Cent. Eur. J. Math. • 8(5) • 2010 • 827-839
DOI: 10.2478/s11533-010-0056-x

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Central European Journal of **Mathematics**

Generators of existence varieties of regular rings and complemented Arguesian lattices

Research Article

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Received 11 May 2010; accepted 14 July 2010

Abstract: We proved in an earlier work that any existence variety of regular algebras is generated by its simple unital Artinian members, while any existence variety of Arguesian sectionally complemented lattices is generated by its simple members of finite length. A characterization of the class of simple unital Artinian members [members of finite length, respectively] of such varieties is given in the present paper.

MSC: 06C20, 16E50, 51D25

Keywords: Complemented modular lattice • Regular ring • Existence variety • Matrix ring

1. Introduction

In [9], existence varieties of (von Neumann) regular rings [sectionally complemented modular lattices] were studied. In the context of [9], a class of regular rings [sectionally complemented modular lattices] is an existence variety (an ∃-variety) if it is closed under homomorphic images, Cartesian products, and substructures which are regular rings [sectionally complemented modular lattices, respectively] themselves. The existence varieties of regular semigroups have been studied in T. E. Hall [6] and in J. Kadourek and M. B. Szendrei [12].

[9, Theorem 16] states that any existence variety of regular rings is generated by its simple unital Artinian members, while [9, Corollary 35] states that any existence variety of Arguesian sectionally complemented lattices is generated by

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its simple members of finite length. More precisely, matrix rings over finite prime fields generate existence varieties of regular rings [with unit], while their lattices of principal right ideals do the same for existence varieties of [sectionally] complemented Arguesian lattices. This implies, in particular, that free algebras in existence varieties of regular rings and [sectionally] complemented modular lattices are residually finite, whence, the equational theory of those varieties (in an extended signature) is decidable.

In the present paper, we give a characterization of the class of generators of existence varieties of regular rings and complemented modular lattices in terms of underlying division rings [see Theorems 4.4 and 5.4].

For all the definitions and general results on lattices which we do not state explicitly here, we refer to G. Birkhoff [1], P. Crawley and R. P. Dilworth [3]. For those on modular lattices and for the coordinatization results, we refer to J. von Neumann [15], B. Jónsson [11], L. A. Skornyakov [16], and P. Jipsen and H. Rose [10], see also C. Herrmann [7].

For all the necessary definitions and results on (regular) rings, we refer to P. M. Cohn [2] and K. R. Goodearl [4]. All the universal algebra results we use here without a reference can be found in A. I. Mal'cev [14]. All the classes we consider here are *abstract*, that is, they are closed under isomorphism.

2. Basic concepts

All the lattices which we consider here have a least element, and we consider them in the signature $\sigma = \{0, \cdot, +\}$, where $\cdot$ denotes the meet operation while $+$ denotes the join operation. In particular, by a sublattice, we always mean a 0-sublattice. A lattice L is *sectionally complemented*, if it satisfies the following first-order sentence φ_l:

$$\text{for all } x \text{ and } y \text{ there exists } z \text{ such that } xyz = 0 \text{ and } xy + z = x.$$

A lattice L with a greatest element 1 is *complemented*, if for any $x \in L$, there is $y \in L$ such that $xy = 0$ and $x + y = 1$.

Remark 2.1.

A modular lattice with a greatest element is complemented if and only if it is sectionally complemented, cf. [3, 4.2].

Let $n \geqslant 0$ be an integer; we say that a lattice L has *length* at most n, and write $\operatorname{lth} L \leqslant n$, if all chains of L have at most $n+1$ elements. If L is modular of length n, then all maximal chains in L have $n+1$ elements. A lattice L is a *finite length* lattice, if $\operatorname{lth} L = n$ for some integer $n \geqslant 0$.

A lattice L is *atomic*, if any non-zero element of L contains an *atom*, that is, such an element $p > 0$ that $0 < x < p$ holds for no $x \in L$. A lattice L is *atomistic*, if any element of L is a join of atoms.

Remark 2.2.

A sectionally complemented lattice is atomistic if and only if it is atomic [3, proof of 4.3].

For a cardinal κ, we denote by M_κ the length-two (Arguesian) lattice with κ many atoms. For a lattice L, $\operatorname{Con} L$ denotes its *congruence lattice*. L is a *simple* lattice, if $\operatorname{Con} L$ is a two-element lattice; L is *subdirectly irreducible*, if $\operatorname{Con} L$ contains a least nontrivial congruence. In particular, any simple lattice is subdirectly irreducible.

Let Λ be a commutative ring with unit. We consider Λ-algebras in the signature $\sigma = \{0, \Lambda, -, +, \cdot\}$ and we consider rings in the signature $\sigma = \{0, -, +, \cdot\}$, cf. [9]. We emphasize that rings do not necessarily have a unit.

An algebra (a ring, respectively) is (*von Neumann*) *regular*, if it satisfies the following first-order sentence φ_r:

$$\text{for all } x \text{ there exists } y \text{ such that } xyx = x.$$

In a regular ring R, all finitely generated (right) ideals in R are principal and are generated by an idempotent.

Remark 2.3.

One can view regular rings as regular $\mathbb{Z}$-algebras with right ideals being subalgebras.

For a (right) module M, let $\operatorname{Sub} M$ denote the lattice of all submodules of M and let $\operatorname{End} M$ denote the endomorphism ring of M. If V_D is a vector space over a division ring, then $\operatorname{Sub} V_D$ is a subdirectly irreducible Arguesian lattice. If V_D is finite dimensional, then $\operatorname{Sub} V_D$ is simple.

For a regular ring R, we denote the lattice of principal (right) ideals by $\mathbb{L}(R)$. Then $\mathbb{L}(R)$ is a sectionally complemented (Arguesian) sublattice of $\operatorname{Sub} R_R$, cf. K.R. Goodearl [4, Theorem 1.1], L.A. Skornyakov [16, Theorem 4]. Moreover, in this case, R is an Artinian ring with unit if and only if $\mathbb{L}(R)$ is a finite length lattice.

For a division ring D and a positive integer n, we denote by D^n_D the nth Cartesian power of D viewed as a right vector space over D, and by $D^{n\times n}$ we denote the ring of $n\times n$ matrices over D (sometimes viewed as an Λ-algebra for a suitable commutative ring Λ); in particular, the ring $D^{n\times n}$ is simple regular Artinian and it is isomorphic to the endomorphism ring $\operatorname{End} D^n_D$ of D^n_D. Moreover, for any division ring D and any positive integer n, $\operatorname{Sub} D^n_D \cong \mathbb{L}(D^{n\times n})$ and $\operatorname{lth} \operatorname{Sub} D^n_D = n$.

Let L be a modular lattice and let $n \geqslant 2$ be a positive integer. A system $\Phi = \{a_i \mid i < n\} \cup \{c_{ij} \mid i,j < n,\ i \neq j\} \subseteq L$ is an *n-frame* in L if the following conditions are satisfied:

(i) $c_{ij} = c_{ji}$ for all $i \neq j$;

(ii) $a_i \sum_{k\neq i} a_k = a_i c_{ij} = a_j c_{ij} = 0_L < \sum_{k<n} a_k$ for all distinct $i,j < n$;

(iii) $a_i + a_j = a_i + c_{ij}$ for all distinct $i,j < n$;

(iv) $c_{ij} = (c_{ik} + c_{kj})(a_i + a_j)$ for all distinct $i,j,k < n$.

An n-frame Φ is *spanning*, if $\sum_{k<n} a_k = 1_L$.

Let L be a modular lattice and let $\Phi = \{a_i \mid i < n\} \cup \{c_{ij} \mid i,j < n,\ i \neq j\}$ be an n-frame in L for a positive integer $n \geqslant 2$. An *auxiliary ring* $\langle R_\Phi;\ 0_\Phi, \ominus, \oplus, \odot\rangle$ is defined as follows:

$$\begin{aligned}
R_\Phi &= \{x \in L \mid x a_2 = 0_L,\ x + a_2 = a_1 + a_2\};\\
\pi(x) &= (x + c_{13})(a_2 + a_3),\\
\pi'(x) &= (x + c_{23})(a_1 + a_3);\\
0_\Phi &= a_1,\\
x \ominus y &= (a_1 + a_2)\big[a_3 + (c_{23} + x)(a_2 + \pi'(y))\big],\\
x \oplus y &= (a_1 + a_2)\big[(x + a_3)(c_{13} + a_2) + \pi(y))\big],\\
x \odot y &= (a_1 + a_2)\big[\pi(x) + \pi'(y))\big].
\end{aligned}$$

The next statement follows e.g. from [11, Theorem 1.7] and [13, XI Anmerkung 3.3]. It captures the classical view on coordinatization of projective spaces due to D. Hilbert, O. Veblen and J.W. Young [17], G. Birkhoff and J. von Neumann.

Theorem 2.4.

Let L be a simple Arguesian complemented lattice of length n, $3 \leqslant n < \omega$. Then L contains a spanning n-frame Φ such that R_Φ is a division ring and $L \cong \mathbb{L}(R_\Phi^{n\times n})$. Moreover, if $L \cong \mathbb{L}(D^{n\times n})$ for a division ring D, then $D \cong R_\Phi$.

The following coordinatization theorem is due to J. von Neumann [15], see also C. Herrmann [7].

Theorem 2.5.

If L is a complemented modular [Arguesian] lattice with a spanning n-frame $\Phi = \{a_i \mid i < n\} \cup \{c_{ij} \mid i,j < n,\ i \neq j\}$, where $4 \leqslant n < \omega$ [$3 \leqslant n < \omega$, respectively], then R_Φ is a regular ring and $L \cong \mathbb{L}(R_\Phi^{n\times n})$. Moreover, if L is a sublattice of $\operatorname{Sub} M_S$ for some S-module M_S, then R_Φ is isomorphic to a subring of $\operatorname{End}(a_0)_S$.

All the classes of structures we consider here are *abstract*; that is, they are closed under isomorphic copies. As usual, S, H, P, and P_u denote the operators of taking substructures, homomorphic images, Cartesian products, and ultraproducts, respectively. Let Σ consist of axioms for modular lattices plus the sentence φ_l in case of lattices and let Σ consist of axioms for algebras (rings) plus the sentence φ_r in case of algebras (rings, respectively). For any $\mathcal{K} \subseteq \operatorname{Mod}\Sigma$, the operator $\mathsf{S}_\exists$ is defined by:

$$\mathsf{S}_\exists(\mathcal{K}) = \mathsf{S}(\mathcal{K}) \cap \operatorname{Mod}\Sigma.$$

A class $\mathcal{K} \subseteq \operatorname{Mod}\Sigma$ is an *existence variety* or an $\exists$-*variety*, if it is closed under H, P, and $\mathsf{S}_\exists$. In particular, any existence variety is closed under ultraproducts and elementary substructures, whence, it forms an axiomatizable class.

The following statement gives a description of existence varieties.

Proposition 2.6 ([9, Proposition 10]).

Let $\mathcal{K}$ be either a class of sectionally complemented modular lattices or a class of regular rings. Then the following holds:

(i) *the class $\mathsf{V}_\exists(\mathcal{K}) = \mathsf{HS}_\exists\mathsf{P}(\mathcal{K})$ is the smallest existence variety containing $\mathcal{K}$;*

(ii) *any subdirectly irreducible algebra from $\mathsf{V}_\exists(\mathcal{K})$ belongs to $\mathsf{HS}_\exists\mathsf{P}_u(\mathcal{K})$;*

(iii) *any existence variety is generated by its finitely generated subdirectly irreducible algebras.*

A class $\mathcal{K} \subseteq \operatorname{Mod}\Sigma$ is $\exists$-*universal*, if it is closed under P_u, and $\mathsf{S}_\exists$. Again, any $\exists$-universal class is closed under ultraproducts and elementary substructures, whence it is an axiomatizable class.

Lemma 2.7.

Let $\mathcal{K}$ be either a class of sectionally complemented modular lattices or a class of regular rings. Then $\mathcal{K}$ is an $\exists$-universal class if and only if $\mathcal{K} = \mathsf{S}_\exists\mathsf{P}_u(\mathcal{K})$.

Proof. Obviously, the class $\mathsf{S}_\exists\mathsf{P}_u(\mathcal{K})$ is closed under $\mathsf{S}_\exists$. Furthermore, if $A_i \in \mathsf{S}_\exists(B_i)$ and $B_i \in \mathsf{P}_u(\mathcal{K})$ for all $i \in I$, U an ultrafilter over I, and $A = \prod_{i\in I} A_i/U$, then $A \in \mathsf{S}\big(\prod_{i\in I} B_i/U\big)$ and $\prod_{i\in I} B_i/U \in \mathsf{P}_u(\mathcal{K})$. Moreover, since $\{A_i \mid i \in I\} \subseteq \operatorname{Mod}\Sigma$, we conclude that $A \models \Sigma$, whence $A \in \mathsf{S}_\exists\mathsf{P}_u(\mathcal{K})$, and the latter class is closed under ultraproducts. $\square$

The following lemma is of extensive use here.

Lemma 2.8 ([9, Lemma 30]).

Let R, S, R_i, $i \in I$, be regular algebras. Then the following holds:

(i) $\prod_{i\in I} \mathbb{L}(R_i) \cong \mathbb{L}\big(\prod_{i\in I} R_i\big)$;

(ii) *if $S \in \mathsf{S}(R)$, then $\mathbb{L}(S) \in \mathsf{S}\big(\mathbb{L}(R)\big)$;*

(iii) *if $S \in \mathsf{H}(R)$, then $\mathbb{L}(S) \in \mathsf{H}\big(\mathbb{L}(R)\big)$.*

Let $\mathbb{P} = \{p > 1 \mid p \text{ is prime}\} \cup \{0\}$. If $p \in \mathbb{P}\backslash\{0\}$, then $\mathbb{F}_p$ denotes a field with p elements. We also put $\mathbb{F}_0 = \mathbb{Q}$. If $p \in \mathbb{P}$, then we say that a modular lattice L has *characteristic* p, and we write $\operatorname{char} L = p$, if any proper 3-frame in L generates a sublattice isomorphic to $\mathbb{L}(\mathbb{F}_p^{3\times 3})$. This notion is due to C. Herrmann and A. Huhn [8], and the authors of [8] have shown that the property of a modular lattice to have characteristic p can be expressed by one identity in the case $p \neq 0$, while it can be expressed by infinitely many identities in the case $p = 0$. The following proposition is also due to C. Herrmann and A. Huhn [8], see also [9, Corollary 22, Corollary 25].

Proposition 2.9.

Any subdirectly irreducible Arguesian sectionally complemented lattice L with $\operatorname{lth} L \geqslant 3$ has a uniquely determined characteristic $\operatorname{char} L$. If $L \in \mathsf{S}\big(\operatorname{Sub} V_D\big)$ for a vector space V_D over a division ring D, then $\operatorname{char} L = \operatorname{char} D$.

For a fixed $p \in \mathbb{P}$, let $\mathcal{V}_p$ denote the variety of Arguesian lattices generated by subdirectly irreducible Arguesian sectionally complemented lattices of characteristic p. According to what was mentioned in the paragraph right before Proposition 2.9, $\mathcal{V}_p$ is finitely based in the case $p \neq 0$ while $\mathcal{V}_0$ is infinitely based.

3. Auxiliary results

In this section, we collect all the statements we need for further proofs.

Lemma 3.1.
Let L be a complemented modular lattice of finite length and let $A \in \mathsf{HS}_\exists(L)$ be subdirectly irreducible. Then $A \in \mathsf{S}_\exists(L)$.

Proof. As $A \in \mathsf{HS}_\exists(L)$, there is $B \in \mathsf{S}_\exists(L)$ such that $A \in \mathsf{H}(B)$; in particular, B is a complemented lattice of finite length and there is $\theta \in \operatorname{Con} B$ such that $A \cong B/\theta$. By [3, 11.10], $B \cong \prod_{i<n} B_i$, where B_i is a simple finite length lattice for all $i < n$. It follows that $\theta = \prod_{i<n} \theta_i$ for some $\theta_i \in \operatorname{Con} B_i$. Since all lattices B_i, $i < n$, are simple, there exists $I \subseteq \{0, \dots, n-1\}$ such that $A \cong B/\theta \cong \prod_{i \in I} B_i \in \mathsf{S}_\exists(B) \subseteq \mathsf{S}_\exists(L)$. □

The following two statements are technical and straightforward to prove.

Lemma 3.2.
Let D_i, $i \in I$, be division rings, let U be an ultrafilter over I, and let $D = \prod_{i \in I} D_i/U$. Then D is a division ring and $D^{n\times n} \cong \prod_{i \in I} D_i^{n\times n}/U$ for any positive integer n.

Proof. If $A_i = \big(a^i_{jk}\big)_{jk} \in D_i^{n\times n}$ for all $i \in I$, then the map

$$\varphi\colon \prod_{i\in I} D_i^{n\times n}/U \to D^{n\times n}, \qquad \varphi\colon (A_i \mid i \in I)/U \mapsto \Big(\big(a^i_{jk} \mid i \in I\big)/U\Big)_{jk}$$

defines an isomorphism. □

Lemma 3.3.
Let I be a set and let U be an ultrafilter over I. If L_i is a simple complemented Arguesian lattice of finite length for any $i \in I$, then $\prod_{i\in I} L_i/U$ is a subdirectly irreducible complemented Arguesian lattice.

Proof. By our assumption, for any $i \in I$, L_i is an atomic lattice, and any two distinct atoms are perspective in L_i. Since these two lattice properties can be expressed by a first-order sentence, we conclude that $\prod_{i\in I} L_i/U$ is also an atomic (whence, atomistic) lattice, where any two distinct atoms are perspective. This implies that for any atom a of $\prod_{i\in I} L_i/U$, the principal congruence $\Theta(0, a)$ generated by the pair $(0, a)$ is the least nontrivial congruence on $\prod_{i\in I} L_i/U$, whence, the latter is subdirectly irreducible. □

Lemma 3.4.
Let n be a positive integer, let F be a division ring, and let $L \in \mathsf{S}_\exists\big(\mathbb{L}(F^{n\times n})\big)$ be a simple lattice of length $m \geqslant 3$. Then there are a division ring D and a positive integer k such that $km \leqslant n$, $D \in \mathsf{S}(F^{k\times k})$, and $L \cong \mathbb{L}(D^{m\times m})$.

Proof. Since L is a simple Arguesian complemented lattice of finite length $m \geqslant 3$, according to Theorem 2.4, there is a spanning frame

$$\Phi = \{a_i, c_{ij} \mid i, j < m,\ i \neq j\}$$

such that a_i is an atom of L for all $i < m$, $\langle R_\Phi;\ 0_\Phi, \ominus, \oplus, \odot\rangle$ is a division ring, and $L \cong \operatorname{Sub}(R_\Phi)^m_{R_\Phi}$. We put $D = R_\Phi$. Moreover, there is $l \leqslant n$ such that $L \cong \operatorname{Sub} D^m_D$ is a $(0,1)$-sublattice of $\mathbb{L}(F^{l\times l}) \cong \operatorname{Sub} F^l_F$. Let $\psi\colon \operatorname{Sub} D^m_D \to \operatorname{Sub} F^l_F$ be the corresponding $(0,1)$-lattice embedding. If k denotes the common value of the dimension of all F-submodules $\psi(a_i)$ of F_F^l, for $i < m$, then $km = l$.

By the definition of the auxiliary ring, the ring

$$R_\Phi = \{x \in L \mid xa_1 = 0,\ x + a_1 = a_0 + a_1\}$$

is a subring of

$$R_{\psi(\Phi)} = \{X \in \operatorname{Sub} F_F^l \mid X \cap \psi(a_1) = 0,\ X + \psi(a_1) = \psi(a_0) + \psi(a_1)\},$$

the latter being isomorphic, by Theorem 2.5, to a subring of $\operatorname{End} \psi(a_0)_F \cong F^{k\times k}$. Therefore, D is isomorphic to a subring of $F^{k\times k}$. □

Remark 3.5.

One can infer that, in the proof of Lemma 3.4, the ring $R_{\psi(\Phi)}$ is isomorphic to $F^{k\times k}$. However for $R_{\psi(\Phi)}$, being isomorphic to a subring of $F^{k\times k}$ is enough to complete that proof.

Lemma 3.6.

Let m, n be positive integers and let F, D be division rings. If $F^{m\times m} \in \mathrm{HS}_\exists(D^{n\times n})$, then there is a positive integer k such that $km \leqslant n$ and $F \in \mathrm{S}_\exists(D^{k\times k})$.

Proof. The argument is similar to the one of Lemma 3.1. Let $B \in \mathrm{S}_\exists(D^{n\times n})$ be a regular ring such that $F^{m\times m} \in \mathrm{H}(B)$. According to Lemma 2.8(ii), the lattice $\mathbb{L}(B)$ embeds into $\mathbb{L}(D^{n\times n}) \cong \operatorname{Sub} D_D^n$, thus $\mathbb{L}(B)$ has finite length. Since B is regular, it is an Artinian ring with unit. Thus, it is semisimple and there is a positive integer k such that $B \cong \bigoplus_{i<k} B_i$, where B_i is a matrix ring over some division ring for any $i < k$. Any ideal of B is therefore a direct sum of ideals of B_i, $i < k$. Since all B_i, $i < n$, and $F^{m\times m}$ are simple rings, there is $i \in \{0, \dots, k-1\}$ such that $F^{m\times m} \cong B_i \in \mathrm{S}_\exists(B) \subseteq \mathrm{S}_\exists(D^{n\times n})$.

Obviously, $F^{3m\times 3m} \in \mathrm{S}_\exists(D^{3n\times 3n})$, whence, $\mathbb{L}(F^{3m\times 3m}) \in \mathrm{S}_\exists\big(\mathbb{L}(D^{3n\times 3n})\big)$ by Lemma 2.8(ii). Applying Lemma 3.4 and using Theorem 2.4, there is a positive integer k such that $km \leqslant n$ and $F \in \mathrm{S}_\exists(D^{k\times k})$. □

Lemma 3.7.

Let D be a finite field and let k, n be positive integers. Then $M_n \in \mathrm{S}_\exists\big(\operatorname{Sub} D_D^{2k}\big)$ if and only if $n \leqslant |D|^k + 1$.

Proof. Let F be a field with $|D|^k$ elements. Since F can be viewed as a k-dimensional vector space over D, $M_{|D|^k+1} \cong \operatorname{Sub} F_F^2 \in \mathrm{S}_\exists\big(\operatorname{Sub} D_D^{2k}\big)$, whence, sufficiency follows.

Conversely, let $M_n \in \mathrm{S}_\exists\big(\operatorname{Sub} D_D^{2k}\big)$ and let A_i, $i < n$, be k-dimensional subspaces of D_D^{2k}, corresponding to the atoms of M_n. We have $|A_i| = |D|^k$ for all $i < n$. Thus, the subset $A = \bigcup_{i<n} A_i$ of D_D^{2k} has $n(|D|^k - 1) + 1$ elements. On the other hand, we obviously have

$$\begin{aligned} n(|D|^k - 1) + 1 = |A| &\leqslant |D|^{2k}; \\ n(|D|^k - 1) &\leqslant |D|^{2k} - 1; \\ n &\leqslant |D|^k + 1, \end{aligned}$$

whence necessity follows. □

4. Sectionally complemented Arguesian lattices

Hereinafter, let $\mathcal{A}_l$ denote the class of simple Arguesian (sectionally) complemented lattices of finite length. For a class $\mathcal{C} \subseteq \mathcal{A}_l$ and for a positive integer n, we define a class $\mathbf{D}_n(\mathcal{C})$ of division rings as follows:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad \operatorname{Sub} D^n_D \in \mathcal{C}.$$

Equivalently, $D \in \mathbf{D}_n(\mathcal{C})$ if and only if $\mathbb{L}(D^{n\times n}) \in \mathcal{C}$.

Definition 4.1.

A class $\mathcal{C} \subseteq \mathcal{A}_l$ is *closed*, if the following conditions are satisfied:

(i) the class $\mathbf{D}_n(\mathcal{C})$ is closed under ultraproducts for all positive $n < \omega$;

(ii) $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all positive $m \leqslant n < \omega$;

(iii) for any positive $n < \omega$, if $n = mk$ and $F \in \mathbf{D}_n(\mathcal{C})$ and $D \in \mathsf{S}(F^{k\times k})$ are division rings, then $D \in \mathbf{D}_m(\mathcal{C})$;

(iv) let $p \in \mathbb{P}$; if for all positive $n < \omega$, there exists $D \in \mathbf{D}_n(\mathcal{C})$ such that $\operatorname{char} D = p$, then $F \in \bigcap_{0<n<\omega} \mathbf{D}_n(\mathcal{C})$ for any division ring F of characteristic p;

(v) if $M_\kappa \in \mathcal{C}$, then $M_\lambda \in \mathcal{C}$ for all cardinals κ and λ such that $2 < \lambda \leqslant \kappa$;

(vi) $\mathbf{D}_1(\mathcal{C})$ is the class of all division rings.

Remark 4.2.

Note that if there exists an $n > 1$ such that $\mathbf{D}_n(\mathcal{C})$ contains an infinite division ring, then it follows from Definition 4.1(i), (ii), and (v) that $\mathbf{D}_2(\mathcal{C})$ contains *all* division rings. Moreover, it also follows from Definition 4.1(ii), (v) that if $D \in \mathbf{D}_n(\mathcal{C})$ for some $n > 1$ and $|F| \leqslant |D|$ for a division ring F, then $F \in \mathbf{D}_2(\mathcal{C})$.

According to Definition 4.1(iii), the class $\mathbf{D}_n(\mathcal{C})$ is closed under (division) subrings for all $n > 0$. Therefore, according to Definition 4.1(i) and Lemma 2.7, for any $n > 0$, $\mathbf{D}_n(\mathcal{C})$ is *relatively universal*, that is, it can be defined by universal sentences within the class of regular rings.

Lemma 4.3.

Modulo all the other conditions of Definition 4.1, (iv) *is equivalent to the following statement: let $p \in \mathbb{P}$; if for all positive $n < \omega$ there exists $D \in \mathbf{D}_n(\mathcal{C})$ such that* $\operatorname{char} D = p$*, then $\mathcal{V}_p \cap \mathcal{A}_l \subseteq \mathcal{C}$.*

Proof. Suppose that (iv) of Definition 4.1 holds and suppose that $p \in \mathbb{P}$ and for any positive $n < \omega$, there exists $D \in \mathbf{D}_n(\mathcal{C})$ such that $\operatorname{char} D = p$. Let $L \in \mathcal{V}_p \cap \mathcal{A}_l$. Without loss of generality, we may assume that $2 \leqslant \operatorname{lth} L < \omega$. If $\operatorname{lth} L = 2$, then $L \cong M_\kappa$ for a cardinal $\kappa \geqslant 3$. Let F be a field of characteristic p such that $\kappa \leqslant |F|$. By Definition 4.1(i), (ii), and (iv), $F \in \mathbf{D}_2(\mathcal{C})$, whence $M_\kappa \in \mathcal{C}$ by (v). Suppose now that $\operatorname{lth} L = n \geqslant 3$. Due to Theorem 2.4, there is a (unique) division ring D such that $L \cong \operatorname{Sub} D^n_D$. By Proposition 2.9, $\operatorname{char} D = p$. Therefore, $L \in \mathcal{C}$ by (iv).

Conversely, suppose that the given condition holds. Let $p \in \mathbb{P}$ be such that for any positive integer n there exists $D \in \mathbf{D}_n(\mathcal{C})$ such that $\operatorname{char} D = p$. Let F be a division ring of characteristic p. Then for any integer $n \geqslant 3$, the lattice $\operatorname{Sub} F^n_F$ is a simple Arguesian complemented lattice of length n and of characteristic p, whence $\operatorname{Sub} F^n_F \in \mathcal{V}_p \cap \mathcal{A}_l \subseteq \mathcal{C}$. Thus, $F \in \mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_1(\mathcal{C}) \cap \mathbf{D}_2(\mathcal{C})$ according to Definition 4.1(ii). Therefore, $F \in \bigcap_{0<n<\omega} \mathbf{D}_n(\mathcal{C})$. $\square$

Theorem 4.4.

A class $\mathcal{C} \subseteq \mathcal{A}_l$ is closed if and only if there is a nontrivial existence variety $\mathcal{V}$ of Arguesian sectionally complemented lattices such that $\mathcal{C} = \mathcal{V} \cap \mathcal{A}_l$.

Proof. Suppose first that $\mathcal{V}$ is an existence variety of Arguesian sectionally complemented lattices and that $\mathcal{C}$ consists of simple finite length members of $\mathcal{V}$. We prove that $\mathcal{C}$ is closed; in other words, we prove that $\mathcal{C}$ satisfies conditions (i)–(vi) of Definition 4.1. We note first that condition (vi) trivially holds on $\mathcal{C}$.

To prove (i), we assume that $n > 1$. Let $D_i \in \mathbf{D}_n(\mathcal{C})$ for all $i \in I$, let U be an ultrafilter over I, and let $D = \prod_{i\in I} D_i/U$. Then $D^{n\times n} \cong \prod_{i\in I} D_i^{n\times n}/U$ by Lemma 3.2. Therefore,

$$\mathbb{L}(D^{n\times n}) \cong \mathbb{L}\left(\prod_{i\in I} D_i^{n\times n}/U\right) \in \mathrm{HP}\big(\mathbb{L}(D_i^{n\times n}) \mid i \in I\big) \subseteq \mathrm{HP}(\mathcal{C}) \subseteq \mathcal{V}$$

by Lemma 2.8. As $\mathbb{L}(D^{n\times n})$ is a simple finite length Arguesian complemented lattice, we conclude that $\mathbb{L}(D^{n\times n}) \in \mathcal{C}$, whence $D \in \mathbf{D}_n(\mathcal{C})$ and the latter is, indeed, closed under ultraproducts.

To establish (ii), we assume that $m \leqslant n < \omega$ and that $D \in \mathbf{D}_n(\mathcal{C})$, that is, $\mathbb{L}(D^{n\times n}) \in \mathcal{C}$. As $D^{m\times m}$ is trivially isomorphic to a regular subalgebra of $D^{n\times n}$, we conclude by Lemma 2.8(ii) that $\mathbb{L}(D^{m\times m}) \in \mathrm{S}_\exists\big(\mathbb{L}(D^{n\times n})\big) \subseteq \mathcal{V}$. Since $\mathbb{L}(D^{m\times m}) \in \mathcal{A}_I$, we get that $D \in \mathbf{D}_m(\mathcal{C})$.

To prove (iii), we assume that n is a positive integer such that $n = mk$, $D \in \mathrm{S}(F^{k\times k})$ is a division ring, and that $F \in \mathbf{D}_n(\mathcal{C})$, that is, $\mathbb{L}(F^{n\times n}) \in \mathcal{C}$. Then $D^{m\times m}$ embeds into $\big(F^{k\times k}\big)^{m\times m} = F^{n\times n}$. By Lemma 2.8(ii) and Proposition 2.6(i), $\mathbb{L}(D^{m\times m}) \in \mathrm{S}_\exists\big(\mathbb{L}(F^{n\times n})\big) \subseteq \mathcal{V}$. Since $\mathbb{L}(D^{m\times m}) \in \mathcal{A}_I$, we get that $D \in \mathbf{D}_m(\mathcal{C})$.

To prove (iv), we fix $p \in \mathbb{P}$ and assume that for all positive $n < \omega$, there is a division ring $D_n \in \mathbf{D}_n(\mathcal{C})$ such that $\operatorname{char} D_n = p$. Since D_n contains the prime field $\mathbb{F}_p$ of characteristic p, we conclude by (iii) that $\mathbb{F}_p \in \mathbf{D}_n(\mathcal{C})$ for all positive integers n. Let F be a division ring of characteristic p. To establish (iv), it suffices to show in view of (ii) that $\operatorname{Sub} F_F^n \in \mathcal{C}$ for any integer $n \geqslant 3$. Let V_F be an infinite dimensional vector space over F. By [9, Theorem 32], $\operatorname{Sub} V_F \in \mathrm{V}_\exists\big(\mathbb{L}(\mathbb{F}_p^{n\times n}) \mid 3 \leqslant n < \omega\big) \subseteq \mathcal{V}$. For all integers $n \geqslant 3$, F_F^n is isomorphic to a subspace of V_F, whence $\operatorname{Sub} F_F^n \in \mathrm{S}_\exists\big(\operatorname{Sub} V_F\big) \subseteq \mathcal{V}$. Since $\operatorname{Sub} F_F^n \in \mathcal{A}_I$, we obtain that $\operatorname{Sub} F_F^n \in \mathcal{C}$.

Finally, we prove that (v) holds on $\mathcal{C}$. Let $D \in \mathbf{D}_2(\mathcal{C})$, that is, $\operatorname{Sub} D_D^2 \in \mathcal{C}$, and let $|F| \leqslant |D|$ for some division ring F. It is folklore that $\operatorname{Sub} D_D^2 \cong M_{|D|+1}$ and $\operatorname{Sub} F_F^2 \cong M_{|F|+1}$. It follows that $M_\kappa \in \mathcal{V}$ for all $2 < \kappa \leqslant |D| + 1$. As $\operatorname{Sub} F_F^2$ is a simple complemented Arguesian finite length lattice, $\operatorname{Sub} F_F^2 \in \mathcal{C}$ and $F \in \mathbf{D}_2(\mathcal{C})$. With this observation, (v) becomes obvious.

Conversely, suppose that a class $\mathcal{C} \subseteq \mathcal{A}_I$ is closed. Let $\mathcal{V}$ denote the existence variety generated by $\mathcal{C}$. By Proposition 2.6(i), $\mathcal{V} = \mathrm{HS}_\exists\mathrm{P}(\mathcal{C})$. Obviously, $\mathcal{C} \subseteq \mathcal{V} \cap \mathcal{A}_I$. To prove the reverse inclusion, let $L \in \mathcal{V}$ be a simple Arguesian finite length complemented lattice and let $\operatorname{lth} L = n$. If $n = 1$, then $L \in \mathcal{C}$ by (vi) of Definition 4.1. Thus, we may assume that $n > 1$. According to Proposition 2.6(ii), $L \in \mathrm{HS}_\exists\mathrm{P}_u(\mathcal{C})$. Therefore, there are a set I and an ultrafilter U over I, together with lattices $L_i \in \mathcal{C}$, $i \in I$, such that $L \in \mathrm{HS}_\exists\big(\prod_{i\in I} L_i/U\big)$. Two cases have to be considered.

Case 1: $n \geqslant 3$. According to Theorem 2.4, there is a unique division ring D such that $L \cong \mathbb{L}(D^{n\times n})$. Let $\operatorname{char} D = p$; by Proposition 2.9, $\operatorname{char} L = p$. Since $n = \operatorname{lth} L \geqslant 3$, we conclude that $\operatorname{lth} \prod_{i\in I} L_i/U \geqslant 3$, whence $J = \{i \in I \mid \operatorname{lth} L_i \geqslant 3\} \in U$. Considering J as the index set and taking $U|_J = \{X \cap J \mid X \in U\}$ instead of U, we may assume without loss of generality that $\operatorname{lth} L_i \geqslant 3$ for all $i \in I$. Again, due to Theorem 2.4, for any $i \in I$, there is a unique division ring D_i such that $L_i \cong \mathbb{L}(D_i^{n_i\times n_i})$, where $n_i = \operatorname{lth} L_i \geqslant 3$. By Lemma 3.3, $\prod_{i\in I} L_i/U$ is a subdirectly irreducible Arguesian lattice, whence $\operatorname{char} \prod_{i\in I} L_i/U$ is defined according to Proposition 2.9. Since the property of a lattice to have certain characteristic can be expressed by identities and $L \in \mathrm{HS}_\exists\big(\prod_{i\in I} L_i/U\big)$, we conclude that $\operatorname{char} L = \operatorname{char} \prod_{i\in I} L_i/U = p$. There are two subcases to consider.

Case 1.1: $p \neq 0$. For any $q \in \mathbb{P}$, we consider two sets:

$$I_0(q) = \{i \in I \mid \operatorname{char} L_i = q\}, \qquad I_1(q) = \{i \in I \mid \operatorname{char} L_i \neq q\}.$$

According to Proposition 2.9,

$$I_0(q) = \{i \in I \mid \operatorname{char} D_i = q\}, \qquad I_1(q) = \{i \in I \mid \operatorname{char} D_i \neq q\}.$$

For any $q \in \mathbb{P}$, either $I_0(q) \in U$ or $I_1(q) \in U$. Since the property of a lattice to have characteristic $p \neq 0$ can be expressed by an identity, we conclude that $I_0(p) \in U$. Taking, if necessary, $I_0(p)$ instead of I and $U|_{I_0(p)}$ instead of U, we may assume without loss of generality that $\operatorname{char} D_i = p$ for all $i \in I$.

Suppose now that $\sup\{n_i \mid i \in I\}$ is infinite. This means that for any $k < \omega$, there is $i \in I$ such that $n_i > k$; that is, using Definition 4.1(ii), for any positive integer $k < \omega$, there is $i \in I$ such that $D_i \in \mathbf{D}_k(\mathcal{C})$. As $\operatorname{char} D_i = p$ for all $i \in I$, we conclude by Definition 4.1(iv) that $D \in \mathbf{D}_n(\mathcal{C})$, whence $L \cong \operatorname{Sub} D_D^n \in \mathcal{C}$.

If $\sup\{n_i \mid i \in I\} = n' < \omega$, then $\operatorname{lth} \prod_{i\in I} L_i/U = m$ for some $n \leqslant m \leqslant n'$. As above, we may assume without loss of generality that $\operatorname{lth} L_i = m$ for all $i \in I$, whence $D_i \in \mathbf{D}_m(\mathcal{C})$ for all $i \in I$. By Lemma 3.2, $L \in \mathsf{HS}_\exists\big(\mathbb{L}(F^{m\times m})\big)$, where $F = \prod_{i\in I} D_i/U$. We have $F \in \mathbf{D}_m(\mathcal{C})$ by Definition 4.1(i). According to Lemma 3.1, $L \in \mathsf{S}_\exists\big(\mathbb{L}(F^{m\times m})\big)$, while according to Lemma 3.4, $L \cong \mathbb{L}(K^{l\times l})$ for a division ring K and a positive integer l such that $K \in \mathsf{S}(F^{k\times k})$ and $kl \leqslant m$. We have $F \in \mathbf{D}_{kl}(\mathcal{C})$ by Definition 4.1(ii), while $K \in \mathbf{D}_l(\mathcal{C})$ by Definition 4.1(iii). Therefore, $L \in \mathcal{C}$.

Case 1.2: $p = 0$. We put for any $q \in \mathbb{P}\backslash\{0\}$:

$$n(q) = \sup\big\{m \mid 1 \leqslant m < \omega,\ \text{there is } D \in \mathbf{D}_m(\mathcal{C}) \text{ with } \operatorname{char} D = q\big\}.$$

There are again two possibilities.

Case 1.2.1: *for any integer $k \geqslant 3$, for any prime $r \in \mathbb{P}\backslash\{0\}$, there is $q \in \mathbb{P}\backslash\{0\}$ such that $q \geqslant r$ and $n(q) \geqslant k$.* Let $m \geqslant 3$ be an arbitrary integer. Then there are an infinite set of primes $\{p_t \in \mathbb{P}\backslash\{0\} \mid t < \omega\}$ and a set of division rings $\{F_t \mid t < \omega\}$ with the following properties:

- $\operatorname{char} F_t = p_t$ for all $t < \omega$;
- for all $t < \omega$, $F_t \in \mathbf{D}_{k_t}(\mathcal{C})$ for some $k_t \geqslant m$.

By Definition 4.1(ii), $F_t \in \mathbf{D}_m(\mathcal{C})$ for all $t < \omega$. By Definition 4.1(i), this means that $F = \prod_{t<\omega} F_t/U \in \mathbf{D}_m(\mathcal{C})$ for any non-principal ultrafilter U over ω. It is obvious that $\operatorname{char} F = 0$. Summarizing, we have proved that for any $m \geqslant 3$ there is a division ring $F \in \mathbf{D}_m(\mathcal{C})$ such that $\operatorname{char} F = 0$. According to Definition 4.1(iv), $\operatorname{Sub} D_D^n \in \mathcal{C}$.

Case 1.2.2: *there are an integer $k \geqslant 3$ and a prime $r \in \mathbb{P}\backslash\{0\}$ such that for all $q \in \mathbb{P}\backslash\{0\}$, $q \geqslant r$, one has $n(q) \leqslant k$.* One of the three sets $I_0 = \{i \in I \mid \operatorname{char} L_i = 0\}$, $I_1 = \{i \in I \mid 0 \neq \operatorname{char} L_i < r\}$, and $I_2 = \{i \in I \mid \operatorname{char} L_i \geqslant r\}$ must belong to U.

Suppose first that $I_0 \in U$. Taking, if necessary, I_0 instead of I and restricting U to $U|_{I_0}$, we may assume that $\operatorname{char} L_i = 0$ for all $i \in I$. If there is a positive integer n' such that $\sup\{\operatorname{lth} L_i \mid i \in I\} = n'$, then repeating the argument of the last paragraph of *Case 1.1*, we can conclude that $L \in \mathcal{C}$. If $\sup\{\operatorname{lth} L_i \mid i \in I\}$ is infinite, then $D \in \mathbf{D}_n(\mathcal{C})$ by Definition 4.1(iv), whence $L \in \mathcal{C}$ again.

Since the property of a lattice to have a nonzero prime characteristic $< r$ can be expressed by a first-order sentence and since $\operatorname{char} L = 0$, the case $I_1 \in U$ cannot occur. Suppose that $I_2 \in U$. Then taking, if necessary, I_2 instead of I and restricting U to $U|_{I_2}$, we may assume that $\operatorname{char} L_i \geqslant r$ for all $i \in I$. It follows by the definition of k that $\operatorname{lth} L_i \leqslant k$ for all $i \in I$, whence $n \leqslant \operatorname{lth} \prod_{i\in I} L_i/U \leqslant k$. We repeat the argument of the last paragraph of *Case 1.1* to conclude that $L \in \mathcal{C}$.

Case 2: $n = 2$. Therefore, $L \cong M_\kappa$ for a cardinal $\kappa \geqslant 3$. Two cases are possible.

Case 2.1: $\kappa < \omega$. We prove that $M_\kappa \in \mathsf{S}(L_i)$ for some $i \in I$. Two subcases are possible.

Case 2.1.1: $\sup\{\operatorname{lth} L_i \mid i \in I\}$ *is finite*. Then $\prod_{i\in I} L_i/U$ is a finite length lattice. Since M_κ is projective within the class of finite length modular lattices, we conclude that $M_\kappa \in \mathsf{S}_\exists\big(\prod_{i\in I} L_i/U\big)$. Furthermore, let φ_κ denote the following existential sentence:

$$\exists x_0, x_1, \ldots, x_{\kappa-1} \bigwedge_{i<\kappa} \neg[x_i = 0] \wedge \bigwedge_{i\neq j}[x_i + x_j = x_0 + x_1] \wedge \bigwedge_{i\neq j}[x_i x_j = 0].$$

Since $M_\kappa \models \varphi_\kappa$, we conclude that $\prod_{i\in I} L_i/U \models \varphi_\kappa$. In particular, there is $i \in I$ such that $L_i \models \varphi_\kappa$; that is, $M_\kappa \in \mathsf{S}_\exists(L_i)$.

Case 2.1.2: $\sup\{\operatorname{lth} L_i \mid i \in I\}$ *is infinite*. This means that there is $i \in I$ such that $\operatorname{lth} L_i \geqslant 2\lceil \log_2 \kappa \rceil + 2$. By Lemma 3.7 we conclude that $M_\kappa \in \mathsf{S}(L_i)$.

So, we have proved that in any case $M_\kappa \in \mathsf{S}(L_i)$ for some $i \in I$. Suppose first that L_i is coordinatizable. Then there are a division ring D and an integer $n \geqslant 2$ such that $L_i \cong \operatorname{Sub} D_D^n$. Then $D \in \mathbf{D}_n(\mathcal{C})$, whence $D \in \mathbf{D}_2(\mathcal{C})$ by Definition 4.1(ii). If D is infinite, then $M_\kappa \in \mathcal{C}$ according to Remark 4.2. Let D be finite; then D is a field. As the image of the unit of M_κ in $\operatorname{Sub} D_D^n$ is a subspace of D_D^n of even dimension, we may assume that $n = 2k$ and that M_κ $(0,1)$-embeds into $\operatorname{Sub} D_D^n$. By Lemma 3.7, $\kappa \leqslant |D|^k + 1$. Let F be a finite field with $|D|^k$ elements. Viewing F as a k-dimensional vector space over D, we get that $F \cong \operatorname{End} F_F$ embeds as a subring into $\operatorname{End} D_D^k \cong D^{k\times k}$. Therefore, $F \in \mathbf{D}_2(\mathcal{C})$ by Definition 4.1(iii), whence $M_{|F|+1} \cong \operatorname{Sub} F_F^2 \in \mathcal{C}$. Recalling that $2 < \kappa \leqslant |D|^k + 1 = |F| + 1$, we get that $M_\kappa \in \mathcal{C}$ by Definition 4.1(v).

If L_i is not coordinatizable, then $\operatorname{lth} L_i = 2$ and $L_i \cong M_n$ for some $n < \omega$. As $M_\kappa \in \mathsf{S}_\exists(M_n)$ in this case and $\kappa \leqslant n$, we conclude that $M_\kappa \in \mathcal{C}$ by Definition 4.1(v).

Case 2.2: $\kappa \geqslant \omega$. In this case, for any $p \in \mathbb{P}\backslash\{0\}$, $M_{p+1} \in \mathsf{S}_\exists\mathsf{H}\mathsf{S}_\exists\mathsf{P}_u(\mathcal{C}) \subseteq \mathsf{V}_\exists(\mathcal{C}) \subseteq \mathcal{V}$. Due to *Case 2.1*, $\mathbb{L}(\mathbb{F}_p^{2\times 2}) \cong M_{p+1} \in \mathcal{C}$, whence $\mathbb{F}_p \in \mathbf{D}_2(\mathcal{C})$ for all $p \in \mathbb{P}\backslash\{0\}$. By Definition 4.1(i), (iii), there is $F \in \mathbf{D}_2(\mathcal{C})$ such that $|F| = \kappa$. Thus, $M_\kappa \cong \mathbb{L}(F^{2\times 2}) \in \mathcal{C}$.

The proof of the theorem is complete. □

5. Regular rings

Let $\mathcal{A}_r$ denote the class of simple Artinian regular rings with unit. For a class $\mathcal{C} \subseteq \mathcal{A}_r$ and for a positive integer n, we define a class $\mathbf{D}_n(\mathcal{C})$ of division rings as follows:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n\times n} \in \mathcal{C}.$$

The following definition is a ring analogue of Definition 4.1.

Definition 5.1.

A class $\mathcal{C} \subseteq \mathcal{A}_r$ is *closed*, if the following conditions are satisfied:

(i) the class $\mathbf{D}_n(\mathcal{C})$ is closed under ultraproducts for all positive $n < \omega$;

(ii) $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all positive $m \leqslant n < \omega$;

(iii) for any positive $n < \omega$, if $n = mk$ and $F \in \mathbf{D}_n(\mathcal{C})$ and $D \in \mathsf{S}(F^{k\times k})$ are division rings, then $D \in \mathbf{D}_m(\mathcal{C})$;

(iv) let $p \in \mathbb{P}$; if for all positive $n < \omega$ there exists $D \in \mathbf{D}_n(\mathcal{C})$ such that $\operatorname{char} D = p$, then $F \in \bigcap_{0<n<\omega} \mathbf{D}_n(\mathcal{C})$ for any division ring F of characteristic p;

(v) $\mathbf{D}_1(\mathcal{C})$ is the class of all division rings.

Remark 5.2.

According to Definition 5.1(iii), the class $\mathbf{D}_n(\mathcal{C})$ is closed under (division) subrings for all $n > 0$.

Before proving Theorem 5.4, the main result of this section, we prove the following statement.

Lemma 5.3.

Let $\mathcal{C} \subseteq \mathcal{A}_r$ be a closed class, let D be a division ring, let n be a positive integer, and let $D^{n\times n} \in \mathsf{H}\mathsf{S}_\exists\mathsf{P}_u(\mathcal{C})$. Then $D^{n\times n} \in \mathcal{C}$.

Proof. Let I be a set and let U be an ultrafilter over I such that

$$D^{n\times n} \in \mathrm{HS}_{\exists}\Big(\prod_{i\in I} R_i/U\Big),$$

where $R_i \in \mathcal{C}$ for all $i \in I$. By the Wedderburn–Artin theorem (see, for example, [2, Theorem 2.21]), there are positive integers n_i, $i \in I$, and division rings $D_i \in \mathbf{D}_{n_i}(\mathcal{C})$, $i \in I$, such that $R_i \cong D_i^{n_i\times n_i}$ for all $i \in I$. Let $\operatorname{char} \prod_{i\in I} R_i/U = p$. Since the property of a ring to have characteristic $p \neq 0$ can be expressed by an identity and since the property of a ring to have characteristic 0 can be expressed by either an infinite set of quasi-identities or by an infinite set of positive sentences, and $D^{n\times n} \in \mathrm{HS}_{\exists}\big(\prod_{i\in I} R_i/U\big)$, we conclude that $\operatorname{char} D^{n\times n} = p$, whence $\operatorname{char} D = p$. There are two cases to consider.

Case 1: $p \neq 0$. For $q \in \mathbb{P}$, define two sets:

$$I_0(q) = \{i \in I \mid \operatorname{char} D_i = q\}, \qquad I_1(q) = \{i \in I \mid \operatorname{char} D_i \neq q\}.$$

Then for any $q \in \mathbb{P}$, either $I_0(q) \in U$ or $I_1(q) \in U$. Since the property of a ring to have characteristic $p \neq 0$ can be expressed by an identity, we conclude that $I_0(p) \in U$. Taking, if necessary, $I_0(p)$ instead of I and $U|_{I_0(p)}$ instead of U, we may assume without loss of generality that $\operatorname{char} D_i = p$ for all $i \in I$.

Suppose now that $\sup\{n_i \mid i \in I\}$ is infinite. This means that for any $k < \omega$ there is $i \in I$ such that $n_i > k$; that is, using Definition 5.1(ii), for any positive $k < \omega$ there is $i \in I$ such that $D_i \in \mathbf{D}_k(\mathcal{C})$. As $\operatorname{char} D_i = p$ for all $i \in I$, we conclude by Definition 5.1(iv) that $D \in \mathbf{D}_n(\mathcal{C})$, whence $D^{n\times n} \in \mathcal{C}$.

If $\sup\{n_i \mid i \in I\} = n' < \omega$, then we may assume without loss of generality that there is a positive integer m such that $n \leqslant m \leqslant n'$ and $n_i = m$ for all $i \in I$, whence $D_i \in \mathbf{D}_m(\mathcal{C})$ for all $i \in I$. By Lemma 3.2, $D^{n\times n} \in \mathrm{HS}_{\exists}(F^{m\times m})$, where $F = \prod_{i\in I} D_i/U$; in particular, $F \in \mathbf{D}_m(\mathcal{C})$ by Definition 5.1(i). By Lemma 3.6, there is a positive integer k such that $kn \leqslant m$ and $D \in \mathrm{S}_{\exists}(F^{k\times k})$. Moreover, $F \in \mathbf{D}_m(\mathcal{C})$ implies that $F \in \mathbf{D}_{kn}(\mathcal{C})$ by Definition 5.1(ii), whence $D \in \mathbf{D}_n(\mathcal{C})$ by Definition 5.1(iii) and $D^{n\times n} \in \mathcal{C}$.

Case 2: $p = 0$. We put for any $q \in \mathbb{P}\backslash\{0\}$:

$$n(q) = \sup\big\{m \mid 1 \leqslant m < \omega,\ \text{there is } D \in \mathbf{D}_m(\mathcal{C}) \text{ with } \operatorname{char} D = q\big\}.$$

There are two possibilities.

Case 2.1: *for any positive integer k, for any prime $r \in \mathbb{P}\backslash\{0\}$ there is $q \in \mathbb{P}\backslash\{0\}$ such that $q \geqslant r$ and $n(q) \geqslant k$.* Let m be an arbitrary positive integer. Then there are an infinite set of primes $\{p_t \in \mathbb{P}\backslash\{0\} \mid t < \omega\}$ and a set of division rings $\{F_t \mid t < \omega\}$ with the following properties:

- $\operatorname{char} F_t = p_t$ for all $t < \omega$;
- for all $t < \omega$, $F_t \in \mathbf{D}_{k_t}(\mathcal{C})$ for some $k_t \geqslant m$.

By Definition 5.1(ii), $F_t \in \mathbf{D}_m(\mathcal{C})$ for all $t < \omega$. By Definition 5.1(i), this means that $F = \prod_{t<\omega} F_t/U \in \mathbf{D}_m(\mathcal{C})$ for any non-principal ultrafilter U over ω. It is obvious that $\operatorname{char} F = 0$. Summarizing, we have proved that for any positive integer m there is a division ring $F \in \mathbf{D}_m(\mathcal{C})$ such that $\operatorname{char} F = 0$. According to Definition 5.1(iv), $D \in \mathbf{D}_n(\mathcal{C})$, whence $D^{n\times n} \in \mathcal{C}$.

Case 2.2: *there are a positive integer k and a prime $r \in \mathbb{P}\backslash\{0\}$ such that for all $q \in \mathbb{P}\backslash\{0\}$, $q \geqslant r$, one has $n(q) \leqslant k$.* One of the three sets $I_0 = \{i \in I \mid \operatorname{char} D_i = 0\}$, $I_1 = \{i \in I \mid 0 \neq \operatorname{char} D_i < r\}$ and $I_2 = \{i \in I \mid \operatorname{char} D_i \geqslant r\}$ must belong to U.

Suppose first that $I_0 \in U$. Taking, if necessary, I_0 instead of I and restricting U to $U|_{I_0}$, we may assume that $\operatorname{char} D_i = 0$ for all $i \in I$. If there is a positive integer n' such that $\sup\{n_i \mid i \in I\} \leqslant n'$, then repeating the argument of the last paragraph of *Case 1*, we can conclude that $D^{n\times n} \in \mathcal{C}$. If $\sup\{n_i \mid i \in I\}$ is infinite, then $D \in \mathbf{D}_n(\mathcal{C})$ by Definition 5.1(iv), whence $D^{n\times n} \in \mathcal{C}$.

Since the property of a ring to have a nonzero prime characteristic $< r$ can be expressed by a first-order sentence and $\operatorname{char} D = 0$, the case $I_1 \in U$ is impossible. Suppose that $I_2 \in U$. Then taking, if necessary, I_2 instead of I and restricting U to $U|_{I_2}$, we may assume that $\operatorname{char} D_i \geqslant r$ for all $i \in I$. It follows by the definition of k that $n_i \leqslant k$ for all $i \in I$. We repeat the argument of the last paragraph of *Case 1* to conclude that $D^{n\times n} \in \mathcal{C}$. □

Theorem 5.4.

A class $\mathcal{C} \subseteq \mathcal{A}_r$ is closed if and only if there is a nontrivial existence variety $\mathcal{V}$ of regular rings such that $\mathcal{C} = \mathcal{V} \cap \mathcal{A}_r$.

Proof. Suppose first that $\mathcal{V}$ is an existence variety of regular rings and that $\mathcal{C}$ consists of simple unital Artinian members of $\mathcal{V}$. We prove that $\mathcal{C}$ is closed; in other words, we prove that $\mathcal{C}$ satisfies conditions (i)–(v) of Definition 5.1. Note that (v) is obvious.

To prove (i), we assume that $n > 1$. Let $D_i \in \mathbf{D}_n(\mathcal{C})$ for all $i \in I$, let U be an ultrafilter over I, and let $D = \prod_{i\in I} D_i/U$. Then $D^{n\times n} \cong \prod_{i\in I} D_i^{n\times n}/U \in \mathcal{V}$ by Lemma 3.2. As $D^{n\times n}$ is a simple Artinian ring with unit, $D^{n\times n} \in \mathcal{C}$, whence $D \in \mathbf{D}_n(\mathcal{C})$ and the latter is, indeed, closed under ultraproducts.

Condition (ii) is straightforward to prove, as $D^{m\times m}$ is trivially isomorphic to a regular subalgebra of $D^{n\times n}$.

To prove (iii), we assume that n is a positive integer such that $n = mk$, $D \in \mathrm{S}(F^{k\times k})$ is a division ring, and that $F \in \mathbf{D}_n(\mathcal{C})$, that is, $F^{n\times n} \in \mathcal{C}$. Then $D^{m\times m}$ embeds into $\left(F^{k\times k}\right)^{m\times m} \cong F^{n\times n}$. By Proposition 2.6(i), $D^{m\times m} \in \mathcal{V}$. Since $D^{m\times m} \in \mathcal{A}_r$, we get that $D \in \mathbf{D}_m(\mathcal{C})$.

To prove (iv), we fix $p \in \mathbb{P}$ and assume that for all positive $n < \omega$ there is a division ring $D_n \in \mathbf{D}_n(\mathcal{C})$ such that $\operatorname{char} D_n = p$. Since D_n contains a prime subfield isomorphic to $\mathbb{F}_p$, we conclude by (iii) that $\mathbb{F}_p \in \mathbf{D}_n(\mathcal{C})$ for all positive integers n. Let F be a division ring with $\operatorname{char} F = p$. Let $B(F)$ denote the regular algebra of $\omega \times \omega$ matrices over F which are both row-finite and column-finite (viewed as an $\mathbb{F}_p$-algebra). By K.R. Goodearl, P. Menal, and J. Moncasi [5, Corollary 2.7] (cf. also [9, Corollary 13]), $B(F) \in \mathrm{V}_\exists\left(\mathbb{F}_p^{n\times n} \mid 3 \leqslant n < \omega\right) \subseteq \mathcal{V}$. Since for all integers $n \geqslant 3$, $F^{n\times n}$ is isomorphic to a regular subalgebra of $B(F)$, we conclude that $F^{n\times n} \in \mathrm{S}_\exists\left(B(F)\right) \subseteq \mathcal{V}$. The ring $F^{n\times n}$ is simple unital Artinian, that is, $F^{n\times n} \in \mathcal{C}$, whence $F \in \mathbf{D}_n(\mathcal{C})$ for all positive $n < \omega$.

Conversely, suppose that a class $\mathcal{C} \subseteq \mathcal{A}_r$ is closed. Let $\mathcal{V}$ denote the existence variety generated by $\mathcal{C}$. By Proposition 2.6(i), $\mathcal{V} = \mathrm{HS}_\exists\mathrm{P}(\mathcal{C})$. Obviously, $\mathcal{C} \subseteq \mathcal{V} \cap \mathcal{A}_r$. To prove the reverse inclusion, let $R \in \mathcal{V}$ be a simple Artinian regular ring with unit. According to Proposition 2.6(ii), $R \in \mathrm{HS}_\exists\mathrm{P}_u(\mathcal{C})$. By the Wedderburn–Artin theorem, there are a division ring F and a positive integer n such that $R \cong F^{n\times n}$. Applying Lemma 5.3, we obtain that $R \cong F^{n\times n} \in \mathcal{C}$. □

Acknowledgements

The second author was supported by the joint DFG-RFBR grant no. 06-01-04002 and by the Presidential Grant Council of Russian Federation, Doctors of Sciences Support grant no. MD-2587.2010.1 and Scientific Schools Support grant no. NSh-3669.2010.1.

We thank the referees for having thoroughly read our paper and making a number of important corrections and useful comments.

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