Complemented modular lattices with involution and Orthogonal Geometry

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Abstract. We associate with each orthogeometry $(P, \perp)$ a CMIL, i.e. a complemented modular lattice with involution, $\mathbb{L}(P, \perp)$ consisting of all subspaces X and $X^\perp$ with $\dim X < \aleph_0$ and study its rôle in decompositions of $(P, \perp)$ as directed resp. disjoint union. We also establish a 1-1-correspondence between $\exists$-varieties $\mathcal{V}$ of CMILs with $\mathcal{V}$ generated by its finite dimensional members and 'quasivarieties' $\mathcal{G}$ of orthogeometries: $\mathcal{V}$ consists of the CMILs representable within some geometry from $\mathcal{G}$ and $\mathcal{G}$ of the $(P, \perp)$ with $\mathbb{L}(P, \perp) \in \mathcal{V}$. Here, $\mathcal{V}$ is recursively axiomatizable if and only if so is $\mathcal{G}$. It follows that the equational theory of $\mathcal{V}$ is decidable provided that the equational theories of the $\{\mathbb{L}(P, \perp) \mid (P, \perp) \in \mathcal{G},\, \dim P = n\}$ are uniformly decidable.

1. Introduction. Part I: The lattice associated with a geometry

In the present note we consider *orthogeometries* $(P, \perp)$, where P is a projective geometry defined in terms of a collinearity relation on the point set (which is also denoted by P) and $\perp$ an *orthogonality* on P, i.e. $p \mapsto p^\perp = \{q \in P \mid p \perp q\}$ is a polarity on P (cf. [5, sect.14.1]). Let $\mathbb{L}(P)$ denote the modular lattice of all subspaces of the projective geometry P and define $X^\perp = \{q \in P \mid \forall p \in X\ p \perp q\}$.

In the literature, two lattice structures have been associated with $(P, \perp)$: The lattice $\mathbb{L}_c(P, \perp)$ of all closed subspaces (which is a complete DAC-lattice [14], i.e. a subprojective lattice [15, 18]) with involution $X \mapsto X^\perp$ and the 'quadratic' lattice (cf. [9]) $\mathbb{Q}(P, \perp)$, namely $\mathbb{L}(P)$ endowed with the unary operation $X \mapsto X^\perp$ (called 'ortholattice' in [5, Def.14.2.1]).

As a step towards an equational theory, in the present note we study complemented lattices which are substructures of $\mathbb{L}_c(P, \perp)$ and $\mathbb{Q}(P, \perp)$, simultaneously. Such are complemented modular lattices with involution (a dual automorphism of order 2), CMILs for short. We consider these as algebraic structures with join, meet, bounds $0, 1$, and involution $x \mapsto x'$ as fundamental operations.

In particular, with each orthogeometry $(P, \perp)$ we associate the atomic CMIL $\mathbb{L}(P, \perp)$ which consists of the X and $X^\perp$ where $X \in \mathbb{L}(P)$ is finite dimensional. This may be viewed as an algebraic version of the subprojective spaces of Markowski and Petrich [15, 18] - such appears as the incidence structure given by atoms and coatoms of $\mathbb{L}(P, \perp)$.

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Conversely, each CMIL L gives rise to an orthogeometry $\mathbb{G}(L)$ the points of which are the atoms of L with p, q, r collinear iff $p + q = p + r = q + r$ and $p \perp q$ iff $p \leq q'$. Let L_f consist of all elements of L having finite dimension or codimension. By an *equivalence* between two classes of structures we mean an equivalence between the corresponding categories where the morphisms are just the isomorphisms. Moreover, the equivalences to be estabished are 'concrete' in the sence of Giudici [6].

A *subgeometry* of $(P, \perp)$ is an orthogeometry $(Q, \perp_Q)$ where Q is a proper projective subgeometry (cf. [5, Def.3.3.11]) and $\perp_Q$ the induced orthogonality. In particular, any non-degenerate subspace of an orthogeometry gives rise to a subgeometry. Let **2** denote the 2-element Boolean algebra.

Theorem 1.1. *There is an equivalence between orthogeometries and atomic CMILs L such that $L = L_f$ given by $L = \mathbb{L}(P, \perp)$ and $(P, \perp) = \mathbb{G}(L)$. In particular, $(P, \perp) \cong \mathbb{G}\,\mathbb{L}(P, \perp)$ for any orthogeometry and $L \cong \mathbb{L}\,\mathbb{G}(L)$ for any finite dimensional CMIL.*

Theorem 1.2. *Any orthogeometry $(P, \perp)$ is the directed union of its finite dimensional subspaces X with $X \oplus X^\perp = P$ and the CMIL $\mathbb{L}(P, \perp)$ is the directed union of the complemented finite dimensional subalgebras $L_X = [0, X] \cup [X^\perp, P]$. In particular, these X are non-degenerate and L_X is isomorphic to the direct product $\mathbb{L}(X, \perp_{|X}) \times \mathbf{2}$ of CMILs (unless $X = P$).*

Theorem 1.3. *$\mathbb{L}(Q, \perp_Q)$ is a directed union of complemented subalgebras of the CMIL $\mathbb{L}(P, \perp)$, for any subgeometry $(Q, \perp_Q)$ of the orthogeometry $(P, \perp)$.*

$(P, \perp)$ is *irreducible* if the projective geometry P is irreducible. An *irreducible dual pair* is an orthogeometry $(P, \perp)$ such that the projective geometry P has decomposition $P = P_1 \cup P_2$ into irreducible components $P_i \subseteq P_i^\perp$.

Proposition 1.4. *Every orthogeometry is uniquely an orthogonal disjoint union of irreducibles and irreducible dual pairs.*

Proofs of the above will be given in Sect.5-7 and require only [5, Ch.1-4,11,14] as background.

Corollary 1.5. *If L is an atomic CMIL then $L_f \cong \mathbb{L}(\mathbb{G}(L))$ and L is subdirectly irreducible if and only if $\mathbb{G}(L)$ is irreducible or an irreducible dual pair. The first case is characterized by L being subdirectly irreducible as a lattice.*

For the finite dimensional case the last two results are due to Schweigert [20]. Prop.8.3, Lemma 10.9, and Cor.19.12 of Petrich [18] contain essential information on the general case. Though, our proofs will not refer to these.

2. Introduction. Part II: Representations and equational theory.

A *geometric representation* of a CMIL, L, is given by an orthogeometry $(P, \perp)$ and a 0-1-lattice embedding $\varepsilon : L \to \mathbb{L}(P)$ such that $\varepsilon(a') = (\varepsilon a)^\perp$ for all $a \in L$. In particular, such represenation is faithful. Any atomic CMIL, L, has a canonical representation in $\mathbb{G}(L)$. In Sect.11 we show the following, which will be the key to equational theory of CMILs.

Theorem 2.1. *With every geometric representation $\eta\colon L \to \mathbb{Q}(P,\perp)$ of a CMIL there is an associated atomic CMIL-extension $\tilde{L}$ of $\eta(L)$ which is a sublattice of $\mathbb{L}(P)$ containing $\mathbb{L}(P,\perp)$ and which is also a subalgebra of $\mathbb{L}_c(P,\perp)$. In particular, $\mathbb{G}(\tilde{L}) = (P,\perp)$.*

In view of Birkhoff's Theorem, equational theory of algebraic structures can be seen as the study of *varieties*, i.e. classes $\mathcal{V}$ closed under the operators H, S, and P where $\mathsf{H}\,\mathcal{C}$, $\mathsf{S}\,\mathcal{C}$, and $\mathsf{P}\,\mathcal{C}$ denote the classes of all homomorphic images, subalgebras, and direct products of members of $\mathcal{C}$, respectively. $\mathsf{V}\,\mathcal{C} = \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathcal{C}$ is the smallest variety containing $\mathcal{C}$. All class operators are assumed to include isomorphic copies.

Dealing with CMILs, to pay heed to complementation (see e.g. Kadourek and Szendrei [13] for a similar approach to regular semigroups), we replace varieties by $\exists$-*varieties*, classes closed under the operators H, $\mathsf{S}_\exists$, and P where $\mathsf{S}_\exists\,\mathcal{C}$ denotes the class of all subalgebras of members of $\mathcal{C}$ which are complemented. Obviously, $\exists$-varieties are axiomatic classes. According to [11, Prop.10], $\mathsf{V}_\exists = \mathsf{H}\,\mathsf{S}_\exists\,\mathsf{P}\,\mathcal{C}$ is the smallest $\exists$-variety containing the class $\mathcal{C}$ of CMILs, the $\exists$-variety *generated* by $\mathcal{C}$.

In [22, 7, 16, 11, 10, 17] it has become apparent that geometric representations are a primary tool in the equational theory of (ortho)complemented modular lattices and regular rings (with involution).

For a class $\mathcal{G}$ of orthogeometries, denote by $\mathsf{P}_\mathrm{u}\,\mathcal{G}$, $\mathsf{U}\,\mathcal{G}$, and $\mathsf{S}_\mathrm{g}\,\mathcal{G}$ the classes of consisting of all ultraproducts, orthogonal disjoint unions, and subgeometries of members of $\mathcal{G}$. $\mathcal{G}$ is a U-*quasivariety* if it is closed under these operators. We show that U-quasivarieties are axiomatic classes, that $\mathsf{V}\,\mathcal{G} = \mathsf{S}_\mathrm{g}\,\mathsf{U}\,\mathsf{P}_\mathrm{u}\,\mathcal{G}$ is the smallest U-quasivariety containing $\mathcal{G}$, and that, for any $\exists$-variety $\mathcal{V}$ of CMILs, the class $\mathbb{G}(\mathcal{V})$ of orthogeometries $(P,\perp)$ with $\mathbb{L}(P,\perp) \in \mathcal{V}$ is a U-quasivariety (consisting of the $\mathbb{G}(L)$, $L \in \mathcal{V}$).

For a class $\mathcal{G}$ of orthogeometries let $\mathbb{L}(\mathcal{G})$ denote the class of all CMILs admitting a geometric representation within some $(P,\perp) \in \mathcal{G}$. For a class $\mathcal{G}$ of orthogeometries resp. $\mathcal{V}$ of CMILs let $\mathcal{G}_n$ ($\mathcal{G}_{<\omega}$) resp. $\mathcal{V}_n$ ($\mathcal{V}_{<\omega}$) denote the class of n-dimensional (finite dimensional) members. In Sect.16. the proofs of the following main results are completed.

Theorem 2.2. *Given a U-quasivariety $\mathcal{G}$ of orthogeometries, the following hold.*

(i) *$\mathbb{L}(\mathcal{G})$ is an $\exists$-variety of CMILs.*
(ii) *Every member of $\mathbb{L}(\mathcal{G})$ admits an atomic extension within $\mathbb{L}(\mathcal{G})$.*
(iii) *$\mathbb{L}(\mathcal{G}) = \mathsf{V}_\exists\{\mathbb{L}(P,\perp) \mid (P,\perp) \in \mathcal{G}_{<\omega}\}$.*

Corollary 2.3. *For any $\exists$-variety $\mathcal{V}$ of CMILs, $\mathcal{V} = \mathsf{V}_\exists\,\mathcal{V}_{<\omega}$ if and only if $V = \mathbb{L}(\mathcal{G})$ for some U-quasivariety $\mathcal{G}$ of orthogeometries, namely $\mathcal{G} = \mathbb{G}(\mathcal{V})$.*

In order the introduce complementation as an operation, given an $\exists$-variety $\mathcal{V}$ of CMILs, consider a formula $\alpha(x,y)$ such that $\forall x \exists y.\alpha(x,y)$ and $\forall x \forall y.\, \alpha(x,y) \Rightarrow x \oplus y = 1$ hold in $\mathcal{V}$. For $\mathcal{C} \subseteq \mathcal{V}$ let $\mathcal{C}^\alpha$ denote the class of all expansions of $L \in \mathcal{C}$ by a unary operation $x \mapsto x^c$ such that $\forall x.\, \alpha(x,x^c)$ holds. Call α an *equational* *definition of complement* for $\mathcal{V}$ if $(\mathsf{V}_{\exists}\,\mathcal{C})^{\alpha} = \mathsf{V}(\mathcal{C}^{\alpha})$ for all $\mathcal{C} \subseteq \mathcal{V}$. In particular, $x \oplus y = 1$ is such α for the class of all CMILs (cf. [11, Prop.10]) and $y = x'$ for the class of orthocomplemented modular lattices. We include also the case $\alpha = \emptyset$ where where no complementation operation is considered and $\mathcal{V}^{\emptyset} = \mathsf{V}\,\mathcal{V}$.

Proposition 2.4. *For any* U*-quasivariety of orthogeometries,* $\mathcal{V} = \mathbb{L}(\mathcal{G})$ *is recursively axiomatizable if and only if* $\mathcal{G}$ *is so. In this case, also* $\mathcal{V}^{\alpha}$ *is recursively aximatizable for any equational definition* α *of complement for* $\mathcal{V}$*.*

Corollary 2.5. *For any* U*-quasivariety* $\mathcal{G}$ *of orthogeometries and equational definition* α *of complement for* $\mathcal{V} = \mathbb{L}(G)$*, if the equational theories of the* $\mathcal{V}_n^{\alpha}$ $(n < \omega)$ *are uniformly decidable then the equational theory of* $\mathcal{V}^{\alpha}$ *is decidable.*

This will be used, elsewhere, to obtain a decision procedure for the equational theory of certain classes of ortolattices including projection ortholattices of type II_1 (resp. the class of all type I_n, $n < \omega$) von Neumann algebra factors.

Problem 2.6. *If the CMIL, L, is subdirectly irreducible as a lattice, does L have a representation within an irreducible orthogeometry? Is every* $\exists$*-variety of CMILs generated by its finite dimensional members?*

Both questions have positive answers for the case of complemented modular lattices (without involution) resp. orthocomplemented modular lattices [11, 10].

Extended preliminaries have been added on request. The typical reader of Algebra Universalis may skip most of these.

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3. Preliminaries: Projective geometries and lattices

Basic notions and results can be found in [5, Ch.1-4,11,14]. Joins in a lattice will be written as $a + b$, meets as $a \cdot b = ab$. We adhere to the usual bracket saving rule that meet has priority over join. We write $a + b = a \oplus b$ if $ab = 0$ and $a \prec b$ if b is an *upper cover* of a, i.e. $a < b$ and $a < x < b$ for no x. An *interval sublattice* is of the form $[a, b] = \{x \in L \mid a \leq x \leq b\}$. The *height* or *dimension* $\dim L$ of a lattice is the supremum of all cardinalities $|C| - 1$, C a chain of L and $\dim u = \dim[0, u]$. A *quotient* is a pair (x, y) with $x \geq y$; we write it as x/y and $\dim x/y = \dim[y, x]$. By L_{fin} (L_{cofin}) we denote the sublattice of all u with $\dim u < \aleph_0$ ($\dim[u, 1] < \aleph_0$). A *lattice homomorphism* is a map ϕ such that $\phi(a+b) = \phi a + \phi b$ and $\phi(ab) = \phi a \phi b$. In a modular lattice one has $[uv, u] \cong [v, u + v]$ via the mutally inverse isomorphisms $x \mapsto x + v$ and $y \mapsto yu$ cf. [5, Prop.1.5.4].

A lattice L with bounds $0, 1$ as constants and endowed with an involution $x \mapsto x'$, i.e. $x'' = x$ and $x \leq y$ iff $y' \leq x'$ for all $x, y \in L$, is called an IL and an MIL if it is modular. A *subalgebra* of an IL, L, is a sublattice M such that $0, 1 \in M$ and $x' \in M$ for all $x \in M$; in particular, M is again an IL. L is a *directed union* of its subalgebras L_i $(i \in I)$ if $L = \bigcup_{i \in I} L_i$ and if for any $i, j \in I$ there is $k \in I$ such that $L_i \cup L_j \subseteq L_k$.

A CML is a complemented modular lattice, i.e. has 0 and 1 and for all a there is b such that $a \oplus b = 1$. Homomorphic images and interval sublattices of CMLs are CMLs. A CMIL is an MIL which is complemented.

For any lattice L with $0, 1$, $P_L = \{p \in L \mid 0 \prec p\}$ denotes the set of *atoms* of L and $H_L = \{h \in L \mid h \prec 1\}$ the set of all *coatoms*. L is *atomic* if for any $a > 0$ there is $p \in P_L$ with $p \leq a$.

Lemma 3.1. *In an atomic CML, if $b \geq p$ for all $p \leq a$ in P_L then $b \geq a$.*

Proof. Assume $b \not\geq a$. Then $ab < a$. Choose c such that $a = ab \oplus c$ and atom $p \leq c$. Then $p \not\leq b$. □

We follow our principal reference [5, Ch.2] calling 'projective geometry' what is traditionally and more correctly called a 'projective space'. We consider a projective geometry, P, to be defined in terms of the collinearity relation $\kappa(p, q, r)$ on the point set, also denoted by P. The subspaces of P form an atomic CML under inclusion, which we denote by $\mathbb{L}(P)$. $\mathbb{L}(P)$ is a complete lattice with join $\sum_{i \in I} X_i = \bigcup\{\sum_{i \in F} X_i \mid F \subseteq I \text{ finite}\}$ and meet $\bigcap_{i \in I} X_i$. Points p are identified with singleton subspaces $\{p\}$, i.e. atoms of $\mathbb{L}(P)$. P is *irreducible* if any two points $p \neq q$ are *perspective*, i.e. there is a third point such that p, q, r are collinear. For any modular lattice L there is a canonical projective geometry on P_L with pairwise distinct points p, q, r collinear if and only if $p + q = p + r = q + r$.

Lemma 3.2. *If P, Q are projective spaces, $\phi \colon \mathbb{L}(P) \to \mathbb{L}(Q)$ a $\sum$-preserving map, and $\phi p \notin \phi X$ for any $p \in P$, $p \notin X \in \mathbb{L}(P)$, then ϕ is a lattice homomorphism.*

Proof. Given Y, we show $\phi X \cap \phi Y = 0$ for all X with $X \cap Y = 0$. For Y of finite dimension we proceed by induction: With $Y = Z \oplus p$, by modularity $(X + p) \cap Z = 0$ and

$$\phi X \cap \phi Y = \phi X \cap \phi(X + p) \cap (\phi Z + \phi p) = \phi X \cap (\phi(X + p) \cap \phi Z + \phi p) = \phi X \cap \phi p = 0.$$

The general case follows since Y is the join of its finite dimensional subspaces Y_i and $\phi Y = \bigcup_i \phi Y_i$. For arbitrary X, Y choose Z with $Y = Z \oplus X \cap Y$ and observe that, by modularity and the preceeding case,

$$\phi X \cap \phi Y = \phi X \cap (\phi(X \cap Y) + \phi Z) = \phi(X \cap Y) + \phi X \cap \phi Z = \phi(X \cap Y)$$

□

Given a set P with a collinearity relation, the *closure* $C_P(X)$ is the smallest subset Y of P such that $r \in Y$ whenever $p, q \in Y$ and p, q, r collinear. If P is a projective geometry then $C_P(X)$ is the smallest subspace containing X.

A *proper projective subgeometry* (cf. [5, Def.3.3.11]) of a projective geometry P is given by a subset Q of P endowed with the restriction of the collinearity relation such that

(i) If p, q, r and s, t, r are collinear triplets in P, but p, q, s are not collinear, and if $p, q, s, t \in Q$ then $r \in Q$.
(ii) If $X = C_Q(X)$ then $X = Q \cap C_P(X)$

Corollary 3.3. *Any proper projective subgeometry Q of a projective geometry P is a projective geometry and $\phi X = C_P(X)$ is a lattice embedding $\phi\colon \mathbb{L}(Q) \to \mathbb{L}(P)$ such that*

$$\dim \phi X = \dim X \quad \textit{and} \quad \dim P/\phi Y = \dim Q/Y \ \textit{for}\ \dim X, \dim Q/Y < \aleph_0.$$

Proof. Q is a projective geometry by (i). Observe that $\phi p = p$ for $p \in P$. ϕ perserves *sum*, obviously, and meets by Lemma 3.2. ϕ is injective by (ii). Preservation of finite dimension now follows from the fact that points are mapped to points. For Y of finite codimension choose X with $X \oplus Y = Q$ and refer to modularity. □

4. Preliminaries: Orthogeometries

A projective geometry P together with a symmetric binary relation $\perp$ between points is an *orthogeometry* (cf. [5, Def.14.1.1]) if the following properties hold true for all $p, q, r, s \in P$

(i) If $p \perp q,\ p \perp r,$ and q, r, s collinear then $p \perp s$
(ii) If $p \neq q$ and $r \not\perp p, q$ then there is t such that $t \perp r$ and p, q, t collinear
(iii) There is t such that $p \not\perp t$

Define $X \perp Y$ iff $p \perp q$ for all $p \in X$ and $q \in Y$. Also, define $X^\perp = \{q \in P \mid X \perp q\}$. Then $X^\perp \in \mathbb{L}(P)$ and

$$X \subseteq Y^\perp \ \text{ iff } \ X \perp Y \ \text{ iff } \ Y \subseteq X^\perp$$

It follows $X \subseteq X^{\perp\perp}$. The lattice $\mathbb{L}(P)$ together with this unary operation is denoted by $\mathbb{Q}(P, \perp)$. Joins in $\mathbb{Q}(P, \perp)$ are turned into meets and the *closed* subspaces $Y = X^\perp$ (equivalently, $Y^{\perp\perp} = Y$) form a complete meet-subsemilattice $\mathbb{L}_c(P, \perp)$ of $\mathbb{Q}(P, \perp)$ with involution $X \mapsto X^\perp$.

Equivalently, one can define orthogeometries as given by a *polarity*, a map $p \mapsto p^\perp$ from P to the set of coatoms (i.e *hyperplanes*) of $\mathbb{L}(P)$ such that for all $p, q \in P$

$$p \leq q^\perp \ \text{ if and only if } q \leq p^\perp$$

Each vector space with non-degenerate orthosymmetric sesquilinear form gives rise to an orthogeometry on the associated projective geometry and each irreducible orthogeometry with desarguean P and $\dim \mathbb{L}(P) \geq 3$ is isomorphic to such [5, Thm.14.1.8].

Given an orthogeometry $(P, \perp)$, a subset Q of P is *non-degenerate*, if $Q \cap Q^\perp = \emptyset$. Orthogeometries are non-degenerate and non-degenerate subspaces of P are orthogeometries under the induced orthogonality. $(P, \perp)$ is *irreducible* if P is so. $(P, \perp)$ is the *directed union* of its non-degnerate subspaces $(P_i, \perp_i)$ $(i \in I)$ if $P = \bigcup_{i \in I} P_i$ and if for any $i, j \in I$ there is $k \in I$ such that $P_i \cup P_j \subseteq P_k$.

Lemma 4.1. *For any orthogeometry $(P, \perp)$*

(i) *If $x \leq y$ in $\mathbb{L}(P)$ such that $\dim y/x < \aleph_0$ then $\dim x^\perp/y^\perp \leq \dim y/x$.*
(ii) *If $u \in \mathbb{L}_c(P, \perp)$ and $u \leq x$ in $\mathbb{L}(P)$ with $\dim x/u < \aleph_0$ then $x \in \mathbb{L}_c(P, \perp)$ and $\dim x/u = \dim u^\perp/x^\perp$.*

Proof. Concering (i) if $\dim y/x = n$ then $y = x + \sum_{i=1}^{n} p_i$ with $p_i \in P$ and $y^\perp = x^\perp \prod_{i=1}^{n} p_i^\perp$. From $\dim 1/p_i^\perp = 1$ it follows $\dim x^\perp/y^\perp \leq n$. In (ii), if also $x \in \mathbb{L}_c(P,\perp)$ then by (i) $\dim u^\perp/x^\perp \leq \dim x/u = \dim x^{\perp\perp}/u^{\perp\perp} \leq \dim u^\perp/x^\perp$ whence $\dim u^\perp/x^\perp = \dim x/u$. In general, it follows $\dim x/u \geq \dim u^\perp/x^\perp \geq \dim x^{\perp\perp}/u^{\perp\perp} = \dim x^{\perp\perp}/u \geq \dim x/u$ and $x = x^{\perp\perp} \in \mathbb{L}_c(P,\perp)$. See also Prop. 14.1.4 and 14.2.4 in [5]. □

For any IL, L, define $\mathbb{G}(L) = (P_L,\perp)$ where $p \perp q$ if and only if $q \leq p'$.

Lemma 4.2. *If L is an MIL then $\mathbb{G}(L)$ is an orthogeometry.*

Proof. We have to show that $p^\perp = \{q \in P_L \mid q \leq p'\}$ is a coatom i.e. $p^\perp \neq P$ and $r + p^\perp = 1$ in $\mathbb{L}(P_L)$ for all $r \notin p^\perp$. The first follows choosing a complement q of p' - q is an atom by modularity. Now, consider $s \neq r$ with $s \notin p^\perp$. Then by modularity $q = (s+r)p' \in P_L$ since p' is a coatom. Now, $q \in p^\perp$ and $s \leq q + r \leq p^\perp + r$. □

5. The atomic CMIL of an orthogeometry

Given an orthogeometry $(P,\perp)$ define

$$\mathbb{L}(P,\perp) = \mathbb{L}(P)_{fin} \cup \{u^\perp \mid u \in \mathbb{L}(P)_{fin}\}.$$

Lemma 5.1. *For any orthogeometry $(P,\perp)$, $\mathbb{L}(P,\perp)$ is a complemented sublattice of $\mathbb{L}(P)$ and a subalgebra of $\mathbb{L}_c(P,\perp)$. In particular, $\mathbb{L}(P,\perp)$ is an atomic CMIL with set P of atoms.*

Proof. By Lemma 4.1, $\mathbb{L}(P,\perp)$ is a subset of $\mathbb{L}_c(P,\perp)$ and, obviously, closed under $^\perp$ and finitary meets - which are taken in $\mathbb{L}(P)$. Hence, $\mathbb{L}(P,\perp)$ is a subalgebra of the IL $\mathbb{L}_c(P,\perp)$ and closed under meets in $\mathbb{L}(P)$.

On the other hand, $\{u^\perp \mid u \in \mathbb{L}(P)_{fin}\}$ is a sublattice of $\mathbb{L}(P)$ since for $u, v \in \mathbb{L}(P)_{fin}$ one has $u^\perp v^\perp = (u+v)^\perp$ and $(u^\perp + v^\perp)^{\perp\perp} = (u^{\perp\perp}v^{\perp\perp})^\perp = (uv)^\perp$ with $u+v$, $uv \in \mathbb{L}(P)_{fin}$. Moreover, $u^\perp v \in \mathbb{L}(P)_{fin}$, trivially, and $u^\perp + v \in \mathbb{L}_c(P,\perp)$ by Lemma 4.1 whence $u^\perp + v = (u^\perp + v)^{\perp\perp} = (u^{\perp\perp}v^\perp)^\perp = (uv^\perp)^\perp$ with $uv^\perp \in \mathbb{L}(P)_{fin}$. Thus, $\mathbb{L}(P,\perp)$ is a sublattice of $\mathbb{L}(P)$ whence modular.

Thus, $\mathbb{L}(P,\perp)$ is an MIL and it suffices to show that each $u \in \mathbb{L}(P)_{fin}$ admits a complement $w^\perp$ with $w \in \mathbb{L}(P)_{fin}$. This is done by induction on $\dim u$. Consider $u > 0$. Since $\bigcap_{p\in P} p^\perp = 0$, there is $p \in P$ such that $u \not\leq p^\perp$. Then $up^\perp < u$ and, by inductive hypothesis, there is $v \in \mathbb{L}(P)_{fin}$ such that $up^\perp \oplus v^\perp = 1$. In particular, $u(p^\perp v^\perp) = 0$ and, by modularity, $u + p^\perp v^\perp = u + up^\perp + p^\perp v^\perp = u + p^\perp(up^\perp + v^\perp) = u + p^\perp = 1$ whence $u \oplus p^\perp v^\perp = 1$. □

Proof. Thm.1.1. By Lemma 5.1, $L = \mathbb{L}(P,\perp)$ is a CMIL and $L_{cofin} = \{u^\perp \mid u \in L_{fin}\}$. Thus, $L = L_f$ and $(P,\perp) = \mathbb{G}(L)$. Conversely, given L, by Lemma 4.2, $(P,\perp) = \mathbb{G}(L)$ is an orthogeometry and we may identify L with a sublattice of $\mathbb{L}(P)$ such that $L_{fin} = \mathbb{L}(P)_{fin}$ and $L = \mathbb{L}(P,\perp)$, □

Proof. Thm.1.2. Define $I = \{v \in \mathbb{L}(P)_{fin} \mid v \oplus v^\perp = 1\}$. Then $L_v = [0,v] \cup [v^\perp, 1]$ is a subalgebra of $\mathbb{L}(P,\perp)$. As a lattice, L_v is the direct product of $[0,v]$ and a 1- or 2-element lattice, whence complemented. For $x \leq v$ and $y \in \{0, v^\perp\}$ one gets

$(x+y)^\perp = x^\perp y^\perp = vx^\perp + v^\perp y^\perp$, i.e. L_v is an MIL direct product. Also, the subspace $\{p \in P \mid p \leq v\}$ is non-degenerate since $vp^\perp \prec v$.

Consider $u \in \mathbb{L}(P)_{fin}$. Choose v as a complement of $u+u^\perp$ in $[u,1]$, in particular $v \in \mathbb{L}(P)_{fin}$. Then $v^\perp \leq u^\perp$ and $vv^\perp = v(u+u^\perp) = 0$ whence $v + v^\perp = 1$. Thus, $v \in I$.

Now, given any finite subset X of $\mathbb{L}(P,\perp)$ let $Y = X \cap \mathbb{L}(P)_{fin}$ and $u = \sum_{y\in Y} y + \sum_{y \in X\setminus Y} y^\perp$ and choose $v \in I$ as above. Then the subalgebra generated by X is contained in L_v. Also $L_{v_1} \cup L_{v_2} \subseteq L_v$ choosing v for $u = v_1 + v_2$, suitably. Therefore, $\mathbb{L}(P,\perp)$ is the directed union of its subalgebras L_v and P the directed union of the non-degenerate subspaces $\{p \in P \mid p \leq v\}$, $v \in I$. □

6. Subgeometries

An orthogeometry $(Q, \perp_Q)$ is a *subgeometry* of the orthogeometry $(P,\perp)$ if Q is a proper projective subgeometry of P and $\perp_Q$ the induced orthogonality.

Lemma 6.1. *A proper projective subgeometry Q of an orthogeometry $(P,\perp)$ induces a subgeometry if and only if*

(iii) *If $p,q,r \in Q$, $q \not\perp p$, and $r \not\perp p$ then there is $s \in Q$ such that $s \perp p$ and r,s,q are collinear.*

In a particular, every non-degenerate subspace of $(P,\perp)$ is a subgeometry.

Proof. For $p,q \in Q$ we have $p^{\perp_Q} = Q \cap p^\perp$ a hyperplane of Q by (iii) and $p \leq q^{\perp_Q}$ iff $p \leq q^\perp$ iff $q \leq p^\perp$ iff $q \leq p^{\perp_Q}$. □

Proof. Thm.1.3. Consider a subgeometry Q such that $C_P(Q) = P$. We claim that $\phi X = C_P(X)$ is an embedding $\phi\colon \mathbb{L}(Q,\perp_Q) \to \mathbb{L}(P,\perp)$. In view of Cor.3.3 it suffices to a show that $\phi(X^{\perp_Q}) = (\phi X)^\perp$ for $X \in \mathbb{L}(Q)_{fin}$. Now with Lemma 4.1

$$\dim P/\phi(X^{\perp_Q}) = \dim Q/X^{\perp_Q} = \dim X = \dim \phi X = \dim P/(\phi X)^\perp < \aleph_0$$

and the claims follows since $\phi(X^{\perp_Q}) \subseteq (\phi X)^\perp$.

Now, let Q be a subspace of P. Consider $X \in \mathbb{L}(Q)_{fin}$ such that $X \oplus X^{\perp_Q} = Q$ and observe that $X^{\perp_Q} = Q \cap X^\perp$ and

$$\dim Q/(Q\cap X^\perp) = \dim X = \dim P/X^\perp < \aleph_0.$$

It follows $Q + X^\perp = P$ whence $X + X^\perp = X + Q\cap X^\perp + X^\perp = Q + X^\perp = P$ and $X \cap X^\perp = 0$. We claim that the subalgebras $L'_X = [0,X] \cup [X^{\perp_Q}, Q]$ of $\mathbb{L}(Q,\perp_Q)$ and $[0,X] \cup [X^\perp, P]$ of $\mathbb{L}(P,\perp)$ are isomorphic. Define

$$\phi(Z) = Z,\ \ \phi(Z^\perp) = Z^\perp \cap Q \quad \text{for } Z \in [0,X].$$

$$\phi\colon [X^\perp, P] \to [Q \cap Y^\perp, Q]$$

is an isomorphism by modularity. Moreover, for $Z \in [0,X]$, $W^\perp \in [X^\perp, P]$

$$\phi(Z^\perp) = Z^\perp \cap Q = \phi(Z)^{\perp_Q}$$

$$\phi Z \cap \phi W^\perp = Z \cap W^\perp = \phi(Z \cap W^\perp).$$

Thus, ϕ is an isomorphism. By Theorem 1.2 we obtain $\mathbb{L}(Q, \perp_Q)$ as a direct union of complemented subalgebras of $\mathbb{L}(P, \perp)$.

Now, given any subgeometry Q, we have Q a subgeometry of the subspace $R = C_P(Q)$ of P and may combine the two cases since $R = C_R(Q)$. $\square$

Consider an orthogeometry $(P, \perp)$ and subspaces $U \subseteq V$ of P with a non-degenerate difference set $Q = V \setminus U$ such that $U \subseteq V^{\perp}$. Then one obtains the *subquotient* V/U having point set $\{p + U \mid p \in Q\}$, orthogonality defined by $p+U \perp' q+U$ if and only if $p+U \perp q+U$, and pairwise distinct $p+U$, $q+U$, $r+U$ collinear if and only if p, q, r are collinear.

Lemma 6.2. *Any subquotient of an orthogeometry is an orthogeometry and* $p+U \perp' q + U$ *if and only if* $p \perp q$. *The lattice* $\mathbb{L}(V/U)$ *is canonically isomorphic to the interval* $[U, V]$ *of* $\mathbb{L}(P)$. *If* V *is irreducible then so is* V/U. *Moreover,* $(V/U, \perp')$ *is isomorphic to the subspace* $(W, \perp_W)$ *for any* $W \in \mathbb{L}(P)$ *with* $V = U \oplus W$.

Proof. As far as projective geometries are concerned we refer to [5, sect.2.6]. From $U \subseteq V^{\perp}$ we have $p \perp q$ if and only if $p + U \perp' q + U$. By modularity, $V \cap (p + U)^{\perp}$ is a coatom of $[U, V]$ for any $p \in Q$ and it follows that $p + U \mapsto V \cap (p + U)^{\perp}$ is a polarity. Thus, V/U is an orthogeometry. Also, if $p + U \neq q + U$ then p, q cannot be collinear with any point of U. Thus, V/U is irreducible if P is.

For the last claim observe that, due for any $p \in Q$ there is unique $p \in W$ such that $p + U = q + U$, namely $q = W \cap (p + U)$. $\square$

7. Orthogonal decomposition

Given orthogeometries $(P_i, \perp_i)$ $(i \in I)$ with pairwise disjoint point sets P_i, the union of the point sets endowed with the union of the collinearity relations is a projective geometry $P = \bigcup_{i\in I} P_i$, the *disjoint union* or *coproduct* (cf. [5, sect.6.4]). Defining

$$p \perp q \text{ if and only if } p \perp_i q \text{ for some } i \in I \text{ or } p \in P_i,\ q \in P_j \text{ with } i \neq j$$

yields the *orthogonal disjoint union* $(P, \perp) = \bigcup^{\perp}_{i\in I}(P_i, \perp_i)$. In general, the point sets P_i have to be replaced by pairwise disjoint copies.

Lemma 7.1. *The orthogonal disjoint union* $(P, \perp)$ *of orthogeometries* $(P_i, \perp_i)$ *is an orthogeometry and one has an isomorphism*

$$\phi\colon\ \mathbb{Q}(P, \perp) \to \prod_{i\in I} \mathbb{Q}(P_i, \perp_i) \text{ given by } \phi(X) = (X \cap P_i \mid i \in I)$$

Proof. The axioms are easily verified. Also, ϕ is order preserving with inverse $\phi^{-1}(X_i \mid i \in I) = \bigcup_{i\in I} X_i$ cf. [5, Prop.2.7.7]. Finally, for $X_i \subseteq P_i$ one has $X_i^{\perp} = X_i^{\perp_i} \cup \bigcup_{j\neq i} P_j$ whence $X_i^{\perp} \cap P_i = X_i^{\perp_i}$. $\square$

A *dual pair* $(P, \perp)$ is given by a pair P_1, P_2 of projective geometries such that P is the disjoint union of P_1 and P_2 and maps $\alpha_i\colon P_i \to H_j$ ({i,j}={1,2}), where H_i is the set of coatoms of $\mathbb{L}(P_i)$, such that

$$\text{for } \{i, j\} = \{1, 2\} \text{ and } p \in P_i,\ q \in P_j: \quad q \leq \alpha_i p \text{ if and only if } p \leq \alpha_j q$$

$$p \perp q \text{ if and only if } \quad p,q \in P_i \quad \text{or} \quad p \in P_i,\ q \in P_j, \text{and } q \leq \alpha_i p \\ \text{for some } i \text{ and } \{i,j\} = \{1,2\}$$

The dual pair is *irreducible* if P_1 and P_2 are irreducible.

For P_i associated with vector spaces V_i over division rings K_i. Any non-singular sesquilinear form w.r.t. an anti-isomorphism from K_2 to K_1 induces a dual pair on the disjoint union $P_1 \cup P_2$. For $\dim V_1 \geq 3$ all dual pairs on $P_1 \cup P_2$ arise in this way cf [5, Prop.11.5.6].

Given CMLs M_1, M_2 and a dual isomorphism $\alpha\colon M_1 \to M_2$ the lattice $M = M_1 \times M_2$ becomes an CMIL with the *exchange involution*

$$(a,b)' = (\alpha^{-1}b, \alpha a)$$

Observe that for points in P_M one has $x \perp y$ if and only if $x,y \in M_1 \times \{0\}$ or $x,y \in \{0\} \times M_2$ or $\{x,y\} = \{(p,0),(0,q)\}$ with $p \in P_{M_1}$, $q \in P_{M_2}$ and $\alpha p \geq q$.

Lemma 7.2. *For any CMIL $M = M_1 \times M_2$ with exchange involution, P_M is a dual pair of $P_1 = P_{M_1} \times \{0\}$ and $P_2 = \{0\} \times P_{M_2}$. Every dual pair is an orthogeometry. An orthogeometry $(P,\perp)$ is a dual pair if and only if P is a disjoint union $P = P_1 \cup P_2$ of subspaces $P_1 \subseteq P_1^\perp$ and $P_2 \subseteq P_2^\perp$.*

Proof. Considering the first claim, the projective geometry P_M is the disjoint union of its subspaces P_1 and P_2. P_i is isomorphic to P_{M_i}. Define $\alpha_1(p,0) = \{(0,q) \in P_2 \mid q \leq \alpha p\}$ for $(p,0) \in P_1$ and $\alpha_2(0,q) = \{(p,0) \in P_2 \mid p \leq \alpha^{-1}q\}$ for $(0,q) \in P_2$. Clearly, $(0,q) \leq \alpha_1(p,0)$ if and only if $(p,0) \leq \alpha_2(0,q)$. We claim that $\alpha_1(p,0)$ is a coatom of $\mathbb{L}(P_2)$. Consider $r \neq s$ in P_{M_2} with $r,s \not\leq \alpha p$. Since $c = \alpha p$ is a coatom of M_2, one has $q = c(r+s) \in P_{M_2}$ whence $s \leq r + q \leq r + \alpha p$ in $\mathbb{L}(P_{M_2})$. By symmetry (using α^{-1}), $\alpha_2(0,q)$ is a coatom of $\mathbb{L}(P_1)$. Thus, $P_1, P_2, \alpha_1, \alpha_2$ define a dual pair on P_M with $p \perp q$ if and only if $p \perp_M q$.

If $(P,\perp)$ is given as a dual pair then $p \mapsto P_i \cup \alpha_{ij}p$ $(p \in P_i)$ is a polarity. Conversely, define $\alpha_{ij}p = p^\perp$ for $p \in P_i$. □

Proof. Prop.1.4. Let Q_i $(i \in I)$ denote the irreducible components of the projective geometry P. We claim that $q_1 \neq q_2$ in P are collinear if there are p_1, p_2 in some Q_i such that $q_j \not\perp p_j$ for $j = 1,2$. More precisely, there is $r \leq p_1 + p_2$ with $r \not\perp q_1, q_2$ whence $r^\perp(q_1 + q_2)$ a third point on $q_1 + q_2$. Indeed, if $p_1 = p_2$ choose $r = p_1$. If $p_1 \neq p_2$ choose a third point $s \leq p_1 + p_2$. If $s \not\perp q_1, q_2$ then put $r = s$. Otherwise, e.g. $s \perp q_1$ whence $p_2 \not\perp q_1$ and we may choose $r = p_2$.

Thus, for each $i \in I$ there is a unique component $Q_{\lambda i} = \{q \mid q \not\perp Q_i\}$ of P and $Q_i \perp \bigcup_{j \neq i, \lambda i} Q_j$. Obviously, if $\lambda i \neq i$ then $\lambda^2 i = i$. Choose $I_0 \subseteq I$ such $i \in I_0$ if $\lambda i = i$ and, otherwise, $i \in I_0$ if and only if $\lambda i \notin I_0$. Then P is an orthogonal disjoint union of the $P_i = Q_i \cup Q_{\lambda i}$ $(i \in I_0)$ with restriction $\perp_i$ of $\perp$ and the $(P_i, \perp_i)$ are non-degenerate since so is $(P,\perp)$. Thus, the $(P_i,\perp_i)$ $(i \in I_0)$ are orthogeometries, irreducible for $\lambda i = i$ and irreducible dual pairs for $\lambda i \neq i$. □

An orthogeometry is *proper* if and only if it is an orthogonal disjoint union of irreducibles.

Corollary 7.3. *An orthogeometry $(P,\perp)$ is proper if and only if $p \not\perp q$ implies that $p = q$ or p,q perspective.*

8. Preliminaries: Congruence relations

Cf. [4, Ch.10]. A *congruence relation* on a lattice is an equivalence relation θ (we write $a\,\theta\,b$ or, equivalently, $(a,b)\in\theta$) such that

$$a_1\,\theta\,b_1 \wedge a_2\,\theta\,b_2 \;\Rightarrow\; (a_1+a_2)\,\theta\,(b_1+b_2) \;\wedge\; (a_1a_2)\,\theta\,(b_1b_2)$$

For a congruence relation on an IL one requires, in addition, that

$$a\,\theta\,b \;\Rightarrow\; a'\,\theta\,b'$$

The congruences of L form a complete lattice under inclusion. For each quotient a/b there is a smallest congruence $\theta(a/b)$ containing a/b. A congruence θ is *minimal* if the identity relation is the only congruence properly contained in θ. A lattice resp. an IL is *subdirectly irreducible* if it has a unique minimal congruence relation, i.e. a smallest congruence relation distinct from the identity relation.

Proposition 8.1. *Let L be a modular lattice.*

(i) *Any congruence relation θ on L is uniquely determined by the set $Q(\theta)$ of quotients in θ: $a\,\theta\,b$ iff $(a+b)/(ab)\in Q(\theta)$.*
(ii) *If Q is a set of quotients of L then $Q=Q(\theta)$ for some congruence relation θ if and only if the following hold: $a/(ab)\in\theta$ iff $(a+b)/b\in\theta$; $c/d\in\theta$ if $a\geq c\geq d\geq b$ and $a/b\in\theta$; $a/b\in\theta$ if $a/c\in\theta$ and $c/b\in\theta$.*
(iii) *If L is atomic then L is subdirectly irreducible iff $\mathbb{L}(P_L)$ is irreducible.*

Proof. If $a\,\theta\,b$ then $ab\,\theta\,aa=a=a+a\,\theta\,a+b$ whence $ab\,\theta\,a+b$. The converse is obvious as is the necessity of the conditions in (ii). Conversely, 10.2 and 10.3 in [4] combine to prove that the conditions are sufficient. (iii) is [4, 13.2]. □

Corollary 8.2. *In an atomic CML resp. CMIL any proper congruence relation contains a minimal one and any minimal congruence relation is generated by a quotient $p/0$ where p is an atom.*

Proof. Lemma 3.1. □

Corollary 8.3. *In any modular lattice M*

$$x\,\mu\,y \quad \textit{iff} \quad \dim(x+y)/(xy)<\aleph_0$$

defines a congruence relation and it holds

$$\begin{aligned} x\,\mu\,y \quad &\textit{iff} \quad \dim z/x<\aleph_0 \textit{ and } \dim z/y<\aleph_0 \textit{ for some } z\geq x,y\\ &\textit{iff} \quad \dim x/u<\aleph_0 \textit{ and } \dim y/u<\aleph_0 \textit{ for some } u\leq x,y\end{aligned}$$

Proof. Prop.8.1(ii) applies to the set of quotients x/y with $\dim x/y<\aleph_0$. The equivalent descriptions are immediate by modularity. □

9. Subdirect decomposition

Proposition 9.1. *An orthogeometry $(P,\perp)$ is irreducible if and only if the lattice $\mathbb{L}(P,\perp)$ is subdirectly irreducible; it is an irreducible dual pair if and only $\mathbb{L}(P,\perp)$ is subdirectly irreducible as a IL but subdirectly reducible as a lattice.*

Proof. If P is irreducible then the lattices $\mathbb{L}(P, \perp)$ and $\mathbb{L}(P)$ are subdirectly irreducible since any two points are perspective. If the lattice $\mathbb{L}(P, \perp)$ is subdirectly irreducible then so is $\mathbb{L}(P)$ by Corollary 8.2 and P is irreducible by Prop.8.1(iii).

If $(P, \perp)$ is an irreducible dual pair, given points $p \neq q$, either p, q are perspective or there is r such that $r^\perp \not\geq q$ and $r = p$ or r perspective to p. But then the pairs $p/0$, $r/0$, $1/r^\perp$, and $q/0$ all generate the same congruence relation of the IL. Thus, there is a unique minimal congruence relation and the IL $\mathbb{L}(P, \perp)$ is subdirectly irreducible. In view of the first claim, $\mathbb{L}(P, \perp)$ is subdirectly reducible as a lattice.

Conversely, assume that the CMIL has unique minimal congruence μ and consider the decomposition according to Prop.1.4 into $(P_i, \perp_i)$ $(i \in I)$. If $p \in P_i$ and $q \in P_j$ with $i \neq j$ then $\mu = \theta(p/0) = \theta(q/0)$ by Lemma 8.2. On the other hand, by Corollary 7.1 there is a surjective homomorphism $\pi_i\colon\ \mathbb{Q}(P, \perp) \to \mathbb{Q}(P_i, \perp_i)$ given by $\pi_i X = X \cap P_i$ and $a\,\theta\,b$ iff $\pi_i a = \pi_i b$ defines a congruence relation θ on the CMIL $\mathbb{L}(P, \perp)$ such that $q/0 \in \theta$ but $p/0 \notin \theta$. By uniqueness of μ it follows $\mu \subseteq \theta$, a contradiction. Thus, $|I| = 1$. $(P, \perp)$ must be a dual pair since the lattice $\mathbb{L}(P, \perp)$ would be subdirectly irreducible, otherwise. $\square$

10. Preliminaries: Representations

For the following compare [4, Thm.13.1]. The proof is an easy consequence of modularity.

Lemma 10.1. *Let M be a modular lattice with $0, 1$ and $a, b, c \in M$ such that $b = c \oplus ab$. Then for any $p \in P_M$ such that $p \leq a + b$, $p \not\leq a$, and $p \not\leq b$ there are atoms $q \leq a$ and $r \leq c$ such that p, q, r are collinear, namely $q = a(p + c)$, $r = c(p + a)$. Moreover, $a \oplus d = 1$ for every complement of $a + b$ in $[c, 1]$ and $q = a(p + d)$ and $r = d(p + a)$.*

A *representation* of a 0-1-lattice L consists of a projective geometry P and a 0-1-lattice embedding $\eta\colon L \to \mathbb{L}(P)$. Given a 0-lattice homomorphism $\phi\colon L \to M$ and a subset Q of P_M we say that

Q is *L-ϕ-closed* if $(\phi a)(p + \phi d) \in Q$ for all $p \in Q$ and $a, d \in L$ such that $p \not\leq \phi d$, $ad = 0$, and $a + d = 1$

Q is *L-ϕ-dense* if for each $c > 0$ in L there is $p \in Q$ with $p \leq \phi c$.

Of course, P_M is L-ϕ-closed for any 0-lattice homomorphism $\phi\colon L \to M$. If ϕ is the identity map we speak just of *L-closed* and *L-dense.*

Lemma 10.2. *Let L be a CML, M a modular lattice, $\phi\colon L \to M$ a 0-lattice homomorphism, and Q an L-ϕ-dense and L-ϕ-closed subspace of P_M with $q \leq \phi 1$ for all $q \in Q$. Then one obtains a representation*

$$\eta\colon L \to \mathbb{L}(Q) \quad \textit{with} \quad \eta a = \{p \in Q \mid p \leq \phi a\}.$$

Proof. We may assume $\phi 1 = 1_M$. Clearly, η is a 0-1-preserving meet homomorphism. Consider $p \leq \eta(a + b)$. Nothing is to be done if $p \leq \phi a$ or $p \leq \phi b$. Choose c such that $b = c \oplus ab$ (whence $ac = 0$) and $d \geq c$ as a complement of a. Since these relations are preserved under ϕ, by Lemma 10.1 there are collinear p, q, r in

P with $q = (\phi a)(p + \phi d)$, $r \leq \phi c \leq \phi b$, and $r = (\phi d)(p + \phi a)$ whence $q, r \in Q$ by L-ϕ-closedness. Thus $p \in \eta a + \eta b$ in all cases. Therefore, η is a homomorphism. If $a < b$ in L, choosing a complement c of a in $[0, b]$ we have $c > 0$ and some $p \leq \phi c$ in Q by L-ϕ-density. It follows $p \leq \phi b$ and $p \not\leq \phi a$ whence $\eta a < \eta b$. □

Corollary 10.3. (Frink). *Every CML, L, admits a representation.*

Proof. Choose M as the lattice of all filters of L, ordered by dual inclusion, and $P = P_M$. Considering L embedded into M via $\phi a = \{x \in L \mid x \geq a\}$, P_M is L-dense due to the Ultrafilter Theorem [12, Lemma 8.5.5]. □

A *geometric representation* of an MIL, L, is a 0-1-lattice representation $\eta\colon L \to \mathbb{L}(P)$ where $(P, \perp)$ is an orthogeometry such that

$$\eta a' = (\eta a)^{\perp} \quad \text{for all } a \in L,$$

i.e. $\eta\colon L \to \mathbb{Q}(P, \perp)$ is an embedding. In particular, each ηa is closed and $\eta\colon L \to \mathbb{L}_c(P, \perp)$ is an embedding, too.

Lemma 10.4. *Each atomic CMIL, L, has the geometric representation*

$$\varepsilon\colon L \to \mathbb{G}(L), \quad \varepsilon a = \{p \in P \mid p \leq a\}$$

Proof. ε is a lattice representation by Lemma 10.2. Clearly, $\varepsilon a' \leq (\varepsilon a)^{\perp}$. Consider $p \perp \varepsilon a$, i.e. $p \leq q'$ for all $q \leq a$. Assume $p \not\leq a'$. Since L is also coatomic, there is some coatom h such that $h \geq a'$ and $h \not\geq p$. With $q = h' \in P$ it follows $q \leq a$ and $p \not\leq q'$, a contradiction. □

Proof. Cor.1.5. In Lemma 10.4 one has $\varepsilon(L_{fin}) = \mathbb{L}(P, \perp)_{fin}$ and this implies $\varepsilon(L_f) = \mathbb{L}(P, \perp)$. The second claim follows directly from Prop.9.1 and Lemmas 10.4 and 8.2. □

Proposition 10.5. *Every CMIL admits a geometric representation in some dual pair.*

Proof. By Cor.10.3 every CML, L admits an embedding $\phi\colon L \to M_1$ into some atomic and coatomic CML M_1. Let M_2 be the dual and $\alpha = \mathsf{id}$. Define $\varepsilon x = (\phi x, \phi x')$ and apply Lemmas 10.4 and 7.2. □

11. Atomic extension

Proof. Thm.2.1. Given any subset L of $M = \mathbb{L}(P)$, $C = \mathbb{L}_c(P, \perp)$, and the congruence μ of Cor.8.3 on M we define

$$\tilde{L} = \{x \in C \mid x \,\mu\, u \text{ for some } u \in L\}$$

Consider the conditions

$$\begin{array}{llr}
(a) & ab \in \tilde{L} & \text{for all } a, b \in L \\
(b) & a + b \in \tilde{L},\ a^{\perp} + b^{\perp} \in C & \text{for all } a, b \in L \\
(c) & a^{\perp} \in \tilde{L} & \text{for all } a \in L
\end{array}$$

We claim that

(a) implies that $\tilde{L}$ is meet-closed in M and C, simultaneously
(b) implies that $\tilde{L}$ is join-closed in M and C, simultaneously
(c) implies that $\tilde{L}$ is closed under $x \mapsto x^\perp$

This is basically Lemma 2 of [2]. For the proof observe that

$$\tilde{L} \quad = \quad \{x \in C \mid \exists a \in L.\, \exists y, z \in C.\ y \leq z,\ a, x \in [y, z] \text{ and } \dim z/y < \aleph_0\}$$

In particular,

$$x, y, z \in C,\ a \in \tilde{L},\ y \leq z,\ a, x \in [y, z],\ \text{and } \dim z/y < \aleph_0 \quad \text{jointly imply} \quad x \in \tilde{L}$$

Indeed, for $x \,\mu\, a$ in C we have also $y = xa$ and, by Lemma 4.1(ii), $z = x + a$ in C and $y \,\mu\, z$. Assuming (c) and $a \in L$, with Lemma 4.1(i) we conclude $y^\perp \,\mu\, z^\perp$ whence $x^\perp \,\mu\, a^\perp$ and so $x^\perp \in \tilde{L}$.

Now, consider $y \leq a \leq z$ and $v \leq b \leq w$ in C, $\dim z/y < \aleph_0$, and $\dim w/v < \aleph_0$. Let $x \in [y, z]$ and $u \in [v, w]$. By the congruence properties of μ one has $xu \,\mu\, ab$ and $x + u \,\mu\, a + b$. By Lemma 4.1(ii) $x, u, xu \in C$. Thus $xu \in \tilde{L}$ if $a, b \in L$ and (a). Moreover $x + u \in C$ provided that $x \geq a$, $u \geq b$ and $a + b \in C$.

Now suppose (b) and $a, b \in L$. We show $y + v \in C$ by induction on $\dim a/y + \dim b/v$. In doing so, by Lemma 4.1(ii) we may assume that we have $y \prec t \leq a$ with t and $t+v$ in C. Considering the sublattice of M generated by y, t, v two cases are possible: firstly, $y + v = t + v$ with nothing left to do; secondly, $y + v \prec t + v$. If we had $v^\perp y^\perp \leq t^\perp$ then by modularity $v^\perp + t^\perp < v^\perp + y^\perp$. Now $a^\perp \leq t^\perp \leq y^\perp$, $b^\perp \leq v^\perp$ and $a^\perp + b^\perp \in C$ by hypothesis. Thus, as shown above, we would have $v^\perp + t^\perp$ and $v^\perp + y^\perp$ in C. It would follow $vt = (v^\perp + t^\perp)^\perp < (v^\perp + y^\perp)^\perp = vy$, a contradiction. So we may choose $p \in P$ such that $p \leq v^\perp y^\perp$, $p \not\leq t^\perp$. Then $p^\perp \geq y + v$, $p^\perp \not\geq t + v$. Consequently, $y + v = (t + v)p^\perp \in C$. With Lemma 4.1(ii) it follows $x + u \in C$ for all $x \in [y, z]$, $u \in [v, w]$ whence $x + u \in \tilde{L}$ since $a + b \in \tilde{L}$ by hypothesis.

Finally, we show by induction on $\dim[a, u]$. that for any $u \in L$ and $u \geq a \,\mu\, u$ there is $b \in \tilde{L}$ such that $u = a \oplus b$. Namely, assuming $u = a \oplus b$ and c a lower cover of a, choose an atom p such that $p \leq a$, $p \not\leq c$. Then, by modularity $u = c \oplus (b + p)$. It follows that for $u \oplus v = 1$ in L and $a \,\mu\, u$ there is $v \leq c \,\mu\, v$ such that $1 = ua \oplus c$. Indeed, choose $u = ua \oplus x$ in $\tilde{L}$ and put $c = v + x$. Dually, there is $v \,\mu\, d \leq v$ such that $(a + u) \oplus d = 1$. Now, choose b as a complement of $ac + d$ in $[d, c]$. Then $a + b = a + ac + d + b = a + c = 1$. By modulartiy, $ac + d = c(a + d)$ whence $ab = 0$ by duality. This shows that $\tilde{L}$ is complemented.

Given the geometric representation $\eta\colon L \to \mathbb{Q}(P, \perp) = M$, identify L with $\eta(L)$. Then (a), (b), and (c) are satisfied, obviously, so that $\tilde{L}$ is an CMIL. Clearly, $\mathbb{G}(\tilde{L}) = (P, \perp)$. □

Corollary 11.1. *A CMIL admits a proper geometric representation if and only if it admits an atomic CMIL-extension which is a subdirect product of factors which are subdirectly irreducible as lattices.*

Proof. This follows from Thms.1.4, 1.5, and 2.1. □

12. Preliminaries: Model Theory

Cf. [12, Ch.1,2,5,8]. Given an infinite set of variables for elements (with specified sort), the *atomic formulas* are of the form $t_1 = t_2$ or $R(t_1, \ldots, t_n)$ where R is a relation symbol and the t_i are terms in the operation symbols. The formulas of the *first order language* are then built using the propositional junctors $\wedge, \vee, \neg, \rightarrow$ and quantifiers $\forall x$ resp. $\exists x$. A *sentence* if a formula without free variables. Considering a structure, A, with relations and operations corresponding to the given symbols one defines, in the obviuos way, *validity* in A of a formula $\sigma(a_1, \ldots, a_n)$ with elements $a_i \in A$ substituted for the free variables. A set of sentences is *valid* in a class $\mathcal{C}$ if each $\sigma \in \Sigma$ is valid in all $A \in \mathcal{C}$. A class $\mathcal{C}$ of structures is *axiomatic* if there is a set Σ of sentences, a set of *axioms* for $\mathcal{C}$, such that a structure A belongs to $\mathcal{C}$ if and only if Σ is valid in A. $\mathcal{C}$ is then the *model class* of Σ.

Given structures A, B, a map $\phi\colon A \to B$ is an *elementary embedding* if for any formula $\sigma(x_1, \ldots, x_n)$ and any $a_i \in A$ one has $\sigma(a_1, \ldots, a_n)$ valid in A if and only if $\sigma(\phi a_1, \ldots, \phi a_n)$ is valid in B.

We rely also on the following concept of saturation. Given a first order structure, A, add all elements of A as constants (also called *parameters*) and consider sets $\Sigma = \Sigma(x_1, \ldots, x_n)$ of first order formulas with free variables among $x_1, \ldots x_n$. Σ is said to be *realized* in A if there are $a_1, \ldots, a_n$ in A such that $\sigma(a_1, \ldots, a_n)$ holds in A for all $\sigma \in \Sigma$. Call Σ *finitely realized* in A if each of its finite subsets is realized in A. Finally, call A *ω-saturated* over a subset B if every Σ which is finitely realized in A and contains only finitely many constants not in B is also realized in A.

Proposition 12.1. *For every first order structure B there is an elementary extension A which is ω-saturated over B.*

Proof. According to [12, Cor.10.2.2] one may choose A as an elementary extension of B which is κ-saturated for some infinite cardinal $\kappa > |B|$. □

Observe that 2-sorted structures can be considered as 1-sorted, assuming the sorts to be disjoint and introducing predicates for the sorts.

Proposition 12.2. *Geometric representations $\varepsilon\colon L \to \mathbb{Q}(P, \perp)$ of MILs may be equivalently described as 2-sorted structures $(L, P, \perp, \leq)$ with underlying sets L and P, carrying the structure of an MIL and an orthogeometry, respectively, and a relation $\leq$ between points and lattice elements such that*

$$\varepsilon a = \{p \in P \mid p \leq a\}.$$

If $\mathcal{G}$ is an axiomatic class of orthogeometries then the associated class of such structures is axiomatic, too. If $(L^*, P^*, \perp^*, \leq^*)$ is an elementary extension of $(L, P, \perp, \leq)$ then $(P^*, \perp^*)$ is an elementary extension of $(P, \perp)$.

Proof. E.g. the fact that $\varepsilon a \in \mathbb{L}(P)$ can be expressed by the axiom

$$\forall p \forall q \forall r.\ \kappa(p, q, r) \wedge p \leq a \wedge q \leq r \Rightarrow r \leq a$$

□

13. Representation of homomorphic images

Theorem 13.1. *If the CMIL, L, has a representation in the orthogeometry $(P, \perp)$ then every homomorphic image has a representation in a subquotient V/U of some elementary extension of $(P, \perp)$ where V is closed and $U = V \cap V^\perp$.*

Lemma 13.2. *Let L, M be CMLs, L a 0-sublattice of M, θ a congruence on L, and $F = \{x \in L \mid x \,\theta\, 1\}$. Then F is a filter of L and for $a \geq b$ in L one has $a \,\theta\, b$ if and only if $b = ac$ for some $c \in F$. Also, the p in P with $p \leq x$ for all $x \in F$ form an L-closed subspace of P_M.*

Proof. Clearly, F is a filter and Q is a subspace. Given $a \geq b$ choose c as a complement of a in $[0, b]$. Then $a \,\theta\, b$ iff $1 \,\theta\, c$ iff $c \in F$. Now, consider $p \in Q$, $p \leq a + d$, $p \not\leq d$, $ad = 0$, and $a + d = 1$. Let $q = a(p + d)$ and $r = d(p + a)$. By Lemma 10.1, p, q, r are collinear. Consider $x \in F$, so $p \leq x$. Let

$$y = (a + xd)(d + x) \geq q, \quad z = (d + xa)(a + x) \geq r$$

By modularity, x, y, z coincide or are the atoms of a sublattice of height 2. In particular, all quotients of that sublattice are in θ whence $z \in F$. From $p \in Q$ it follows $p \leq z$ and thus $q \leq p + r \leq z$. Hence $q \leq yz \leq x$ and $q \leq x$. This holds for all $x \in F$ whence $q \in Q$. □

For the proof of Thm.13.1, consider a geometric representation $(L, P_0, \perp_0, \leq_0)$ in the sense of Prop.12.2. Let $(L^*, P, \perp, \leq)$ be an elementary extension which is ω-saturated over (L, P_0). Such exists by Prop.12.1. Let θ be a nontrivial congruence on L with associated filter $F = \{a \in L \mid a \,\theta\, 1\}$ and ideal $I = \{a \in L \mid a \,\theta\, 0\} = \{a' \mid a \in F\}$. Observe that $F \cap I = \emptyset$. Consider L a sublattice of $\mathbb{L}(P)$ such $a = \sum\{p \in P \mid p \leq a\}$ and $a' = a^\perp$. Define

$$\varepsilon F = \{p \in P \mid p \leq a \text{ for all } a \in F\}, \quad \varepsilon I = \{p \in P \mid p \leq a \text{ for some } a \in I\}$$

$$U = \varepsilon F \cap \varepsilon I, \quad Q = \varepsilon F \setminus U.$$

Claim 13.3. *$\varepsilon F \in \mathbb{L}_c(P, \perp)$, $U = \varepsilon F \cap (\varepsilon F)^\perp$, and εF is an L-closed subspace of P.*

Proof. $\varepsilon I \in \mathbb{L}(P)$, obviously and $\varepsilon F \in \mathbb{L}_c(P, \perp)$ as the intersection of the closed subspaces $\varepsilon a \, (a \in F)$. εF is L-closed by Lemma 13.2. Consider $p \in \varepsilon I$ and $q \in \varepsilon F$. Then $p \leq a$ for some $a \in I$ and $q \leq a'$ since $a' \in F$. Thus, $q \leq p^\perp$ and it follows $\varepsilon F \perp \varepsilon I$.

We will show $U = \varepsilon F \cap (\varepsilon F)^\perp$. Consider $p \in \varepsilon F$, $p \notin \varepsilon I$. Then, given $a \in F$ we have $p \not\leq a'$, i.e. $a \not\leq p^\perp$ and there is $q \in P$ with $q \leq a$ and $q \not\leq p^\perp$. Consider the following set of first order formulas (where x is a variable for elements of sort P)

$$\Sigma_p(x) = \{x \not\perp p\} \cup \{x \leq a \mid a \in F\}$$

For a finite subset $\Phi(x)$ of $\Sigma_p(x)$ there is finite $C \subseteq F$ such that C contains all constants occuring in $\Phi(x)$. Then, as observed, we may choose $q \leq a = \prod C \in F$, $q \not\perp p$ so that q realizes $\Phi(x)$. By saturation, there is q realizing $\Sigma_p(x)$, i.e $q \in \varepsilon F$ and $q \not\perp p$. □

Claim 13.4. *$\eta\colon L/\theta \to L(\varepsilon F)$ with $\eta(a/\theta) = \{p \in \varepsilon F \mid p \leq a\}$ is a well defined 0-1-lattice homomorphism.*

Proof. If $a \geq b$ and $a\,\theta\,b$ then by Lemma 13.2 $b = ac$ for some $c \in F$ whence $a \geq p \in \varepsilon F$ implies $p \leq b$. Thus, η is well defined. Perservation of meets and 0-1 is obvious. The proof that η preserves joins follows that of Lemma 10.2. Given $a, b \in L$ choose c, d as in Lemma 10.1. Consider $p \in \eta((a+b)/\theta)$, $p \notin \eta(a/\theta)$ and $p \notin \eta(b/\theta)$. Then $p, a(p+d)$, and $d(p+a) \leq c$ are collinear points in P. εF being L-closed, these points are in εF, whence $p \in \eta(a/\theta) + \eta(c/\theta) \subseteq \eta(a/\theta) + \eta(b/\theta)$. ☐

Claim 13.5. *For any $a \notin I$ there is $p \in Q$ with $p \in \eta(a/\theta)$.*

Proof. Given $c \in F$ we have $a\theta ac$ and so $ac \notin I$. In particular, $0 < ac \not\leq d$ for all $d \in I$ and there is $p \in P_0$ with $p \leq ac$. Thus, since F is closed under finite meets, for any finite $C \subseteq F$ and $D \subseteq I$ we have $p \in P_0$ such that $p \leq ac$ for all $c \in C$ and $p \not\leq d$ for all $d \in D$. In other words, for a variable x of sort P, the set

$$\Phi_a(x) = \{x \leq ac \mid c \in F\} \cup \{x \not\leq d \mid d \in I\}$$

of formulas is finitely realized in $(L, P, \leq)$. By saturation, it is realized, i.e. we get $p \in Q$ with $p \leq a$. Thus $p \in \eta(a/\theta)$. ☐

Define

$$\gamma(a/\theta) = U + \eta(a/\theta)$$

and consider the subquotient geometry $\varepsilon F/U$.

Claim 13.6. *$\gamma\colon L/\theta \to L(\varepsilon F/U)$ is a 0-1-lattice representation.*

Proof. Preservation of joins and 0-1 follows from Claim 13.4, immediately. Considering meets let $X \in \gamma(a/\theta) \cap \gamma(b/\theta)$. By definition we have

$$X = p + U = q + U \quad \text{with some } p, q \in Q,\ p \leq a,\ q \leq b$$

In particular, $p \leq q + r$ for some $r \in U$, i.e $r \leq c$ for some $c\,\theta\,0$. It follows $b\,\theta\,b + c$. Now $p \leq b + c$, whence $p \in \eta((b+c)/\theta) = \eta(b/\theta)$, i.e. $p \leq b$. Thus $p \leq ab$ and $X \in \gamma((ab)/\theta) = \gamma(a/\theta \cdot b/\theta)$. This shows that γ is a 0-1-lattice homomorphism which is injective by Claim 13.5 and Lemma 10.2. ☐

.

Claim 13.7. *$\varepsilon F/U$ is an orthogeometry according to Lemma 6.2.*

Proof. If $p \in U$ then $p \leq a'$ for some $a \in F$ since $p \in \varepsilon I$. If $q \in \varepsilon F$ then $q \leq a$, hence $q \leq a = a'' = (a')^\perp \leq p^\perp$ and $q \perp p$. This proves $U \subseteq (\varepsilon F)^\perp$. We will be done if we show that Q is non-degenerate. So, let $p \in Q$ and $p \leq p^\perp$. Consider $a \in F$. Then $p \leq a$ and $p \not\leq a' \in I$, whence $a = a^{\perp\perp} \not\leq p^\perp$ and $ap^\perp \prec a$. Hence there is $v_a \succ p$ in $\mathbb{L}(P)$ with $a = v_a + ap^\perp$ and $p = v_a p^\perp$. Choose $q_a \in P$ with $q_a + p = v_a$. Then

$$q_a \not\leq p,\ q_a \leq a,\ q_a \not\leq a'$$

since $q_a a' \leq q_a p^\perp \leq q_a a p^\perp \leq q_a p = 0$. Now, for fixed p and variable x of sort P consider the following set of first order formulas

$$\Sigma_p(x) = \{x \not\leq p\} \cup \{x \leq a \mid a \in F\} \cup \{x \not\leq b \mid b \in I\}$$

For a finite subset $\Phi(x)$ of $\Sigma_p(x)$ we have finite $C \subseteq F$ and $D \subseteq I$ such that $C \cup D$ contains all constants from L occuring in $\Phi(x)$. Then with

$$a = \prod C(\sum D)'$$

the above q_a realizees $\Phi(x)$. In other words, $\Sigma_p(x)$ is finitely realized in the structure $(L, P, \leq)$. By saturation, $\Sigma_p(x)$ is realized by some $q \in P$. But then $q \not\leq p^\perp$ and $q \in Q$. □

Claim 13.8. *$\gamma\colon L/\theta \to L(\varepsilon F/U)$ is a geometric MIL-representation.*

Proof. In view of Claims 13.6 and 13.7 we are left to show that

$$U + \eta(a'/\theta) = \gamma(a'/\theta) = \gamma(a/\theta)^\perp = U + \eta(a/\theta)^\perp$$

Recall that $a' = a^\perp$. Thus, for any $p \in Q$, if $p \in \eta(a'/\theta)$ then $p \perp q$ for all $q \in \eta(a)$ proving '$\subseteq$' and we are left to prove '$\supseteq$'. Consider fixed $a \in L$ and $p \in Q$ with $p \perp \eta(a/\theta)$. Now, for given $b \in F$ assume $p \not\leq a' + b'$. Then $ab = (a' + b')' > 0$ and there is some $q \in P$ with $q \leq ab$ and $q \not\perp p$. Namely, otherwise we had $p \leq (ab)^\perp = (ab)' = a' + b'$. Thus, with variable x of sort P,

$$\Psi_{a,p}(x) = \{p \not\leq a' + b' \;\to\; (x \leq ab \;\wedge\; x \not\perp p) \mid b \in F\}$$

is finitely realized. By saturation we have $q \in P$ such that

$$p \not\leq a' + b' \text{ implies } q \leq ab \text{ and } q \not\perp p \quad \text{for all } b \in F$$

Now, assume $p \not\leq a' + b'$ for all $b \in F$. It follows that $q \leq ab$ for all $b \in F$ and $q \not\perp p$. But also $q \in \varepsilon F$ and $q \leq a$ so $q \in \eta(a/\theta)$ whence $p \perp q$, a contradiction.

Hence $p \leq a' + b'$ for some $b \in F$, i.e. $p \leq a' + c$ for some $c \in I$. We have $p \not\leq c$ by definition of Q and are done if $p \leq a'$. So assume $p \not\leq a'$. By Lemma 10.1 there are collinear points p, q, r in P with $q \leq a'$ and $r \leq c$. Since εF is L-closed (Claim 13.3), q and r have to belong to εF, Thus, $q \in \eta(a'/\theta)$, $r \in U$, and $p \in \gamma(a'/\theta)$. □

14. Preliminaries: Universal Algebra

Cf.[8, Ch.1,2] and [12, Ch.1,2,4,8]. Assume that a first order language is given and consider appropriate structures, only. Observe that any formula $\sigma(x_1, \ldots, x_n)$ *defines* an n-ary relation on A: the set of all $(a_1, \ldots, a_n)$ such that $\sigma(a_1, \ldots, a_n)$ is valid in A.

Given structures A, B, a map $\phi\colon A \to B$ is an *homomorphism* (*embedding*) if for any atomic formula $\sigma(x_1, \ldots, x_n)$ and any $a_i \in A$ one has $\sigma(\phi a_1, \ldots, \phi a_n)$ valid in B if (if and only if) $\sigma(a_1, \ldots, a_n)$ is valid in A. If the identity map $\phi\colon A \to B$ is an embedding and if $f^B(a_1, \ldots, a_n) \in A$ for all fundamental operations f^B of B and $a_i \in A$, then A is a *substructure* of B. A is a *directed union* of substructures A_i $(i \in I)$ if $A = \bigcup_{i \in I} A_i$ and if for all $i, j \in I$ there is $k \in I$ such that $A_i \cup A_j \subseteq A_k$.

Given structures A_i $(i \in I)$ and an filter $\mathcal{U}$ on I there is a unique equivalence relation on $\prod_{i \in I} A_i$ with classes $(a_i \mid i \in I)/\mathcal{U}$ forming a well defined structure A such that for any atomic formula $\sigma(x_1, \ldots, x_n)$ and $a_k = (a_k(i) \mid i \in I)/\mathcal{U} \in A$, $\sigma(a_1, \ldots, a_n)$ is valid in A if and only if $\{i \in I \mid \sigma(a_1(i), \ldots, a_n(i)) \text{ valid in } A_i\} \in \mathcal{U}$. This is called the *$\mathcal{U}$-reduced product* $A = \prod_{\mathcal{U}} A_i$ cf. [12, Ch.8.5] or [8, Ch.1.2.2].

If $\mathcal{U}$ is the set of all subsets of I then A is the *direct product* $\prod_{i\in I} A_i$. If $\mathcal{U}$ is an ultrafilter then A is an *ultraproduct.*

Given a class $\mathcal{C}$ of structures, we denote by $\mathsf{H}\,\mathcal{C}$, $\mathsf{P}\,\mathcal{C}$, $\mathsf{P}_\mathrm{u}\,\mathcal{C}$, $\mathsf{S}\,\mathcal{C}$, $\mathsf{S}_\mathrm{e}\,\mathcal{C}$, and $\mathsf{L}\,\mathcal{C}$ the classes of all homomorphic images, direct products, ultraproducts, substructures, elementary substructures, and directed unions of members of $\mathcal{C}$.

Proposition 14.1. *For any ultrafilter $\mathcal{U}$ the following hold*

(i) *Given $a_k = (a_k(i) \mid i \in I)/\mathcal{U} \in \prod_{\mathcal{U}} A_i$, $\sigma(a_1, \ldots, a_n)$ is valid if and only if $\{i \in I \mid \sigma(a_1(i), \ldots, a_n(i))$ valid in $A_i\} \in \mathcal{U}$.*
(ii) *If $\Phi(x)$ is a set of formulas in free variable x, then $\{a \in \prod_{\mathcal{U}} A_i \mid \Phi(a)\} \cong \prod_{\mathcal{U}}\{a_i \in A_i \mid \Phi(a_i)\}$ w.r.t. all definable relations.*
(iii) *$\mathcal{C}$ is axiomatic if and only if $\mathsf{P}_\mathrm{u}\,\mathcal{C} \subseteq \mathcal{C}$ and $\mathsf{S}_\mathrm{e}\,\mathcal{C} \subseteq \mathcal{C}$.*
(iv) *If $B \in \mathsf{S}_\mathrm{e}\,A$ then $A \in \mathsf{S}_\mathrm{e}\,\mathsf{P}_\mathrm{u}\,B$.*
(v) *$\mathsf{P}_\mathrm{u}\,\mathsf{P}_\mathrm{u}\,\mathcal{C} \subseteq \mathsf{P}_\mathrm{u}\,\mathcal{C} \subseteq \mathsf{H}\,\mathsf{P}\,\mathcal{C}$ and $\mathsf{L}\,\mathcal{C} \subseteq \mathsf{S}\,\mathsf{P}_\mathrm{u}\,\mathcal{C}$.*
(vi) *Consider axiomatic classes $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{C}$ are where $\mathcal{C}$ consists of 2-sorted structures (L, P) with $L \in \mathcal{L}$ and $P \in \mathcal{G}$. If $(L, P) = \prod_{\mathcal{U}}(L_i, P_i)$ then $L \cong \prod_{\mathcal{U}} L_i$ and $P \cong \prod_{\mathcal{U}} P_i$.*

Proof. Ad (i): This is Łoś' Theorem [12, Thm.8.5.3]. (ii) follows, immediately. Ad (iii): See Cor.8.5.4 and 8.5.13 [12]. Ad (iv): A and B are elementarily equivalent hence they have isomorphic ultrapowers A^* and B^* (cf. [12, Thm.8.5.10]; also, A has a canonical elementary embedding into A^* [12, Cor.8.5.4]. Ad (v): See [8, Thm.1.2.12] an observe that $(a(i) \mid i \in I) \mapsto (a(i) \mid i \in I)/\mathcal{U}$ is a homomorphism. Ad (vi): This follows from (ii), considering L and P as predicates. ☐

15. U-quasivarieties of orthogeometries

Given a class $\mathcal{G}$ of orthogeometries, denote by $\mathsf{U}\,\mathcal{G}$ and $\mathsf{S}_\mathrm{g}\,\mathcal{G}$ the class of all orthogonal disjoint unions and subgeometries, resp., of members of $\mathcal{G}$. Call $\mathcal{G}$ a U-*quasivariety* if it is closed under these operators.

Lemma 15.1. *The class of all projective geometries P resp. orthogeometries $(P, \perp)$ with designated subset Q, where Q is a proper projective subgeometry of P resp. subgeometry of $(P, \perp)$, is an axiomatic class. In particular, $\mathsf{S}_\mathrm{e}\,\mathcal{G} \subseteq \mathsf{S}_\mathrm{g}\,\mathcal{G}$ for any class $\mathcal{G}$ of orthogeometies.*

Proof. Conditions (i) and (iii) of the definition are first order, obviuosly. Concerning (ii), define the formula $\alpha_0^Q(x)$ as $x = x$ and, recurively, $\alpha_n^Q(x, x_1, \ldots, x_n)$ as

$$\alpha_{n-1}^Q(x, x_1, \ldots, x_{n-1}) \vee \exists y.\ Q(y) \wedge \alpha_{n-1}^Q(y/x, x_1, \ldots, x_{n-1}) \wedge \kappa(x, y, x_n)$$

where Q is a new unary predicate symbol interpreted as the subset Q. Then $\alpha_n^Q(x, x_1, \ldots x_n)$ holds for $x, x_1, \ldots, x_n$ in Q if and only if $x \in C_Q\{x_1, \ldots, x_n\}$. Similarly, we have $\alpha_n^P(x, x_1, \ldots, x_n)$. Then (ii) is eqivalent to the following set of axioms ($n = 1, 2 \ldots$)

$$\forall x \forall x_1 \ldots \forall x_n.\ Q(x) \wedge \bigwedge_{i=1}^{n} Q(x_i) \wedge \alpha^P(x, x_1, \ldots, x_n) \Rightarrow \alpha^Q(x, x_1, \ldots, x_n)$$

$\mathsf{S}_{\rm e}\,\mathcal{G} \subseteq \mathsf{S}_{\rm g}\,\mathcal{G}$ follows with Prop.14.1(iii). □

Theorem 15.2. *Every* U*-quasivariety of orthogeometries is an axiomatic class.* $\mathsf{V}\,\mathcal{G} = \mathsf{U}\,\mathsf{S}_{\rm g}\,\mathsf{P}_{\rm u}\,\mathcal{G} = \mathsf{S}_{\rm g}\,\mathsf{U}\,\mathsf{P}_{\rm u}\,\mathcal{G}$ *is the smallest* U*-quasivariety containing* $\mathcal{G}$*.*

Proof. By Lemma 7.1

$$(1) \qquad \mathbb{G}(\prod_{i\in I}\mathbb{L}(P_i,\perp_i)) \cong \bigcup_{i\in I}^{\perp}(P_i,\perp_i)$$

Similary, by Prop.14.1(ii), for ultraproducts

$$(2) \qquad \mathbb{G}(\prod\nolimits_{\mathcal{U}}\mathbb{L}(P_i,\perp_i)) \cong \prod\nolimits_{\mathcal{U}}(P_i,\perp_i).$$

Obviously, any subgeometry Q of an orthogonal disjoint union $\bigcup_i^{\perp}(P_i,\perp_i)$ is an orthogonal disjoint union of subgeometries and vice versa, namely $Q \cong \bigcup_i^{\perp}(Q\cap P_i)$. In view of Lemma 15.1 and Prop.14.1(i),(ii), any ultraproduct of subgeometries is isomorphic to a subgeometry of an ultraproduct.

Observe that an orthogonal disjoint union $\bigcup_j^{\perp}(Q_j,\perp_j)$ can be viewed as given by an equivalence relation θ on P the classes $p[\theta]$ of which are the Q_j. Thus, a member of $\mathsf{P}_{\rm u}\,\mathsf{U}\,\mathcal{G}$ may be understood as an ultraproduct $\prod_{\mathcal{U}}(P_i,\perp_i,\theta_i)$ which is again such structure $(P,\perp,\theta)$. Given $p = (p(i) \mid i \in I)/\mathcal{U} \in P$, again by Prop.14.1(ii), the θ-class $p[\theta]$ of p consists of the $q \in P$ such $\{i \in I \mid q(i)\,\theta_i\,p(i)\} \in \mathcal{U}$ and one obtains

$$p[\theta] \cong \prod\nolimits_{\mathcal{U}} p_i[\theta_i]$$

which proves $\mathsf{P}_{\rm u}\,\mathsf{U}\,\mathcal{G} \subseteq \mathsf{S}\,\mathsf{P}_{\rm u}\,\mathcal{G}$. Finally, $\mathsf{U}\,\mathsf{U}\,\mathcal{G} = \mathsf{U}\,\mathcal{G}$, $\mathsf{S}_{\rm g}\,\mathsf{S}_{\rm g}\,\mathcal{G} = \mathsf{S}_{\rm g}\,\mathcal{G}$, obvioulsy, and $\mathsf{P}_{\rm u}\,\mathsf{P}_{\rm u}\,\mathcal{G} = \mathsf{P}_{\rm u}\,\mathcal{G}$ by Prop.14.1(v). It follows that $\mathsf{U}\,\mathsf{S}_{\rm g}\,\mathsf{P}_{\rm u}\,\mathcal{G} = \mathsf{S}_{\rm g}\,\mathsf{U}\,\mathsf{P}_{\rm u}\,\mathcal{G}$ and that this is a U-quasivariety. By Lemma 15.1, and Prop.14.1(iii), any U-quasivariety is an aximomatic class. □

16. ∃-varieties of CMILs

Cf. [11]. Consider a first order language with operations symbols $+$ and $\cdot$ for joins and meets, constants 0 and 1, and additional operation symbols. Let $\mathcal{C}_0$ denote the class of all such structures which are modular lattices w.r.t. $+$ and $\cdot$, with $0,1$ and, moreover, complemented - but complementation ist not necessarily considered as an operation.

Given a class $\mathcal{L} \subseteq \mathcal{C}_0$, we denote by $\mathsf{S}_\exists\,\mathcal{L}$ the class of all complemented members of $\mathsf{S}\,\mathcal{L}$. If $\mathsf{H}\,\mathcal{L}$, $\mathsf{P}\,\mathcal{L}$, and $\mathsf{S}_\exists\,\mathcal{L}$ are subclasses of $\mathcal{L}$ then $\mathcal{L}$ is called an *∃-variety* (a similar approach is possible in the sectionally complemented case). Define

$$\mathsf{V}_\exists\,\mathcal{L} = \mathsf{H}\,\mathsf{S}_\exists\,\mathsf{P}\,\mathcal{L}$$

Proposition 16.1. *Given any class* $\mathcal{L} \subseteq \mathcal{C}_0$

(i) $\mathsf{V}_\exists$ *is the smallest ∃-variety containing* $\mathcal{L}$*.*
(ii) $L \in \mathsf{H}\,\mathsf{S}_\exists\,\mathsf{P}_{\rm u}\,\mathcal{L}$ *for any subdirectly irreducible* $L \in \mathsf{V}_\exists\,\mathcal{L}$*.*
(iii) *Any ∃-variety* $\mathcal{V} \subseteq \mathcal{C}_0$ *is an axiomatic class.*

Proof. (i) and (ii) are [11, Prop.10(i),(iii)]. (iii) follows with Prop.14.1(iii). □

Given any class $\mathcal{V}$ of CMILs, let $\mathbb{G}(\mathcal{V})$ consist of all orthogeometries $(P,\perp)$ with $\mathbb{L}(P,\perp)\in\mathcal{V}$. Given a class $\mathcal{G}$ of orthogeometries, let $\mathbb{L}(\mathcal{G})$ denote the class of CMILs, L, for which there is $(P,\perp)\in\mathcal{G}$ and a geometric representation $\eta\colon L\to\mathbb{Q}(P,\perp)$.

Theorem 16.2. *If $\mathcal{V}$ is an $\exists$-variety of CMILs then $\mathbb{G}(\mathcal{V})$ is a U-quasivariety of orthogeometries.*

Proof. By Thm.1.3 and Prop.14.1(v)

$$\mathbb{L}(\mathsf{S}\,\mathbb{G}(\mathcal{V}))\subseteq \mathsf{S}_\exists\,\mathsf{P}_\mathrm{u}\,\mathcal{V}.$$

From (1) and (2) in the proof of Thm.15.2 and Cor.1.5 it follows $\mathbb{L}(\mathsf{U}\,\mathbb{G}(\mathcal{V}))\subseteq\mathcal{V}$ and $\mathbb{L}(\mathsf{P}_\mathrm{u}\,\mathbb{G}(\mathcal{V}))\subseteq\mathcal{V}$. ☐

Theorem 16.3. *$L\in\mathsf{V}_\exists\,L_f$ for any atomic CMIL, L.*

Proof. For a similar construction see [1]. By Prop.12.1 there is an elementary extension L^* of L_f which is ω-saturated over L_f. In particular, L^* is a CMIL, too, having L_f as a subalgebra. For $x\in L^*$ and $a\in L$ define

$$x\sim a\ \Leftrightarrow\ \forall p\in P_L\,\forall h\in H_L\,((p\leq a\Leftrightarrow p\leq x)\ \wedge\ (h\geq a\Leftrightarrow h\geq x))$$

and observe that

$$x\sim a\ \Leftrightarrow\ \forall p\in P_L\,\forall h\in H_L\,((p\leq a\Rightarrow p\leq x)\ \wedge\ (h\geq a\Rightarrow h\geq x))$$

Indeed, if $x\leq h$ for all $h\in H_L$ with $h\geq a$ then also $p\leq x$ implies $p\leq h$ for all these h whence $p\leq a$ by the dual of Lemma 3.1. Define

$$S=\{x\in L^*\mid \exists a\in L.\ x\sim a\}$$

We claim that $S\in\mathsf{S}_\exists\,L^*$ and $L\in\mathsf{H}\,S$. Consider $x\sim a$ and $y\sim b$. If $p\in P_L$ and $p\leq a+b$ then by Lemma 10.1 there are $q,r\in P_L$ such that $p\leq q+r$ and $q\leq a$, $r\leq b$. If follows $q\leq x$, $r\leq y$, and $p\leq x+y$. If $h\in H_L$ and $h\geq a+b$ then $h\geq a$ and $h\geq b$ whence $h\geq x$ and $h\geq y$ which yields $h\geq x+y$. Thus, $x+y\sim a+b$ and, by duality, $xy\sim ab$. Also, $x'\sim a'$ since $p\in P_L$ iff $p'\in H_L$ and $p\leq a'$ iff $a\leq p'$ iff $x\leq p'$ iff $p\leq x'$.

Given $x\sim a$ we have to find $y\in S$ with $x\oplus y=1$. Choose $b\in L$ with $a\oplus b=1$. Given $u\in L_{fin}$ and $v\in L_{cofin}$ with $u\leq b\leq v$ there is y in the CML L^* such that $u\leq y\leq v$ and $x\oplus y=1$. Indeed, choose y as a complement of $v(x+u)=u+xv$ in $[u,v]$. This implies that the following set $\Phi(y)$ of formulas in parameters $x\in L^*$ and $p\in P_L$, $h\in H_L$

$$\{x\oplus y=1\}\cup\{p\leq y\mid b\geq p\in P_L\}\cup\{h\geq y\mid b\leq h\in H_L\}$$

is finitely realized in L^*, whence realized in L^* by some y. If follows $y\sim b$, i.e. y is a complement of x in S.

Again by Lemma 3.1, $\phi x=a$ iff $x\sim a$ is a well defined map $\phi\colon S\to L$ and a homomorphism according to the above observations. To show that ϕ is surjective, given $a\in L$ consider the set $\Psi(x)$ of formulas given as

$$\{p\leq x\mid a\geq p\in P_L\}\cup\{h\geq x\mid a\leq h\in H_L\}$$

Each finite subset $\Delta(x)$ of $\Psi(x)$ is realized by the join u of all $p \in P_L$ occuring in Δ. Thus, $\Psi(x)$ is realized by some $x \in L^*$. By construction, $x \sim a$ whence $x \in S$ and $\phi x = a$. $\square$

Proof. Thm.2.2. Ad (i). Clearly, $\mathsf{S}_\exists \mathbb{L}(\mathcal{G}) \subseteq \mathbb{L}(\mathcal{G})$ and, by Lemma 7.1, $\mathsf{P}\,\mathbb{L}(\mathcal{G}) \subseteq \mathbb{L}(\mathsf{U}\,\mathcal{G})$. By Thm.13.1 and Lemma 15.1 one has $\mathsf{H}\,\mathbb{L}(\mathcal{G}) \subseteq \mathbb{L}(\mathsf{S}_\mathrm{e}\,\mathsf{P}_\mathrm{u}\,\mathcal{G}) \subseteq \mathbb{L}(\mathsf{S}_\mathrm{g}\,\mathsf{P}_\mathrm{u}\,\mathcal{G})$.

Ad (ii). This is immediate by Thm.2.1. Ad (iii). Consider $L_0 \in \mathbb{L}(\mathcal{G})$. By Thm.2.2(iii) L_0 admits an atomic extension $L \in \mathbb{L}(\mathcal{G})$. By Thm.16.3 $L \in \mathsf{V}_\exists\, L_f$. In view Thm.1.1 and Prop.1.2, L_f is a directed union of finite dimensional complemented subalgebras L_i $(i \in I)$ whence $L_f \in \mathsf{S}\,\mathsf{P}_\mathrm{u}\{L_i \mid i \in I\}$ by Prop.14.1(v). Since L_f is complemented, it follows $L_f \in \mathsf{S}_\exists\,\mathsf{P}_\mathrm{u}\{L_i \mid i \in I\}$ and $L_0 \in \mathsf{V}_\exists\{L_i \mid i \in I\}$. On the other hand, $L_i \in \mathbb{L}(\mathcal{G})$ by (i) and so $L_i = \mathbb{L}(\mathbb{G}(L_i)$ with $\mathbb{G}(L_i) \in \mathcal{G}_{<\omega}$. $\square$

Proof. Cor.2.3. Assume that $\mathcal{V} = \mathsf{V}_\exists\,\mathcal{V}_{<\omega}$ and put $\mathcal{G} = \mathbb{G}(\mathcal{V})$ which is a U-quasivariety by Thm.16.2. Then $\mathcal{V}_{<\omega} = \mathbb{L}(\mathcal{G}_{<\omega})$ and it follows $\mathcal{V} \subseteq \mathbb{L}(\mathcal{G})$ since this is an $\exists$-variety by Thm.2.2(i). The converse inclusion follows from Thm.2.2(iii). $\square$

Corollary 16.4. (i) $\mathsf{V}_\exists\,\mathbb{L}(\mathcal{G}) \subseteq \mathbb{L}(\mathsf{U}\,\mathsf{S}_\mathrm{g}\,\mathsf{P}_\mathrm{u}\,\mathcal{G})$ *for any class* $\mathcal{G}$ *of orthogeometries and* $L \in \mathbb{L}(\mathsf{S}_\mathrm{g}\,\mathsf{P}_\mathrm{u}\,\mathcal{G})$ *for any subdirectly irreducible* $L \in \mathsf{V}_\exists\,\mathbb{L}(\mathcal{G})$.

(ii) *Let* $\mathcal{L}$ *consist of CMILs which have proper representations, e.g. atomic CMILs which are subdirecty irreducible as lattices. Then every member of* $\mathsf{V}_\exists\,\mathcal{L}$ *admits a proper geometric representation.*

Proof. The first claim follows from Thms.15.2 and 2.2(i). If $L \in \mathsf{V}_\exists\,\mathbb{L}(\mathcal{G})$ is subdirectly irreducible, then by Prop.16.1(ii), Prop.14.1(vii) applied to structures as in Prop.12.2, Thm.2.1, and Prop.14.1(v)

$$L \in \mathsf{H}\,\mathsf{S}_\exists\,\mathsf{P}_\mathrm{u}\,\mathbb{L}(\mathcal{G}) \subseteq \mathsf{H}\,\mathsf{S}_\exists\,\mathbb{L}(\mathsf{P}_\mathrm{u}\,\mathcal{G}) \subseteq \mathsf{H}\,\mathbb{L}(\mathsf{P}_\mathrm{u}\,\mathcal{G}) \subseteq \mathbb{L}(\mathsf{S}_\mathrm{e}\,\mathsf{P}_\mathrm{u}\,\mathsf{P}_\mathrm{u}\,\mathcal{G}) \subseteq \mathbb{L}(\mathsf{S}_\mathrm{g}\,\mathsf{P}_\mathrm{u}\,\mathcal{G})$$

(ii) follows with Cor.7.3 Prop.11.1. $\square$

17. Preliminaries: Logic

For the following we refer to [19, Ch.1,5] and [21, Ch.6]. A function $f\colon \mathbb{N} \to \mathbb{N}$, $\mathbb{N}$ the natural numbers, is *recursive* if and only if it can be computed by a Turing machine. A set $M \subseteq \mathbb{N}$ is *recursively enumerable* if it is empty or the image of a recursive function. $M \subseteq \mathbb{N}$ is *recursive* if its characteristic function is recursive.

Now, consider a first order language Λ with only finitely many operation and relation symbols. One can effectively associate with each formula σ a natural number $\lceil\sigma\rceil$, its *Gödel number*, such $\sigma \mapsto \lceil\sigma\rceil$ is a bijection of the set Λ of all sentences onto a recursive subset of $\mathbb{N}$ cf. [21, Ch.6.6]. We say that a set Γ of sentences is *recursive* resp. *recursively enumerable* if its image is. Recursive sets of sentences are also called *decidable.* According to Church' Thesis, these are believed to be exactly the sets admitting a 'decision procedure' for membership. A family $\Gamma_n\,(n \in \mathbb{N})$ is *uniformly decidable* if $\{\tau(n,m) \mid m = \lceil\sigma\rceil \text{ for some } \sigma \in \Gamma_n\}$ is recursive where τ is the 'pair coding function' $\tau(x,y) = \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$.

Λ_u consists of all *universal sentences* of the form $\forall x_1 \ldots \forall x_n.\sigma$ where σ is a quantifier free formula. $\Lambda_{eq} \subseteq \Lambda_u$ consists of all *equations* or *identities*, i.e. where σ is of the form $t_1(x_1,\ldots,x_n) = t_2(x_1,\ldots,x_n)$ with terms $t_i(x_1,\ldots,x_n)$.

A sentence α is a *consequence* of a set Γ of sentences if it is valid in all models of Γ. Let $\mathsf{C}(\Gamma)$ denote the set of all consequences of Γ. The *theory* resp. the *equational theory* of a class $\mathcal{C}$ of structures is *decidable* if the set of all sentences resp. equations valid in $\mathcal{C}$ is decidable. A model class of a recursive axiom set is *recursively axiomatizable.*

Proposition 17.1. (i) *A subset Γ of a recursive set Σ is recursive if and only if Γ and $\Sigma \setminus \Gamma$ are both recursively enumerable.*

(ii) *If Γ is recursively enumerable and Σ recursive then $\Gamma \cap \Sigma$ is recursively enumerable. If the family Γ_n ($n \in \mathbb{N}$) is uniformly decidable then $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is recursively enumerable.*

(iii) *Any class with a recursively enumerable axiom set is recursively axiomatizable.*

(iv) *If Γ is recursively enumerable and Σ recursive, then $\mathsf{C}(\Gamma)\cap\Sigma$ is recursively enumerable.*

(v) *Λ_u and Λ_{eq} are recursive.*

(vi) *Consider axiomatic classes $\mathcal{L} = \mathsf{S}\,\mathcal{L}$, $\mathcal{G}$, and $\mathcal{C}$ are where $\mathcal{C}$ consists of 2-sorted structures (L, P) with $L \in \mathcal{L}$ and $P \in \mathcal{G}$. Then the $M \in \mathcal{L}$ with $M \in \mathsf{S}\,L$ for some $(P, L) \in \mathcal{C}$ form an axiomatic class $\mathbb{P}_{\mathcal{L}}(\mathcal{C})$. This class is recursively axiomatizable if so are $\mathcal{L}$ and $\mathcal{C}$.*

Proof. Ad (i): $\Lambda \setminus \Sigma$ is recursively enumerable by [19, Thm.5.II] whence also $\Lambda \setminus \Gamma$ by [19, Thm.5.XIII]. It follows that Γ is recursive, again with [19, Thm.5.II]. Ad (ii): These are Thm.5.XIII and Cor.5.XI in [19]. Ad (iii): This is Craig's trick cf. [12, Exerc.5.1.3]. Ad (iv): By Gödel's Completeness Theorem, $\mathsf{C}(\Gamma)$ is recursively enumerable (cf. [21, 6.6] and (iii)). Now appyly (ii). Ad (v): This follows by inspection of the definition of Gödel numbers cf. [21, 6.6]. Ad (vi): Given axiom sets Γ for $\mathcal{C}$ and Θ for $\mathcal{L}$ define $\Sigma = \Lambda_u \cap \mathsf{C}(\Gamma \cup \Theta)$. Now, $\mathbb{P}_{\mathcal{L}}(\mathcal{C})$ is closed under substructures, obviously, and a PC'_Δ-class in the sense of [12, Thm.5.5.5]. By the proof given there, it is axiomiatized by Σ. If Γ and Θ are recursive then Σ is recursively enumerable by (ii) and (iv) and $\mathbb{P}_{\mathcal{L}}(\mathcal{C})$ is recursively axiomatizable by (iii). $\square$

Consider a language with equality but no relation symbols. The appropriate structures are then *algebraic structures.* A class $\mathcal{V}$ of such is a *variety* if it is closed under H, S, and P. Define $\mathsf{V}\,\mathcal{C} = \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathcal{C}$.

Proposition 17.2. (i) *A class $\mathcal{V}$ of algebraic structures is a variety if and only it it is the model class of a set of equations.*

(ii) *$\mathsf{V}\,\mathcal{C}$ is the smallest variety containing $\mathcal{C}$. In particular, an equation is valid in $\mathsf{V}\,\mathcal{C}$ if and only if it is valid in $\mathcal{C}$.*

(iii) *If a variety $\mathcal{V}$ is recursively axiomatizable then $\mathcal{V}$ is the model class of a recursively enumerable set of equations.*

Proof. Ad (i) cf. [3, Thm.11.9]. (ii) follows with [3, Thm.9.5]. Ad (iii): This follows with Prop.17.1(iv),(v). □

18. Equational theory of CMILs

Proof. Prop.2.4. Viewing representations as 2-sorted structures as in Prop.12.2 and apply Prop.17.1(vi) one obtains axioms for the class of MILs having a representation within $\mathcal{G}$. Addition of $\forall x \exists y.\, x \oplus y = 1$ yields an axiom set for $\mathbb{L}(\mathcal{G})$. Conversely, axioms for $\mathbb{L}(\mathcal{G})$ apply to the $\mathbb{L}(P, \perp)$ with $(P, \perp) \in \mathcal{G}$ and translate into axioms for $\mathcal{G}$. □

Proof. Cor.2.5. Prop.2.4 and Prop.17.2(iii) imply that the set Σ of identities valid in $\mathcal{V}^\alpha$ is recursively enumerable. By hypothesis and Prop.17.1(i), the set Δ_n of identities failing in $\mathcal{V}^\alpha_n$ is decidable. Hence, $\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n$ is recursively enumerable by Prop.17.1(ii). But, in view of Cor.2.3 $\mathcal{V}^\alpha = \bigvee \bigcup_{n \in \mathbb{N}} \mathcal{V}^\alpha_n$ and by Prop.17.2(ii) it follows that Δ is the complement of Σ. Thus, Σ is decidable by Prop.17.1(i). □

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