A Note on the "Third Life of Quantum Logic"

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The purpose of this note is to discuss some of the questions raised by

Dunn, J. Michael; Moss, Lawrence S.; Wang, Zhenghan

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Let us first recall some facts about modular ortholattices (MOLs) L of finite height. Page references are to the above mentioned paper.

1) Up to isomorphism, L is the subspace lattice $L(P)$ of a finite dimensional projective space P with an anisotropic polarity (providing the involution on $L)P$).
2) L is isomorphic to a direct product of simple ones. The factors are given by the irreducible components of P with the induced polarity.
3) Simple L of height ≥ 3 are infinite. This is due to Baer.
4) Simple L of height ≥ 4 are Arguesian (since so are the associated projective spaces)
5) Simple Arguesian L of height $d \geq 3$ are up to isomorphism exactly the lattices of linear subspaces of d-dimensional vector spaces over division rings F with involution, endowed with an anisotropic "hermitean" form w.r.t. this involution. This is in essence Birkhoff and von Neumann. Cf. the book of Faure and Froelicher. F and V are determined by L up to "isomorphism". F may be quite far away from the complex number field (p.449), e.g. of prime characteristic.
6) As remarked on p.466 and 450, according to Hans Keller, lattices of closed subspaces of inner product spaces are modular if an only if the space has finite dimension, Though, there are non-modular orthomodular lattices in which all maximal chains are of the same finite cardinality.
7) In a remarkable paper, Roddy has constructed a simple modular ortholattice L_{Rod} of height 14 which interprets an unsolvable word problem for division rings and used this to show that there is a finite ortholattice presentation which has unsolvable word problem in any variety of modular ortholattices which contains L_{Rod}. This implies unsolvability of the equational theory for any variety of n-distributive modular ortholattices (n fixed) containing L_{Rod}.
8) The word problem is unsolvable for any variety of modular ortholattices containing as subreducts the subspace lattices of P^d, $d < \infty$ for a fixed prime field. In particular, the universal Horn theory of the ortholattices $L(\mathbb{C}^d), d < \infty$ is undecidable.
9) It remains an open problem whether the equational theory of the class of all modular ortholattices is decidable (p.452)
10) (p.452) For fixed $d \leq 3$, an axiomatization of the first order theory of $L(\mathbb{C}^d)$ is obtained as follows: There is a single ortholattice equation $\Phi_d(\bar{z})$ such that $\Phi_d(\bar{a})$ holds for a system $\bar{a}$ in $L(\mathbb{C}^d)$ if that is of the form $\mathbb{C}\vec{v}_i(i \leq d) : \mathbb{C}(\vec{v}_i - \vec{v}_j)(i \neq j)$ where the $\vec{v}_1, \dots \vec{v}_d$ form an orthonormal basis. Given such, the field $\mathbb{C}$ with conjugation as involution can be recovered in $L(\mathbb{C}^d)$ a subset $R(\bar{a})$ defined by ortholattice equations and with operations given by ortholattice terms - both with constants from $\bar{a}$. By this interpretation, any formula ψ in the language of fields with involution translates into $\hat{\psi}(\bar{z})$.
As a field with involution, the first order theory of $\mathbb{C}$ can be axiomatized as that an extension by imaginary unit of the real closed formally real field of its hermitean elements (there is an obvious scheme behind this). Axiomatize $L(\mathbb{C}^d)$ by $\forall \bar{z}.\Phi_d(\bar{z}) \Rightarrow \hat{\psi}(\bar{z})$ with ψ from the axiomatization of $\mathbb{C}$, together with $\exists \bar{z}, \Phi_d(\bar{z})$ and the axioms defining "Arguesian ortholattice of height d".
11) (p.452) The first order theory of $L(\mathbb{C}^d)$ cannot be finitely axiomatized - Compactness Theorem would say that the above axioms suffice where one requires zeros only for polynomials of degree $\leq N$, some fixed N. Take a prime $p > N$ and take the closure F of $\mathbb{Q}$ adjoining zeros of irreducible polynomials of degree not a multiple of p. $L(F^d)$ provides a counterexample.
12) Satisfiability of equations in $L(\mathbb{C}^d)$ is NP-complete for $d = 2$: for $d > 3$ it is complete for $\mathcal{BP}(\mathsf{NP}^0_{\mathbb{R}})$: a natural complexity class between NP and PSPACE (cf [Ren92a]). It is also P-time equivalent to validity of Σ_1-sentences.
13) For fixed d, any quantifier free formula is equivalent to an existentially (alternatively: universally) quantified equation $t(\bar{x}) = \mathbf{1}$. In particular, the decision problem for the equational theory of $L(\mathbb{C}^d)$ is in PSPACE and complete for $\mathrm{co}\mathcal{BP}(\mathsf{NP}^0_{\mathbb{R}})$
14) The following are equivalent for any OL equation φ
    a) φ is valid in all $L(\mathbb{C}^d)$
    b) φ is valid in the continuous geometry CG of [Neum36] (John Harding, p.453)
    c) φ is valid in the projection ortholattice of some/any finite type II_1 von Neumann algebra factor ([Her10a]) e.g. of the hyperfinite factor $\mathcal{R}$ (cf [Neum58])
    d) φ is valid in the projection ortholattice of all finite Rickart C^*-algebras ([HeSe14]).

    The decision problem is in PSPACE

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