

EXISTENCE VARIETIES OF REGULAR RINGS AND COMPLEMENTED MODULAR LATTICES

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ABSTRACT. Goodearl, Menal, and Moncasi [11] have shown that free regular rings with unit are residually artinian. We extend this result to the case without unit (in case of rings without unit, ‘artinian’ is used in the sense that the lattice of right ideals is of finite height) and use it to derive that free regular rings as well as free complemented (sectionally complemented) Arguesian lattices are residually finite. Here, quasi-inversion for rings and complementation (sectional complementation, respectively) for lattices are considered as fundamental operations in the appropriate signature. It follows that the equational theory of each of the classes listed above is decidable. The approach is *via* so called existence varieties in ring or lattice signature. Those are classes closed under operators H , S , and P within the class of all regular rings or the class of all sectionally complemented modular lattices. We show that any existence variety in the considered classes is generated by its artinian or finite height members.

Dedicated to the memory of George Hutchinson

1. INTRODUCTION

Dealing with (von Neumann) regular rings (sectionally complemented modular lattices), more precise information is obtained, if the concept of a variety is modified to that of an existence variety; that is, such a class which is closed under homomorphic images, direct products, and substructures which are regular rings (sectionally complemented modular lattices, respectively) themselves. Existence varieties have been studied in the context of regular semigroups, see Hall [12] and Kadourek and M. Szendrei [22].

For regular rings, the most prominent result on existence varieties states that free regular rings are residually artinian, see Goodearl, Menal, and Moncasi [11]. The key fact is that a regular algebra of

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countable dimension over a commutative field F is within the existence variety generated by matrix algebras $F^{n \times n}$, $n < \omega$. This had been shown in Tyukavkin [31] for $*$ -regular algebras and is the basis for the study of (existence) varieties of $*$ -regular rings, cf. Micol [25].

Based on the close relationship between regular rings and their (sectionally complemented Arguesian) lattices of principal right ideals and the above mentioned result of Goodearl, Menal, and Moncasi, we shall prove that any existence variety of regular algebras over a given commutative ring is generated by its artinian members (in case of rings without unit, ‘artinian’ is used in the sense that the lattice of right ideals is of finite height), and that any existence variety of (sectionally) complemented modular lattices is generated by its finite height members. Since free objects in these existence varieties are subdirect products of generators, it follows that they are residually artinian, of finite height, respectively.

Considering the existence variety of all regular rings (with unit), or of (sectionally) complemented Arguesian lattices, the matrix rings over finite prime fields and their lattices of right ideals, respectively, already provide sets of generators. Hence, free objects, with quasi-inverse, or with (sectional) complementation as a fundamental operation, are residually finite and have a solvable word problem. In contrast, the word problem for finite presentations in the ring (or lattice) signature is unsolvable for existence varieties admitting no bound on the height of subdirect irreducibles (see Hutchinson [18]).

The lattice results can be derived from the ring results via coordinatization. The more direct access, used here, is based on Frink’s embedding of a complemented modular lattice into the subspace lattice of a projective space, see Frink [8], and Jónsson’s fine analysis of that embedding, see Jónsson [20, 21].

We also analyse atomic complemented modular lattices, the Frink extension in particular, in terms of existence varieties and deal with the class of sublattices of complemented modular lattices. It is an open problem whether that class is a variety. We can show that it is, at least, closed under formation of ideal lattices.

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2. SECTIONALLY COMPLEMENTED MODULAR LATTICES

Joins and meets in lattices are written as $a + b$ and ab , respectively. If $a, b \in L$, L has a least element 0 , and $ab = 0$, then we write $a \oplus b$ instead of $a + b$. In a lattice with least element 0 and greatest element 1 , the element b is a *complement* of a , if $a \oplus b = 1$. Elements a and b are *perspective* ($a \sim b$ in symbol), if they have a common complement within the interval $[0, a + b]$.

A *complemented modular lattice* (CML, for short) is a modular lattice L with 0 and 1 considered as constant operations, where any element admits a complement. A modular lattice L with least element 0 considered as a constant operation is *sectionally complemented* (SCML, for short), if for any $u \in L$, the interval $[0, u]$ is a complemented lattice. Obviously, any ideal of an SCML is an SCML. Any CML L can be considered an SCML, since for any $a \in L$ and for any $u \in L$ such that $a \leq u$, ub is a complement of a in $[0, u]$, whenever b is a complement of a in L .

For a lattice L , we denote its ideal lattice by $\text{Id } L$. Also, let $\text{ht } L$ denote the *height* of L , so that $\text{ht } L$ equals the supremum of the set $\{|C| - 1 \mid C \text{ is a chain in } L\}$, whenever it is finite; $\text{ht } L = \infty$, whenever the latter supremum is infinite. For a modular lattice L with least element 0 , let L_{fin} denote the sublattice of all elements of finite height in L .

An element a of a lattice L is *neutral*, if for any $x, y \in L$, the sublattice of L generated by a, x , and y is distributive. An ideal I of a lattice L is *neutral*, if it is a neutral element in the ideal lattice $\text{Id } L$. If L is an SCML, then this is equivalent to I being closed under perspectivity, see Birkhoff [1, Chapter V, Theorem 3.2]. Moreover, the map

$$\varphi: \theta \mapsto \{x \in L \mid x\theta 0\}, \quad \theta \in \text{Con } L$$

establishes an isomorphism between the congruence lattice $\text{Con } L$ and the lattice of neutral ideals of L . The following statement is the content of Lemma 1.5 and Lemma 2.2 in Jónsson [21].

Proposition 1. *Let L be a SCML and let $a \in L$. Then the following holds:*

- (i) *The neutral ideal generated by a consists of all finite sums of elements which are perspective to some elements from $\downarrow a$;*
- (ii) *If L is simple and $I \in \text{Id } L$, then I is a simple SCML.*

Proposition 2. *Let L be a subdirectly irreducible SCML and let $I \in \text{Id } L$ be nonzero neutral. Then the following holds:*

- (i) *L 0-embeds into $\text{Id } I$;*

- (ii) Let I be the minimal neutral ideal. If $\text{ht } I < \infty$, then $L = I$ is a simple CML. If $\text{ht } L = \infty$, then I is a simple SCML of infinite height;
- (iii) Any $J \in \text{Id } L$ is a subdirectly irreducible SCML with minimal neutral ideal $I \cap J$, where $I \in \text{Id } L$ is minimal neutral.

Proof. (i) Let $\varphi: L \rightarrow \text{Id } L$, $\varphi: x \mapsto \downarrow x$, be the canonical embedding. Since I is a neutral element in $\text{Id } L$, $\text{Id } L$ is a subdirect product of $\varphi_0(\text{Id } L)$ and $\varphi_1(\text{Id } L)$, where $\varphi_0(J) = I \cap J$ and $\varphi_1(J) = I + J$ for any $J \in \text{Id } L$. As L is subdirectly irreducible, there is $i < 2$ such that $\varphi_i \circ \varphi$ is an embedding. Suppose that $i = 1$. For any $a \in I$, $\varphi_1 \varphi(a) = I$ is the zero element of $\varphi_1(\text{Id } L) = [I, \text{Id } L]$. As $\varphi_i \circ \varphi$ is an embedding, it follows that $I = \{0\}$, a contradiction. Therefore, $i = 0$ and L embeds into the interval $[0, I]$ of $\text{Id } L$ which is $\text{Id } I$.

(ii) Suppose that I has finite height. Then $I = \downarrow u$ for a neutral element $u \in L$. Consider the set $J = \{x \in L \mid ux = 0\}$. Since u is neutral, J is an ideal of L . To prove that this ideal is neutral, let $x \in J$ and $x \sim y$. Then $yu \sim z$ for some $z \leq x$. Neutrality of I implies that $z \in I$, whence $z \leq ux = 0$. Therefore, $yu = 0$ and $y \in J$.

On the other hand, $I \cap J = \{0\}$. Since L is subdirectly irreducible, this yields $J = \{0\}$. Now, consider any $a \geq u$ in L . If x is a complement of u in $[0, a]$, then $ux = 0$, whence $x \in J = \{0\}$. Thus, $x = 0$, $a = u$, and u is the greatest element of L .

(iii) Clearly, $I \cap J$ is a neutral ideal in J . Let $0 \neq a \in J$ and let $b \in I \cap J$. By minimality of I , b belongs to the neutral ideal generated by a in L . By Proposition 1(i), $b = \sum_{i < n} b_i$, where for all $i < n$, there is $x_i \leq a$ such that $b_i \sim x_i$. Since $a \in J \in \text{Id } L$, we conclude that $x_i \in J$, whence $y_i \leq b_i + x_i \in J$ for all $i < n$. This means that b is in the neutral ideal generated by a in J . \square

Let $n < \omega$ be a positive integer, let $\mathbf{a} = \{a_i \mid 0 \leq i < n\}$ and $\mathbf{c} = \{c_{i,j} \mid 0 \leq i, j < n, i \neq j\}$ be sequences of elements of a SCML L such that $c_{i,j} = c_{j,i}$ for all $i, j < n$. We say that the pair $\langle \mathbf{a}, \mathbf{c} \rangle$ is an n -frame, if the following conditions are satisfied:

- (i) $a_j \cdot \sum_{i \neq j} a_i = \prod_{i=0}^{n-1} a_i < \sum_{i=0}^{n-1} a_i$ for all $j < n$;
- (ii) $a_i + a_j = a_i + c_{i,j}$ and $a_i c_{i,j} = \prod_{i=0}^{n-1} a_i$ for all distinct $i, j < n$;
- (iii) $c_{i,j} = (c_{i,p} + c_{p,j})(a_i + a_j)$, for all distinct $i, j, p < n$.

An n -frame $\langle \mathbf{a}, \mathbf{c} \rangle$ is an n -frame at 0, if $\prod_{i=0}^{n-1} a_i = 0_L$. This frame is *spanning* in L , if it is at 0 and $\sum_{i=0}^{n-1} a_i = 1_L$. For a right R -module M_R , let $\mathbb{L}(M_R)$ denote the lattice of right submodules of M_R . The

canonical spanning n -frame in $\mathbb{L}(M_R^n)$ is given by

$$\begin{aligned} a_i &= \{(x_0, \dots, x_{n-1}) \in M_R^n \mid x_j = 0 \text{ for all } j \neq i\} \\ c_{i,j} &= \{(x_0, \dots, x_{n-1}) \in M_R^n \mid x_j = -x_i, x_h = 0 \text{ for all } h \neq i, j\}. \end{aligned}$$

Corollary 3. *A subdirectly irreducible SCML of height at least n contains an n -frame at 0.*

Proof. For any u in the minimal neutral ideal of L with $\text{ht } u \geq n$, Jónsson [21, Theorem 1.7] applies to yield a “large partial n -frame” in $[0, u]$, the $\langle \mathbf{a}, \mathbf{c} \rangle$ part of which forms an n -frame at 0. \square

3. REGULAR ALGEBRAS

Let Λ be a commutative ring which is associative and with unit. An associative Λ -algebra R (with or without unit) is (*von Neumann*) *regular*, if for any $a \in R$, there is $b \in R$ such that $a = aba$. If Λ is the ring of integers, then we get a definition of a regular ring. See von Neumann [27], Goodearl [10], and Skornyakov [30] for basic results about regular rings. In particular, according to [9, Lemma 2], for any $a \in R$, there is an idempotent $e \in R$ such $a = ea = ae$. Thus, if R is a regular Λ -algebra, then any ideal is also a Λ -subalgebra. Moreover, if $e \in R$ is an idempotent, then eRe is a regular Λ -subalgebra of R with unit e .

For any regular ring R , let $\mathbb{L}(R)$ denote the set of principal (right) ideals of R . It is well-known that for any $a \in R$, there is an idempotent $e \in R$ such that $aR = eR$ ($Ra = Re$, respectively). Ordered by inclusion, $\mathbb{L}(R)$ is an SCML sublattice of the lattice of all right ideals of R . Indeed, for any two idempotents $e, f \in R$,

$$eR + fR = (e + g_0)R, \quad eR \cap fR = (f - fg_1)R,$$

where g_0 is an idempotent such that $g_0R = (f - ef)R$ and g_1 is an idempotent such that $Rg_1 = R(f - ef)$. Moreover, $eR \subseteq fR$ for idempotents $e, f \in R$ if and only if $fe = e$, and then $(f - ef)R$ is a complement of eR in $[0, fR]$.

If R is regular and has a unit 1, then $R \in \mathbb{L}(R)$ and $\mathbb{L}(R)$ is a CML. In this case, $(1 - e)R$ is obviously a complement of eR for any idempotent $e \in R$. Moreover, R is artinian if and only if $\mathbb{L}(R)$ is of finite height (and thus has a unit). The analogous results hold for left ideals.

Proposition 4. *Any regular Λ -algebra R is the directed union of its subalgebras of the form eRe , $e = e^2 \in R$. In particular, R embeds into an ultraproduct of those. and so in a regular Λ -algebra with unit.*

Proof. We adapt the proof of Lemma 2 in Fuchs and Halperin [9]. Given $a_0, a_1 \in R$, there is an idempotent $f \in R$ such that $a_i \in fR$ for all $i < 2$; in particular, $a_i = fa_i$ for all $i < 2$. On the other hand, there is an idempotent $g \in R$ such that $Rg = R(a_0 - a_0f) + R(a_1 - a_1f)$. In particular, $g = r_0(a_0 - a_0f) + r_1(a_1 - a_1f)$ for suitable $r_i \in R$, $i < 2$. Straightforward calculation shows that $gf = 0$. Put $e = f + g - fg$. Then $e^2 = e$, $ea_i = efa_i = fa_i = a_i$, and $a_ie = a_if + (a_i - a_if)g = a_if + a_i - a_if = a_i$ since $a_i - a_if \in Rg$. \square

The following statement is due to Jónsson [21, proof of Lemma 8.2] and Wehrung [32, Theorem 4.3], respectively.

Proposition 5. *Let R be a regular ring. Then the following holds:*

- (i) $\mathbb{L}(eRe) \cong [0, eR] \subseteq \mathbb{L}(R)$ for any idempotent $e \in R$;
- (ii) *The map*

$$\varphi: I \mapsto \{J \in \mathbb{L}(R) \mid J \subseteq I\}, \quad I \in \text{Id } R,$$

establishes an isomorphism between the lattice $\text{Id } R$ of two-sided ideals of the regular ring R and the lattice of neutral ideals of the lattice $\mathbb{L}(R)$.

Corollary 6. *For a regular algebra R , the lattice $\mathbb{L}(R)$ is subdirectly irreducible if and only if R is. In that case, eRe is subdirectly irreducible for any idempotent $e \in R$ and eRe is simple, if e is in the minimal ideal of R .*

Proof. This follows with Proposition 2 and Proposition 5. \square

Proposition 7. *Any subdirectly irreducible Λ -algebra is, naturally, an F -algebra, where F is the quotient field of Λ modulo some prime ideal. Such F is unique up to isomorphism.*

Proof. For algebras with unit, the statement was verified within the proof of [11, Proposition 1.5]. Following the idea there, we observe that for any idempotent $e \in R$, the center $Z(eRe)$ of the algebra eRe is a field, cf. [10, Corollary 1.15]. Given an idempotent $e \in R$, we define the homomorphism

$$\varphi_e: \Lambda \rightarrow eRe, \quad \varphi_e: \lambda \mapsto \lambda e,$$

whose image is contained in $Z(eRe)$. Also, for any idempotent $f \in eRe$ we define

$$\psi_f: Z(eRe) \rightarrow Z(fRf), \quad \psi_f: a \mapsto faf.$$

which is an embedding since $Z(eRe)$ is a field. Now, $\psi_f \varphi_e(\lambda) = f\lambda e f = \lambda f e f = \lambda f = \varphi_f(\lambda)$, whence $\ker \varphi_e = \ker \varphi_f$. From Proposition 4 it follows that all φ_e have the same kernel P which is a prime ideal, since

the image is contained in a field. Again in view of Proposition 4, the action of Λ on R has kernel P and induces an action of the quotient field $F = \Lambda/P$ as observed in the proof of [11, Proposition 1.5]: if λ is invertible modulo P and $a \in eRe$, we define $(\lambda + P)^{-1}a = \lambda^{-e}a$, where λ^{-e} is the inverse of λe in $Z(eRe)$. This definition is correct, since $\lambda^{-f} = \lambda^{-e}f$ for any $f \in eRe$. Also, F is isomorphic to the subfield of $Z(eRe)$ generated by the image of φ_e , whence it is unique up to isomorphism. \square

4. EXISTENCE VARIETIES OF REGULAR RINGS AND SCMLs

Let \mathcal{K} be a class of algebraic structures of the same finite similarity type σ . Then $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$, $\mathbf{P}(\mathcal{K})$, $\mathbf{P}_s(\mathcal{K})$, and $\mathbf{P}_u(\mathcal{K})$ denote the class of structures isomorphic to homomorphic images, substructures, direct products, subdirect products, and ultraproducts of structures from \mathcal{K} , respectively. Let also $\mathbf{V}(\mathcal{K})$ denote the variety generated by \mathcal{K} ; due to Birkhoff's theorem, $\mathbf{V}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$.

In the case of lattices, $\sigma = \{\cdot, +, 0\}$; in the case of rings, σ is just the ring signature. In the case of an Λ -algebra R , the elements of Λ are considered as unary operations on R . Additional operations of interest are the constant 1 for top element in lattices and for unit in algebras, as well as an involution operation for algebras. Let Σ consist either of the axioms defining regular Λ -algebras or of those defining SCMLs. In particular, we include the following axiom α

$$\begin{aligned} \forall x \exists y \, xyx = x & \quad (\text{for rings and algebras}); \\ \forall x \forall y \exists z \, (xyz = 0) \& (xy + z = x) & \quad (\text{for lattices}). \end{aligned}$$

CMLs are considered as SCMLs with 1.

Observe that the model class $\text{Mod } \Sigma$ is closed under \mathbf{H} and \mathbf{P} . Define the following operators:

$$\mathbf{S}_{\exists}(\mathcal{K}) = \text{Mod } \Sigma \cap \mathbf{S}(\mathcal{K}), \quad \mathbf{P}_{s\exists}(\mathcal{K}) = \text{Mod } \Sigma \cap \mathbf{P}_s(\mathcal{K})$$

A class \mathcal{K} is an *existence variety*, shortly \exists -*variety*, if it is closed under \mathbf{H} , \mathbf{P} , and \mathbf{S}_{\exists} . In particular, \mathcal{K} is closed under \mathbf{P}_u and elementary substructures, whence it is an axiomatizable class. Due to the above observations and the definition of an existence variety, $\text{Mod } \Sigma$ is an \exists -variety.

For rings and Λ -algebras, let $\sigma^* = \sigma \cup \{\prime\}$, where \prime is a unary operation symbol standing for the quasi-inverse operation. For lattices, let $\sigma^* = \sigma \cup \{\backslash\}$, where \backslash is a binary operation symbol standing for sectional

complementation. Then the above axiom α translates into identity α^* :

$$\begin{aligned} \forall x \, xx'x = x & \quad (\text{for rings and algebras}); \\ \forall x \forall y \, (xy(x \setminus y) = 0) \& (xy + (x \setminus y) = x) & \quad (\text{for lattices}). \end{aligned}$$

If $\langle A, \sigma \rangle \in \text{Mod } \Sigma$, then $\langle A, \sigma^* \rangle$ is a *companion* of $\langle A, \sigma \rangle$, if $\langle A, \sigma^* \rangle \models \alpha^*$; this concept is due to Thoralf Skolem. Then for any class $\mathcal{K} \subseteq \text{Mod } \Sigma$, the *companion* $\mathbb{T}(\mathcal{K})$ of \mathcal{K} consists of all companions $\langle A, \sigma^* \rangle$, where $\langle A, \sigma \rangle \in \mathcal{K}$. The following lemma is straightforward to prove.

Lemma 8. *For any $\mathcal{K} \subseteq \text{Mod } \Sigma$, the following statements hold:*

$$\text{ST}(\mathcal{K}) = \text{TS}_{\exists}(\mathcal{K}), \quad \text{PT}(\mathcal{K}) \subseteq \text{TP}(\mathcal{K}), \quad \text{HT}(\mathcal{K}) \subseteq \text{TH}(\mathcal{K}).$$

In particular, if \mathcal{K} is an \exists -variety, then $\mathbb{T}(\mathcal{K})$ is a variety in the extended signature. In this case, for any set X , by $F_{\mathbb{T}(\mathcal{K})}(X)$ we denote an algebraic structure freely generated in $\mathbb{T}(\mathcal{K})$ by X and call it *\mathbb{T} -freely generated in \mathcal{K} by X* . So far, we have just reviewed what can be said about existence varieties relative to axioms which are quantified conjuncts of identities in general. It is in the following lemma that we have to deal with our particular cases of regular rings, algebras and SCMLs.

Lemma 9. *For any $\mathcal{K} \subseteq \text{Mod } \Sigma$, the following holds: $\text{TH}(\mathcal{K}) = \text{HT}(\mathcal{K})$, $\text{S}_{\exists}\text{H}(\mathcal{K}) \subseteq \text{HS}_{\exists}(\mathcal{K})$, $\text{TS}_{\exists}\text{H}(\mathcal{K}) \subseteq \text{HST}(\mathcal{K})$, $\text{TS}_{\exists}\text{P}(\mathcal{K}) = \text{SPT}(\mathcal{K})$.*

Proof. Let $A \in \mathcal{K}$ and let $\varphi: A \twoheadrightarrow B$ be a surjective homomorphism. Let B^* be a companion of B in $\mathbb{T}(\mathcal{K})$. We prove that there exists a companion A^* of A in $\mathbb{T}(\mathcal{K})$ such that $\varphi: A^* \rightarrow B^*$ is a homomorphism.

For regular algebras, the proof of this statement essentially goes as in [11, Lemma 1.4]. Indeed, in the case of regular algebras, the two-sided ideal $I = \ker \varphi$ is regular. Let $a \in A$ and let $b = \varphi(a)'$. There is $y \in A$ such that $\varphi(y) = b$. Then $a - aya \in I$. Since I is regular, there is $u \in I$ such that $(a - aya)u(a - aya) = a - aya$. It follows from the latter that $aua - ayaua - auaya + ayauaya + aya = a$. Taking $x = u - uay - yau + yauay + y$, we get $axa = aua - auaya - ayaua + ayauaya + aya = a$ and $x - y = u - uay - yau + yauay \in I$, whence $\varphi(x) = b$. So, we may put $a' = x$ in A^* .

We prove now the statement for SCMLs. Let $I = \ker \varphi$. Given $a \leq u$ in A and a complement $b + I$ of $a + I$ in $[I, u + I] \subseteq A/I$, we must find a complement a' of a in $[0, u] \subseteq A$ such that $a' + I = b + I$. After replacing b by bu , we may assume that $b \leq u$. Since $a + b + I = u + I$ and $a + b \leq u$, there is $c \in I$ such that $a + b + c = u$. Let a' be a complement of $a(b + c)$ in $[0, b + c]$. Then $a + a' = a + a(b + c) + a' = a + b + c = u$ and $aa' = a(b + c)a' = 0$, whence a' is a complement of a in $[0, u]$.

Moreover, since $b + c + I = b + I$, we get using neutrality of I that $a(b + c) + I = (a + I)(b + c + I) = (a + I)(b + I) = I$. This implies $a' + I = a' + a(b + c) + I = b + c + I = b + I$, as desired.

This proves that $\text{TH}(\mathcal{K}) \subseteq \text{HT}(\mathcal{K})$. The reverse inclusion follows from Lemma 8. Hence, $\text{TH}(\mathcal{K}) = \text{HT}(\mathcal{K})$.

To prove the second inclusion, suppose that $C \in \mathcal{S}_{\exists}(B)$, where A and B are as above. If $C^* \in \mathcal{T}(C)$, then by Lemma 8, there is $B^* \in \mathcal{T}(B)$ such that $C^* \in \mathcal{S}(B^*)$. The statement we have proved above implies that there is $A^* \in \mathcal{T}(A)$ such that $B^* \in \mathcal{H}(A^*)$. Hence $C^* \in \mathcal{SHT}(A) \subseteq \mathcal{HST}(A) = \mathcal{HTS}_{\exists}(A) = \mathcal{THS}_{\exists}(A)$, whence $C \in \mathcal{HS}_{\exists}(A)$. This shows that $\mathcal{S}_{\exists}\mathcal{H}(\mathcal{K}) \subseteq \mathcal{HS}_{\exists}(\mathcal{K})$.

To prove the next inclusion, we note that by the above and by Lemma 8, $\mathcal{TS}_{\exists}\mathcal{H}(\mathcal{K}) \subseteq \mathcal{THS}_{\exists}(\mathcal{K}) = \mathcal{HTS}_{\exists}(\mathcal{K}) = \mathcal{HST}(\mathcal{K})$.

To prove the last statement, we note that by Lemma 8, $\mathcal{SPT}(\mathcal{K}) \subseteq \mathcal{STP}(\mathcal{K}) = \mathcal{TS}_{\exists}\mathcal{P}(\mathcal{K})$. For the reverse inclusion, let $A \in \mathcal{TS}_{\exists}\mathcal{P}(\mathcal{K})$ and let A_{σ} be its σ -reduct. Then $A_{\sigma} \in \mathcal{S}_{\exists}\mathcal{P}(\mathcal{K})$. This means that $A_{\sigma} \models \Sigma$ and there is a set I and there are $B_i \in \mathcal{K}$, $i \in I$, such that A_{σ} embeds into $\prod_{i \in I} B_i$. Let π denote the corresponding embedding and let π_i denote the i th projection from $\prod_{i \in I} B_i$ onto B_i , $i \in I$. For any $i \in I$, the homomorphism $\pi_i \circ \pi: A_{\sigma} \rightarrow \pi_i \circ \pi(A_{\sigma})$ gives rise, in a natural way, to a homomorphism $\rho_i: A \rightarrow \pi_i \circ \pi(A_{\sigma})^*$. Since $\pi_i \circ \pi(A_{\sigma}) \in \mathcal{S}(B_i)$ and $\pi_i \circ \pi(A_{\sigma}) \models \Sigma$, we conclude that $\pi_i \circ \pi(A_{\sigma}) \in \mathcal{S}_{\exists}(B_i)$, whence $\pi_i \circ \pi(A_{\sigma})^* \in \mathcal{TS}_{\exists}(B_i) \subseteq \mathcal{TS}_{\exists}(\mathcal{K}) = \mathcal{ST}(\mathcal{K})$. Moreover, the map $\rho: A \rightarrow \prod_{i \in I} \pi_i \circ \pi(A_{\sigma})^*$, $\rho: a \mapsto \langle \rho_i(a) \mid i \in I \rangle$ is a σ^* -embedding, whence $A \in \mathcal{SPST}(\mathcal{K}) = \mathcal{SPT}(\mathcal{K})$. Therefore, $\mathcal{TS}_{\exists}\mathcal{P}(\mathcal{K}) \subseteq \mathcal{SPT}(\mathcal{K})$. \square

Proposition 10. *Let $\mathcal{K} \subseteq \text{Mod } \Sigma$.*

- (i) *The class $\mathcal{V}_{\exists}(\mathcal{K}) = \mathcal{HS}_{\exists}\mathcal{P}(\mathcal{K})$ is the smallest \exists -variety containing \mathcal{K} , and $\mathcal{TV}_{\exists}(\mathcal{K}) = \mathcal{VT}(\mathcal{K})$.*
- (ii) *The σ -reduct of any \mathcal{T} -free algebraic structure in $\mathcal{V}_{\exists}(\mathcal{K})$ belongs to $\mathcal{P}_{s\exists}(\mathcal{K})$ – the corresponding subdirect decomposition giving rise to one in $\mathcal{TV}_{\exists}(\mathcal{K})$, too.*
- (iii) *Any subdirectly irreducible structure from $\mathcal{V}_{\exists}(\mathcal{K})$ belongs to the class $\mathcal{HS}_{\exists}\mathcal{P}_u(\mathcal{K})$.*
- (iv) *Any \exists -variety is generated by its finitely generated subdirectly irreducible members.*

Proof. (i) By Lemma 9, the class $\mathcal{HS}_{\exists}\mathcal{P}(\mathcal{K})$ is closed under \mathcal{S}_{\exists} . Since $\mathcal{HS}_{\exists}\mathcal{P}(\mathcal{K})$ is obviously closed under \mathcal{H} and \mathcal{P} , it forms an \exists -variety, the smallest one containing \mathcal{K} . Furthermore, by Lemma 9, $\mathcal{TV}_{\exists}(\mathcal{K}) = \mathcal{THS}_{\exists}\mathcal{P}(\mathcal{K}) = \mathcal{HTS}_{\exists}\mathcal{P}(\mathcal{K}) = \mathcal{HSPT}(\mathcal{K}) = \mathcal{VT}(\mathcal{K})$.

(ii) Since any free algebraic structure in the variety generated by $\mathcal{T}(\mathcal{K})$ belongs to $\mathcal{P}_s\mathcal{T}(\mathcal{K})$, its σ -reduct belongs to $\mathcal{P}_{s\exists}(\mathcal{K})$.

(iii) Let $A \in \mathbf{H}(B)$ be subdirectly irreducible, where $B \in \mathbf{S}_\exists \mathbf{P}(\mathcal{K})$. Both SCMLs and regular rings are congruence distributive. Hence Jónsson's Lemma implies the existence of $C \in \mathbf{SP}_u(\mathcal{K})$ such that $A \in \mathbf{H}(C)$ and $C \in \mathbf{H}(B)$. The last relation implies that $C \in \mathbf{S}_\exists \mathbf{P}_u(\mathcal{K})$.

(iv) Let \mathcal{K} be an \exists -variety, let \mathcal{K}_{SI}^ω denote the class of its finitely generated subdirectly irreducible members, and let $\mathbf{T}(\mathcal{K})_{SI}^\omega$ denote the class of finitely generated subdirectly irreducible members of the variety $\mathbf{T}(\mathcal{K})$. Then the class of σ -reducts of structures from $\mathbf{T}(\mathcal{K})_{SI}^\omega$ is a subclass of \mathcal{K}_{SI}^ω . Given $A \in \mathcal{K}$, let A^* be a companion of A in $\mathbf{T}(\mathcal{K})$. According to Mal'cev [23], A^* embeds into an ultraproduct B^* of finitely generated substructures of A^* . All those belong to $\mathbf{SP}(\mathbf{T}(\mathcal{K})_{SI}^\omega)$ by Birkhoff's Theorem. Therefore, $A^* \in \mathbf{SP}_u \mathbf{SP}(\mathbf{T}(\mathcal{K})_{SI}^\omega)$. When passing to σ -reducts, we get $A \in \mathbf{S}_\exists \mathbf{P}_u \mathbf{S}_\exists \mathbf{P}(\mathcal{K}_{SI}^\omega) \subseteq \mathbf{V}_\exists(\mathcal{K}_{SI}^\omega)$. \square

Proposition 11. *The following statements hold:*

- (i) *Any SCML L embeds into a CML which is the ultraproduct of intervals $[0, u]$, $u \in L$;*
- (ii) *If L is a CML and $u \in L$, then $[0, u] \in \mathbf{HS}_\exists(L)$;*
- (iii) *If R is a regular Λ -algebra with unit and $e^2 = e \in R$, then $eRe \in \mathbf{HS}_\exists(R)$.*

Consequently, if $\mathcal{K} \subseteq \mathbf{Mod} \Sigma$ is a class of CMLs and L is a CML such that $L \in \mathbf{V}_\exists(\mathcal{K})$ in the sense of SCMLs, then $L \in \mathbf{V}_\exists(\mathcal{K})$ also in the sense of CMLs. The same statement holds, if we consider regular rings with unit instead of CMLs.

Proof. The statement (i) is obvious. To prove (ii), let v be a complement of u in L . Then $C = [0, u] \cup [v, 1]$ is a complemented sublattice of L , and $\varphi: C \rightarrow [0, u]$, $\varphi: x \mapsto ux$, is a homomorphism from C onto $[0, u]$, that is, $[0, u] \in \mathbf{HS}_\exists(L)$.

(iii) $eRe + (1 - e)R(1 - e)$ is a regular subalgebra of R with unit having eRe as a homomorphic image. \square

5. FREE REGULAR ALGEBRAS

For the case of algebras with unit, the following is the main result of Goodearl, Menal, and Moncasi [11, Theorem 2.5].

Theorem 12. *For any commutative ring Λ , the \exists -variety of regular Λ -algebras (with or without unit) is generated by the Λ -algebras $F^{n \times n}$, $n < \omega$, where F ranges over quotient fields of Λ modulo prime ideals. In particular, free regular Λ -algebras are residually artinian.*

Proof. Due to Proposition 4, every \exists -variety of regular Λ -algebras is generated by its members with unit. To those, [11, Theorem 2.5] applies

to yield the desired conclusion. The claim about free objects follows with Proposition 10. \square

Corollary 13. *Let R be a regular F -algebra over a commutative field F . Then $R \in \mathbf{V}_{\exists}(F^{n \times n} \mid n_0 \leq n < \omega)$ for all $n_0 < \omega$.*

The crucial step in the proof of the Theorem is the instance of the Corollary where R has countable dimension over F , see [11, Proposition 2.2] and [31]. For countable F , the Theorem can be derived from this using Proposition 10(iv). The general case would require to consider 2-sorted \exists -varieties. Actually, the proof of [11, Theorem 2.4] is based on a 2-sorted approach.

Let \mathbb{P} consist of all primes and 0 and let \mathbb{F}_p denote the prime field $\mathbb{Z}/p\mathbb{Z}$ for $p \in \mathbb{P} \setminus \{0\}$; we also put $\mathbb{F}_p = \mathbb{Q}$, if $p = 0$. Let \mathcal{R} denote the \exists -variety of all regular rings (with or without unit)

Corollary 14. *For any $n_0 < \omega$, $\mathcal{R} = \mathbf{V}_{\exists}(\mathbb{F}_p^{n \times n} \mid n_0 \leq n < \omega, p \in \mathbb{P} \setminus \{0\})$. In particular, free regular rings (with or without unit) are residually finite, and the equational theory of regular rings with quasi-inversion, considered as a fundamental operation, is decidable.*

Proof. Taking $\Lambda = \mathbb{Z}$ and applying Theorem 12, one sees that any regular ring (with or without unit) belongs to $\mathbf{V}_{\exists}(\mathbb{F}_p^{n \times n} \mid n_0 \leq n < \omega, p \in \mathbb{P})$. Moreover, $\mathbb{Q}^{n \times n}$ embeds into the ultraproduct of $\mathbb{F}_p^{n \times n}$, $p \in \mathbb{P} \setminus \{0\}$, over a non-principal ultrafilter on $\mathbb{P} \setminus \{0\}$. Indeed, such an ultraproduct is an algebra over a field of characteristic 0 and contains the set of $(n \times n)$ -matrix units. Thus, the first statement follows.

Due to J. C. C. McKinsey [24], residual finiteness implies decidability of the equational theory. \square

Corollary 15. *Let F be a field. The \exists -variety of regular F -algebras is generated by its members of the form $F^{n \times n}$, $n < \omega$. If F is recursive, then the equational theory of regular F -algebras with quasi-inversion is decidable. If F is finite, then free regular F -algebras are residually finite.*

Proof. The first claim is immediate by Corollary 13. If F is recursive, then the theory of regular F -algebras is recursively axiomatizable. On the other hand, in this case, the set of all equations falsified in some $F^{n \times n}$ (with a quasi-inversion) is recursively enumerable, since any equation is equivalent to a universal sentence in the signature of F -algebras. \square

6. ARTINIAN GENERATORS OF \exists -VARIETIES

Theorem 16. *Let R be a subdirectly irreducible non-artinian regular Λ -algebra. Then there exists a field F such that $\mathbf{V}_{\exists}(R) = \mathbf{V}_{\exists}(F^{n \times n} \mid n_0 \leq n < \omega)$ for all $n_0 < \omega$.*

Proof. By Proposition 7, R is an F -algebra for a quotient field F of Λ . By Corollary 13, $R \in \mathbf{V}_{\exists}(F^{n \times n} \mid n_0 \leq n < \omega)$. Further, by Corollary 6, the lattice $\mathbb{L}(R)$ is subdirectly irreducible and of infinite height. By Corollary 3, $\mathbb{L}(R)$ has an n -frame at 0 for any $0 < n < \omega$. Fixing $0 < n < \omega$, there is an idempotent $e \in R$ such that the lattice $\mathbb{L}(eR)$ contains a spanning n -frame. By Proposition 5(i), $\mathbb{L}(eRe) \cong \mathbb{L}(eR)$. Putting $S = eRe$, we conclude that $\mathbb{L}(S)$ contains a spanning n -frame. We prove that $F^{n \times n} \in \mathbf{S}(S)$, which implies $F^{n \times n} \in \mathbf{S}_{\exists}(R)$.

Indeed, since $\mathbb{L}(S)$ contains a spanning n -frame, there are nonzero right ideals S_i , $i < n$, in $\mathbb{L}(S)$ which are independent and pairwise perspective (thus, isomorphic) such that $S = \bigoplus_{i < n} S_i$. Now

$$\text{End}(S_S) = \text{End}\left(\bigoplus_{i < n} S_i\right) \cong \left(\text{End}(S_0)\right)^{n \times n},$$

cf. [3, Corollary 2.20]. Associating with every $\lambda \in F$ the left multiplication $x \mapsto \lambda x$, one obtains an embedding of F into $\text{End}(S_0)$. Thus, $F^{n \times n}$ embeds into $\text{End}(S_S) \cong S$. \square

Corollary 17. *Any \exists -variety \mathcal{K} of regular Λ -algebras (with or without unit) is generated by its artinian members. The \top -freely generated algebras in \mathcal{K} are subdirect products of artinian members in $\top(\mathcal{K})$,*

Proof. By Theorem 16, \mathcal{K} is generated by its artinian members. The claim about free objects follows by Proposition 10(ii). \square

Corollary 18. *If R is a subdirectly irreducible regular Λ -algebra and I is the minimal ideal of R , then $R \in \mathbf{V}_{\exists}(I)$. More precisely, $R \in \mathbf{V}_{\exists}(eRe \mid e^2 = e \in I)$.*

Proof. By Proposition 5(ii), $[0, I]$ is a minimal neutral ideal in $\mathbb{L}(R)$. By Proposition 2(ii), either $I = R$ or $\mathbb{L}(I)$ is a simple SCML of infinite height. In the latter case, applying Theorem 16 to I , we get that $F^{n \times n} \in \mathbf{V}_{\exists}(I)$ for all $0 < n < \omega$. Again by Theorem 16, $R \in \mathbf{V}_{\exists}(I)$. \square

7. PROJECTIVE SPACES AND FRINK'S EMBEDDING

For what follows, we also refer to [4, 5, 19, 20].

An *atom* in a lattice L with zero 0 is an element $p > 0$ such that $p > x > 0$ holds for no $x \in L$. By P_L , we denote the set of all atoms in L . We say that L is *atomic*, if for any $a > 0$, there is $p \in P_L$ such that

$p \leq a$. We say that L is *upper continuous*, if for any upward directed set $X \subseteq L$ such that $\sum X$ exists and for any $a \in L$, $a \cdot \sum X = \sum \{a \cdot x \mid x \in X\}$.

A *projective space* is defined by a set P of *points* together with a distinguished set Δ of 3-element subsets of P called *collinear triplets*, such that the *triangle axiom* holds:

if $\{p, s, q\}, \{q, t, r\} \in \Delta$, and $\{p, q, r\} \notin \Delta$,

then there is a unique $u \in P$ such that $\{p, u, r\}, \{s, u, t\} \in \Delta$.

$Q \subseteq P$ is a *subspace* of P , if $p, q \in Q$ and $\{p, q, r\} \in \Delta$ imply $r \in Q$. The set of all subspaces of P forms an atomic upper continuous CML under inclusion, which we denote by $\mathbb{L}(P)$.

We say that P is *irreducible*, if for any $p, q \in P$, $p \neq q$, there is $r \in P$ such that $\{p, q, r\} \in \Delta$. Each projective space is (uniquely) represented as a disjoint union of its irreducible subspaces, so called *components*. If $\{P_i \subseteq P \mid i \in I\}$ is a partition of P into irreducible components, then the maps $X \mapsto X \cap P_i$ provide a direct decomposition of $\mathbb{L}(P)$; that is, $\mathbb{L}(P) \cong \prod_{i \in I} \mathbb{L}(P_i)$. We call the $\mathbb{L}(P_i)$ the *components* of $\mathbb{L}(P)$. Then $\mathbb{L}(P)$ is subdirectly irreducible if and only if P is irreducible.

For a modular lattice M with 0 , P_M is a projective space, where a triplet $\{p, q, r\}$ is collinear if and only if $p + q = p + r = q + r$. If M is a CML of finite height, then $M \cong \mathbb{L}(P_M)$. If M is atomic, then $M_{fin} \cong \mathbb{L}(P_M)_{fin}$.

Let D be a division ring. For any (right) vector space V_D over D , we denote the lattice of all vector subspaces of V_D by $\mathbb{L}(V_D)$. Then $M = \mathbb{L}(V_D)$ is a subdirectly irreducible CML which also satisfies the Arguesian identity, see Jónsson [20]. Moreover, $\mathbb{L}(P_M) \cong M$. Conversely, if P is irreducible and $\text{ht } \mathbb{L}(P) = n \geq 4$ (or $n \geq 3$ and $\mathbb{L}(P)$ is Arguesian), then $\mathbb{L}(P) \cong \mathbb{L}(V_D)$ for some division ring D and some vector space V_D over D such that $\dim V_D = n$.

According to Sachs [29], for any lattice L , its ideal lattice $\text{Id } L$, as well as its filter lattice $\text{Fil } L$ (ordered by inverse inclusion), belongs to $\mathcal{V}(L)$. The following result is well known, cf. Freese [7].

Lemma 19. *Let $\varphi: L \rightarrow N$ be a lattice homomorphism and let N be an upper continuous lattice. Then φ extends to a homomorphism $\bar{\varphi}: \text{Id } L \rightarrow N$ by setting $\bar{\varphi}(I) = \sup \varphi(I)$.*

The following result is contained in Frink [8] implicitly, cf. Herrmann and Roddy [17].

Lemma 20. *Let L be an SCML, let M be a modular lattice, and let $\varepsilon: L \rightarrow M$ be a 0-preserving homomorphism. Then*

$$\varphi(a) = \{p \in P_M \mid p \leq \varepsilon(a)\}$$

defines a 0-preserving lattice homomorphism from L into $\mathbb{L}(P_M)$. Moreover, φ is an embedding provided that for any $a > 0$ in L , there is $p \in P_M$ with $p \leq \varepsilon(a)$.

Thus by Lemma 20, for any SCML L , the canonical embedding, $a \mapsto \uparrow a$, of L into the filter lattice $\text{Fil } L$ (ordered by reverse inclusion) defines the embedding

$$\varphi_{\text{Fr}}: L \rightarrow \mathbb{L}(P_{\text{Fil } L}), \quad a \mapsto \{F \in P_{\text{Fil } L} \mid a \in F\}.$$

Putting $\text{Fr } L = \mathbb{L}(P_{\text{Fil } L})$, we call the pair $\langle \text{Fr } L, \varphi_{\text{Fr}} \rangle$ the *Frink extension* of L .

Theorem 21. *Let L be an SCML. Then the following holds:*

- (i) $\text{Fr } L \in \mathbf{V}(L)$;
- (ii) $L \in \mathbf{S}_{\exists}(\text{Fr } L)$ and $\varphi_{\text{Fr}}: L \rightarrow \text{Fr } L$ is the corresponding 0-lattice embedding;
- (iii) $M \in \mathbf{HS}_{\exists}(L)$ for any finite height component M of $\text{Fr } L$.

Proof. Most of this is due to Jónsson [20]. By Lemma 19, $\text{Fr}(L) \in \mathbf{H}(\text{Id Fil } L)$, whence $\text{Fr } L \in \mathbf{V}(L)$, and (i) holds. By Lemma 20, L embeds into $\text{Fr}(L)$, whence (ii) holds.

As to (iii), suppose now that $M = \mathbb{L}(Q)$ is a finite height component of $\text{Fr } L$. Recalling that the elements of Q are given as maximal filters of L , let $F = \bigcap Q \in \text{Fil } L$ and let $u \in F$. Of course, $Q = \pi \varphi_{\text{Fr}}(u)$, where π is the projection of $\text{Fr } L$ onto M . For each lower cover Q' of Q in M , there is $a \in \bigcap Q' \setminus \bigcap Q$ such that $a \leq u$. As $\pi \varphi_{\text{Fr}}(a) = Q'$, $\pi \circ \varphi_{\text{Fr}}$ maps the interval $[0, u]$ of L onto M . Since $[0, u] \in \mathbf{S}_{\exists}(L)$ for any $u \in L$, we get that $M \in \mathbf{HS}_{\exists}(L)$. \square

Corollary 22. *Every subdirectly irreducible SCML L with $\text{ht } L \geq 4$ is Arguesian. For any subdirectly irreducible Arguesian SCML L with $\text{ht } L \geq 3$, there are a division ring D and a vector space V_D over D such that $L \in \mathbf{S}_{\exists}(\mathbb{L}(V_D))$ and $\mathbb{L}(V_D) \in \mathbf{V}(L)$.*

Proof. Since L is subdirectly irreducible, it embeds into some component M of $\text{Fr } L$ by Lemma 20. In particular, $\text{ht } M \geq \text{ht } L$. Thus, if $\text{ht } L \geq 4$, then $M \cong \mathbb{L}(V_D)$ for some vector space V_D over a division ring D , whence both M and L are Arguesian. Further, if L is Arguesian and $\text{ht } L \geq 3$, then the same holds for M and, again, $M \cong \mathbb{L}(V_D)$. The rest follows from Theorem 21(i)-(ii). \square

8. FRAMES IN SCMLS

Lemma 23. *Frames at 0 are projective configurations within the class of SCMLs.*

Proof. Let L and M be SCML, let $f: L \rightarrow M$ be a homomorphism of SCMLs, and let $\langle \mathbf{a}, \mathbf{c} \rangle$ be an n -frame at 0 in M . Since n -frames are projective configurations within the class of modular lattices, there is an n -frame $\langle \mathbf{u}, \mathbf{v} \rangle$ in M such that $\langle f(\mathbf{u}), f(\mathbf{v}) \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$. Let $u = \prod_{i=0}^{n-1} u_i$, let $u' = \sum_{i=0}^{n-1} u_i$, and let w be a complement of u in the interval $[0, u']$. Then the pair $\langle \mathbf{r}, \mathbf{s} \rangle$, where $r_i = wu_i$ and $s_{i,j} = wv_{i,j}$ for all $i, j < n$, is an n -frame at 0 in L , and $\langle f(\mathbf{s}), f(\mathbf{r}) \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$. \square

We cite several results on frames in subspace lattices, which are due to Herrmann and Huhn, see [14] and [16].

Proposition 24. [14] *Let D be a division ring with prime subfield P , let V_D be a vector space over D , and let $n \geq 3$. Then every n -frame in the lattice $\mathbb{L}(V_D)$ generates a 0-sublattice isomorphic to $\mathbb{L}(P_P^n)$.*

If $p \in \mathbb{P}$ and $n < \omega$, we put $\mathbf{F}_p^n = (\mathbb{F}_p)_{\mathbb{F}_p}^n$. We say that a modular lattice L has *characteristic p* , if all proper 3-frames in L generate a sublattice isomorphic to $\mathbb{L}(\mathbf{F}_p^3)$. This property can be expressed by one identity, if $p \neq 0$, and by infinitely many identities, if $p = 0$, see [15]. Therefore, Corollary 22 implies

Corollary 25. *Every subdirectly irreducible Arguesian SCML L with $\text{ht } L \geq 3$ has a uniquely determined characteristic p . If $L \in \mathbf{S}(\mathbb{L}(V_D))$ then p is the characteristic of D .*

Proposition 26. [14] *Let $\forall \mathcal{V}$ be a lattice variety generated by SCMLs and let $n \geq 3$. If $L \in \forall \mathcal{V}$ is generated by an n -frame and if either $n \geq 4$ or L Arguesian, then either $L \cong \prod_{p \in I} \mathbb{L}(\mathbf{F}_p^n)$ for some finite $I \subseteq \mathbb{P}$ or $\{\mathbb{L}(\mathbf{F}_p^n) \mid p \in I\} \subseteq \mathbf{H}(L)$ for some infinite $I \subseteq \mathbb{P}$.*

Proposition 27. [16] *Let D be a division ring with prime subfield P , let V_D be an infinite dimensional vector space over D , and let V_P be the induced P -vector space. Then for any $n_0 < \omega$,*

$$\begin{aligned} \{\mathbb{L}(P_P^n) \mid n < \omega\} &\subseteq \mathbf{S}_{\exists}(\mathbb{L}(V_D)) \subseteq \mathbf{S}_{\exists}(\mathbb{L}(V_P)); \\ \mathbf{V}(\mathbb{L}(V_D)) &= \mathbf{V}\{\mathbb{L}(P_P^n) \mid n_0 \leq n < \omega\}. \end{aligned}$$

Lemma 28. *Let L be an SCML, let $n \geq 4$, and let P be a prime field. If $\mathbb{L}(P_P^n) \in \mathbf{V}(L)$, then $\mathbb{L}(P_P^n) \in \mathbf{S}_{\exists} \mathbf{P}_u(L)$.*

Proof. By assumption, $\mathbb{L}(P_p^n) \in \mathbf{HSP}(L)$. Since $\mathbb{L}(P_p^n)$ is a SCML, it suffices to show that $\mathbb{L}(P_p^n) \in \mathbf{SP}_u(L)$. If P is finite, then according to Freese [6], the lattice $\mathbb{L}(P_p^n)$ is a projective modular lattice, whence $\mathbb{L}(P_p^n) \in \mathbf{SP}(L)$. Since $\mathbb{L}(P_p^n)$ is simple, the inclusion $\mathbb{L}(P_p^n) \in \mathbf{SP}(L)$ implies that $\mathbb{L}(P_p^n) \in \mathbf{S}(L)$, and we are done.

Suppose that P is infinite, so that $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n) \in \mathbf{HSP}(L)$. Then there is $M \in \mathbf{SP}(L)$ such that $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n) \in \mathbf{H}(M)$. By Lemma 23, there is an n -frame Φ at 0 in M mapped onto the canonical n -frame of $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n)$. Taking the sublattice M' of M generated by Φ instead of M , we may assume that Φ is a spanning n -frame in M . By Proposition 26, either $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n)$ is a direct factor of M , or there is an infinite set $I \subseteq \mathbb{P}$ of primes such that $\mathbb{L}(\mathbf{F}_p^n) \in \mathbf{H}(M)$ for all $p \in I$. In the first case, $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n) \in \mathbf{S}(M) \subseteq \mathbf{SP}(L)$. Since $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n)$ is simple, we get immediately $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n) \in \mathbf{S}(L)$, whence $\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n) \in \mathbf{S}_{\exists}(L)$.

In the second case, $\mathbb{L}(\mathbf{F}_p^n) \in \mathbf{H}(M)$ for any $p \in I$. By the first case, one gets $\mathbb{L}(\mathbf{F}_p^n) \in \mathbf{S}(L)$, for any prime $p \in I$. Therefore,

$$\mathbb{L}(\mathbb{Q}_{\mathbb{Q}}^n) \in \mathbf{SP}_u(\mathbb{L}(\mathbf{F}_p^n) \mid p \in I) \subseteq \mathbf{SP}_u\mathbf{S}(L) \subseteq \mathbf{SP}_u(L).$$

So, $\mathbb{L}(P_p^n) \in \mathbf{SP}_u(L)$ in any case. \square

Corollary 29. *Let L be a subdirectly irreducible Arguesian SCML and let $\text{ht } L \geq 3$. There are a prime field P , unique up to isomorphism, and a vector space V_P over P such that $L \in \mathbf{S}_{\exists}(\mathbb{L}(V_P))$. If $\text{ht } L = \infty$, then $\mathbb{L}(P_p^n) \in \mathbf{S}_{\exists}(L)$ for all $n < \omega$ and $\mathbf{V}(\mathbb{L}(V_P)) = \mathbf{V}(L)$.*

Proof. By Corollary 22 and Proposition 27, $L \in \mathbf{S}_{\exists}(\mathbb{L}(V_D)) \subseteq \mathbf{S}_{\exists}(\mathbb{L}(V_P))$ for some V_D with prime subfield P of D which has the same characteristic as L . Suppose that $\text{ht } L = \infty$. By Corollary 3, for any $n \geq 3$, there is a nontrivial n -frame in L . Since L embeds into $\mathbb{L}(V_P)$, by Proposition 24 we get that L has a 0-sublattice isomorphic to $\mathbb{L}(P_p^n)$; the latter contains $\mathbb{L}(P_p^m)$ as a 0-sublattice, for all $m \leq n$. Applying Proposition 27 with $D = P$ we get $\mathbf{V}(\mathbb{L}(V_P)) = \mathbf{V}(L)$. \square

9. FINITE HEIGHT GENERATORS FOR \exists -VARIETIES OF SCMLs

The main result of this section can also be derived from Theorem 16 and Lemma 30. This would mean using coordinatization theory; hence, we rather give a more direct proof.

Lemma 30. *Let $R, S, R_i, i \in I$, be regular algebras. Then $\prod_{i \in I} \mathbb{L}(R_i) \cong \mathbb{L}(\prod_{i \in I} R_i)$. Moreover, $\mathbb{L}(S) \in \mathbf{H}(\mathbb{L}(R))$ whenever $S \in \mathbf{H}(R)$, and $\mathbb{L}(S) \in \mathbf{S}(\mathbb{L}(R))$ whenever $S \in \mathbf{S}(R)$.*

Proof. By the observation at the beginning of Section 3, $eR + fR$ and $eR \cap fR$ can be defined as subsets of R by existentially quantified conjunctions of identities with parameters e, f . Those sentences are preserved under products and surjective homomorphisms which accounts for the first two claims (the second one also follows from Proposition 5(ii)).

Now, let S be a regular subalgebra of R . For all $I \in \mathbb{L}(S)$, put $\varphi(I) = IR$. Then $\varphi(eS) = eR$ for any idempotent e , and φ is a join-homomorphism. Let idempotents $e, f, g_1 \in S$ be such that $Sg_1 = S(f - ef)$. Then $Rg_1 = RSg_1 = RS(f - ef) = R(f - ef)$. Therefore, $(eS \cap fS)R = (f - fg)SR = (f - fg)R = eR \cap fR = eSR \cap fSR$, whence φ is also a meet-homomorphism. Finally, φ is one-to-one, since $eS \subseteq fS$ if and only if $fe = e$, which is equivalent to $eSR = eR \subseteq fR = fSR$. Thus, $\mathbb{L}(S) \in \mathcal{S}(\mathbb{L}(R))$. \square

From Corollary 13 and Lemma 30, we get immediately the following

Corollary 31. *If F is a commutative field and R is a regular F -algebra, then for all $n_0 < \omega$,*

$$\mathbb{L}(R) \in \mathbf{HS}_{\exists} \mathbf{P} \{ \mathbb{L}(F^n) \mid n_0 \leq n < \omega \}.$$

Theorem 32. *Let L be a subdirectly irreducible SCML (CML) of infinite height. Then there is a prime field P , unique up to isomorphism (namely, the prime subfield of D , if $L \in \mathcal{S}(\mathbb{L}(V_D))$), such that for all $n_0 < \omega$,*

$$\mathbf{V}_{\exists}(L) = \mathbf{V}_{\exists}(\mathbb{L}(P^n) \mid n_0 \leq n < \omega)$$

Proof. By Corollary 22, L is Arguesian. By Corollary 29, there is a unique prime field P and there is a vector space V_P over P such that $L \in \mathcal{S}_{\exists}(\mathbb{L}(V_P))$ and $\mathbb{L}(P^n) \in \mathcal{S}_{\exists}(L)$ for all $n < \omega$. Since $\mathbb{L}(V_P) \cong \mathbb{L}(\text{End } V_P)$ and $\text{End } V_P$ is a regular P -algebra, $\mathbb{L}(\text{End } V_P) \in \mathbf{V}_{\exists}(\mathbb{L}(P^n) \mid n_0 \leq n < \omega)$ by Corollary 31. Hence $L \in \mathbf{V}_{\exists}(\mathbb{L}(P^n) \mid n_0 \leq n < \omega)$. \square

Corollary 33. *Any \exists -variety of SCMLs (CMLs) is generated by its finite height members.*

Corollary 34. *The \exists -variety of Arguesian SCMLs (CMLs) is generated by lattices of the form $\mathbb{L}(\mathbf{F}_p^n)$, where $n < \omega$ and $p \in \mathbb{P} \setminus \{0\}$ is a finite prime field. In particular, free Arguesian SCMLs (CMLs) are residually finite.*

Proof. It suffices to note that $\mathbb{L}(\mathbb{Q}_0^n)$ belongs to $\mathcal{S}_{\exists} \mathbf{P}_u(\mathbb{L}(\mathbf{F}_p^n) \mid n < \omega, p \in \mathbb{P} \setminus \{0\})$. \square

Corollary 35. *The equational theory of Arguesian SCMLs (CMLs) with sectional complementation as a fundamental operation is decidable. The same is true for a fixed characteristic.*

Proof. This follows from Corollary 34. In the case of characteristic 0, we observe that the set of sentences falsified in some $\mathbb{L}(\mathbb{Q}_0^n)$ is recursively enumerable. We also may refer to [24]. \square

Corollary 36. *The equational theory of SCMLs (CMLs), where sectional complementation is a fundamental operation (complementation and unit 1 are fundamental operations, respectively), is decidable.*

Proof. It suffices to provide a decision procedure for those of height at most 3. This is done by giving an effective description of the finite partial substructures containing 0 and 1, which is trivial for height 1. For height 2, we just have a finite lattice M_n , where $0 < n < \omega$, with n atoms and a binary relation ρ on atoms, where $a\rho b$, if $b = a'$. If there is only one atom a , add a new atom b , and put $a' = b$, $b' = a$. Otherwise, choose any atom $a' \neq a$, if $a\rho b$ for no b .

In height 3, one has a partial projective plane, and $a\rho b$ for some pairs where a is a point and b a line or vice versa. Any such plane has a free completion to a (possibly degenerated) projective plane, see Pickert [28]. If a is a point and $a\rho b$ for no b , then choose a line b , which is not incident with a , and put $a' = b$. For lines, the procedure is similar. \square

10. THE \exists -VARIETY GENERATED BY AN ATOMIC SCML

Corollary 37. *If L is an atomic SCML, then*

$$\mathbf{V}_{\exists}(L) = \mathbf{V}_{\exists}([0, a] \mid a \in L_{fin}) = \mathbf{V}_{\exists}(\mathbb{L}(P_L)).$$

Proof. By Lemma 20, $L \in \mathbf{S}_{\exists}(\mathbb{L}(P_L))$ and by Proposition 11, $\mathbf{V}_{\exists}([0, u] \mid u \in L_{fin}) \subseteq \mathbf{V}_{\exists}(L)$. Hence, it suffices to show that $\mathbb{L}(P_L) \in \mathbf{V}_{\exists}([0, u] \mid u \in L_{fin})$. Let M be a component of $\mathbb{L}(P_L)$. If $\text{ht } M < \infty$, then $M \cong [0, u]$ for some $u \in L_{fin} \cong \mathbb{L}(P_L)_{fin}$. If $\text{ht } M = \infty$, then, by Corollary 22, $M \cong \mathbb{L}(V_D)$ for a vector space V_D over a division ring D . By Theorem 32, $\mathbf{V}_{\exists}(M) = \mathbf{V}_{\exists}(\mathbb{L}(P_P^n) \mid n < \omega)$, where P is the prime subfield of D . Now, if $\text{ht}[0, u] = n$, then $[0, u] \cong \mathbb{L}(D_D^n)$, into which $\mathbb{L}(P_P^n)$ embeds by tensoring with D . Since $\mathbb{L}(P_L)$ belongs to the \exists -variety generated by its components, we are done. \square

Corollary 38. *If L is a SCML, then $\mathbf{V}_{\exists}(\text{Fr } L) = \mathbf{V}_{\exists}(L)$.*

Proof. By Theorem 21(ii), $L \in \mathbf{S}_{\exists}(\text{Fr } L)$, whence $\mathbf{V}_{\exists}(L) \subseteq \mathbf{V}_{\exists}(\text{Fr } L)$. Let M be a component of $\text{Fr } L$. If $\text{ht } M < \infty$, then by Theorem 21(iii),

$M \in \mathbf{HS}_{\exists}(L) \subseteq \mathbf{V}_{\exists}(L)$. If $\text{ht } M = \infty$, then by Theorem 32 and Theorem 21(i), there is a prime field P such that

$$\mathbf{V}_{\exists}(\mathbb{L}(P_P^n) \mid 4 \leq n < \omega) = \mathbf{V}_{\exists}(M) \subseteq V(\text{Fr } L) \subseteq \mathbf{V}(L).$$

By Lemma 28, $\mathbb{L}(P_P^n) \in \mathbf{V}_{\exists}(L)$. Thus, $M \in \mathbf{V}_{\exists}(L)$ for all components M of $\text{Fr } L$, whence $\text{Fr } L \in \mathbf{V}_{\exists}(L)$ and $\mathbf{V}_{\exists}(\text{Fr } L) \subseteq \mathbf{V}_{\exists}(L)$. \square

11. AN EMBEDDING RESULT FOR IDEAL LATTICES

Let \mathcal{C} denote the class of CMLs. The class $\mathbf{S}(\mathcal{C})$ is obviously a quasivariety. It is still an open question whether $\mathbf{S}(\mathcal{C})$ is a variety, cf. [13] for a failed approach. Due to Sachs [29], for any lattice L , the ideal lattice $\text{Id } L$ belongs to $\mathbf{V}(L)$. Therefore, it is of some interest to know whether the ideal lattice of a lattice which embeds into a (sectionally) complemented modular lattice, also does. In this section, we prove that this is, indeed, the case. An analogous result for lattices of permuting equivalence relations has been obtained by Nation [26].

Consider a structure $\langle A, \sigma \rangle$ of finite signature σ . Let $\bar{\sigma}$ denote the extension of σ by elements of A as constants, that is,

$$\bar{\sigma} = \sigma \cup \{c_a \mid a \in A\},$$

where $c_a \notin \sigma$, for all $a \in A$. Call a set $\Sigma(x)$ of formulas of signature $\bar{\sigma}$ (with free variables in $\{x\}$) *satisfiable* in A , if there is a in A such that $\langle A, \bar{\sigma} \rangle \models \Phi(a)$ for all $\Phi(x) \in \Sigma(x)$ (under the natural interpretation of the new constants). Call such a set $\Sigma(x)$ of formulas *finitely satisfiable* in A if any of its finite subsets is satisfiable in A . The structure $\langle A, \sigma \rangle$ is *saturated* in cardinality κ , if any finitely satisfiable set $\Sigma(x)$ of formulas of signature $\bar{\sigma}$, which contains less than κ new constants, is satisfiable in A . Due to [2, Lemma 5.1.4], for any structure $\langle B, \sigma \rangle$ and any cardinal κ such that $\kappa \geq \max\{|\sigma|, \aleph_0\}$ and $\aleph_0 \leq |B| \leq 2^\kappa$, there exists a κ^+ -saturated elementary extension $\langle A, \sigma \rangle$ of $\langle B, \sigma \rangle$; this extension can be chosen as an elementary substructure of an ultrapower of $\langle B, \sigma \rangle$. In particular, $A \in \mathbf{S}_{\exists} P_u(B)$, in the case B is a (sectionally) complemented lattice.

Theorem 39. *For any SCML C , there is $C' \in \mathbf{V}_{\exists}(C)$ such that $\text{Id } L \in \mathbf{S}(C')$ for all $L \in \mathbf{S}(C)$.*

Proof. First of all, we may assume that C is infinite, as for finite C , the conclusion is trivial. In view of Corollary 38, $\text{Fr } C \in \mathbf{V}_{\exists}(C)$. Let $\kappa > |C|$ and let C^* be a κ^+ -saturated elementary extension of $\text{Fr } C$. As $\text{Fr } C$ is an atomic CML, C^* also is. Moreover, $C^* \in \mathbf{V}_{\exists}(C)$. Since $C \in \mathbf{S}_{\exists}(\text{Fr } C)$, we get that $C \in \mathbf{S}_{\exists}(C^*)$, that is, $\mathbf{V}_{\exists}(C^*) = \mathbf{V}_{\exists}(C)$. Put $Q = P_{C^*}$ and $C' = \mathbb{L}(Q)$. By Corollary 37, $\mathbb{L}(Q) \in \mathbf{V}_{\exists}(C)$.

Now, if $L \in \mathbf{S}(C)$, then $L \in \mathbf{S}_{\exists}(C^*)$. Let ε embed L into C^* . One may assume ε to be identical embedding. Let $\varphi: L \rightarrow \mathbb{L}(Q)$ be the embedding given by Lemma 20. Then the map $\bar{\varphi}: \text{Id } L \rightarrow \mathbb{L}(Q)$ defined as in Lemma 19, is a lattice homomorphism, and it remains to show that it is one-to-one. For ideals $J \not\subseteq I$ of L , we choose $b \in J \setminus I$. It suffices to show that there is an atom $p \in C^*$ such that $p \in \bar{\varphi}(J)$ and $p \notin \bar{\varphi}(I)$.

Consider the following set of formulas:

$$\Sigma(x) = \{\Psi(x)\} \cup \{x \leq c_b\} \cup \{x \not\leq c_a \mid a \in I\},$$

where $\Psi(x)$ is the formula $\neg(x = 0) \ \& \ \forall y[y \leq x \rightarrow (y = 0) \vee (x = y)]$. Obviously, for any lattice K and any $a \in K$, one has $K \models \Psi(a)$ if and only if a is an atom.

Then $\Sigma(x)$ is finitely satisfiable in C^* . Indeed, let a_1, \dots, a_n in I . Then $a = a_1 + \dots + a_n \in I$. Now, $ab < b$, since $b \notin I$. Let d be a complement of ab in $[0, b] \subseteq C^*$. Obviously, $d > 0$. Since the lattice C^* is atomic, there is an atom $p \in C^*$ such that $p \leq d$. In particular, $p \leq b$ and $p \not\leq a_i$ for any $i \leq n$. Since C^* is κ -saturated, the set $\Sigma(x)$ is satisfiable in C^* , whence there is an atom $p \in C^*$ such that $p \leq b$ and $p \not\leq a$ for all $a \in I$. This implies that $p \in \varphi(b) \subseteq \bar{\varphi}(J)$ and $p \notin \varphi(a)$ for all $a \in I$, whence $p \notin \bigcup_{a \in I} \varphi(a) = \bar{\varphi}(I)$. Therefore, $\bar{\varphi}$ is an embedding. \square

Corollary 40. *If L and C are SCMLs such that L 0-embeds into $\text{Id } C$ then $L \in \mathbf{V}_{\exists}(C)$. In particular, $L \in \mathbf{V}_{\exists}(I)$, whenever L is subdirectly irreducible and I is the minimal neutral ideal of L .*

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