7. *The subdirectly irreducible members of $\mathcal{L}_\infty$ generated by a frame of order $n \geq 3$ are exactly the lattices isomorphic to some $L(K_p^n)$. The lattice freely generated by a frame of order $n \geq 3$ in $\mathcal{L}_\infty$ is a subdirect product of the $L(K_p^n)$, p prime. Moreover, for each square-free m there is a direct decomposition u_m, v_m such that u_m (v_m) is the direct sum (meet) of the u_p (v_p), p a prime dividing m, and $[0, u_p] \cong [v_p, 1] \cong L(K_p^n)$.*

Lemma 9.8. *For modular lattices L_1 and L_2 of finite height, $E \subseteq L_1 \times L_2$ with $\pi_1(E)$ generating L_1, and a join preserving map $\sigma\colon L_1 \to L_2$ with $\sigma\pi_1 x \leq \pi_2 x$ for all $x \in E$, it follows that $\sigma = 0$ if $(1, 0)$ belongs to the sublattice L generated by E.*

Proof. Consider such σ. The map $\gamma x = \inf\{y \in L \mid \pi_1 y = x\}$ is a meet preserving map $\gamma\colon L_1 \to L$ and $\pi_1\gamma x = x$ for all $x \in L_1$. Thus

$$L' = \bigcup_{x \in L_1} [(x, \sigma x), \gamma x]$$

is a sublattice of $L_1 \times L_2$. For $x \in E$ we have $(\pi_1 x, \sigma\pi_1 x) \leq (\pi_1 x, \pi_2 x) = x \leq \gamma\pi_1 x$ whence $E \subseteq L'$ and $L \subseteq L'$. In particular, $(1, 0) \in L'$ whence $(x, \sigma x) \leq (1, 0) \leq \gamma x$ for some $x \in L_1$. Now $\gamma x \leq (x, 1)$ whence $x = 1$ and so $\sigma 1 = 0$. □

10. Perfect elements

The representations of S considered in representation theory are order preserving maps $\rho\colon S \to L(V)$ where $\dim V < \infty$. Two elements of $F\mathcal{L}(S)$ are *K-equivalent* if $\rho s = \rho t$ for all representations $\rho\colon S \to L(V)$ over K with $\dim V < \infty$. By Proposition 9.2, this depends only on the characteristic p of K; we write $s \sim_p t$ and speak of *p-equivalence*. Elements s, t are *linearly equivalent* ($s \sim t$) if they are p-equivalent for all $p < \infty$. Corollary 9.5 and Proposition 9.6 imply the following.

Corollary 10.1. *Consider $\mathcal{L} \supseteq \mathcal{L}_p$. Then $\sim_p$ is the kernel of the canonical homomorphism of $F\mathcal{L}(S)$ onto $F\mathcal{L}_p(S)$. In particular, $F\mathcal{L}_p(S)$ is $FM(S)$*

modulo p-linear equivalence. On the free lattice over S, the relation of p-linear equivalence is decidable.

A finite dimensional representation is *indecomposable* if it admits the trivial decomposition, only.

Corollary 10.2. *$F\mathcal{L}_p(S)$ is a subdirect product of the* $\mathsf{im}\rho$, *$\rho\colon S \to L(K^n)$ indecomposable and $n < \infty$, where K is any fixed field of characteristic p if $p < \infty$, and K ranges over all K_p, p prime, if $p = \infty$.*

Adapting terminology of Gel'fand and Ponomarev [11], we call an element u of $F\mathcal{L}(S)$ *K-perfect* if $\rho(u) \in \{0, V\}$ for each indecomposable representation $\rho\colon F\mathcal{L}(S) \to L(V)$ over K with $\dim V < \infty$. Equivalently, for every representation $\rho\colon F\mathcal{L}(S) \to L(V)$ over K with $\dim V < \infty$ there is a direct decomposition $\rho u, \overline{U}_\rho$. We call u *perfect* if it is K-perfect for all K.

Corollary 10.3. *The perfect, resp. K-perfect, elements of a lattice $F\mathcal{L}(S)$ form a sublattice. For perfect elements of $F\mathcal{L}_p(S)$ one has $u \leq v$ if and only if $\rho v = 0$ implies $\rho u = 0$ (equivalently, $\rho u = 1$ implies $\rho v = 1$) for all indecomposable $\rho\colon S \to L(K_q^n)$ with $n < \infty$ and q dividing p.*

Gel'fand and Ponomarev [11] also noticed the relation to neutrality. They called a subspace U of V *admissible* for $\rho\colon S \to L(V)$ if $U + \rho(xy) = (U + \rho x)(U + \rho y)$, equivalently $U\rho(x+y) = U\rho x + U\rho y$, for all $x, y \in F\mathcal{L}(S)$. If U, W provide a decomposition of ρ, then they are admissible for ρ; cf. Lemma 2.2. Moreover, they are unique complements of each other in the sublattice they generate together with $\mathsf{im}\rho$; cf. [7]. The converse is not true even if one of U, W is an admissible element of $\mathsf{im}\rho$: viz. the representation of $\{a, b\}$ in V with basis e_1, e_2, e_3, e_4 and $U = \rho a$, ρb, and W spanned by e_1, e_3, by $e_1 + e_2$, and by e_2, e_4 respectively. Also, Poguntke [26] has given an example of a representation of the 5-element antichain and a U such that there is no decomposition U, W. But, as shown by Dilworth [7], for a neutral element u of complemented modular lattice L, the unique complement v yields a decomposition u, v of L. Also, from Corollary 9.3 and Proposition 9.6 we get the following.

Corollary 10.4. *Let $u \in F\mathcal{M}(S)$. Then ρu is admissible for all finite dimensional representations, resp. all those over K, if and only if its image in $F\mathcal{L}_\infty(S)$, resp. $F\mathcal{L}_p(S)$, is neutral, where p is the characteristic of K. Both properties are decidable. If u is perfect, resp. K-perfect, then its image in $F\mathcal{L}_\infty(S)$ (resp. in $F\mathcal{L}_p(S)$) is neutral.*

Lemma 10.5. *If v/u and $\tilde{v}/\tilde{u}$ are quotients in a distributive sublattice B of $F\mathcal{L}(S)$ generating the same congruence relation in B, then $\rho u = 0$ and $\rho v = 1$ if and only if $\rho\tilde{u} = 0$ and $\rho\tilde{v} = 1$ for any representation ρ.*

Proof. Without loss of generality, we have $u = v\tilde{u}$ and $\tilde{v} = v + \tilde{u}$. Assume $\rho u = 0$ and $\rho v = 1$. Then $\rho\tilde{v} \geq \rho v = 1$ whence $\rho\tilde{u} = \rho\tilde{u}\rho v = \rho\tilde{u}v = \rho u = 0$. The converse follows by duality. □

In the other direction, we have to discuss strengthenings of the concept of perfect elements. Replacing $\mathcal{C}$ by $\mathcal{C} \cap \mathcal{L}$ in the definition of perfect pairs, we arrive at the concept of *$\mathcal{L}$-perfect pairs*. Observe that $\mathsf{V}(\mathcal{C} \cap \mathcal{L}_p) = \mathcal{L}_p$.

Corollary 10.6. *$u, \overline{u}$ in $F\mathcal{L}_pU(S)$ with $u \in F\mathcal{L}_p(S)$ form a $\mathcal{L}_p$-perfect pair if and only if $\rho u, \rho\overline{u}$ is a direct decomposition of ρ for each representation $\rho\colon S \to L(K^n)$, $n < \infty$ and $K = K_p$ if $p < \infty$, and K any finite prime field if $p = \infty$.*

Proof. [22, Cor. 34] carries over to the classes considered, here: the lattices in $\mathcal{L}_p$ with complementation operation are obtained as homomorphic images of complemented sublattices of direct products of lattices K^n, $K = K_p$ if $p < \infty$ and K ranging over all finite prime fields if $p = \infty$. Also, Proposition 10 in [22] generalizes to projective systems of generators, posets in particular. Thus, $F\mathcal{L}_pU(S)$ is a subdirect product of lattices $L(K^n)$ with complementation. □

As defined in [17], $u \in F\mathcal{L}(S)$ is *strongly perfect* if for each homomorphism $\rho\colon F\mathcal{L}(S) \to L$ into a complemented modular lattice there is $v \in L$ such that $\rho x = \rho u \rho x + v \rho x$ for all $x \in F\mathcal{L}(S)$ (equivalently, all $x \in S$; cf. the proof of Lemma 2.2). Considering the identical representation in $F\mathcal{L}U(S)$ we get the following corollary.

Corollary 10.7. *For $\mathcal{L} \subseteq \mathsf{V}\mathcal{C}$, $u \in F\mathcal{L}(S)$ is strongly perfect if and only if there is $\overline{u} \in F\mathcal{L}U(S)$ such that $u, \overline{u}$ is an $\mathcal{L}$-perfect pair.*

Proposition 10.8. *The strongly perfect elements in $F\mathcal{L}(S)$ form a sublattice.*

Proof. We show that, for decompositions $u, \overline{u}$ and $v, \overline{v}$ of ρ with $u, v \in \mathsf{im}\rho$, also uv, $\overline{u} + u\overline{v}$ is a decomposition. Namely, $uv(\overline{u} + u\overline{v}) = v(u\overline{u} + u\overline{v}) \leq v\overline{v} = 0$ and $uv + \overline{u} + u\overline{v} = (u + \overline{u})(v + \overline{u}) + u\overline{v} = v + \overline{u} + u\overline{v} = \overline{u} + (v + u)(v + \overline{v}) = \overline{u} + v + u = 1$. Moreover, for $x \in \mathsf{im}\rho$ one has $uv(x + \overline{u} + u\overline{v}) = v(u(x + \overline{u}) + u\overline{v}) = v(ux + u\overline{v}) \leq v(ux + \overline{v}) \leq ux$ since $ux \in \mathsf{im}\rho$. □

Obviously, the decomposition to be associated with uv is not unique. Also, contrary to the claim in [17, 6.2], uv, $\overline{u} + \overline{v}$ is not even a pair of complements, in general; M_3 with atoms $u = v$, $\overline{u}$, and $\overline{v}$ yields a counterexample.

To summarize, we have the following implications for $u \in F\mathcal{L}_p(S)$:

$$\text{there is } \overline{u} \text{ with } u, \overline{u} \text{ a } \mathcal{L}_p\text{-perfect pair} \;\Leftrightarrow\; u \text{ strongly perfect}$$
$$\Rightarrow\; u \text{ perfect} \;\Rightarrow\; u \text{ neutral} \;\Leftrightarrow\; \rho u \text{ admissible for all } \rho\colon F\mathcal{L}_p(S) \to L(V).$$

11. Dependence on the base field

Consider a field extension K' of K. With any representation $\rho\colon S \to L(V)$ over K tensoring yields a representation $\rho \otimes K'$ in $L(V \otimes K')$ over K' since $L(V)$ is embedded into $L(V \otimes K')$, canonically. If $\rho \otimes K'$ is indecomposable then so is ρ but the converse may fail as witnessed by the quadruples associated with the classification of endomorphisms. On the other hand, in general there are representations over K' not isomorphic to any $\rho \otimes K'$. Therefore, it remains open how the sets of K-perfect and K'-perfect are related.

In contrast, by Corollary 9.3, strongly perfect elements in $F\mathcal{L}_p(S)$ are K-perfect for all K of characteristic dividing p. Though, in $F\mathcal{L}_\infty(S)$, the notion of strongly perfect depends on characteristic, substantially.

Proposition 11.1. *Let u, v be (strongly) perfect elements in $F\mathcal{L}_\infty(S)$ such that the interval $[u, v]$ is generated by an n-frame with $n \geq 3$ and m a square-free number. Then there is a (strongly) perfect element u_m, $u \leq u_m \leq v$ such that in any representation $\rho\colon P \to L(V)$ over K with $\rho u = 0$ and $\rho v = 1$, one has $\rho u_m = 1$ if and only if the characteristic of K divides m, and $\rho u_m = 0$ otherwise. Moreover, if $[u, v]$ is freely generated in $\mathcal{L}_\infty$ by Φ_n, then there are infinitely many non-neutral elements in $[u, v]$ that are K-perfect for all these K.*

Proof. Choose u_m, v_m as a direct decomposition of $[u, v]$ according to Proposition 9.7. Then u_m is perfect. If $w \in [u, u_q]$ with a prime q not dividing m, then by Proposition 9.2 it follows that $\rho w = \rho u$ for any indecomposable representation over a field K of characteristic p dividing m, i.e., w is K-perfect. For any prime q, the neutrality of u and u_q implies that $\rho x = u_q x + u$ is a representation $\rho\colon S \to [u, u_q] \cong L(K_q^n)$ such that $\rho x = x$ for all $x \in [u, u_q]$. Since ρ is onto, it must be indecomposable. Thus, u and u_q are the only neutral elements in $[u, u_q]$.

Now, assume that $u, \overline{u}$ and $v, \overline{v}$ are perfect pairs. Then $u_m, v_m\overline{u}$ is perfect under the proviso $v = 1$. Indeed, $u_m v_m \overline{u} = u\overline{u} = 0$ and $u_m + x v_m \overline{u} \geq u_m + x v_m \geq x$ for $x \leq v$. Thus, u_m is strongly perfect by Lemma 4.5. □

In order to introduce a notion of perfectness capturing independence of the base field we recall some facts about the Auslander-Reiten translation quiver $\Gamma_K(S)$. See [9, 27, 4, 28, 30] for more detailed explanation.

A morphism ϕ between representations $\rho\colon S \to L(V)$ and $\rho'\colon S \to L(V')$ over K is a linear map $\phi\colon V \to V$ such that $\phi\rho x \leq \rho' x$ for all $x \in S$. $\mathsf{Hom}(\rho, \rho')$ denotes the set of all such ϕ. The indecomposable projectives are the $\rho_x\colon S \to L(K)$ $(x \in S)$ with $\rho_x y = K$ if and only if $x \leq y$ and $\rho_\omega\colon S \to L(K)$ with $\rho_\omega x = 0$ for all x.

Lemma 11.2. *Given a field K of characteristic p, a variety $\mathcal{L} \supseteq \mathcal{L}_p$, a K-perfect element $u \in F\mathcal{L}(S)$, and finite dimensional representations ρ, ρ' of S over K such that $\rho u = 1$ and $\mathsf{Hom}(\rho, \rho') \neq 0$, it follows that $\rho' u = 1$.*

Proof. Given ϕ, by induction one proves $\phi\rho t \leq \rho' t$ for all $t \in F\mathcal{L}(S)$. In particular, passing from t_1, t_2 to $t = t_1 t_2$ one has $\phi\rho(t_1 t_2) = \phi(\rho t_1 \cdot \rho t_2) \leq \phi\rho t_1 \cdot \phi\rho t_2 \leq \rho' t_1 \cdot \rho' t_2 = \rho'(t_1 t_2)$. □

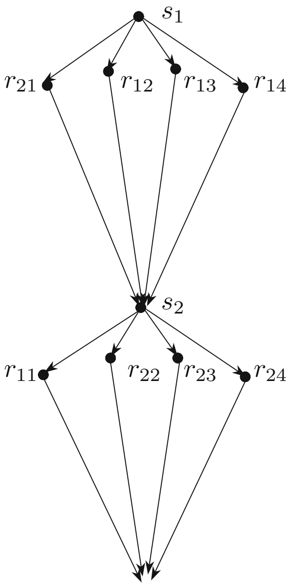

FIGURE 2. Auslander-Reiten quiver: $\Gamma^+(1+1+1+1)$

A translation quiver is a locally finite directed graph together with an injective partial map τ such that the number of arrows from τz to y equals the number of arrows from y to z. The vertices of $\Gamma_K(S)$ are the isomorphism types of finite dimensional irreducible representations of S over K. There are no loops. For $\rho \neq \rho'$ there is at most one arrow from ρ to ρ'; if so then $\mathsf{Hom}(\rho, \rho') \neq 0$. τ is given by the Coxeter functor Φ^+. Exactly the projectives are not in the image of τ.

ρ is preprojective if it belongs to the τ-orbit of some projective. The set $\Gamma_K^+(S)$ of preprojectives is a connected component of $\Gamma_K(S)$ and closed under τ, thus it forms a translation quiver, too. Defining $\rho \geq \rho'$ if $\rho = \rho'$ or if there is a path from ρ to ρ' turns $\Gamma_K^+(S)$ into a poset with largest element ρ_ω and all intervals $[\rho, \rho_\omega]$ finite. Moreover, $\mathsf{Hom}(\rho, \rho') \neq 0$ implies $\rho \geq \rho'$ for $\rho, \rho' \in \Gamma_K^+(S)$. On the other hand, if $\rho \in \Gamma_K^+(S)$ and $\rho' \in \Gamma_K(S) \setminus \Gamma_K^+(S)$, then $\mathsf{Hom}(\rho, \rho') \neq 0$ and $\mathsf{Hom}(\rho', \rho) = 0$.

Given fields K, K' there is a unique isomorphism from $\Gamma_K^+(S)$ onto $\Gamma_{K'}^+(S)$ matching the projectives ρ_x over K and ρ'_x over K'; cf. [28, 29]. This allows us to speak of the preprojective translation quiver $\Gamma^+(S)$ and its isomorphisms π_K onto $\Gamma_K^+(S)$. Let X_ω denote the largest element. Figure 2 shows the uppermost part of $\Gamma^+(S)$ where S is the 4-element antichain.

Dually (with injectives and τ^{-1}) we have the preinjective translation quiver $\Gamma^-(S)$ and its isomorphisms onto $\Gamma_K^-(S)$. Either $\Gamma(S) = \Gamma^+(S) = \Gamma^-(S)$ is finite or $\Gamma^+(S)$ and $\Gamma^-(S)$ are disjoint and $\Gamma^+(S)$ has no minimal, $\Gamma^-(S)$ no maximal element.

Lemma 11.3. *The set of all pairs (X, u) with $X \in \Gamma^+(S)$ and $u \in F\mathcal{L}_\infty(S)$ perfect such that*

$$\rho \in [\pi_K X, \rho_\omega] \Leftrightarrow \rho u = 0, \qquad \text{for all fields } K \text{ and all } \rho \in \Gamma_K(S),$$

defines a partial order embedding γ^+ from $\Gamma^+(S)$ into $F\mathcal{L}_\infty(S)$.

Compare Figures 1 and 2.

Proof. Consider perfect u, v such that $\rho v = 0 \Rightarrow \rho u = 0$ for all primes p and $\rho \in \Gamma_{K_p}(S)$. Then $u \leq v$ by Corollary 10.3. Thus, γ^+ is well defined and order preserving. Conversely, consider $X, Y \in \Gamma^+(S)$ and perfect $u \leq v$ such that $\rho \in [\pi_K X, \rho_\omega] \Leftrightarrow \rho u = 0$ and $\rho \in [\pi_K Y, \rho_\omega] \Leftrightarrow \rho v$ for some K and all $\rho \in \Gamma_K^+(S)$. In particular, for $\rho = \pi_K Y$ one has $\rho v = 0$ whence $\rho u = 0$ and $\rho \in [\pi_K X, \rho_\omega]$. Then $X \leq Y$ since π_K is an order isomorphism. □

We say that γ^+ is a Γ^+-*system of perfect elements* for S if γ^+ is defined on all of $\Gamma^+(S)$. $B^+(S)$ denotes the sublattice (with 1 but without 0) of $F\mathcal{L}_\infty(S)$ generated by the image of γ^+. By Corollary 10.3, $B^+(S)$ consists of perfect elements.

Proposition 11.4. *Suppose that γ^+ is a Γ^+-system of perfect elements for S. Then γ^+ is an order embedding of $\Gamma^+(S)$ into $F\mathcal{L}_\infty(S)$ and may be defined referring to the $K = K_p$, p prime, only. The map $F \mapsto \gamma^+ F = \prod_{X \in F} \gamma^+ X$ is an isomorphism of the lattice of finite filters of $\Gamma^+(S)$ onto $B^+(S)$.*

Proof. Define $\tilde{\gamma}$ as in Lemma 11.3 but referring to the $K = K_p$, only. Then according to the proof of the lemma, $\tilde{\gamma}$ is a well-defined map. By definition, $\gamma^+ X = u$ implies $\tilde{\gamma} X = u$. Thus $\tilde{\gamma} = \gamma^+$ proving the first claim.

Next, we show that for $\rho \in \Gamma_K(S)$ and a finite filter F of $\Gamma^+(S)$ one has $\rho\gamma^+ F = 0$ if and only if $\rho \in \pi_K(F)$. Indeed, if $\rho = \pi_K X$ for some $X \in F$ then $\rho\gamma^+ X = 0$. Conversely, if $\rho \notin \pi_K(F)$ then, for all $X \in F$, $\rho \notin [\pi_K X, \rho_\omega]$ whence $\rho\gamma^+ X = V_\rho$. Thus, $\rho\gamma^+ F = V_\rho$. This proves that γ^+ defines an order embedding of the lattice of finite filters. Now, consider $u = \gamma^+ F_1$, $v = \gamma^+ F_2$ and $w = \gamma^+(F_1 \cap F_2)$. Trivially, $u + v \leq w$. If $\rho(u+v) = 0$ for $\rho \in \Gamma_K(S)$, then $\rho u = \rho v = 0$, whence $\rho \in \pi_K(F_1) \cap \pi_K(F_2) = \pi_K(F_1 \cap F_2)$ and $\rho w = 0$. By Corollary 10.3 it follows that $u + v = w$. Thus, the image under this embedding is a sublattice, whence equal to $B^+(S)$ by definition. □

Corollary 11.5. *If γ^+ is a Γ^+-system of perfect elements for S, then Γ^+ is isomorphic to the poset of meet irreducibles of $B^+(S)$. Every interval $[u, 1]$ of $B^+(S)$ is finite. For $u \in B^+(S)$ and finite filter F of $\Gamma^+(S)$, one has $u \geq \gamma^+ F$ if and only if $u = \gamma^+ X$ for some $X \in F$. For any $u, v \in B^+(S)$ and p prime or $p = 0$, if $u \sim_p v$ then $u = v$.*

Corollary 11.6. *If γ^+ is a Γ^+-system of perfect elements for S, then for any $w \in B^+(S)$ the interval $[w, 1]$ of $F\mathcal{L}_\infty(S)$ is a subdirect product of the intervals $[u, v]$ where $w \leq u$ and v/u prime in $B^+(S)$. These in turn are subdirect products of the $\mathsf{im}\pi_{K_p} X$ where $X \in \Gamma^+(S)$ with $\pi_{K_p} X u = 0$ and*

$\pi_{K_p}Xv = 1$ *and their image in* $F\mathcal{L}_p(S)$ *is isomorphic to* $\mathsf{im}\rho$ *for a unique* ρ *(associated with the meet irreducible* $x \geq u$, $x \not\geq v$ *in* $B^+(S)$*). Thus, the image of* $[w, 1]$ *in* $F\mathcal{L}_p(S)$, $p < \infty$, *is of finite height.*

Dually, we have γ^-, the notion of a Γ^--system of perfect elements for S, and the lattice $B^-(S)$. Put $B(S) = B^+(S) \cup B^-(S)$.

Proposition 11.7. *If* $\Gamma(S)$ *is finite and* γ^+ *a* Γ^+*-system of perfect elements of* S, *then* γ^- *is a* Γ^-*-system of perfect elements for* S *and* $B^+(S) = B^-(S)$. *For infinite* $\Gamma(S)$, *if* γ^+ *is a* Γ^+*- and* γ^- *a* Γ^-*-system of perfect elements for* S, *then* $B(S)$ *is a sublattice of* $F\mathcal{L}_\infty(S)$ *such that* $u < v$ *for all* $u \in B^-(S)$ *and* $v \in B^+(S)$.

In the first case we call γ^+, in the second $\gamma^+ \cup \gamma^-$, a GP-*system of perfect elements* for S.

Proof. In case of finite $\Gamma(S)$ one has $\gamma^- X = v$ where $v \in B^+(S)$ is the unique join-irreducible element which is minimal such that $v \not\leq u = \gamma^+ X$. Assume $\Gamma(S)$ is infinite and consider $X \in \Gamma^+(S)$, $u = \gamma^+ X$, $Y \in \Gamma^-(S)$, and $v = \gamma^- Y$. For any K and $\rho \in \Gamma_K(S)$, if $\rho v = 1$, then $\rho \in \Gamma_K^-(S)$, whence $\rho \notin [\pi_K X, \rho_\omega]$ and so $\rho u = 1$. Thus $v \leq u$ for all $u \in B^+(S)$ and $v \in B^-(S)$ by Corollary 10.3 and $v < u$ since there is $w \in B^+(S)$ with $v \leq w < u$. □

A one-to-one function which associates with each isomorphism type of finite dimensional preprojective representations ρ an n-tuple $f(\rho)$ of natural numbers will be called a Γ^+-*dimension function* if, for any $X \in \Gamma^+(S)$, $f(\pi_K X)$ does not depend on K. Such are the coordinate vector and the dimension vector as defined in [30] 5.1, resp. 11.11.

The perfect element $u \in F\mathcal{L}_\infty(S)$ is *compatible* with the Γ^+-dimension function f if

$$\{f(\rho) \mid \rho \in \Gamma^+_{K_p}(S),\, \rho u = 0\} = \{f(\rho) \mid \rho \in \Gamma^+_{K_q}(S),\, \rho u = 0\} \text{ for all } p, q < \infty.$$

A sublattice B^+ of perfect elements of $F\mathcal{L}_\infty(S)$ is Γ^+-*complete* if, for all $p < \infty$, $\rho u = 0$ with $\rho \in \Gamma_{K_p}(S)$ and $u \in B^+$ implies $\rho \in \Gamma^+_{K_p}(S)$ and if, for any $\rho \in \Gamma^+_{K_p}(S)$, there is $u \in B^+$ such that, for all $\rho' \in \Gamma^+_{K_p}(S)$, $\rho' u = 0$ if and only if $\rho' \geq \rho$. The following is obvious.

Proposition 11.8. *Given any* Γ^+*-dimension function, there exists a* Γ^+*-complete sublattice* B^+ *of compatible perfect elements of* $F\mathcal{L}_\infty(S)$ *if and only if* γ^+ *is a* Γ^+*-system of perfect elements for* S. *In this case,* $B^+ = B^+(S)$.

GP-systems of perfect elements have been established by Gel'fand and Ponomarev [11] for antichains, Bünermann in his diploma thesis (and maybe by others) for posets of finite representation type, and by Stekolshchik [31, 32] for 2+2+2, as revealed by inspection of the proofs. Also, Cylke's constructions [6] suggest that a GP-system of perfect elements exists for any poset of tame representation type (and finite growth). On the other hand, given a GP-system, in general there are many perfect elements not belonging to $B(S)$.

Corollary 11.9. *If a* GP*-system of perfect elements for* S *exists and if* $u, v \in B(S)$ *such that* $[u, v]$ *is freely generated in* $\mathcal{L}_\infty$ *by* Φ_n*, then in* $[u, v]$ *there are infinitely many perfect elements not in* $B(S)$*.*

Proof. Choose u_q as in Proposition 11.1. Then one has $u \sim_p u_q$ if and only if $p \neq q$, hence $u_q \notin B(S)$. □

12. Completeness of systems of perfect elements

Theorem 12.1. *For a poset* S *admitting a* GP*-system of perfect elements and neutral element* u *of* $F\mathcal{L}_p(S)$*,* $p < \infty$*, either* $u \in B_p(S)$ *or* $w < u < v$ *for all* $w \in B_p^-(S)$ *and* $v \in B_p^+(S)$*.*

Here, $B_p^+(S)$ denotes the image of $B^+(S)$ in $F\mathcal{L}_p(S)$. For antichains and perfect elements, this result has been attributed to Gel'fand and Ponomarev [11, 12, 13] by Dlab and Ringel [8].

Proof. Claim 1. If w/v is prime in $B_p^+(S)$ and $u \in [v, w]$ neutral, then $u \in \{v, w\}$.

By Corollary 11.6 there is unique $\rho \in \Gamma^+_{K_p}(S)$ such that $\rho v = 0$ and $\rho w = 1$. Define

$$\rho' x = \rho u + \rho x, \quad x \in F\mathcal{L}_p(S).$$

By neutrality of ρu this is a representation $\rho' \colon S \to \rho u + V_\rho$ of S satisfying $\rho' v = 0$ and $\rho' w = \rho u + V_\rho$. The indecomposable summands satisfy these relations, too, so they are copies of ρ. It follows that $\rho' = \rho$ or $\rho' = 0$ since $\dim V_{\rho'} \leq \dim V_\rho$. Thus $\rho u \in \{\rho v, \rho w\}$ and $u \in \{v, w\}$ by Corollary 11.6.

Claim 2. If u is neutral and $u_0 < u < u_2$ with prime quotients u_1/u_0 and u_2/u_1 in $B_p^+(S)$, then $u \in B_p^+(S)$.

Assume $u \neq u_1$. Then $uu_1 = u_0$ and $u{+}u_1 = u_2$ by Claim 1. By neutrality it follows that u, u_1 is a direct decomposition of $[u_0, u_2]$. First, assume that u_0 is meet irreducible in $B_p^+(S)$. With ρ_0 and ρ_1 in $\Gamma^+_{K_p}(S)$ corresponding according to Corollary 11.6 to u_1/u_0 and u_2/u_1, respectively, it follows $\rho_1 > \rho_0$ and there is $\phi \in \mathsf{Hom}(\rho_1, \rho_0)$, $\phi \neq 0$. But ϕ induces a nontrivial join homomorphism $\sigma \colon [u_1, u_2] \to L(V_{\rho_0})$ contradicting Lemma 9.8.

In general, there is a (unique) meet irreducible v_0 in $B_p^+(S)$ such that $v_0 \geq u_0$ and $v_0 \not\geq u_1$. If $u_2 v_0 = u_0$, then $[u_0, u_2]$ is isomorphic to $[v_0, v_0 + u_2]$. Thus, $u{+}v_0 \in B_p^+(S)$ by the case already considered, whence $u = u_2(u{+}v_0) \in B_p^+(S)$. Otherwise, $u_2 v_0 > u_0$ and $u_2 v_0$ is a complement of u_1 in the height 2 interval $[u_0, u_2]$ of $B_p^+(S)$. By the distributivity of the lattice of neutral elements it follows $u = u_2 v_0 \in B_p^+(S)$.

Claim 3. If u is neutral and $u \leq v$ for some $v \in B_p^+(S)$, then $u \in B_p^+(S)$.

We show $u \in B_p^+(S)$ by induction on the height of $[v, w]$ in $B(S)$ such that $u \in [v, w]$. Choose a maximal chain $w = v_0 > v_1 \cdots > v_k = v$. Assume

$u \notin B_p^+(S)$. Then $v_1 \not\geq u \not\geq v_{k-1}$. Applying the inductive hypothesis we have uv_1 and $u+v_{k-1}$ in $B_p^+(S)$. By distributivity, $u+v_{k-1}/c$ and c/uv_1 are prime quotients in $B_p^+(S)$ where $c = v_1(u+v_{k-1})$. By Claim 2 it follows that $u = c$, a contradiction.

Claim 4. Either $u \geq v$ for some $v \in B_p^+(S)$ or $u \leq v$ for some $v \in B_p^-(S)$ or $w \leq u \leq v$ for all $w \in B_p^-(S)$ and $v \in B_p^+(S)$.

Assume otherwise. Then there are $w \in B_p^-(S)$ and $v \in B_p^+(S)$ such that $uv < u < u+w$. By Claim 3 and its dual we have $u+w \in B_p^+(S)$ and $uv \in B_p^-(S)$ and we may assume that $(u+w)/w$ and v/uv are prime quotients in $B(S)$. By distributivity, $c = (u+w)v = uv+w$ is a complement of u in $[uv, u+w]$ and we get a contradiction as in the proof of Claim 2.

The theorem follows from Claims 3 and 4. □

Corollary 12.2. *Assume that S admits a* GP*-system of perfect elements and that, for a given K of characteristic p, for any $X, Y \in \Gamma_K(S)$ which are neither preprojective nor preinjective there are $X = X_0, X_1, \ldots, X_n = Y$ for some n with $\mathsf{Hom}(X_i, X_{i+1}) \neq 0$ for $i < n$. Then in $F\mathcal{L}_p(S)$ there are at most two perfect elements $u_0, u_1 \notin B_p(S)$. Namely, $\rho u_1 = 0$ if and only if $\rho \in \Gamma_K^+(S)$, and $\rho_0 u = 1$ if and only if $\rho \in \Gamma_K^-(S)$.*

Dlab and Ringel [8, Thm 2] verified the hypothesis for antichains with more than 4 elements and derived the claim for perfect elements [8, Thm 1]. In view of Corollary 10.2, their proof of the latter applies to prove the corollary.

We say that a GP-system of perfect elements for S is *complete* if, for any perfect element u of $F\mathcal{L}_\infty(S)$ and prime p, there is $v \in B(S)$ such that $u \sim_p v$.

For quadruples, the GP-system is complete [17]; cf. Theorem 13.1 below. For any poset of tame representation type with a chain of strongly coupled critical subposets, Theorem 1(ii) of Cylke [6] implies that a GP-system of perfect elements has to be complete.

Proposition 12.3. *A* GP*-system of perfect elements for S is complete if and only if for any perfect element u and any field K (equivalently, any $K = K_p$, p prime) one of the sets $\{\rho \in \Gamma_K(S) \mid \rho u = 0\}$ and $\{\rho \in \Gamma_K(S) \mid \rho u = V_\rho\}$ is finite.*

Proof. Assume that $\{\rho \in \Gamma_{K_p}(S) \mid \rho u = 0\}$ is finite. Then $\{\rho \in \Gamma_{K_p}(S) \mid \rho u = 0\} = \pi_K(F)$ for some finite filter in $\Gamma^+(S)$. Choose $v = \gamma^+ F \in B^+(S)$. Then $\rho u = \rho v$ for all $\rho \in \Gamma_{K_p}(S)$ whence $u \sim_p v$ by Corollary 10.2. The converse is trivial. □

A poset $S_1 \cup \{c\} \cup S_2$, where S_1 and S_2 are of infinite type and $x < c < y$ for all $x \in S_2$ and $y \in S_1$, has c as a strongly perfect element such that $\rho c = 0$ for all preprojectives and $\rho c = V_\rho$ for all preinjectives. Thus, such a poset does not admit a complete GP-system of perfect elements.

13. Results on quadruples

Theorem 13.1. *For a quadruple, S, there is a complete* GP*-system of perfect elements. Moreover, for all $p \leq \infty$, considering $B(S)$ as subset of $F\mathcal{L}_p(S)$, the following hold:*

(i) *Every element of $B(S)$ is strongly perfect.*
(ii) *If v covers u in $B(S)$, then the interval $[u, v]$ of $F\mathcal{L}_p(S)$ is isomorphic to D_2 or M_3 or freely generated within $\mathcal{L}_p$ by some n-frame, $n \geq 3$.*
(iii) *If v covers u in $B(S)$ and if ρ is any indecomposable representation of S in $L(V)$ over F, then one of the following takes place:*
   (a) *$[u, v] \cong D_2$ and $\dim V = 1$;*
   (b) *$[u, v] \cong M_3 \cong \mathsf{im}\rho$ and $\dim V = 2$;*
   (c) *$3 \leq \dim V < \infty$ and $\mathsf{im}\rho \cong L(K_p^n) \cong [u, v]/{\sim_p}$, where p is the characteristic of F.*
(iv) *$\mathsf{im}\rho$ is subdirectly irreducible for any $\rho \in \Gamma(S)$. If ρ is a finite dimensional representation such that $\mathsf{im}\rho$ is subdirectly irreducible, then $\mathsf{im}\rho \cong M_4$ or ρ is indecomposable.*
(v) *For any neutral $x \in F\mathcal{L}_\infty(S)$ there is a prime quotient v/u in $B(S)$ such that x is a strongly perfect element of $[u, v]$. In particular, there is a finite set I of primes such that $x \sim_p u$ for all $p \notin I$ or $x \sim_p v$ for all $p \notin I$.*

Corollary 13.2. *For a quadruple S, and within $F\mathcal{L}_p(S)$, the concepts of neutral, perfect, and strongly perfect elements are equivalent. For $p < \infty$, the image of $B(S)$ under the canonical homomorphism is the sublattice of all neutral elements of $F\mathcal{L}_p(S)$.*

$F\mathcal{L}_\infty(4)$ admits a sublattice $L \neq B(S)$ of strongly perfect elements such that the canonical homomorphism π_p onto $F\mathcal{L}_p(4)$, $p < \infty$, induces an isomorphism of L onto the lattice of all perfect elements in $F\mathcal{L}_p(4)$. Indeed, for some prime p choose u_{pi}, v_{pi} in $[t_3, p_{3i}]$ according to Proposition 9.7 and replace in $B(S)$ the sublattice generated by the p_{3i} by the sublattice generated by $u_{p1} + v_{p2}$, $v_{p1} + u_{p2}$, p_{33}, p_{34}.

Proposition 13.3. *Fix $n \geq 2$ and a division ring K. Up to isomorphism there is exactly one indecomposable representation ρ_0 of $F\mathcal{L}_\infty(\Phi_n)$ over K. $\dim \rho_0 = n$. Any nontrivial representation ρ (also in infinite dimension) is isomorphic to a direct multiple of ρ_0 and $\mathsf{im}\rho \cong \mathsf{im}\rho_0 \cong L(K_p^n)$, p the characteristic of K.*

Proof. [21, 2.2]. □

Proposition 13.4. *A lattice freely generated in a modular variety $\mathcal{L}$ by a quadruple satisfying $t_n = 1$, $s_{n+1} = 0$ ($p_{ni} = 1$, $t_n = 0$) is isomorphic to D_2 for $n = 0$ (D_2, resp. M_3, for $n = 1$, resp. $n = 2$) and, for $n \geq 1$ ($n \geq 3$) freely generated in $\mathcal{L}$ by a $(2n-1)$-frame (n-frame) in analogy to a $(2n-1)$-dimensional (n-dimensional) indecomposable preprojective defect*

-2 (-1) *representation* ρ *of a quadruple.* $\mathsf{im}\rho$ *satisfies these relations for all such* ρ*. Over* K *of characteristic dividing* p*, exactly these indecomposable representations will occur in case* $\mathcal{L} = \mathcal{L}_p$*.*

Proof. Satz 2.1 and 3.3 of [17] and Proposition 13.3. □

Proof of Theorem 13.1. Define $B^+ = \{1\} \cup \{\sum_{i \in I} p_{ni} \mid \emptyset \neq I \subseteq \{1,2,3,4\}\}$. From equations (5.12) and (5.13) it follows that the p_{ni} are independent in $[t_n, s_n]$ and generate a Boolean sublattice with bottom t_n and top s_n. Since $s_{n+1} \leq t_n \leq s_n$ by equation (5.9), B^+ is a linear sum of these sublattices, whence a sublattice, too. The existence of representations according to Proposition 13.4 proves that p_{ni} are distinct atoms of the Boolean lattice and $s_{+1} < t_n$ and that the same holds for the images in $F\mathcal{L}_p(4)$.

Being freely generated by the antichain, $F\mathcal{L}(4)$ admits a unique dual automorphism $t \mapsto t^*$ such that $e_i^* = e_i$ and by [17, 1.3] it holds that $s_m^* \leq s_n$ for all n, m. Defining $B^- = \{t^* \mid t \in B^+\}$ and $B = B^+ \cup B^-$ it follows $u < v$ and $[0, u]$ and $[v, 1]$ finite for all $u \in B^-$ and $v \in B^+$.

By Theorem 6.1, Proposition 10.8, and duality each element of B is strongly perfect, whence neutral. If v covers u in B, then up to translation and duality, $v = t_n, u = s_{n+1}$ or $v = p_{ni}, u = t_n$. By neutrality, Propositions 13.4 and 13.3 can be applied to prove (ii) and (ii).

The Γ^+-system γ^+ is given by matching the representations $\pi_k(X_n)$ such that $t_n = 1, s_{n+1} = 0$ with s_{n+1} and the representations $\pi_K(Y_{ni})$ such that $p_{ni} = 1, t_n = 0$ (equivalently, $s_n = 1, r_{ni} = 0$) with r_{ni}. Recall that $\Gamma^+(S)$ has exactly the arrows $X_n \to Y_{ni} \to X_{n+1}$ with $n \geq 1$. (iv) follows from inspecting the list of indecomposables of defect 0 and the associated lattices $\mathsf{im}\rho$; cf. case (i) below. (v) is Satz 8.1 in [17]. □

According to Proposition 9.6 the word problem for $F\mathcal{L}_p(4)$ is solvable. This even extends to all systems of 4 generators and finitely many relations [14]; cf. [1].

By neutrality of the elements of B, the interval sublattices $[s_n, 1]$ of $F\mathcal{L}_p(4)$ are a subdirect product of the $[u, v]$ (which are well understood), v/u ranging over the prime quotients in maximal chain in the interval $[s_n, 1]$ of B. Under the additional relation $a + b = 1$, the structure of these subdirect products has been analysed in [19] to such extent that neutrality could be proved requiring only the classification of frame generated lattices according to Proposition 9.7. The general structure of $F\mathcal{L}_p(4)$ is still to be determined, based on the atomic elements of Stekolshchik [32].

Considering all modular lattices with 4 generators, it has been shown in [16] that any subdirectly irreducible either

(i) is of breadth 2, satisfies $s_n = 1$ and $s_n^* = 0$ for all n, and is in a list containing 2 infinite height lattices dual to each other, M_4, and the finite height n lattices $S(n, 4) = \mathsf{im}\rho$ associated with n-dimensional non-homogeneous defect 0 indecomposables ρ, or

(ii) satisfies $s_n = 0$ or $s_n^* = 1$ for some n.

For $\emptyset \neq I \subseteq \{1,2,3\}$ define $u_{nI} = \prod_{i \in I} \phi^n q_i$ and $v_{nI} = \sum_{i \in I} (\phi^n q_i)^*$.

Corollary 13.5. *For any representation ρ of $F\mathcal{L}_\infty(4)$ in V of $\dim V = n < \infty$ such that $\rho s_m = V$ and $\rho s_m^* = 0$ for all m, one has the decompositions $\rho u_{nI}, \rho v_{nI}$, and the $\rho u_{nI}, \rho v_{nI}$ for $I \subseteq \{1,2,3\}$ form a Boolean sublattice of $\mathsf{im}\rho$. Moreover, for $I = \{1,2,3\}$, $[0, \rho u_{nI}] \cong M_4$ or trivial and $[0, \rho v_{nI}]$ is a subdirect product of some $S(k,4)$'s with $k \leq n$.*

Proof. Each $S(k,4)$ has a unique set of generators which can be labelled by a, b, c, d such that $s_n = 1$ and $s_n^* = 0$ for all n. And there are, up to isomorphism, six such labellings. In each, either $q_i = 1$ or q_i is a coatom such that $[0, q_i] \cong S(k-1, 4)$. The claim follows easily. □

By [23, 6.3] the $S(k,4)$ are acyclic and have unique representation (and that in dimension k) in analogy to Proposition 13.3. Finite subdirect products of $S(n,k)$'s are acyclic, too, and their indecomposable representations factor through some $S(k,4)$. A lattice theoretic proof of the classification of quadruples of subspaces is easily derived. Observe that the $v_{n\{i\}}$ comprise two series of inhomogeneous defect 0 quadruples.

The filters, resp. ideals, generated by the u_{nI}, resp. v_{nI}, with $n \to \infty$ yield neutral elements in the ideal lattice of the filter lattice of $F\mathcal{L}_p(4)$, forming a 16-element Boolean algebra — corresponding to the 16 candidates for perfect elements according to Dlab and Ringel [8].

In contrast, in [19, Sect. 11] there has been given an example of a quadruple in a subgroup lattice of an abelian group of exponent 4 generating a sublattice in which $t_2 = 1$ and $s_5 = 0$ but t_3 and s_4 are not neutral and which is subdirectly irreducible but not generated by a frame. This is behind the undecidability of the word problem for $F\mathcal{M}(4)$ shown in [18] and indicates that the varieties $\mathcal{L}_p$ are the proper lattice theoretic framework for the study representations of posets.

14. Résumé

It has become clear that the varieties $\mathcal{L}_\infty$ ($\mathcal{L}_p$) generated by subspace lattices of vector spaces (over fields of characteristic p) are the proper ones to discuss perfect elements. The concepts of neutral and strongly perfect elements are primarily lattice theoretic while perfect elements belong to representation theory. We expect that, in general, only the obvious implications from strongly perfect to perfect to neutral will take place.

We suggest that one should study perfect elements primarily related to preprojective and preinjective representations and consider such which do not depend on the base field. Being independent of the base field, the preprojective and the preinjective component of the Auslander–Reiten quiver of a poset provide the proper framework for this and allow us to define the notion of a GP-system of perfect elements.

We conjecture that every poset S admits a GP-system of strongly perfect elements and that (ii) and (iii) of Theorem 13.1 hold, while (v) holds for neutral x in $[w, 1]$ with $w \in B^+(S)$, resp. $[0, w]$, with $w \in B^-(S)$. We conjecture also that, for posets of finite representation type and posets of tame representation type, which are critical or have their critical subposets forming a strongly coupled chain, any neutral element of $F\mathcal{L}_p(S)$ is in $B_p(S)$; moreover, that any neutral element of $F\mathcal{L}_\infty(S)$ is strongly perfect and $F\mathcal{L}_\infty(S)$ a subdirect product of the interval sublattices $[u, 1]\,(u \in B^+(S))$ and $[0, u]\,(u \in B^-(S))$.

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Christian Herrmann
TUD FB4, Schloßgartenstr. 7, D 64289 Darmstadt, Germany