Proatomic modular ortholattices: Representation and equational theory

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Abstract

We study modular ortholattices in the variety generated by the finite dimensional ones from an equational and geometric point of view. We relate this to coordinatization results.

1 Introduction

Modular lattices endowed with an orthocomplementation, MOLs for short, were introduced by Birkhoff and von Neumann [7] as abstract anisotropic orthogonal geometries. The cases of particular interest were the finite dimensional [7] and the continuous (von Neumann [37]) ones. These include the projection lattices of type I_n resp. type II_1 factors of von Neumann algebras. According to Kaplansky [29], completeness implies continuity and, in particular, the absence of infinite families of pairwise perspective and orthogonal elements (*finiteness*). This implies that in general there is no completion. In particular, there is no obvious analogue to ideal and filter lattices, the basic tool in the equational theory of lattices.

In our context, the most relevant result of that theory is Frink's [12] embedding of a complemented modular lattice in a subspace lattice of a projective space and Jónsson's [26] supplement that lattice identities are preserved under this construction. It easily follows that the lattice variety generated by complemented modular lattices is generated by its finite dimensional members (cf [19]). The rôle of finite resp. finite dimensional MOLs for the equational theory of MOLs was discussed in Bruns [8] and in Roddy [39] focussing on a description of the lower part of the lattice of MOL-varieties.

In this paper our main objective are the members of the variety generated by finite dimensional MOLs. These will be called *proatomic* in view of the following (where 'geometric representation' refers to a projective space with an anisotropic polarity).

*Supported by NSERC Operating Grant 0041702

Theorem 1.1 *The following are equivalent for an MOL L*

(1) L is proatomic

(2) L has an atomic MOL-extension

(3) L has a geometric representation

Our main tools are the MOL-construction method from Bruns and Roddy [9] and the concept of orthoimplication from Herrmann and Roddy [20]. The most prominent examples are the continuous geometries constructed by von Neumann [38] from finite dimensional inner product spaces. Also, we construct subdirectly irreducible proatomic MOLs generated by an orthogonal 3-frame and of arbitrarily large finite as well as infinite height.

Quite a few questions remain unanswered - notably, whether there is a non-proatomic MOL and whether every proatomic MOL has an atomic extension within its variety. Also, how to characterize $*$-regular rings with a proatomic lattice of principal right ideals. These and related questions are discussed in the final section.

As general references see [6, 10, 14, 22, 28, 31, 34, 35, 37, 41, 43]. An excellent survey of complemented modular lattices has been presented by Wehrung [44]. The most important concepts and results will be recalled in the sequel.

2 Structure and coordinatization of MOLs

2.1 Complemented modular lattices

All lattices will have smallest element 0, treated as a constant. Joins and meets will be written as $a + b$ and ab. The *dimension* or *height* of a lattice L is the minimal length of maximal chains where length of a chain is cardinality with one element deleted. P_L denotes the set of atoms of L. L is *atomic* if for every $a > 0$ there is $a \geq p \in P_L$. And L is *atomless* if it has no atoms, equivalently if for all $a > 0$ there is $a > b > 0$.

Elements a, b of a lattice form a *quotient* a/b if $a \geq b$. Then we have the interval sublattice $[a, b] = \{x \in L \mid a \leq x \leq b\}$ and we write $\dim[a, b] = \dim a/b$. The *height* of an element a is $\dim a/0$. a/b *transposes* down to c/d and c/d up to a/b if $a = b+c$ and $d = bc$. Quotients in the equivalence relation generated by transposed quotients are called *projective* to each other. Each lattice congruence is determined by its set of quotients and closed under projectivity.

A lattice is *complemented* if it has bounds $0, 1$ and if for every a there is a *complement* b such that $ab = 0$ and $a + b = 1$. A lattice is *relatively complemented* if each of its interval sublattices is complemented. Any modular complemented lattice is such.

Elements a, b of a lattice are *perspective*, $a \sim b$, via c if c is a common complement of a, b in $[0, a + b]$. In a complemented modular lattice, $a \sim b$ via d in $[ab, a + b]$ iff

$a \sim b$ via c where c is a complement of ab in $[0,d]$ resp. $d = ab + c$. Also, according to Lemma 1.4 in Jónsson [27], if $a \sim c \sim b$ and $a > 0$ then there are $a \geq \tilde{a} > 0$ and $b \geq \tilde{b} > 0$ such that $\tilde{a} \sim \tilde{b}$.

An ideal is called *neutral* or a *p(erspectivity)-ideal*, if it is closed under perspectivity. According to [6] p.78, for complemented modular lattices the neutral ideals I are precisely the 0-classes $I(\theta)$ of lattice congruence relations θ - and determine those, uniquely:

$$a/b \in \theta \text{ iff } a = b + c \text{ for some } c \in I \quad \text{resp. } ab' \in I$$

Let $I(a)$ consist of all finite sums of x_i perspective to some $y_i \leq a$. By [27] Lemma 1.5 we have that $I(a)$ is the neutral ideal associated with the congruence generated by $a/0$. A lattice is *finitely subdirectly irreducible* if the meet of any two nontrivial congruences is nontrivial.

Proposition 2.1 *Let M be a subdirectly irreducible complemented modular lattice with minimal congruence μ. Then*

$I(a) = I(\mu)$ for all $a/0 \in \mu$

$[0,b]$ is a simple lattice for each $b/0 \in \mu$

For each $a > 0$ there is $0 < \tilde{a} \leq a$ with $\tilde{a}/0 \in \mu$.

A complemented modular lattice is finitely subdirectly irreducible iff

For all $a, b > 0$ there are $a \geq \tilde{a} > 0$ and $b \geq \tilde{b} > 0$ with $\tilde{a} \sim \tilde{b}$.

Every such is either atomic or atom-less.

Proof. Ad 1: μ is generated by any of its quotients. Ad 2. Let $y < x \leq b$. Choose a as complement of y in $[0,x]$. Since $b \in I(\mu) = I(a)$ we have $b/0$ in the congruence of $[0,b]$ generated by $a/0$, i.e. by x/y cf Lemma 2.2 in [27].
Ad 3. Let c/d a generating quotient of μ, w.l.o.g. $d = 0$. Then $c/0 \in con(a/0)$, i.e. $c/0$ has a proper sub-quotient projective to a sub-quotient x/y of $a/0$. But then $\tilde{a}/0 \in \mu$ with $a \geq \tilde{a} > 0$ and a relative complement $\tilde{a}$ of y in $[0,x]$.

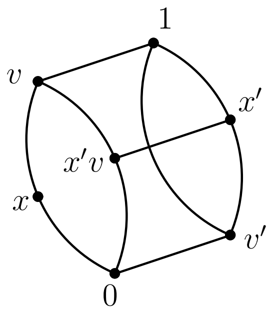
Now, assume that M is finitely subdirectly irreducible. Then given $a, b > 0$ we have $I(a) \cap I(b) \neq 0$ whence there $a \geq a_1 \sim c_1 \in I(b)$ and then $b \geq b_2 \sim c_2 \leq c_1$ and, by modularity, $a_1 \geq a_2 \sim c_2$ whence $\tilde{a} \sim \tilde{b}$ for some $a_2 \geq \tilde{a} > 0$ and $b_2 \geq \tilde{b} > 0$. If L has an atom a, then each b contains an atom perspective to a in two steps. Conversely, we have $0 < \tilde{c} \in I(a) \cap I(b)$. $\square$

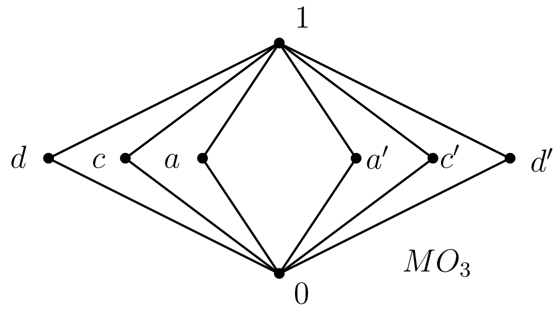
2.2 Ortholattices

An *ortholattice* is a bounded lattice, $L = (L; +, \cdot, 0, 1)$, together with an orthocomplementation, i.e. a unary operation $' : L \mapsto L$ satisfying, for all $x, y \in L$,

$$x + x' = 1,\ x \cdot x' = 0,\ x = x'' \text{ and } x \leq y \text{ implies } y' \leq x'.$$

Since the last property, in the presence of the other ones, is equivalent to DeMorgan's laws ($(x+y)' = x' \cdot y'$ and its dual), this class of algebras forms a variety, or equational class. *Modular* ortholattices will be called MOLs, for short. Examples are Boolean algebras, the height 2 lattice MO_κ with atoms a_α, a'_α ($\alpha < \kappa$) and orthocomplemented non-desarguean planes, e.g. arising by a free construction.





Orthomodular lattices satisfy only a special case of modularity: $x = y + xy'$ for $y \leq x$. It follows that $y \leq x$ generate a Boolean subalgebra and that lattice congruences are ortholattice congruences. In particular, subdirect irreducibility depends only on the lattice structure and we have Prop.2.1 for MOLs, too.

Let $V(L)$ denote the ortholattice variety generated by L. Any interval $[0, u]$ of an orthomodular lattice is itself an orthomodular lattice with complementation $x \mapsto x'u$ which is a homomorphic image of the subalgebra $[0, u] \cup [u', 1]$ of L whence in $V(L)$. Hence, by duality so are the intervals $[v, u]$. We refer to these as *interval subalgebras*. A *relative orthomodular lattice* is a lattice with an orthomodular complementation on each of its interval sublattices, such that each subinterval has the induced complementation. Thus, each orthomodular lattice L can be considered as a relative one and we have $M \in V(L)$ if and only if M belongs to the relative variety of L. In particular, an MOL, L, has the relative sub-MOL L_{fin} which in turn can be considered as directed union of the $[0, u]$, $u \in L_{fin}$.

Lemma 2.2 *Let $\sim$ be a reflexive binary relation on an orthomodular lattice L which is compatible with the lattice operations (i.e. a sublattice of L^2). If $\sim$ is symmetric or compatible with the orthocomplement (i.e. a subalgebra of L^2) then $\sim$ is a congruence relation of L*

Proof. If $\sim$ is also symmetric (i.e. a lattice *tolerance*) then we have $a \sim b$ iff $a + b \sim ab$. Namely, $a + b \sim b + b = b$ and $a + b \sim a + a = a$ from $a \sim b$ resp. $b \sim a$ whence $a + b = (a + b)(a + b) \sim ab$. Conversely, from $a + b \sim ab$ if follows $a + b = a + b + b \sim ab + b = b$ and similarly $a + b \sim a$ whence $a \sim a + b$ and $a = a(a + b) \sim (a + b)b = b$. Therefore, from $a \sim b$ with $c = a + b \sim ab = d$ it follows $cd' \sim 0$ and, since $d' = cd' + c'$ by orthomodularity, $a' + b' = d' \sim c' = a'b'$ whence $a' \sim b'$.

This means that $\sim$ is a subalgebra of L^2, in any case. Now, recall that $p(x, y, z) = (x + ((y + z)y')(z + ((x + y)y')$ is a Mal'cev term for orthomodular lattices, i.e.

$p(x,x,z) = z$ and $p(x,z,z) = x$. Thus, according to the Goursat-Lambek Lemma [32] p.10 we have symmetry and transitivity, too. Indeed, from $y \sim y$, $x \sim y$, and $x \sim x$ it follows $y = p(y,x,x) \sim p(y,y,x) = x$ and from $x \sim y$, $y \sim y$, and $y \sim z$ it follows $x = p(x,y,y) \sim p(y,y,z) = z$. $\Box$

Corollary 2.3 *A set $\mathcal{Q}$ of quotients in an orthomodular lattices is the set of quotients of a congruence relation (i.e. $a\theta b$ iff $(a+b)/(ab) \in \mathcal{Q}$) if and only if it contains all a/a and is closed under subquotients, transposes and*

$$a/c, b/c \in \mathcal{Q} \textit{ implies } (a+b)/c \in \mathcal{Q}, \quad c/a, c/b \in \mathcal{Q} \textit{ implies } c/(ab) \in \mathcal{Q}$$

Proof. According to [5] θ is a lattice tolerance. Also, the transitivity of $\mathcal{Q}$ is immediate from the existence of relative complements. $\Box$ The most prominent example of a congruence on an MOL and its neutral ideal are

$$a\,\theta_{fin}\,b \;\Leftrightarrow\; \dim[\,ab, a+b] < \infty \qquad I = L_{fin} = \{a \in L \mid \dim[0,a] < \infty\}$$

2.3 Review of coordinatization

Let $n \geq 3$ fixed. An *n-frame*, in the sense of von Neumann [37], in a lattice L is a list $\boldsymbol{a} : a_i, a_{ij}, 1 \leq i,j \leq n, i \neq j$ of elements of L such that for any 3 distinct j,k,l

$$a_j \sum_{i \neq j} a_i = \prod_i a_i = a_j a_{jk}, \;\; a_j + a_{jk} = a_j + a_k, \;\; a_{jl} = a_{lj} = (a_j + a_l)(a_{jk} + a_{kl}).$$

The frame is *spanning* in L if $\prod_i a_i = 0_L$ and $\sum_i a_i = 1_L$. The *coordinate domains* associated with the frame $\boldsymbol{a}$ are

$$R_{ij} = R(L,\boldsymbol{a})_{ij} = \{r \in L \,|\, ra_j = a_i a_j, \;\; r + a_j = a_i + a_j\} \quad i \neq j.$$

Now assume that L is modular and $n \geq 4$ or in case $n = 3$ assume the Arguesian law of Jónsson [26]. According to von Neumann [37] and Day and Pickering [11], using lattice polynomials $\oplus_{ij}$, $\ominus_{ij}$, $\otimes_{ij}$ in $\boldsymbol{a}$, each of these can be turned into a ring with zero a_i and unit a_{ij} such that there are ring isomorphism of R_{ij} onto R_{ik} and R_{kj} respectively

$$\pi_{ijk} r = r_{ik} = (r + a_{jk})(a_i + a_k), \quad \pi_{jik} r = r_{kj} = (r + a_{ik})(a_k + a_j).$$

Thus, we can speak of the *ring* $R(L,\boldsymbol{a})$. The operations on R_{ij} can be defined with just one auxiliary index k and the result does not depend on the choice of k. In particular, the multiplication on R_{ik} is given by

$$(s \cdot r)_{ik} = (r_{ij} + s_{jk})(a_i + a_k)$$

The invertible elements of R_{ij} are those which are also in R_{ji}, i.e. $(r^{-1})_{ij} = r_{ji}$. It follows that every s lattice homomorphism induces a homomorphism of coordinate rings. If L is complemented, then surjectivity is preserved.

For a right module M_R let $\mathcal{L}(M_R)$ denote the lattice of all right R-submodules. A *von Neumann regular ring* is an associative ring with unit such that for each $r \in R$ there is a *quasi-inverse* $x \in R$ such that $rxr = r$ (so homomorphic images are also regular). Equivalently, the principal right ideals form a complemented sublattice $\overline{L}(R_R)$ of the lattice $\mathcal{L}(R_R)$ of all right ideals - consisting precisely of the compact elements. And, equivalently, each principal right ideal has an idempotent generator (resp. the same on the left). The lattice structure is given in terms of idempotents e, f, g by

$$\begin{array}{lcll} eR + fR & = & (e+g)R & \text{with} \quad gR = (f - ef)R \\ eR \cap fR & = & (f - fg)R & \text{with} \quad Rg = R(f - ef) \\ Re \oplus R(1-e) & = & R & \end{array}$$

Corollary 2.4 *If R is regular and $\phi : R \twoheadrightarrow S$ a surjective homomorphism then there is a surjective homomorphism $\overline{\phi} : \overline{L}(R) \twoheadrightarrow \overline{L}(S)$ such that $\overline{\phi}(aR) = \phi(a)S$.*

This is part of the following result of Wehrung [42]

Theorem 2.5 *For a regular ring R there is a 1-1-correspondence between two-sided ideals of R and neutral ideals of $\overline{L}(R)$ given by*

$$I = \{a \in R \mid aR \in \mathcal{I}\}, \qquad \mathcal{I} = \{aR \mid a \in I\}$$

We say that a lattice L is *coordinatized* by the regular ring R, if L is isomorphic to $\overline{L}(R_R)$ - and then Arguesian, in particular. Of course, a height 2-lattice is coordinatizable if an only if it is infinite or has $p^k + 1$ atoms for some k and some prime p. From Jónsson [27] Cor.8.5, Lemma 8.2, and Thm.8.3 and von Neumann [37] (see [16] for a short proof) we have

Theorem 2.6 *Every complemented modular lattice which is simple of height $\geq n$ or has a spanning frame of order n, $n \geq 4$ resp. $n \geq 3$ and L Arguesian, can be coordinatized by a regular ring. Every interval $[0, u]$ of a coordinatizable lattice is coordinatizable.*

We need more information about frames and an alternative view of coordinatization. Recall, that the ring R_n of $n \times n$-matrices over a regular ring R is itself regular. Assume $n \geq 3$ and let e_i denote the i-th unit vector in the module R^n.

(1) Given a ring R, the right submodules of R^n form a modular lattice $\mathcal{L}(R^n_R)$. For regular R, the finitely generated ones form a complemented sublattice $\overline{L}(R^n_R)$. Moreover, the $E_i = e_i R$, $i \leq n$ and $E_{ij} = (e_i - e_j)R$ form a spanning (*canonical*) frame $\boldsymbol{E}$. For $n \geq 3$, the lattice $\overline{L}(R^n_R)$ is generated by $\boldsymbol{E}$ and its coordinate ring.

(2) For every complemented modular L with spanning n-frame $\boldsymbol{a}$ there is regular ring R and an isomorphism ϕ of $\overline{L}(R^n_R)$ onto L with $\phi(\boldsymbol{E}) = \boldsymbol{a}$. Moreover, $R(L, \boldsymbol{a})_{ij}$ is a regular ring with zero a_i, unit a_{ij}, $\oplus_{ij}$, $\ominus_{ij}$, and $\otimes_{ij}$ and an isomorphic image of R via $r_{ij} \mapsto \phi((e_i - e_j r)R)$.

(3) The lattices $\mathcal{L}(R_{nR_n})$ and $\mathcal{L}(R^n_R)$ are isomorphic with an ideal I corresponding to a submodule U iff the columns in U are exactly the columns of matrices in I. The canonical idempotent matrices with all entries 0 but one diagonal entry 1 correspond to the canonical basis vectors. This isomorphism takes $\overline{L}(R_{nR_n})$ to $\overline{L}(R^n_R)$.

2.4 Coordinatization of ortholattices

An *involution* $*$ on a ring R is an involutory anti-automorphism

$$(r+s)^* = r^* + s^*, \quad (rs)^* = s^* r^*, \quad r^{**} = r \quad \text{for all } r, s \in R.$$

An element such that $r^* = r$ is called *hermitian.* A $*$*-ring* is an associative ring R with 1 endowed with an involution. A $*$-ring is $*$*-regular* if it is von Neumann regular and if

$$r^* r = 0 \text{ implies } r = 0 \text{ for all } r \in R.$$

Equivalently, each principal right ideal is generated by an hermitian idempotent. On a $*$-regular ring R, $x \perp y \Leftrightarrow x^* y = 0$ defines an anisotropic symmetric relation compatible with addition and right scalar multiplication, whence an anisotropic orthogonality on $\mathcal{L}(R_R)$. In particular

$$X \mapsto X^\perp = \{y \in R \mid \forall x \in X.\ x \perp y\} \in \mathcal{L}(R_R)$$

turns $\overline{L}(R_R)$ into an MOL - again this characterizes $*$-regularity. This MOL satisfies the same orthoimplications as $\mathcal{L}(R_R)$ and is said to be *coordinatized* by R. If e is a hermitian idempotent we also have $eR^\perp = (1-e)R$ and eRe is $*$-regular if e is, in addition, central.

Corollary 2.7 *If R is $*$-regular and I an ideal of R then $I^* = I$ and R/I is $*$-regular, too. Homomorphic images of coordinatizable MOLs are coordinatizable.*

Proof. I is generated by $\{e \mid e^* = e,\ eR \in \mathcal{I}\}$, whence closed under the involution. Thus, R/I is a $*$-ring, naturally, and $*$-regular since every principal right ideal is generated by a hermitean idempotent. Thus, $\overline{L}(R/I)$ with involution $^\perp$ is an MOL and the lattice homomorphism $\overline{\phi}$, associated with the canonical homomorphism of R onto R/I according to Cor.2.4, preserves orthocomplementation, as well. The second claim follows by Thm.2.5. □ From von Neumann [37] II, Thms.4.3 -4.5 and 2.6 we have

Theorem 2.8 *Every MOL coordinatized as a lattice by a regular ring is coordinatized by a $*$-regular ring - having the given ring as reduct. In particular, every MOL L with spanning frame of order $n \geq 4$ ($n \geq 3$ for Arguesian L) can be coordinatized by a $*$-regular ring.*

Now, assume we are given an MOL L and $n \geq 3$. A frame $\boldsymbol{a}$ in L is *orthogonal*, if $a_j \leq a'_k$ for all $j \neq k$ cf [40]. According to Maeda [33, 34] we can add to the above description

(1) Given a $*$-regular ring R and invertible elements $\alpha_1, \ldots, \alpha_n$ of R such that $\alpha_i^* = \alpha_i$ then $L = \overline{L}(R_R^n)$ is a MOL with orthogonal frame $\boldsymbol{E}$ and

$$X' = \{(y_1, \ldots, y_n) \mid \sum_{i=1}^{n} y_i^* \alpha_i x_i = 0 \quad \text{for all } (x_1, \ldots, x_n) \in X\}.$$

(2) For every MOL L with spanning orthogonal frame $\boldsymbol{a}$ there is an isomorphism ϕ of an MOL as in (1) (and w.l.o.g. $\alpha_1 = 1$) onto L with $\phi(\boldsymbol{E}) = \boldsymbol{a}$. Moreover, $R(L, \boldsymbol{a})_{12}$ is $*$-regular

(3) The matrix ring R_n of $*$-regular ring R is $*$-regular if and only if there are α_i as in (1). Then, the involution is given by

$$(x_{ij})^* = (\alpha_i^{-1} x_{ji}^* \alpha_j)$$

and the isomorphism between $\overline{L}(R_{nR_n})$ and $\overline{L}(R_R^n)$ is an MOL-isomorphism, too.

Lemma 2.9 *Let S be a $*$-regular ring such that $\overline{L}(S_S)$ contains an orthogonal n-frame $\boldsymbol{a}$. Then choosing hermitian idempotents e_i generating a_i there is a $*$ regular ring R with invertible hermitian $1 = \alpha_1, \ldots, \alpha_n$ such that S is isomorphic to the $*$-ring R_n as above and the induced MOL-isomorphism maps $\boldsymbol{a}$ onto the canonical frame.*

Proof. The case $n = 2$ is illustrative enough. We may assume that $S = R_n$ as a ring and

$$e_1 = E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

To define the involution on R consider

$$A = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Form $A \perp E_2$ we get $A^* E_2 = 0$ and $c = d = 0$. From $A = AE_1$ we get $A^* = E_1 A^*$ and $b = 0$. Thus, we get an involution of R such that

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} r^* & 0 \\ 0 & 0 \end{pmatrix}$$

Using orthogonality to E_2 resp. E_1 we get

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} a & 0 \\ \beta & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ \beta & 0 \end{pmatrix}^* = \begin{pmatrix} a^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}$$

whence $a^* = 0$ and $a = 0$. Thus, with a similar argument, we have β and α in R such that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

Hence

$$\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}^* = (\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})^* = \begin{pmatrix} 0 & 0 \\ \beta r^* & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}^* = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix})^* = \begin{pmatrix} o & r^*\alpha \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}^* = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix})^* = \begin{pmatrix} 0 & 0 \\ 0 & \beta r^*\alpha \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^* = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2)^* = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}^2 = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \beta\alpha \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \beta^*\alpha \\ \beta\alpha^* & 0 \end{pmatrix}$$

whence $\beta = \alpha^{-1} = \beta^*$. $\square$

Given a right R-module V, a map $\Phi : V^2 \to R$ is $*$-*sesqui-linear* if $\Phi(x, y)$ is linear in y, additive in x, and $\Phi(rx, y) = r^*\Phi(x, y)$. It is $*$-*hermitian* if also $\Phi(y, x) = (\Phi(x, y))^*$. Defining

$$U^\perp = \{x \in V \mid \forall u \in U.\ \Phi(x, u) = 0\}, \quad L_\Phi(V) = \{U \in L(V_R) \mid U^{\perp\perp} = U\}$$

one obtains a complete lattice with involution $^\perp$ which is an ortholattice if an only if Φ is anisotropic: $\Phi(x, x) \neq 0$ for $x \neq 0$. If also $\dim V_R < \infty$ them it is an MOL. Of course, with respect to an orthogonal basis, one obtains a description by a diagonal matrix as in (1) above. Now, the results of Baer [4] and Birkhoff and von Neumann [7] can be formulated as follows

Theorem 2.10 *Every finite MOL is a direct product of Boolean algebras and MO_n's. Every finite dimensional MOL is a direct product of MOLs of height ≤ 3 and MOLs arising from finite dimensional vector spaces with anisotropic $*$-hermitian form resp. matrix $*$-rings over skew fields.*

3 MOLs in projective spaces

3.1 Projective spaces

If a modular lattice, M, is algebraic (i.e. complete with a join-dense set of compact elements) and atomistic (equivalently: M is complemented resp. 1_M is a join of atoms) we speak of a *geomodular* lattice. By M_{fin} we denote the neutral ideal of elements of finite height in M. For geomodular M, these are the elements which are joins of finitely many atoms.

By a projective space we understand a set P of points together with a distinguished set of 3-element subsets, the *collinear triplets*, such that the following 'triangle axiom' holds: If p, s, q and q, t, r are collinear but p, q, r are not then there is unique u such that p, r, u and s, t, u are collinear. A *subspace* of P is a subset U of P such that if $p, q \in U$ and p, q, r collinear then $r \in U$. The subspaces form a geomodular lattice $\mathcal{S}(P)$ where meet is intersection and the join of X and Y consists of all r collinear with some $p \in X$ and $q \in Y$. Singleton subspaces and points are identified. P is *irreducible* if for any two points there is a third one collinear with them. If P is irreducible and $\mathcal{S}(P)$ of height $n \geq 4$ then, by the Coordinatization Theorem of Projective Geometry, there is a vector space V such that $\mathcal{S}(P)$ is isomorphic to the lattice $\mathcal{L}(V)$ of linear subspaces of V.

Now, let M be any modular lattice and $P = P_M$ be the set of *points*, i.e. atoms, of M. Then P is turned into a projective space where p, q, r are collinear if $p+q = p+r = q+r$. We will refer to this as the projective space P_M of M. The subspace lattice $\mathcal{S}(P)$ is canonically isomorphic to the ideal lattice of the sublattice of L consisting of all elements which are joins of finitely many atoms.

If M is algebraic and $P = P_M$, then $\mathcal{S}(P)$ is isomorphic to the interval sublattice $[0, \sum P]$ of M in the following manner: If $u \in M$ then $U = \{p \in P | p \leq u\}$ is a subspace. Conversely, if S is a subspace then $\sum S \in M$, and $S = \{p \in P | p \leq \sum S\}$. It will sometimes be convenient to consider $u \in M$ as a subspace and, when we do, we will do so without changing notation. A *subgeometry* Q of a projective geometry P is just a relatively complemented 0-sublattice of $\mathcal{S}(P)_{fin}$ with set $Q \subseteq P$ of atoms. In other terms, Q is a subset of P with the induced collinearity and closed under the operation given by the triangle-axiom: If p, q, r, s, t are in Q and p, s, q and q, t, r are collinear, but p, q, r are not, then there is u in Q such that p, r, u and s, t, u are collinear.

The disjoint union of projective spaces P_i constitutes a projective space P. Conversely, on the point set P of a geomodular lattice, perspectivity is transitive and P splits into connected *irreducible components* P_i which are projective spaces in their own right. The subspace lattice $\mathcal{S}(P_i)$ forms an interval $[0, \sum P_i]$ in $\mathcal{S}(P)$ and $\mathcal{S}(P)$ is isomorphic to the direct product of the $\mathcal{S}(P_i)$ via

$$X \mapsto (X \cap P_i \mid i \in I)$$

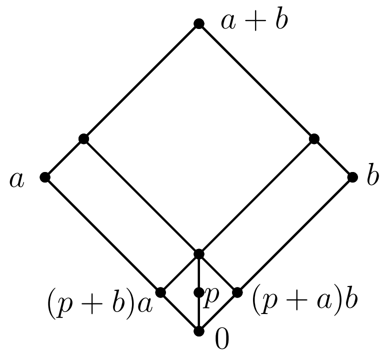
In particular, we have projections which are lattice homomorphisms

$$\pi_i : \mathcal{S}(P) \to \mathcal{S}(P_i), \quad \pi_i X = X \cap P_i$$

The following are due to Frink [12] (cf [10]).

Lemma 3.1 *Let $ab = 0$ in M and p an atom of M such that $p \leq a + b$, $p \not\leq a$, and $p \not\leq b$. Then $p, a(p + b), b(p + a)$ are collinear atoms of M. In a complemented modular lattice, if p an is atom of M such that $p \leq a + b$, $p \not\leq a$, and $p \not\leq b$ then there are atoms $q \leq a$ and $r \leq b$ such that p, q, r are collinear.*

Proof. The first is done by a direct calculation. In the second let $\tilde{b}$ a complement of ab in $[0, b]$ and apply the first. □



For any map $\gamma : L \to M$ of and $Q \subseteq P_M$ there is a natural map $\gamma_Q : L \to \mathcal{S}(Q)$ given by

$$\gamma_Q a = \sum\{q \in Q \mid q \leq \gamma a\}$$

Lemma 3.2 *Let M, L be modular lattices, Q a subgeometry of P_M, L complemented, and $\gamma : L \to M$ a lattice homomorphism. Then $\gamma_Q : L \to \mathcal{S}(Q)$ is a lattice homomorphism provided that $\gamma 0 = 0_M$ and*

for all $a, b \in L$ with $ab = 0$ and all $p \in Q$ with $p \leq \gamma a + \gamma b$ but $p \not\leq \gamma a$ and $p \not\leq \gamma b$ one has also $(p + \gamma b)\gamma a \in Q$ and $(p + \gamma a)\gamma b \in Q$

Moreover, γ_Q is a lattice embedding if γ is such and for all $a > 0$ in L there is $p \in Q$ with $p \leq \gamma a$

The *Frink embedding* of a complemented modular lattice arises by Lemma 3.2 from the principal embedding $\gamma_L : L \to \mathcal{F}(L)$. The points are the maximal filters of L - we also speak of the *Frink space* of L.

If $Q \subseteq P$ is closed under perspectivity, i.e. a union of components then

$$\pi_Q : \mathcal{S}(P) \to \mathcal{S}(Q), \quad \pi_Q x = x \cdot \sum Q = \{q \in Q \mid q \leq x\}$$

is a surjective lattice homomorphism. For any 0-lattice homomorphism $\varepsilon : L \to M$ and $Q \subseteq P_M$ there is a natural map $\varepsilon_Q : L \to \mathcal{S}(Q)$ given by

$$\varepsilon_Q a = \sum\{q \in Q \mid q \leq \varepsilon a\}$$

and this the 0-lattice homomorphism $\pi_Q \circ \varepsilon$, if $M = \mathcal{S}(P)$ and Q is closed under perspectivity.

Now, let ε be an embedding - so consider L as a sublattice of $\mathcal{S}(P)$. If P is the disjoint union of subspaces P_i then the projections π_i provide a subdirect decomposition of L. Thus, if L is subdirectly irreducible then there exists a component Q of P such that ε_Q is an embedding.

3.2 Orthogonalities

By an *orthogonality on a lattice* M we understand a binary relation such that $0 \perp u$ for all u and

$$u \perp v \text{ implies } v \perp u$$

$$u \perp v \text{ and } w \leq v \text{ together imply } u \perp w$$

$$u \perp v \text{ and } u \perp w \text{ together imply } u \perp v + w$$

The orthogonality *induced* on a subset Q of M is given by

$$x \perp_Q y \text{ iff } x \perp y, \quad x, y \in Q$$

Given M_i with orthogonality $\perp_i$, the product M has the orthogonality

$$(a_i \mid i \in I) \perp (b_i \mid i \in I) \text{ iff } \forall i \in I.\, a_i \perp_i b_i$$

If $\phi : M \to N$ is a surjective homomorphism, then N has the orthogonality

$$a \perp_N b \text{ iff } a = \phi c,\ b = \phi d \text{ for some } c \perp_M d$$

On the filter lattice $\mathcal{F}(M)$ we obtain the *canonical* orthogonality

$$F \perp_{\mathcal{F}} G \text{ iff } a \perp b \text{ for some } a \in F,\ b \in G$$

An orthogonality is *anisotropic* if $u \perp v$ implies $uv = 0$. This property is preserved under forming products, sublattices, homomorphic images, and filter lattices. An orthogonality is *non-degenerate* if $u \perp v$ for all v implies $u = 0$. This is obviously so in the anisotropic case. For any ortholattice we have a canonical orthogonality: $x \perp y$, iff $x \leq y'$.

Now, let M be algebraic and P a join-dense set of compact elements such that for any $u, v \in M$ and $p \in P$ with $p \leq u + v$ there are $q \leq u$ and $r \leq v$ in P such that $p \leq q + r$. This applies with P the set of all compact elements of any M resp. P the set of points in a geomodular M. By an *orthogonality* on P we understand a binary symmetric relation $\perp$ on P such that

$$p \perp q,\ p \perp r, \text{ and } s \leq q + r \text{ together imply } p \perp s$$

We obtain an orthogonality on M defining

$$u \perp v \quad \text{iff } p \perp q \text{ for all } p \leq u,\ q \leq v$$

Namely, the first two properties are obvious. In the third we may assume $u \in P$. Now, if $v + w \geq p \in P$ then there are $q \leq v, r \leq w$ such that $p \leq q + r$ whence $u \perp q$, $u \perp r$ and so $u \perp p$. Defining

$$u^\perp = \sum \{q \in P \mid q \perp u\}$$

we get

$$u \leq v^\perp \text{ iff } u \perp v \text{ iff } v \leq u^\perp$$

Namely, if $u \geq p \in P$ and $u \perp v$, then $v^\perp \geq p$ and there are finitely many $q_i \in P$ with $q_i \perp v$ and $p \leq \sum q_i$ whence $p \perp v$. It follows that $x \mapsto x^\perp$ is a self adjoint Galois connection on the lattice M. In particular the map is order reversing and the map $x \mapsto x^{\perp\perp}$ is a closure operator on M. To wit

$$u \leq v \text{ implies } v^\perp \leq u^\perp, \quad u \leq u^{\perp\perp}$$

$$(\textstyle\sum_{i \in I} u_i)^\perp = \prod(u_i^\perp), \quad u^\perp = \prod\{p^\perp \mid u \geq p \in P\}.$$

The closed elements of M, endowed with the partial order inherited from M and the restriction of $^\perp$, form a complete meet sublattice K of M and a complete ortholattice containing L as a subalgebra. Indeed, $a \vee_K b = a^{\perp\perp} \vee_K b^{\perp\perp} = (a^\perp b^\perp)^\perp = (a'b')^\perp = (a'b')' = a + b$ for $a, b \in L$. Moreover K is atomistic if P consists of atoms p such that $p = p^{\perp\perp}$. K satisfies the *covering property.* if $u \vee p = u + p$ covers u for any atom $p \not\leq u$.

Let Q be join-dense in M. Then any orthogonality on M is determined by the orthogonality induced on Q. It is anisotropic if $p \not\perp p$ for all $p \in Q$ and non-degenerate if for each $p \in Q$ there is $q \in Q$ such that $p \not\perp q$. For a direct product $M = \mathcal{S}(P)$ of $M_i = \mathcal{S}(P_i)$ with the product orthogonality, we have P the disjoint union of the P_i and speak of the *orthogonal disjoint union* of the $P_i, \perp_i$.

Proposition 3.3 *Let L be a bounded lattice with anisotropic orthogonality $\perp$ and assume that for each x there is x' such that*

$$x' \perp x \quad \textit{and } \forall y.\ y \leq x + (x+y)x'$$

Then $x' = \sup\{z \mid z \perp x\}$ is uniquely determined and L with $x \mapsto x'$ is an orthomodular lattice.

Proof. With $y = 1$ we get $1 \leq x + x'$. Now, if $x \leq y$ and $y \perp x'$ then $y \leq x + yx' = x$ whence $x = \sup\{y \mid y \perp x'\}$. Since $x'' \perp x'$, it follows $x'' \leq x$. For $z \geq x'$ and $z \perp x$ we get $z \leq x' + x''z \leq x' + xz = x'$ whence $x' = \sup\{z \mid z \perp x\}$. It follows $x \leq x''$, thus $x = x''$. Moreover, $x \leq y'$ implies $x \perp y$ whence $y \leq x'$ and we have an ortholattice, indeed. □

Lemma 3.4 *Every 0-1-lattice embedding $\eta : L \to \mathcal{S}(P)$ of an MOL induces an anisotropic orthogonality on P*

$$p \perp q \quad \textit{iff } p \leq \eta a \textit{ and } q \leq \eta(a') \textit{ for some } a \in L$$

Moreover,

$$\eta(a') \leq (\eta a)^\perp \quad \textit{for all } a \in L$$

Proof. For convenience, we think of η as id_L. Consider $p \perp q, r$ and $s \leq q + r$. Then $p \leq a, b$ and $q \leq a, r \leq b$ for some $a, b \in L$ whence $p \leq ab$ and $s \leq a' + b' = (ab)'$. Thus we obtain an anisotropic orthogonality. Moreover, if $p \leq \eta(a')$ then $p \leq a'$, i.e. $p \perp a$ and so $p \perp \eta a$. Thus, $\eta(a') \leq (\eta a)^{\perp}$. Now, $\eta a + \eta a' = \eta(a + a') = 1_M$ by embedding, $\eta a \cdot (\eta a)^{\perp} = 0$ by anisotropicity, and $\eta a' \leq (\eta a)^{\perp}$ by hypothesis, whence $\eta a' = (\eta a)^{\perp}$ by modularity. □

3.3 Polarities

An orthogonality on a geomodular lattice resp. its projective space is a *polarity* if it is nondegenerate and if $p^{\perp}$ is a coatom for each atom p.

Lemma 3.5 *A nondegenerate orthogonality $\perp$ on a geomodular lattice is a polarity if and only if*

$$p^{\perp}(q + r) > 0 \quad \textit{for all points } q \neq r \textit{ with } p \not\perp q,\ p \not\perp r$$

Proof. If $\perp$ is a polarity then $p^{\perp}$ is a coatom whence the claim follows by modularity. Conversely, consider $q \not\leq p^{\perp}$. We have to show that $q + p^{\perp} = 1$, i.e. $r \leq q + p^{\perp}$ for all $r \neq q$. But, by hypothesis, if $r \not\leq p^{\perp}$ then $0 < s = p^{\perp}(q + r) < q + r$, so $s \leq p^{\perp}$ is a point and $r \leq q + s \leq q + p^{\perp}$. □

Corollary 3.6 *For an anisotropic orthogonality $\perp$ on a geomodular lattice the following are equivalent*

(1) *$\perp$ is a polarity*

(2) *$p + p^{\perp} = 1$ for all atoms p*

(3) *$(p + r)p^{\perp} > 0$ for all atoms $p \neq r$*

Proof. As observed above, $pp^{\perp} = 0$. Thus for $r \neq p$ we have (3) trivially, if $r \perp p$, and by the Lemma, otherwise. If (2) holds, then $p^{\perp}$ is a coatom by modularity. Thus, by modularity, $p + p^{\perp} = 1$ if and only if $p^{\perp}$ is a coatom. □

Corollary 3.7 *For each MOL M there is a canonical anisotropic polarity on P_M given by*

$$p \perp q \ \textit{ if and only if } \ p \leq p'$$

Corollary 3.8 *A projective space with anisotropic polarity is the orthogonal disjoint union of its irreducible components. Conversely, the orthogonal disjoint union of spaces with polarity yields a space with polarity.*

Proof. In view of (3) $p \neq r$ and $p \not\perp r$ jointly imply that there is a q collinear with p, r. □ According to Maeda [33] an orthogonality $\perp$ on a desarguean irreducible projective space P (so $L(P) \cong L(V_D)$ for some vector space) is a polarity if and only if there is an anti-automorphism $*$ of D and $*$-hermitian form Φ on V_D such that

$$p = vD \perp q = wD \;\text{ if and only if } \Phi(v, w) = 0$$

and Φ is anisotropic if and only if so is $\perp$. For such, the lattice $L_\Phi(V)$ of closed elements is modular if and only if V_D is finite dimensional (Keller [30]).

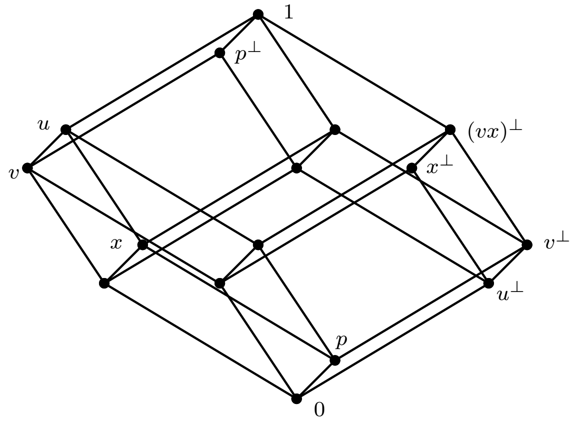
Lemma 3.9 *Let $\perp$ be an anisotropic polarity on the geomodular lattice M. Then*

(1) $u^{\perp\perp} = u$ *and* $u + u^\perp = 1$ *for all* $u \in M_{fin}$

(2) *Each interval* $[0, u] \subseteq M_{fin}$ *with the induced orthogonality is an MOL with orthocomplementation*

$$x \mapsto x^{\perp u} = \sum\{q \leq u \mid q \perp x\} = ux^\perp.$$

Proof by induction on the height of u. For $u = 0$ nothing is to be done. So let v a lower cover of u. By inductive hypothesis. $v + v^\perp = 1$, whence by modularity $p = uv^\perp \in P$ and $u = v + p$. It follows, with modularity again, $u + u^\perp = v + p + v^\perp p^\perp = (v + v^\perp)(p + p^\perp) = 1$. Since $u^{\perp\perp} \geq u$ and $u^\perp u^{\perp\perp} =$ we have $u^{\perp\perp} = u$ by another application of modularity. Finally, choose $p \leq x$ and let $v = up^\perp$. v is a lower cover of u. Then



$$x = p + vx, \quad x^\perp = p^\perp(vx)^\perp, \quad x^{\perp u} = (vx)^{\perp v}$$

whence by induction

$$x + x^{\perp u} = p + vx + x^{\perp v} = p + v = u. \quad \square$$

A *geometric representation* of an MOL is a 0-1-lattice embedding $\eta : L \to M = \mathcal{S}(P)$ into the subspace lattice of a projective space P with anisotropic polarity $\perp$ such that

$$\eta(a) \perp \eta(a') \quad \text{ for all } a \in L$$

By modularity it follows

$$\eta(a') = \eta(a)^\perp \quad \text{ for all } a \in L$$

Indeed, $\eta(a)^\perp \geq \eta(a')$, $\eta(a) \cdot \eta(a)^\perp = 0$, and $\eta(a) + \eta(a') = \eta(a + a') = 1$.

Corollary 3.10 *Every subalgebra L of an atomic MOL M has a geometric representation $\eta : L \to \mathcal{S}(P_M)$ with $\eta(a) = \{p \in P_M \mid p \leq a\}$.*

3.4 Geometric MOL construction

For each polarity on a geomodular lattice M the following hold

(i) If $x \leq y \in M$ such that $\dim y/x < \aleph_0$ then $\dim x^\perp/y^\perp \leq \dim y/x$

(ii) If $x \leq y \in C$ such that $\dim y/x < \aleph_0$ then $\dim_C y/x = \dim x^\perp/y^\perp = \dim y/x$

(iii) If $u \in C$ and $x \geq u$ in M such that $\dim x/u < \aleph_0$ then $x \in C$

Namely, consider $x \leq y$ in M with $\dim y/x < \aleph_0$. Then $y = x + \sum p_i$ with $\dim y/x$ many $p_i \in P$ and $y^\perp = x^\perp \prod_i p_i^\perp$. This proves (i). Now, if $x, y \in C$ then $\dim y/x = \dim x^\perp/y^\perp$. If $x \prec_C y$ is a covering in C, then $y^\perp < x^\perp$ and we may choose $p \leq x^\perp$, $p \not\leq y^\perp$. Then $y \not\leq p^\perp$ and $yp^\perp \in C$. It follows $x = yp^\perp \prec_M y$ whence (ii). Finally, if $u \in C$ and $x \geq u$ in M then $\dim x/u \geq \dim u^\perp/x^\perp \geq \dim x^{\perp\perp}/u^{\perp\perp} = \dim x^{\perp\perp}/u \geq \dim x/u$. Thus (iii).

An important congruence relation μ on any modular lattice M (cf [10]) is given by

$$\begin{array}{lll} x \,\mu\, y & \text{iff} & \dim(x+y)/(xy) < \aleph_0 \\ & \text{iff} & \dim z/x < \aleph_0 \text{ and } \dim y/z < \aleph_0 \text{ for some } z \geq x, y \\ & \text{iff} & \dim x/u < \aleph_0 \text{ and } \dim y/u < \aleph_0 \text{ for some } u \leq x, y \end{array}$$

Given any subset L of M we define

$$\hat{L} = \{x \in C \mid x \,\mu\, u \text{ for some } u \in L\}$$

Consider the conditions

$$\begin{array}{lrl} (a) & ab \in \hat{L} & \text{for all } a, b \in L \\ (b) & a+b \in \hat{L},\ a^\perp + b^\perp \in C & \text{for all } a, b \in L \\ (c) & a^\perp \in \hat{L} & \text{for all } a \in L \end{array}$$

Lemma 3.11 *Let $\perp$ be a polarity on the geomodular lattice M. Then*

- (a) *implies that $\hat{L}$ is meet-closed in M and C, simultaneously*
- (b) *implies that $\hat{L}$ is join-closed in M and C, simultaneously*
- (c) *implies that $\hat{L}$ is closed under $x \mapsto x^\perp$*

In particular, $\hat{L}$ is a modular ortholattice if $\perp$ is anisotropic and $(a), (b), (c)$ hold.

This is basically Lemma 2 of [9]. *Proof.* Observe that

$$\hat{L} \;=\; \{x \in C \,|\, \exists a \in L.\, \exists y, z \in C.\, y \leq z,\ a, x \in [y, z] \text{ and } \dim z/y < \aleph_0\}$$

In particular,

$$x, y, z \in C,\ a \in \hat{L},\ y \leq z,\ a, x \in [y, z], \text{ and } \dim z/y < \aleph_0 \quad \text{jointly imply} \quad x \in \hat{L}$$

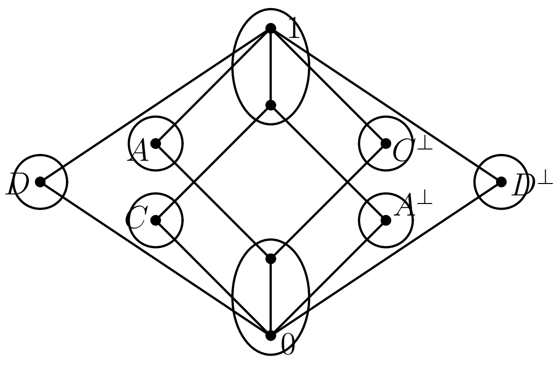
Indeed, for $x\,\mu\,a$ in C we have also $y = xa$ and, by (iii), $z = x + a$ in C and $y\,\mu\,z$. Assuming (c), for $x \in \hat{L}$ with (i) we conclude $y^\perp\,\mu\,z^\perp$ whence $x^\perp\,\mu\,a^\perp$ and so $x^\perp \in \hat{L}$.

Now, consider $y \leq a \leq z$ and $v \leq b \leq w$ in C, $\dim z/y < \aleph_0$, and $\dim w/v < \aleph_0$. Let $x \in [y, z]$ and $u \in [v, w]$. By the congruence properties of μ one has $xu\,\mu\,ab$ and $x + u\,\mu\,a + b$. By (iii) $x, u, xu \in C$. Thus $xu \in \hat{L}$ if $a, b \in L$ and (a). Moreover $x + u \in C$ provided that $x \geq a$, $u \geq b$ and $a + b \in C$.

Now suppose (b) and $a, b \in L$. We show $y + v \in C$ by induction on $\dim a/y + \dim b/v$. In doing so, by (iii) we may assume that we have $y \prec t \leq a$ with t and $t+v$ in C. Considering the sublattice of M generated by y, t, v two cases are possible: firstly, $y + v = t + v$ with nothing left to do; secondly, $y + v \prec t + v$. If we had $v^\perp y^\perp \leq t^\perp$ then by modularity $v^\perp + t^\perp < v^\perp + y^\perp$. Now $a^\perp \leq t^\perp \leq y^\perp$, $b^\perp \leq v^\perp$ and $a^\perp + b^\perp \in C$ by hypothesis. Thus, as shown above, we would have $v^\perp + t^\perp$ and $v^\perp + y^\perp$ in C. It would follow $vt = (v^\perp + t^\perp)^\perp < (v^\perp + y^\perp)^\perp = vy$, a contradiction. So we may choose $p \in P$ such that $p \leq v^\perp y^\perp$, $p \not\leq t^\perp$. Then $p^\perp \geq y + v$, $p^\perp \not\geq t + v$. Consequently, $y + v = (t + v)p^\perp \in C$. With (iii) it follows $x + u \in C$ for all $x \in [y, z]$, $u \in [v, w]$ whence $x + u \in \hat{L}$ since $a + b \in \hat{L}$ by hypothesis. $\Box$

Proposition 3.12 *For any geometric representation $L \subseteq M$ of an MOL, there is a sub-MOL $\hat{L}$ of the ortholattice K of closed elements of M containing L and all atoms of M. In particular, $\hat{L}$ is an atomic MOL containing L as a sub-MOL.*

Proof. Apply 3.11. $\Box$ The original example in [9] was based on a separable real Hilbert space (H, Φ) and $L = \{0, H, A, A^\perp, C, C^\perp, D, D^\perp\} \subseteq L_\Phi(H)$ such that $A^\perp + C \in L_\Phi(H)$ coatom, $X + Y = H$ for $X \neq Y$ in $L \setminus \{0\}$, else. Thus $\hat{L}/\,\theta_{fin} \cong MO_3$ whence $\hat{L}$ is not coordinatizable. On the other hand, $\hat{L}$ contains an infinite set of orthogonal perspective elements and is not normal in the sense of Wehrung [43]. The same holds for the subalgebra generated by A, C, D.



3.5 Topological MOL construction

In his paper [12] Frink pointed out that his embedding can be seen as a generalization of Stone's representation of Boolean algebras as rings of sets. In [26] Jónsson established as much of a duality as appears possible without an orthogonality. Topological representations for orthomodular lattices have been given by Iturrioz [24, 25]. But modularity hardly can be characterized within that approach. Therefore, we prefer to work on a projective space at the price of using a more general concept of 'topology', as explained in Abramsky and Jung [1].

An abstract characterization of the Frink embedding has been given by Jónsson [26]: Considering L as a sublattice of $M = \mathcal{S}(P)$ it is a *regular sublattice* which means that L is a complemented 0-1-sublattice of the geomodular lattice M such that

for all $X \subseteq L$ with $0 = \prod_M X$ then there is finite $Y \subseteq X$ with $\prod_M Y = 0$
for any $u \in M_{fin}$ and $q \in P$ with $uq = 0$ there are $a, b \in L$ with $a \geq u$, $b \geq q$, and $ab = 0$.

A *subspace topology* $\mathcal{O}$ on a projective space P is a 0-1-sublattice of $\mathcal{S}(P)$ closed under arbitrary joins. The members of $\mathcal{O}$ are referred to as *open subspaces*. The space is *strongly Hausdorff* if for any finite n and $p \neq q_i, (1 \leq i \leq n$ in P there are $U, V \in \mathcal{O}$ such that $p \in U$, $q_i \in V(i \leq n)$ and $U \cap V = \emptyset$. The space is *Hausdorff* if this holds for $n = 1$.

An *s-basis* $\mathcal{B}$ of $\mathcal{O}$ is a 0-sublattice such that each member of $\mathcal{O}$ is a directed sum (i.e. union) of members of $\mathcal{B}$.

Call a subspace A *s-compact* if for any covering $A \subseteq \bigcup_{i \in I} U_i$ with a directed system of open subspaces U_i there is $i \in I$ such that $A \subseteq U_i$. Observe that if U, V are s-compact subspaces then so is $U + V$.

A *MOL-space* is a projective space P endowed with an anisotropic orthogonality $\perp$ and a s-compact subspace topology $\mathcal{O}$ having a s-basis $\mathcal{B}$ such that $U^\perp \in \mathcal{O}$ and $U + U^\perp = P$ for all $U \in \mathcal{B}$.

If the collinearity relation on P is empty, then $U + V = U \cup V$ and $U^\perp = P \setminus U$ which means that in this case MOL-spaces are just Boolean spaces.

Proposition 3.13 *A MOL-space P has a unique s-basis, namely the s-compact open subspaces. These form a subalgebra L of $(\mathcal{S}(P),^\perp)$ which is an MOL. If $\perp$ is a polarity, the Hausdorff property implies its strong variant.*

Proof. Let X be a subspace of a MOL-space P such that X and $X^\perp$ are open and $X + X^\perp = P$. Then X is s-compact and $X = X^{\perp\perp}$. Namely, let $X = \bigcup U_i$ and $X^\perp = \bigcup V_j$ directed unions of basic sets, each including $\emptyset$. Then $P = \bigcup(U_i + V_j)$ is also a directed union of basic sets. S-compactness of P yields that $P = U_i + V_j$ for some i, j. By $X^{\perp\perp} \cap X^\perp = 0$ and modularity, one derives $U_i = X = X^{\perp\perp}$.

It follows that the basic sets are s-compact open - the converse being trivial. Also, if U is basic, then $U = U^{\perp\perp}$. Thus, applying the above to $X = U^\perp$ and $X^\perp = U$ we get that $X^\perp$ is s-compact whence basic. In particular, $L = \mathcal{B}$ is an MOL.

Now, assume $\perp$ a polarity. For $u \in M_{fin}$ and $q \in P$ with $p \perp u$ there is $a \in L$ such that $u \leq a$ and $q \leq a'$. We show this by induction on the height of u. For $u = 0$ this is trivial. So let $u > 0$ and v a lower cover of u. Then $p = uv^\perp \in P$ and $p \perp v$. Hence, by inductive hypothesis we have $a \in L$ such that $a \geq v$ and $a^\perp \geq p$. Since $p \perp q$ we have $b \in \mathcal{O}$ such that $b \geq p$ and $b^\perp \geq q$. Since L is a basis, we may choose $b \in L$. Then $a + b \in L$ with $a + b \geq p + v = u$ and $(a + b)^\perp = ap\, b^\perp \geq q$.

Consider $0 < u \in M_{fin}$ and $q \in P$ with $uq = 0$. Then $v = uq^\perp$ is a lower cover of u whence $p = uv^\perp \in P$ and $v \perp q$ as well as $v \perp p$. As just shown, we have $a, b \in L$

such that $a, b \geq v$, $a^\perp \geq q$ and $b^\perp \geq p$ and we may assume $a \leq b$ and $b^\perp \leq a^\perp$. By the Hausdorff property we have $c, d \in L$ such that $p \leq c$, $q \leq d$ and $cd = 0$. We may assume $c \leq b^\perp$ and $d \leq a^\perp$. It follows $u = p + v \leq a + c$ and, by modularity, $(a+c)d \leq (a+b^\perp)a^\perp = b^\perp$ whence $(a+c)d = (a+c)b^\perp d = (ab^\perp + c)d = cd = 0$. □

An *MOL-space* is *Frinkian* if it is strongly Hausdorff and if $P = U_i$ for some $i \in I$ whenever $P = (\bigcup_{i \in I} U_i)^{\perp\perp}$ for a directed system of open subspaces.

Theorem 3.14 *Frink spaces of MOLs with canonical orthogonality and basic open subspaces*

$$U(a) = \{p \in P \mid p \leq \varepsilon a\}, \quad a \in L.$$

are Frinkian MOL-spaces. Moreover, $a \mapsto U(a)$ provides an (object)duality between MOLs and Frinkian MOL-spaces.

Proof. Consider a Frinkian MOL-space, By the Proposition, $L = \mathcal{B}$ is a MOL. We claim that L is a regular sublattice of $M = \mathcal{S}(P)$. If we have $a_i \in L$ such that $\prod_{i \in I} a_i = 0$ then $(\sum_{i \in I} a_i^\perp)^\perp = 0$ and $P = (\sum_{i \in I} a_i^\perp)^{\perp\perp}$. Hence $P = \sum_{i \in J} a_i^\perp$ for some finite $J \subseteq I$ and $0 = \prod_{i \in J} a_i$.

Conversely, let M be the Frink-extension of the MOL L. Then the $U(a)$, $a \in L$ form a s-basis of s-compact open subspaces. Namely, observe that $U(a)^\perp = U(a')$ and suppose that a directed set $\{a_i \in L \mid i \in I\}$ is given such that $U(a) = \bigcup_{i \in I} U(a_i)$. Then in M we have $a = \sum_{i \in I} a_i$. Also $\prod_{i \in I} aa'_i = a(\sum_{I \in I} a_i)^\perp = 0$. Thus, by regularity there is $j \in I$ with $aa'_j = 0$. It follows $a' + a_j = 1$ whence $a = a_j$ by modularity and $a \leq a_i$.

Similarly, if we have $P = (\bigcup_{i \in I} U(a_i))^{\perp\perp}$ with directed $a_i \in L$ then $0 = \prod_{i \in I} a'_i$ whence, by regularity of the embedding, $0 = a'_i$ for some i and so $P = U(a_i)$.

Regularity implies the strong Hausdorff property, immediately. Also if $p \perp q$ then $p \in U(a)$ and $q \in U(a')$ for some a.

This shows that we have a Frinkian MOL-space, indeed, and that $a \mapsto U(a)$ is an isomorphism of L onto the algebra of s-compact open subspaces.

On the other hand, starting with a Frinkian MOL-space P, as we have seen above, the embedding of L into $\mathcal{S}(P)$ is regular and Thm. 2.6 of Jónsson [26] applies to show that

$$\psi(p) = \{a \in L \mid p \in a\}, \quad \psi(x) = \sum \{\psi p \mid p \leq x\}$$

is a lattice isomorphism of $\mathcal{S}(P)$ onto the subspace lattice of the Frink-space such that $\psi|L$ is the Frink-embedding. Moreover, in P we have, by hypothesis, $p \perp q$ iff $p \leq a$ and $q \leq a^\perp$ for some basic a. Thus, ψ is also an isomorphism with respect to orthogonality. Since it matches bases and it is a homeomorphism, indeed. □

Let us take the opportunity to point out an error in A.Day and C.Herrmann, Gluings of modular lattices, Order **5** (1988), 85-101. It is claimed there that the direct limits of the lattices $(\mathcal{IF})^n(L)$ resp. $(\mathcal{FI})^n(L)$ (taken over the canonical embeddings) are isomorphic - here $\mathcal{I}(L)$ denotes the ideal lattice. Yet, the map α offered, fails to be an isomorphism - and we suspect that there is none. Nethertheless their Lemma 2.1 can be proved directly.

4 Equational theory

4.1 Orthoimplications and varieties

Let M be a lattice with 0 and an orthogonality (actually, for the generalities we only need that $a \perp b$, $c \leq a$, and $d \leq b$ imply $c \perp d$). Considering M as structure $(M; +, \cdot, 0, \perp)$, the *orthoimplication* given by a lattice term f (in two sorts of variables, x_i and y_i) is the first order formula,

$$x_1 \perp y_1 \wedge ... \wedge x_n \perp y_n \to f(x_1, y_1, ..., x_n, y_n) = 0.$$

Lemma 4.1 *Orthoimplications are preserved under formation of direct unions, products, sublattices, homomorphic images, and filter lattices - with the induced orthogonalities. Also, they are preserved under weakening of the orthogonality.*

Proof. Formation of direct unions, products, and substructures (weak with respect to the relation symbols) preserves any universal sentences of the above type. Now, let $\phi : L \to M$ a surjective homomorphism. Assume $a_i \perp b_i$ in M. Then there are $c_i \perp d_i$ in L with $a_i = \phi c_i$ and $b_i = \phi d_i$. By hypothesis $f(c_1, d_1, \ldots, c_n, d_n) = 0$ whence $f(a_1, b_1, \ldots, a_n, b_n) = 0$. If $F_i \perp_{\mathcal{F}} G_i$ then $a_i \perp b_i$ for some $a_i \in F_i, b_i \in G_i$ whence

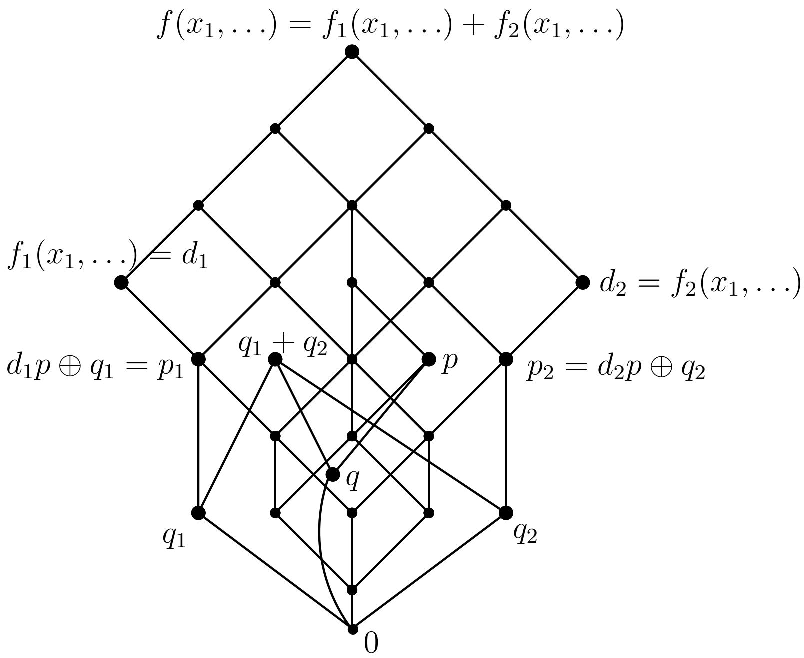
$$0 = f(a_1, b_1, \ldots, a_n, b_n) \in f(F_1, G_1, \ldots, F_n, G_n) \quad \square$$

Lemma 4.2 *Let M be an algebraic lattice and I its set of compact elements or M a complemented modular lattice and I a neutral ideal. For each lattice polynomial $f(z_1, ..., z_m)$ with constants in M and $c_1, ..., c_m$ in M, and for each $p \in I$ one has: $f(c_1, ..., c_m) \geq p$ iff $f(u_1, ..., u_m) \geq p$, for some some $u_i \in I$ with $u_i \leq c_i$, $i = 1, ..., m$.*

Proof. We proceed by induction on the complexity of f. The claim is trivially true if f is a single variable or constant.

Suppose $f = f_1 f_2$. Then $p \leq f(c_1, ..., c_m)$ implies $p \leq f_k(c_1, ..., c_m)$, for $k = 1, 2$. By the inductive hypothesis, there exist $u_{k1}, ..., u_{km} \in I$ with $u_{ki} \leq c_i$, $i = 1, ..., m$, and $p \leq f_k(u_{k1}, ..., u_{km})$, for $k = 1, 2$. Set $u_i = u_{1i} + u_{2i}$, for $i = 1, ..., m$.

Now suppose $f = f_1 + f_2$, and, for convenience, define $d_k = f_k(c_1, ..., c_m)$, for $k = 1, 2$. In the first case, we have $d_i = \sum Q_i$ with directed $Q_i \subseteq I$ whence by compactness $p \leq \sum P$ with finite $P \subseteq Q_1 \cup Q_2$ and $p_i = \sum P \cap Q_i \in I$. In the second case let $p_i = d_i(p + d_j)$ and q_i a complement of $d_i p$ in $[0, p_i]$ and $q = p(q_1 + q_2)$. Then $qq_i \leq d_i p q_i = 0$, $p + q_i = p + d_i(p + d_j) \geq q_j$ whence $q + q_i = (q_1 + q_2)(p + q_i) = q_1 + q_2$. It follows that $q/0$ is projective to $q_i/0$ whence $q_i \in I$. Then also $p_i = q_i + d_i p \in I$. Thus, in both cases by the inductive hypothesis, there exist $u_{k1}, ..., u_{km} \in I$ with $u_{ki} \leq c_i$, $i = 1, ..., m$, and $p_k \leq f_k(u_{k1}, ..., u_{km})$ for $k = 1, 2$. Set $u_i = u_{1i} + u_{2i}$, for $i = 1, ..., m$. and notice that $p \leq p_1 + p_2 \leq f(u_1, ..., u_m)$. The converse follows from monotonicity of lattice polynomials. $\square$



Corollary 4.3 *Let M be a complemented modular lattice with orthogonality $\perp$ and I a neutral ideal such that for each $a > 0$ there is $p \in I$, $a \geq p > 0$. Then an orthoimplication holds in M if and only if it holds in all $[0,u]$, $u \in I$.*

Proof. Consider an orthoimplication given by f which is not valid in M. There exist $x_1 \perp y_1, ..., x_n \perp y_n$ so that $f(x_1, y_1, ..., x_n, y_n) = a > 0$ whence $0 < p \leq a$ with $p \in I$. By 4.2, there exist $u_i, v_i \in I$ with $u_i \leq x_i$, $v_i \leq y_i$, for $i = 1, ..., n$ and $f(u_1, v_1, ..., u_n, v_n) \geq p > 0$. But $u_i \leq x_i$ and $v_i \leq y_i$ give $u_i \perp v_i$, for $i = 1, ..., n$. Let $u = \sum_{i=1}^n u_i + \sum_{i=1}^n v_i$. Then the orthoimplication does not hold in $[0,u]$. $\Box$

Corollary 4.4 *Let M be a algebraic lattice with orthogonality. Then an orthoimplication holds in M if and only if it holds for all substitutions with compact elements.*

Lemma 4.5 *Within the variety of orthomodular lattices, each ortholattice identity is equivalent to an orthoimplication in terms of the canonical orthogonality.*

Proof. Considering an identity $g = h$ in the language of ortholattices we may replace the constants $0, 1$ by uu' resp. $u + u'$, u a new variable. Also, we may assume that $g \leq h$ is valid in all ortholattices. Due to DeMorgan's Laws and $x'' = x$, there is a lattice term $f(x_1, y_1, ..., x_n, y_n)$ such that $hg'(x_1, ..., x_n) = f(x_1, x_1', ..., x_n, x_n')$ holds in all ortholattices. If $g = h$ holds in the orthomodular lattice L, and $x_i \perp y_i$, $i = 1, ..., n$, then $0 = f(x_1, x_1', ..., x_n, x_n') \geq f(x_1, y_1, ..., x_n, y_n)$ and the orthoimplication

$$x_1 \perp y_1 \wedge ... \wedge x_n \perp y_n \to f(x_1, y_1, ..., x_n, y_n) = 0$$

holds in L. Conversely, if this orthoimplication holds, then $f(x_1, x_1', ..., x_n, x_n') = 0$ holds in L and, consequently, $g = h$ holds in L. $\Box$ With 4.3, 4.5, and 2.1 one gets

Corollary 4.6 *For a subdirectly irreducible MOL L with minimal congruence μ the variety $V(L)$ is generated by the simple interval subalgebras $[0,u]$ of L, $u/0 \in \mu$. In particular, every variety of MOLs is generated by its simple members.*

Corollary 4.7 *The variety $V(L)$ of an atomic MOL L is generated by the interval subalgebras $[0,u]$, $u \in L_{fin}$.*

MOLs in the variety generated by atomic MOLs (i.e. by finite dimensional MOLs) will be called *proatomic*.

Corollary 4.8 *If $L \subseteq M$ is a geometric representation of an MOL then the orthoimplications of M are valid in L and L belongs to the variety generated by $\hat{L}$ resp. the set of interval subalgebras $[0,u]$ of M, $u \in M_{fin}$*

Proof. Use Lemma 4.1 and the fact that L is a weak substructure of M. Also, use 4.3, 3.9, and 4.5. □

4.2 Atomic extension

For the proof of Thm 1.1 we need the following Lemma. The concept of *neutral filter* is the dual of "neutral ideal". We write $p \leq F$ if $p \leq x$ for all $x \in F$.

Lemma 4.9 *Let L, M be MOLs, L a subalgebra of M, and F a neutral filter of L. Consider $a, b \in L$ and $p \in P_M$ such that $ab = 0$, $p \leq a + b$, and $p \leq F$. Then $a(p+b)$, $b(p+a) \leq F$.*

Proof. In view of restriction to interval subalgebras, we may assume $a + b = 1$. Let $q = a(p+b)$ and $r = b(p+a)$ and θ the congruence associated with F. Consider $x \in F$, i.e. $x\,\theta\,1$ and $p \leq x$. Let

$$y = (a + xb)(b + x) \geq q, \quad z = (b + xa)(a + x) \geq r$$

By modularity, x, y, z coincide or are the atoms of a sublattice of height 2. In particular, all its quotients are in θ whence $1/y \in \theta$ and $y \in F$. From $p \leq F$ it follows $p \leq y$ and thus $r \leq p + q \leq y$. Hence $r \leq yz \leq x$ and $q \leq x$, symmetrically. □

Proof of Thm. 1.1. (2) and (3) are equivalent by Cor.3.10 and Prop.3.12, and imply (1) by Cor.4.7. The class of MOLs admitting an atomic extension contains all finite dimensional ones and is closed under subalgebras and direct products. Thus, to prove that (1) implies (2) we have to show that this class is closed under homomorphic images, too. Consider a subalgebra L of an atomic MOL M and congruence θ on L with associated neutral filter F. Define

$$Q = \{p \in P_M \mid p \leq F\}, \quad \eta : L/\theta \to L(Q), \quad \eta(a/\theta) = \{p \in Q \mid p \leq a\}$$

Then Q is a subgeometry of P_M with polarity $\perp$, obviously, η is meet preserving and $\eta(a/\theta) \perp \eta(a'/\theta)$. If $a/b \in \theta$ then $b = ac$ for some $c \in F$ whence $a \geq p \in Q$ implies $p \leq b$; thus, η is well defined. The proof that η preserves joins follows Frink: Given $a, b \in L$ choose $\tilde{b}$ such that $a + b = a + \tilde{b}$ and $a\tilde{b} = 0$. Consider $p \in \eta((a + b)/\theta)$, $p \notin \eta(a/\theta)$ and $p \notin \eta(\tilde{b}/\theta)$. Then by Lemma 3.1 $p, a(p + \tilde{b})$, and $\tilde{b}(p + a)$ are collinear elements of P_M. By Lemma 4.9 they are in Q, whence $p \in \eta(a/\theta) + \eta(\tilde{b}/\theta) \subseteq \eta(a/\theta) + \eta(b/\theta)$.

Finally, consider $a/0 \notin \theta$ which means $ac > 0$ for all $c \in F$. Thus, since F is closed under finite meets, for any finite $C \subseteq F$ we have $x \in M$ such that $x \leq c$ for all $c \in C$. In other words, the set

$$\Phi_a(x) = \{0 < x \leq ac \mid c \in F\}$$

of formulas with parameters in L is finitely satisfiable in M. By the Compactness Theorem of First Order Logic, M has an elementary extension M' such that each $\Phi_a(x)$ is satisfiable in M', i.e. there is $x \in M'$ with $0 < x \leq ac$ for all $c \in F$. Replacing M, we may assume $M = M'$. Since M is atomic, we get $p \in P_M$ with $p \leq x$ and then $p \in Q$ by definition. Thus $\eta(a/\theta) > 0$ which proves that η is a geometric representation. $\Box$ With Cor.3.8 we obtain

Corollary 4.10 *Every proatomic MOL has a geometric representation in an orthogonal union of spaces P_i, each of is given by a vector space V_i over a $*$-division-ring D_i with anisotropic $*$-hermitian form Φ_i - or possibly of height 3 if L is not Arguesian. Every subdirectly irreducible proatomic MOL has a representation with a single $P_i = P$.*

Von Neumann [38] constructs a continuous, simple, atomless MOL as the metric completion of a direct union of finite dimensional MOLs. Since the metric completion amounts to a homomorphic image of a subalgebra of a direct power, this MOL is proatomic. The finite dimensional MOLs are the $\mathcal{L}(\mathbb{R}^{2^n}_{\mathbb{R}})$ and the union is formed with respect to the canonical embedding maps

$$\phi_n : \mathcal{L}(\mathbb{R}^{2^n}_{\mathbb{R}}) \rightarrowtail \mathcal{L}(\mathbb{R}^{2^{n+1}}_{\mathbb{R}}), \quad \dim \phi_n X = 2 \cdot \dim X$$

4.3 Interpretation of $*$-ring identities

Frames have played a crucial rôle in the equational theory of modular lattices - due to the fact that the modular lattice freely generated by an n-frame is projective with respect to onto-homomorphisms. The analogous result holds according to Mayet and Roddy [36] for orthogonal n-frames within the variety of relative MOLs. The following is the basis for connecting the equational theories of MOLs and $*$-regular rings.

Lemma 4.11 *There exist ortholattice-polynomials $t(x)$ and $x^{\circledast}$ with constants from $\boldsymbol{a}$ such that for any MOL with spanning orthogonal n-frame ($n \geq 4$ or Arguesian)*

$$(r^*)_{12} = (r_{12})^{\circledast}, \quad \forall x.\ t(x) \in R_{12}, \quad t(r_{12}) = r_{12}$$

Proof. Indeed, $e_1 - e_2 r \perp e_1 r^* \alpha_2 + e_2$ whence $(e_1 r^* \alpha_2 + e_2)R \leq r'_{12}(a_1 + a_2) \in R_{21}$ and equality follows by modularity. Thus

$$(-r^*\alpha_2)_{21} = r'_{12}(a_1 + a_2), \ \ (-\alpha_2)_{21} = a'_{12}(a_1 + a_2)$$

$t(x)$ is provided by the lattice term

$$l(x, x', z_1, \ldots, z_n) = (x' + x(x' + \sum_{j \neq 2} z_j))(x + x' \sum_{j \neq 2} z_j)(z_1 + z_2)$$

Observe that $\hat{x} = (x' + x(xz_2)')(x + (x + z_2)')$ is a complement of x in $[x(xz_2)', x + (x + z_2)']$ whereas $x(xz_2)'$ is a complement of xz_2 in $[0, x]$ and $x + (x + z_2)'$ a complement of $x + z_2$ in $[x, 1]$. Therefore, $\hat{x}$ is a complement of z_2 and $\hat{x}(z_1 + z_2)$ a complement of z_2 in $[0, z_1 + z_2]$. Now, for any given spanning orthogonal frame $\boldsymbol{a}$ one has $l(x, a_1, \ldots, a_n) = \hat{x}(a_1 + a_2)$ and it follows $l(x, x', a_1, \ldots, a_n) \in R_{12}$. $\square$ Combining this with the Mayet-Roddy terms providing the orthogonal frame and the polynomials yielding the structure of the coordinate ring, one obtains the following.

Theorem 4.12 *For every $*$-ring identity α there is an MOL-identity $\hat{\alpha}$ such that for every $*$-ring R associated with a $*$-regular matrix ring R_3, the identity α holds in R if and only if $\hat{\alpha}$ holds in $\overline{L}(R_3)$.*

4.4 Generating frames

The subdirectly irreducible frame generated objects have been determined for $n \geq 4$ (resp. Arguesian) modular lattices ([16]). The analogous task appears intractable for MOLs. The starting point was the construction of a 3-frame generated height 6 MOL by B.Müller.

Let $\boldsymbol{E}^3$ denote the canonical 3-frame of $L = \mathcal{L}((R_2)^3_{R_2})$. The canonical isomorphism between $\mathcal{L}(R^2_R)$ and $\mathcal{L}(R_{2R_2})$ gives rise to an isomorphism of $\mathcal{L}(R^6_R)$ onto L mapping the canonical 6-frame $\boldsymbol{E}^6$ onto $\tilde{\boldsymbol{E}}^6 : \tilde{e}^6_i R_2, (\tilde{e}^6_i - \tilde{e}^6_j)R_2$ where

$$\tilde{e}^6_i = e^3_i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \tilde{e}^6_{i+3} = e^3_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ \text{ for } i = 1, 2, 3$$

Let $\mathbb{Q}$ be the field of rational numbers.

Lemma 4.13 *Let R be a finite dimensional $\mathbb{Q}$-algebra and a, b invertible elements of R such that all $a + (1 - \frac{1}{2^k})b$ are invertible. Let S be generated by a, b under ring operations and inversion (as far as inverses exist) and M be sublattice of L generated by $\boldsymbol{E}^3$ and A_{12}, B_{13} where*

$$A = \begin{pmatrix} a+b & b \\ b & 2b \end{pmatrix}, \ \ B = \begin{pmatrix} b & b \\ b & 2b \end{pmatrix}$$

Then A,B, and all $A + (1 - \frac{1}{2^k})B$ are invertible in R_2, $\tilde{\boldsymbol{E}}^6 \subseteq M$ and $C_{ij} \in M$ for every matrix $C \in S_2$.

Proof.

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}(A+(1-\frac{1}{2^k})B) = \begin{pmatrix} a+(1-\frac{1}{2^{k+1}})b & 0 \\ (2-\frac{1}{2^k})b & 2(2-\frac{1}{2^k})b \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} B = \begin{pmatrix} b & b \\ 0 & b \end{pmatrix}$$

Calculating in M resp. $R(M,\boldsymbol{E}^3)$ we get

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = A - B, \quad \tilde{E}^6_2 = E^3_2 \cap (E^3_1 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}_{12}), \quad \tilde{E}^6_4 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}_{12} \cap E^3_1$$

whence $\tilde{E}^6_i = E_i \cap (\tilde{E}^6_2 + E_{2i})$ and $\tilde{E}^6_{ij} = E_{ij} \cap (\tilde{E}^6_i + \tilde{E}^6_j)$ for $i,j \leq 3$ and, similarly, for $i,j \geq 4$. In particular

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{12} = (\tilde{E}^6_{12} + \tilde{E}^6_4) \cap (E^3_1 + E^3_2) \in R(M,\boldsymbol{E}^3)_{12}$$

Thus we have in $R(M,\boldsymbol{E}^3)$

$$\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } c \in S$$

since

$$\begin{pmatrix} c^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{12} = (\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}_{21} + \tilde{E}^6_4 + \tilde{E}^6_5) \cap (E^3_1 + \tilde{E}^6_2)$$

Now, we get in $R(M,\boldsymbol{E}^3)$

$$\begin{pmatrix} 0 & b \\ b & 2b \end{pmatrix} = B - \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ b & 2b \end{pmatrix}\begin{pmatrix} b^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ b & 2b \end{pmatrix}$$

whence all of S_2. Moreover, we have $\tilde{\boldsymbol{E}}^6 \subseteq M$ from

$$\tilde{E}^6_{15} = (\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_{12} + \tilde{E}^6_4) \cap (\tilde{E}^6_1 + \tilde{E}^6_5) \in M. \qquad \square$$

Define $R(1) = \mathbb{Q}$, $A_1 = B_1 = (1)$ and, inductively,

$$R(k+1) = R(k)_2, \quad A_{k+1} = \begin{pmatrix} A_k + B_k & B_k \\ B_k & 2B_k \end{pmatrix}, \; B_{k+1} = \begin{pmatrix} B_k & B_k \\ B_k & 2B_k \end{pmatrix}.$$

Lemma 4.14 *The lattice $\mathcal{L}(R(k)^3_{R(k)})$ is generated by its canonical 3-frame together with $(A_k)_{12}$ and $(B_k)_{13}$.*

Proof. The case $k = 1$ is well known, cf [6]. Now, in the inductive step $k \rightsquigarrow k+1$ we use 4.13 with $R = R(k)$, $a = A_k$, $b = B_k$. We have $\mathcal{L}(R^3_R)$ embedded via ϕ into L with the canonical 3-frame mapped onto $\tilde{E}^6_i, \tilde{E}^6_{ij}$, $i, j \leq 3$, all contained in M. Also,

$$\phi a_{12} = (\tilde{E}^6_1 + \tilde{E}^6_2) \cap (\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}_{12} + \tilde{E}^6_4), \quad \phi b_{13} = (\tilde{E}^6_1 + \tilde{E}^6_3) \cap (\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}_{13} + \tilde{E}^6_4)$$

belong to M. By the inductive hypothesis, $\mathcal{L}(R^3_R)$ is generated by a_{12}, b_{13} together with the canonical 3-frame. Thus, all of the image belongs to M and so does

$$\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}_{12} = (\phi c_{12} + \tilde{E}^6_4) \cap (E^3_1 + E^3_2), \quad \text{where } c \in R$$

As above, we get $R(M, \boldsymbol{E}^3) = R_2 = R(k+1)$ and it follows $M = L$. □

Proposition 4.15 *For all $n = 2^k$ there is a positive definite symmetric form on the vector space $\mathbb{Q}^{\nVdash\ltimes}$ such that the image of the canonical frame of $\mathcal{L}((\mathbb{Q}_{\ltimes})^{\nVdash}_{\mathbb{Q}_{\ltimes}})$ is a generating orthogonal 3-frame in the MOL $\mathcal{L}(\mathbb{Q}^{\nVdash\ltimes}_{\mathbb{Q}})$.*

Proof. Start with $a, b > 0$ in $\mathbb{Q}$ and consider the above defined A_k, B_k as $2^k \times 2^k$-matrices over $\mathbb{Q}$. Induction and the congruence transformations

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{1}{2}b & 0 \\ 0 & 2b \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} B \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

show that both are positive definite symmetric matrices. Endow $\mathbb{Q}^{\nVdash\ltimes}$ with the form given by the positive definite block matrix

$$\begin{pmatrix} I_k & O & O \\ O & A_k^{-1} & O \\ O & O & B_k^{-1} \end{pmatrix}$$

and $\mathcal{L}(\mathbb{Q}^{\nVdash\ltimes})$ with the induced orthocomplementation. Under the isomorphism $\psi : \mathcal{L}((\mathbb{Q}_{\ltimes})^{\nVdash}_{\mathbb{Q}_{\ltimes}}) \to \mathcal{L}(\mathbb{Q}^{\nVdash\ltimes}_{\mathbb{Q}})$ the image $\tilde{\boldsymbol{E}}^3$ of the canonical 3-frame $\boldsymbol{E}^3$ consists of

$$\tilde{E}^3_i = \sum_{t=1}^{n} e_{t+(i-1)n}\mathbb{Q}, \quad \tilde{E}^{\nVdash}_{\beth\beth} = \sum_{\approx=\nVdash}^{\ltimes} (_{\approx+(\beth-\nVdash)\ltimes} - _{\approx+(\beth-\nVdash)\ltimes})\mathbb{Q}.$$

The $\tilde{E}^3_i$, $i = 1, 2, 3$ are pairwise orthogonal. Moreover, in $R(\mathcal{L}(\mathbb{Q}^{\nVdash\ltimes}), \tilde{\boldsymbol{E}}^{\nVdash})$ we have

$$\psi(A_k)_{12} = \ominus_{12}((\tilde{E}^3_{12})' \cap (\tilde{E}^3_1 + \tilde{E}^3_2)), \quad \psi(B_k)_{13} = \ominus_{13}((\tilde{E}^3_{13})' \cap (\tilde{E}^3_1 + \tilde{E}^3_3)).$$

By 4.14 $\mathcal{L}((\mathbb{Q}_{\ltimes})^{\nVdash}_{\mathbb{Q}_{\ltimes}})$ is generated as a lattice by $\boldsymbol{E}^3$ and $(A_k)_{12}, (B_k)_{13}$. Hence, the MOL $\mathcal{L}(\mathbb{Q}^{\nVdash\ltimes}_{\mathbb{Q}})$ is generated by $\tilde{\boldsymbol{E}}^3$. □ Let L a non-principal ultraproduct of the L_k, $k \geq 1$, and let $\boldsymbol{a}$ correspond to the $\boldsymbol{a}_k$. Then in the sublattice generated by $\boldsymbol{a}$, for any k one has $x_1 > \ldots > x_k$ with the x_i/x_{i+1} pairwise projective. Hence

Corollary 4.16 *There is a subdirectly irreducible MOL of infinite height generated by an orthogonal 3-frame.*

4.5 Word problems

Finally, we consider quasi-identities $\bigwedge_i s_i = t_i \to s = t$ resp. their model classes, called quasi-varieties. Recall that there is a MOL [9] not in the quasi-variety generated by finite dimensional MOLs. The *word problem* for a quasi-variety requires an algorithm dealing with all finite presentations, i.e. a decision procedure for quasi-identities.

Proposition 4.17 *Let Q be any quasi-variety of modular (ortho)lattices containing all $\mathcal{L}(\mathbb{Q}^{\ltimes}_{\mathbb{Q}})$, $n \geq 4$ (with orthogonality given by the identity matrix). Then Q has unsolvable word problem.*

Proof. Let Λ denote the set of all quasi-identities in the language of semigroups. According to Gurevich and Lewis [15] there is no recursive $\Gamma \subseteq \Lambda$ such that Γ contains all ϕ valid in all semigroups but none falsified in some finite semigroup. Associate with each ϕ in Λ a lattice quasi-identity $\hat{\phi}$ expressing that the semigroup variables correspond to elements of the coordinate ring of a 4-frame and translating semigroup relations into lattice relations (cf [18] for a similar translation). Here, 4-frames with a family of elements of the coordinate ring have to be considered as systems of lattice generators and relations (as defined by von Neumann [37]). By the Coordinatization Theorem, the coordinate ring is indeed a ring under the intended operations. Therefore, if Γ is the set of all ϕ with $\hat{\phi}$ valid in Q then Γ contains all ϕ valid in all semigroups. On the other hand, if ϕ is falsified in the finite semigroup S, we represent S as a subsemigroup of some matrix ring $\mathbb{Q}_{\ltimes}$, i.e. a subsemigroup of the coordinate ring of $\mathcal{L}((\mathbb{Q}_{\ltimes})^{\not\trianglerighteq}_{\mathbb{Q}_{\ltimes}})$ with canonical 4-frame. The lattice may be turned into an MOL transferring the canonical orthogonality of $\mathcal{L}(\mathbb{Q}^{\not\trianglerighteq\ltimes}_{\mathbb{Q}})$ via an isomorphism. Thus, $\hat{\phi}$ is falsified in L which means $\phi \notin \Gamma$. Now, assuming that Q has decidable quasi-identities would yield that Γ is recursive, a contradiction. $\square$
In the case of modular lattices, $\mathbb{Q}$ can be replaced by any prime field. The task of finding a particular finite presentation with unsolvable word problem is substantially more demanding. It has been completed for modular lattices with 5 generators by Hutchinson [23] under the same assumption, for MOLs with 3 generators in [40] for each quasi-variety containing all subdirectly irreducible MOLs of height 14.

5 Discussion

The Frink space of an MOL, L, is endowed with a canonical anisotropic orthogonality $\perp$ satisfying all orthoimplications of L according to Lemma 4.1. So, if $\perp$ is a polarity, Prop.3.12 provides a canonical atomic extension within the variety of L. Unfortunately, the direct union of MOLs in the von Neumann example constitutes a counterexample, already.

Problem 5.1 *Characterize the MOLs for which the Frink embedding provides a geometric representation.*

Problem 5.2 *Does every MOL admit an atomic extension?*

Problem 5.3 *Does every proatomic MOL admit an extension within its variety?*

For the last problem, the following concept might be helpful. Call a geometric representation $\eta : L \to M = \mathcal{S}(P)$ *orthogonally separating* if for all $u \perp v$ in the same component of M_{fin} there is some $a \in L$ such that $u \leq \eta(a)$ and $v \leq \eta(a')$. Observe that then L and $\mathcal{S}(L)$ satisfy the same orthoimplications and Prop.3.12 provides an atomic extension in $V(L)$. Also, the class of MOLs admitting such a representation is closed under subdirect products. Thus, a positive answer to the following problem would also imply that for 5.3.

Problem 5.4 *Is the class of MOLs admitting an orthogonally separating representation closed under homomorphic images?*

Actually, the original motivation for this research was the following question partly answered in Roddy [39].

Problem 5.5 *Which MOL varieties, not generated by an MO_κ, do contain a projective plane?*

G.Bruns [8] conjectured that it is true for all varieties. But the answer for proatomic varieties is open as well. Results of [40] suggest that the equational theory of MOLs with suitable bound on the height of irreducible factors should be undecidable.

Problem 5.6 *Is the equational theory of (proatomic) Arguesian MOLs decidable?*

Conjecture 5.7 *The von Neumann example of a continuous geometry admits a geometric representation over an elementary extension of the reals.*

As we have seen, the von Neumann example is proatomic. How far does this extend to abstract continuous geometries - a positive answer could be seen as a kind of construction for these. Recall, that by Kaplansky [29] and Amemiya and Halperin [2] every countably complete MOL is continuous and every continuous MOL is 'finite'.

Problem 5.8 *Is every 'finite' (continuous, countably complete, complete) MOL proatomic? Do such even belong to the quasivariety generated by finite dimensional MOLs?*

Recall, that the quasivariety generated by a class consists of the subalgebras of products of ultraproducts. In a quasivariety generated by modular lattices of finite height, no quotient may be projective to a proper subquotient - a property shared with modular lattices admitting a dimension function. Wehrung [43] calls a lattice *normal* if projective a, b with $ab = 0$ are perspective. Bruns and Roddy [9] provide an atomic MOL which is not normal.

Problem 5.9 *Is normality inherited by sub-MOLs?*

Also, the representing space is of interest. According to Gross [14] p.65 every hermitian vector space of countable dimension admits an orthogonal basis.

Problem 5.10 *Can every Arguesian proatomic MOL be represented by means of spaces having orthogonal bases?*

Concerning coordinatization, one has to ask how far Jónsson's results [27] for complemented modular lattices extend to MOLs. Jónsson constructed an example of a simple coordinatizable lattice with no spanning n-frame ($n \geq 3$) which lead him to consider 'large partial n-frames', $n \geq 3$. He showed that every complemented modular lattice L with such frame ($n \geq 4$ or L Arguesian) is coordinatizable and that every simple L of height ≥ 4 contains such a frame. We suggest the following definition of an *orthogonal large partial n-frame*: For given $m \geq n \geq 3$ it is constituted by orthogonal elements a_i ($1 \leq i \leq m$) such that

$$\sum_{i=1}^{m} a_i = 1, \quad a_i \sim a_1 \text{ for } i \leq n, \quad a_i \sim y_i \leq a_1 \text{ for } n < i \leq m$$

Conjecture 5.11 *The analogues of Jónsson's results hold for MOLs with orthogonal large partial n-frames.*

This would imply that every MOL-variety is generated by members of height ≤ 3 and members of the form $\overline{L}(R)$ with simple R. Yet, even so one might fail to characterize coordinatizability.

Conjecture 5.12 *There are subdirectly irreducible coordinatizable MOLs of height* ≥ 3 *not containing an orthogonal large partial n-frame.*

Problem 5.13 *Can every Arguesian MOL be embedded into the interval* $[0, a_1]$ *of an MOL with orthogonal large partial* 3*-frame?*

Problem 5.14 *Does every* $*$*-regular ring belong to the* $*$*-ring variety generated by* $*$*-rings* R *associated with* $*$*-regular matrix rings* R_3*?*

For $*$-regular rings, the following concept appears to reflect geometric representation of MOLs. A *representation* of $*$-regular ring R is given by a vector space V_D, a ring embedding $\iota : R \to \mathrm{End}(V_D)$, and a $*$-hermitian form Φ on V_D such that $\iota(r^*)$ is the adjoint of $\iota(r)$ for all $r \in R$ The following is due to Kaplansky (cf [21])

Theorem 5.15 *Primitive* $*$*-regular rings with minimal left ideal are representable.*

Characterizing representability in terms of proatomic MOLs could provide a construction of representable rings from artinian $*$-regular rings and shed light on the type I_n and II_1 factors of von Neumann algebras.

Conjecture 5.16 *Every subdirectly irreducible representable $*$-regular ring can be embedded into a homomorphic image of a regular $*$-subring of an ultraproduct of artinian $*$-regular rings.*

Problem 5.17 *Is every $*$-regular ring representable?*

The following two concepts are quite important in the theory of regular rings: A ring is *unit regular* if for every a there is a unit u such that $aua = a$. A ring is *directly finite* if $xy = 1$ always implies $yx = 1$. Observe that every artinian regular ring is unit regular and every unit regular ring is directly finite. Moreover, a regular ring R with n-frame ($n \geq 2$) in $\overline{L}(R)$ is unit regular if and only if perspectivity is transitive in this lattice. The following is due to Handelman (see [13])

Problem 5.18 *Is every $*$ regular ring directly finite or even unit regular?*

Conjecture 5.19 *If R is $*$-regular and $\overline{L}(R)$ proatomic with orthogonal large partial n-frame then R is unit regular. Every representable ring is directly finite - and unit regular, if simple.*

Some positive evidence is given by the following results of Ara and Menal [3] and of Kaplansky [29] and Amemiya and Halperin [2].

Theorem 5.20 *If R is $*$-regular, then $xx^* = 1$ implies $x^*x = 1$. If, in addition, $\overline{L}(R)$ is $\aleph_0$-complete then R is unit regular.*

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