A Note On the Equational Theory of Modular Orthollattices

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today

ABSTRACT. We prove that every atomic modular ortholatice is in the variety generated by its finite dimensional members.

1 Introduction

An *ortholattice*, abbreviated OL, is an algebra $(L;+,\cdot,',0,1)$ where $(L;+,\cdot,0,1)$ is a bounded lattice and $' : L \to L$ is an *orthocomplementation*, ie. $x+x'=1$, $x\cdot x'=0$[1], and $x \le y$ implies $y' \le x'$, for all $x,y \in L$. Since the last condition, in the presence of the other two, is equivalent to DeMorgan's laws ($(x+y)'=x'y'$ and $(xy)'=x'+y'$), the class of ortholattices forms a variety. An OL, L, is an *orthomodular lattice*, abbreviated OML, iff it satisfies the identity $y(xy+y')=xy$. This is a weak, or 'orthogonal', version of the modular law. An OML is a *modular ortholattice*, abbreviated MOL, iff it is modular. For background on these classes of algebras the reader is referred to [4], and for background on modular lattices to [2], for example.

The *height* of a modular lattice is the length of any maximal chain in the lattice. For our purposes this height is a non negative integer or ∞, with $n < \infty$ for all non negative integers n. In, [1], Bruns made the following conjecture (stated slightly differently),

Conjecture 1 *(Bruns' Conjecture). Every variety of MOLs which contains a subdirectly irreducible algebra of height greater than two, contains a subdirectly irreducible algebra of height 3.* brconj

A partial confirmation of Bruns' Conjecture is given in [5]. In this note we prove,

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[1]We will follow the common convention of usually writing the meet operation as juxtaposition, ie. '$x \cdot y$' as 'xy'.

Proposition 2 *Every variety of MOLs which is generated by its atomistic members is already generated by its finite dimensional members.* main

We present this result as motivation for the following strengthening of Bruns' Conjecture,

Conjecture 3 *Every variety of MOLs is generated by its finite dimensional members.* hrconj

2 A Lemma

We begin with a lemma, the proof of which is essentially in the proof of Frink's Embedding Theorem, [3].

Lemma 4 *Let L be an atomic complemented modular lattice. Then, for any lattice polynomial $p = p(y_1, ..., y_n)$ and $b_i \in L$, $i = 1, .., n$, if $0 < p(b_1, ..., b_n)$, then there exist $c_i \in L$, $i = 1, ..., n$, of finite height, so that $c_i \leq b_i$, $i = 1, ..., n$, and $0 < p(c_1, ..., c_n)$.* frlemma

Proof. We actually prove the following stronger statement by induction on the length of the polynomial p.

> If a is an atom of L and $a \leq p(b_1, ..., b_n)$, then there exist $c_i \leq b_i$ of finite height so that $a \leq p(c_1, ..., c_n)$.

If p is one of the constants then the claim is vacuously true. If p is a single variable then setting $c_1 = a$ does the trick.

If $p = p_1 p_2$ then $a \leq p(b_1, ...b_n)$[2] implies $a \leq p_k(b_1, ..., b_n)$, $k = 1, 2$. By inductive hypothesis, there exist $c_{ki} \leq b_i$, $k = 1, 2$, of finite height, with $a \leq p_k(c_{k1}, ..., c_{kn})$, $k = 1, 2$. Setting $c_i = c_{1i} + c_{2i}$, $i = 1, ..., n$, gives $a \leq p_1(c_{11}, ..., c_{1n}) \cdot p_2(c_{21}, ..., c_{2n}) \leq p_1(c_1, ..., c_n) \cdot p_2(c_1, ..., c_n) = p(c_1, ..., c_n)$.

If $p = p_1 + p_2$ then, for convenience, we set $d_k = p_k(b_1, ..., b_n)$, $k = 1, 2$. Choose e_1 as a relative complement of $d_1 d_2$ in $[0, d_1]$. Set $e_2 = d_2$ and, for $k = 1, 2$, $a_k = e_k(a + e_l)$, where $\{k, l\} = \{1, 2\}$. One easily computes using modularity that $\{0, a, a_1, a_2, a_1 + a_2\}$ form an M_3 in $[0, a_1 + a_2]$ and, consequently, the a_k, $k = 1, 2$, are atoms of L. Now, for $k = 1, 2$, $a_k \leq e_k \leq d_k = p_k(b_1, ..., b_n)$, so by inductive hypothesis, there exist $c_{ki} \leq b_i$, $i = 1, ..., n$, of finite height, with $a_k \leq p_k(c_{k1}, ..., c_{kn})$. Again, set $c_i = c_{1i} + c_{2i} \leq b_i$, $i = 1, ..., n$. This gives $a \leq a_1 + a_2 \leq p_1(c_{11}, ..., c_{1n}) + p_2(c_{21}, ..., c_{2n}) \leq p_1(c_1, ..., c_n) + p_2(c_1, ..., c_n) = p(c_1, ..., c_n)$.

[2] Formally every polynomial is a polynomial on the whole countable set of variables, with all but finitely many set to 0. Our notation is a matter of convenience then, and not part of the induction.

3 Orthoimplications

Let L be an ortholattice. Elements $x, y \in L$ are *orthogonal*, written $x \perp y$, iff $x \leq y'$. More generally, two sequences $(x_1, ..., x_n), (y_1, ..., y_n)$ of elements of L are *orthogonal*, written $(x_1, ..., x_n) \perp (y_1, ..., y_n)$, iff $x_i \perp y_i$, $i = 1, .., n$. An *orthoimplication* is a sentence formed by the universal quantification of a formula of the form,

$$(x_1, ..., x_n) \perp (y_1, ..., y_n) \text{ implies } r(x_1, y_1, ..., x_n, y_n) = 0,$$

where r is a bounded lattice term.

Lemma 5 *For any two ortholattice terms $p(x_1, ..., x_n)$ and $q(x_1, ..., x_n)$, there is a bounded lattice term $r(x_1, y_1, ..., x_n, y_n)$ such that for all orthomodular lattices the equation*

$$p(x_1, ...x_n) = q(x_1, ..., x_n)$$

holds in L iff the orthoimplication

$$(x_1, ..., x_n) \perp (y_1, ..., y_n) \textit{ implies } r(x_1, y_1, ..., x_n, y_n) = 0$$

holds in L. oimp

Proof. By orthomodulararity the ortholattice identity $p = q$ holds in an OML L iff the identity $p(p' + q') + q(p' + q') = 0$ holds in L. Repeated application of De Morgan's laws (which hold in any OL) allow one to bring all occurences of $'$ inside all brackets, so that any ortholattice term $t(x_1, ..., x_n)$ is equivalent to a bounded lattice term $r(x_1, x_1', ..., x_n, x_n')$. These two observations are easily combined to prove the lemma.

4 Proof of the Proposition

If $c_1, ..., c_n$ are elements of finite height in an MOL L, then $u = \sum_{i=1}^n c_i$ is of finite height, $[0, u] \times [u', 1]$ is a subalgebra of L, containing $c_1, ..., c_n$, and $[0, u]$ is a homomorphic image of this subalgebra. These elementary facts will be used in our proof of Proposition 2 which we are now in a position to give.

Proof of Proposition 2.
Let $u \in L$ of finite height. From the above comments, $[0, u]$ is in the variety generated by L. Let $p = q$ be an ortholattice identity which does not hold in L and let $(x_1, ..., x_n) \perp (y_1, ..., y_n)$ implies $r(x_1, y_1, ..., x_n, y_n) = 0$ be its associated orthoimplication. By Lemma 5, there exist $(x_1, ..., x_n) \perp (y_1, ..., y_n)$ in L so that $r(x_1, y_1, ..., x_n, y_n) > 0$. By Lemma 4, there exist $c_i, d_i \in L$, $i = 1, ..., n$, of finite height, so that $c_i \leq x_i$, $d_i \leq y_i$ for each i, and $r(c_1, d_1, ..., c_n, d_n) > 0$. Let $u = \sum_{i=1}^n (c_i + d_i)$ and note that $c_i \perp d_i$ in $[0, u]$, so the orthoimplication $(x_1, ..., x_n) \perp (y_1, ..., y_n)$ implies $r(x_1, y_1, ..., x_n, y_n) = 0$ fails in $[0, u]$. By Lemma 5, the identity $p = q$, does not hold in $[0, u]$.

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