On varieties of modular ortholattices which are generated by their finite-dimensional members

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ABSTRACT. We prove that the following three conditions on a modular ortholattice, L, with respect to a given variety of modular ortholattices, $\mathcal{V}$, are equivalent: L is in the variety of modular ortholattices generated by the finite-dimensional members of $\mathcal{V}$; L can be embedded in an atomistic member of $\mathcal{V}$; L has an orthogeometric representation in an anisotropic orthogeometry $(Q, \perp)$, where $[0, u] \in \mathcal{V}$, for all $u \in L_{\mathrm{fin}}(Q)$.

1. Introduction

As far as we know modular orthocomplemented lattices, MOLs, first explicitly appear in the Birkoff and von Neumann paper, [2], which was an initial attempt at a setting for a Propositional (Quantum) Logic, one whose lattice reducts are not necessarily distributive. In a loosely connected manner they also arise as the projection ortholattices of some important rings of operators, cf. [13]. However, from a purely algebraic viewpoint, the structure of the lattice of varieties of MOLs has been a matter of interest to the present authors for some decades. This interest was stimulated by the work of G. Bruns, [3], the Doctoral supervisor of the second author of this paper.

Recall, that an MOL is an algebra $(L; +, \cdot, 0, 1)$, where $(L; +, 0, 1)$ is a 0,1-modular lattice (we generally just use juxtaposition for the meet operation) and $'\colon L \to L$ is an orthocomplementation; $x + x' = 1, xx' = 0, (x + y)' = x'y', (xy)' = x' + y'$ and $x'' = x$. The useful property, $x \leq y$ iff $y' \leq x'$ follows directly.

We mention here two classes of MOLs that play a central role in this paper. The first is the class of simple, non-Boolean, 2-distributive MOLs, the MO_n, n a cardinal greater than one. For a given n, MO_n consists of the bounds and a pair of disjoint sets of n atoms; a suitable orthocomplementation on the atoms just resulting from a bijection between these two sets. The second is the class of simple modular otholattices of finite dimension[1] greater than two; each of which, except for those arising from non-Arguesian projective planes, comes from a finite-dimensional projective geometry over a (perhaps noncommutative) field equipped

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[1]Since complemented modular lattices are often associated with lattices of subspaces of vector spaces there is some temptation to use the term dimension in place of height in this setting. Height is more usual, and certainly more reasonable, for general lattices. In [12] we used dimension and, partly in the interest of continuity, we continue to do so in this paper.

with an anisotropic Hermitian form. The MOLs we have listed above, the two-element Boolean Algebra and the simple non-Arguesian MOLs of dimension 3 are all the subdirectly irreducible MOLs of finite-dimension.

One of the first interesting observations[2] about MOLs was that the unique cover of the variety of Boolean Algebras is the variety generated by $M0_2$. In [3], Bruns showed that the unique cover of the variety generated by MO_2 in the lattice of varieties of MOLs is the variety generated by MO_3. Baer's classical result, [1], that there is no finite projective plane whose lattice of subspaces admits an orthocomplementation, was perhaps partly behind a Conjecture of Bruns in [3]: Every variety of MOLs which is different from the Boolean Algebras and each of the varieties generated by an MO_n, $2 \leq n \leq \omega$, must contain an MOL whose lattice reduct is the lattice of subspaces of a projective plane. This interesting question remains open. However, in her review of [3], [15], Kalmbach asked the question of whether every variety of MOLs different from each of the varieties generated by the MO_n's and the Boolean Algebras must contain MO_ω. This question was answered affirmatively in [17]. From the Theorem of Baer mentioned above this would also be implied by an affirmation of Bruns' Conjecture.

We note that sitting above the variety generated by MO_ω is the variety generated by all the simple MOLs of dimension 3, PG_2 (geometric dimension is complemented modular lattice dimension minus one). And above this, the variety generated by all the simple MOLs of dimension 4, PG_3, and so on for each $n \in \mathbb{N}$. We call the union of this chain of varieties PG_ω. We do not know of an MOL which is not in PG_ω, and this begs, in particular, the question of whether every variety of MOLs is generated by its finite-dimensional members - but we do not know any nontrivial example of a variety such that any of its subvarieties is generated by its finite-dimensional members.

A more modest and probably more realistic project is to look for conditions under which a variety of MOLs is generated by its finite-dimensional members. In [10], we proved that this is the case for varieties generated by atomistic MOLs. It is also true for the Continuous Geometries arising from the classical von Neumann construction, [16] (this was observed independently in [6] and [11]), and for the projection ortholattices arising from finite von Neumann factors, [8].

Roughly speaking, an anisotropic orthogeometry is a projective geometry whose points are endowed with an orthogonality relation which is strong enough to produce an orthocomplementation on the lattice of closed subspaces with the property that the orthocomplement of every point is a hyperplane.

Now back to the atomistic situation. If M is an atomistic MOL then the set of atoms of M, P_M, and the set of elements of M of dimension 2 form the points and lines, respectively, of an anisotropic orthogeometry $(P_M, \perp)$. If L is a subalgebra of M then there is a natural embedding from L into the lattice of subspaces of P_M which 'preserves' orthogonality. This is a special case of a representation of L in an

[2]This special initial step is also true of the larger variety of Orthomodular lattices, cf. [14], page 123.

anisotropic orthogeometry, in this case $(P_M, \perp)$. These concepts were tied together by the following Theorem, stated in [9].

Theorem 1.1. *Let L be an MOL. The following are equivalent:*
1. $L \in PG_\omega$.
2. *L can be embedded in an atomistic MOL.*
3. *L has a repesentation in an anisotropic orthogeometry.*

Here we relativize Thm 1.1 to an arbitrary variety of MOLs to get,

Theorem 1.2. *Let L be an MOL and let $\mathcal{V}$ be a variety of MOLs. The following are equivalent:*
1. *L is in the variety of MOLs generated by the finite-dimensional members of $\mathcal{V}$.*
2. *L can be embedded in an atomistic MOL in $\mathcal{V}$.*
3. *L has a repesentation in an anisotropic orthogeometry, $(Q, \perp)$, where $[0, u] \in \mathcal{V}$ for all $u \in L_{\mathrm{fin}}(Q)$.*

To some extent, the above results are contained in [7] in a much more general setting, and with substantially more complicated proofs than we provide here. On the other hand, [9] appeared as an unpublished conference proceeding, is not very polished, and is hardly available. In the present paper we rely only on the references [10], [11] and [12] and give self-contained proofs otherwise for this range of material.

2. More preliminaries

First we provide slightly more detailed information about modular lattices and MOLs. The *dimension* of a modular lattice L is the cardinality of a maximal chain in L minus one. For our purposes this is either finite or ∞. We will explicitly use the following elementary fact about modular lattices.

Lemma 2.1. *Let L be a complemented modular lattice, let M be a modular 0-lattice and let $\varphi\colon L \to M$ be a map with $\varphi(0) = 0$, which preserves meets, and so that $\varphi(a + b) = \varphi(a) + \varphi(b)$, for all $a, b \in L$ with $ab = 0$. Then φ is a 0-lattice homomorphism.*

Proof. Since φ preserves meets it is order-preserving. We have to show φ preserves arbitrary binary joins. Let $x, y \in L$. Let s be a complement of xy in $[0, x]$. Then, $sy = 0$ because $s \leq x$ and so $\varphi(s) + \varphi(y) = \varphi(s + y) = \varphi(s + xy + y) = \varphi(x + y)$. Now, $\varphi(x + y) = \varphi(s) + \varphi(y) \leq \varphi(x) + \varphi(y)$, because φ is order-preserving. The reverse inequality holds because φ is order-preserving. □

A lattice is *atomistic* iff every non-zero element is a join of atoms. For MOLs this is equivalent to being *atomic*; every non-zero element is above an atom. Other (ortho)lattice-theoretic terminology and elementary, or well-known, facts are given in [12].

If L is an MOL and $[a, b]$ is an interval in L then $[a, b]$ inherits a natural MOL structure from L where the orthocomplementation, $x \mapsto x^{\#}$, for $x \in [a, b]$, is given by $x^{\#} = a + x'b$. It is well-known that, thus equipped, $[a, b]$ is in the MOL-variety generated by M (in fact it is a homomorphic image of a subalgebra of M). In particular, if the interval is a *section*, ie. $a = 0$, then $x^{\#} = x'b$. A slightly modified form of the main result, Proposition 1.2, of [10] is,

Theorem 2.2. *Let L be an MOL and let $\mathcal{V}$ be a variety of MOLs. If L can be embedded in an atomistic MOL, M, where $M \in \mathcal{V}$ then L is in the variety generated by the finite-dimensional members of $\mathcal{V}$.*

For the remainder of this section we briefly recall the geometric framework presented in detail in [12]. For further information on projective geometries we refer the reader to, for example, [4] or [5].

Let P be a projective geometry. Then $L(P)$ is the geomodular lattice of all subspaces of P. Notationally we identify the elements of P with the (one-element) subspaces they determine, and we do the same for lines. That is, if $p, q \in P$ are distinct we denote the line they determine by $p + q$. A *Baer subgeometry* of a projective geometry is a subset, Q of P, which is itself a projective geometry under the restriction of the collinearity relation on P and, moreover, the natural map from $L(Q)$ to $L(P)$ given by $a \mapsto \Sigma_{L(P)} a$, for all $a \in L(Q)$, is a lattice embedding.

An *anisotropic pre-orthogonality* on P is an antireflexive, symmetric, binary relation, $\perp$, on P satisfying the following:

$$p \perp q \text{ and } p \perp r \implies p \perp s \text{ for all } s \leq q + r$$

If $u \in L(P)$ then $u^{\perp} = \{p \in P \mid p \perp q \text{ for all } q \in u\} \in L(P)$ and the map $u \mapsto u^{\perp}$ is a Galois operator on $L(P)$. The subspaces, $u^{\perp}, u \in L(P)$, equivalently $u = u^{\perp\perp}$, are called *closed*. The closed subspaces, $L_c(P)$, form a complete lattice. However, this is not in general a sublattice of $L(P)$; the meet operation is still intersection but the join operation is the usual one associated with a closure operator, in this case ${}^{\perp\perp}$. An anisotropic pre-orthogonality is an *anisotropic orthogonality* if $p + p^{\perp} = 1$ in $L(P)$, or equivalently, $p^{\perp} \prec 1$ in $L(P)$, for all $p \in P$. In this case, we call $(P, \perp)$ an *anisotropic orthogeometry*. We recall some basic facts: $L(P)$ is a geomodular lattice. $L_c(P)$ is an ortholattice but generally not modular (nor even orthomodular). $L_{\text{fin}}(P)$ is a sub-0-lattice of $L_c(P)$ and $L_{\text{fin}}(P) \cup \{u^{\perp} \mid u \in L_{\text{fin}}(P)\}$ is an MOL. In particular, the sections $[0, u]$, $u \in L_{\text{fin}}(P)$ are MOLs with the orthocomplementation, as above, given by $x \mapsto x^{\perp}u$, for all $x \in [0, u]$. The following calculating rules are useful. For all $u, v \in L(P)$,

$$u \leq v^{\perp} \text{ iff } v \leq u^{\perp}, \quad u \leq v \text{ implies } v^{\perp} \leq u^{\perp}, \quad u \leq u^{\perp\perp}$$
$$(u + v)^{\perp} = u^{\perp}v^{\perp} \text{ (de Morgan)}, \quad (uv)^{\perp} \geq u^{\perp} + v^{\perp} \text{ (weak de Morgan)}.$$

A *faithful representation* of an MOL, L, in an anisotropic orthogeometry $(P, \perp)$ is a $0,1$-lattice embedding $\psi \colon L \to L(P)$ which preserves orthogonality in the sense that $\psi(a) \perp \psi(a')$, for all $a \in L$. In this situation $\psi(L) \leq L_c(P)$ as an ortholattice as well (see the remark below Lemma 6.2 of [12]).

In particular, returning to the atomistic case, if an MOL L is embedded, via φ, in an atomistic MOL, M, then $\eta \colon L \to L(P_M)$ given by $\eta(a) = \{p \in P_M \mid p \leq \varphi(a)\}$ is a faithful representation of L in $(P_M, \perp)$. And referring to the relevant case of Lemma 2.4 of [12], where M is atomistic, $L_{\text{fin}}(P) \simeq M_{\text{fin}}$.

We recall some technical results needed for the proof of the Main Theorem; the first three are well-known.

Proposition 2.3. *Let* $(P, \perp)$ *be an anisotropic orthogeometry, let* $a \leq b$ *and, for* $i \in \{1,2\}$, $a_i \leq b_i$ *in* $L(P)$ *such that* $\dim([a,b]) < \infty$ *and* $\dim([a_i,b_i]) < \infty$. *Then,*

(1) $\dim([a_1a_2, b_1b_2]) < \infty$ *and* $\dim([a_1+a_2, b_1+b_2]) < \infty$.
(2) $\dim([b^\perp, a^\perp]) \leq \dim([a,b])$.
(3) *If, in addition* $a \in L_c(P)$, *then* $\dim([b^\perp, a^\perp]) = \dim([a,b])$ *and* $b \in L_c(P)$.
(4) *If, for* $i \in \{1,2\}$, $a_i, b_i \in L_c(P)$ *and* $a_1+a_2, a_1^\perp + a_2^\perp \in L_c(P)$, *then* $b_1+b_2, b_1^\perp + b_2^\perp \in L_c(P)$.

Proof. In any modular lattice defining a related to b iff $\dim([ab, a+b]) < \infty$ is a congruence relation, and thus (1) holds.

For both (2) and (3) it suffices to deal with $a \prec b$ and then appeal to a simple induction argument, which we do not include.

If $a \prec b$ then $b = a + p$ for some $p \in P$, whence, $b^\perp = a^\perp p^\perp$, by de Morgan's Law. Since $p^\perp \prec 1$ we have $b^\perp = a^\perp p^\perp = a^\perp$ or $b^\perp = a^\perp p^\perp \prec a^\perp$. Thus (2) holds for covers.

Now assume $a \in L_c(P)$. Then, as above, we get $b^\perp = a^\perp p^\perp = a^\perp$ or $b^\perp = a^\perp p^\perp \prec a^\perp$. But, the first of these gives $a^\perp \leq p^\perp$ and $p = p^{\perp\perp} \leq a^{\perp\perp} = a$, contradicting $b = a+p$. Hence, $b^\perp \prec a^\perp$. This proves the first part of (3) for covers.

Now, since $b^\perp, a^\perp \in L_c(P)$ and we have proved the first part of (3) for covers, $a = a^{\perp\perp} \prec b^{\perp\perp}$. But $a \prec b \leq b^{\perp\perp}$ now gives $b = b^{\perp\perp}$. This proves (3) for covers.

The proof of (4) requires a little more work. First we assume $a_1 \prec b_1$ and $a_2 = b_2$ (we'll refer to this element for this case as a_2) and will prove $b_1 + a_2$ and $b_1^\perp + a_2^\perp$ are in $L_c(P)$. The first of these follows directly from (3); $a_1 \prec b_1$ gives $\dim([a_1+a_2, b_1+a_2]) \leq 1$ and, either $b_1 + a_2 = a_1 + a_2 \in L_c(P)$ or, by (3), $a_1 + a_2 \in L_c(P)$ gives $b_1 + a_2 \in L_c(P)$.

We now show that $b_1^\perp + a_2^\perp \in L_c(P)$. If $b_1^\perp + a_2^\perp = a_1^\perp + a_2^\perp$ then we are done trivially. So we may assume $b_1^\perp + a_2^\perp \prec a_1^\perp + a_2^\perp$.

We first show that if there exists $p \in P$ with $p \leq b_1a_2$ and $p \nleq a_1$, then $b_1^\perp + a_2^\perp \in L_c(P)$. Observe that $p \leq b_1a_2$ gives $b_1^\perp + a_2^\perp \leq p^\perp$ and, since $a_1 \in L_c(P)$, $a_1^\perp \nleq p^\perp$, so $a_1^\perp + a_2^\perp \nleq p^\perp$. This gives, $b_1^\perp + a_2^\perp \leq (a_1^\perp + a_2^\perp)p^\perp < a_1^\perp + a_2^\perp$. But $\dim([b_1^\perp + a_2^\perp, a_1^\perp + a_2^\perp]) = 1$, hence $b_1^\perp + a_2^\perp = (a_1^\perp + a_2^\perp)p^\perp$, and as the meet of closed elements, $b_1^\perp + a_2^\perp$ is closed.

It remains to show that there exists such a $p \in P$ ($p \leq b_1a_2$ and $p \nleq a_1$). Assume there does not. Since $L(P)$ is atomistic we would have $b_1a_2 \leq a_1 < b_1$. Modularity now gives $a_1 + a_2 < b_1 + a_2$ and, by assumption and from the first part of the proof for this case of (4), we have both these elements in $L_c(P)$. Hence $b_1^\perp a_2^\perp < a_1^\perp a_2^\perp$; we will obtain a contradiction to this.

Since $b_1^\perp + a_2^\perp \prec a_1^\perp + a_2^\perp$, we have $a_1^\perp \nleq b_1^\perp + a_2^\perp$ and $b_1^\perp \leq a_1^\perp(b_1^\perp + a_2^\perp) < a_1^\perp$. But now, since $a_1, b_1 \in L_c(P)$, $b_1^\perp \prec a_1^\perp$ and so $b_1^\perp = a_1^\perp(b_1^\perp + a_2^\perp)$. This gives, $b_1^\perp a_2^\perp = a_1^\perp(b_1^\perp + a_2^\perp)a_2^\perp = a_1^\perp a_2^\perp$ which is the desired contradiction.

This completes the proof for the case $a_1 \prec b_1$ and $a_2 = b_2$. If $a_1 = b_1$ and $a_2 \prec b_2$ then a symmetric argument works. Thus (4) holds by induction on $\dim([a_1, b_1]) + \dim([a_2, b_2])$. □

3. Proof of the main Theorem

For convenience we repeat the statement of the main Theorem.

Let L be an MOL and let $\mathcal{V}$ be a variety of MOLs. The following are equivalent:

(1) L is in the variety of MOLs generated by the finite-dimensional members of $\mathcal{V}$.
(2) L can be embedded in an atomistic MOL in $\mathcal{V}$.
(3) L has a repesentation in an anisotropic orthogeometry, $(Q, \perp)$, where $[0, u] \in \mathcal{V}$ for all $u \in L_{\text{fin}}(Q)$.

(2) $\Rightarrow$ (1) follows directly from Theorem 2.2. We first show that (2) and (3) are equivalent and, based on these, that (1) and (3) are equivalent.

Proof of (2) $\Rightarrow$ (3). We refer the reader to the comment on page 4 immediately above the preamble to Proposition 2.3. If M is an atomistic extension of L in $\mathcal{V}$ then $\eta\colon L \to L(P_M)$ given by $\eta(a) = \{p \in P_M \mid p \leq a\}$, for all $a \in L$, is a faithful representation of L in the anisotropic orthogeometry $(P_M, \perp)$. Because M is in $\mathcal{V}$, and $L_{\text{fin}}(P_M) \simeq M_{\text{fin}}$, the sections $[0, u], u \in L_{\text{fin}}(P_M)$, are in the variety generated by M, and since $M \in \mathcal{V}$ we have $[0, u] \in \mathcal{V}$ for all $u \in L_{\text{fin}}(P_M)$. This completes the proof of (2) $\Rightarrow$ (3). □

Proof of (3) $\Rightarrow$ (2). For simplicity we assume that L is simultaneously a $0,1$-sublattice of L(P) and a subalgebra of $L_c(P)$, that is, id_L is representation of L in $(P, \perp)$. Let M consist of all $x \in L_c(P)$ such that there are $c \in L$, $y, z \in L_c(P)$ such that $y \leq x \leq z$, $y \leq c \leq z$, and $\dim([y, z]) < \infty$.

By Proposition 2.3 part (2), one has M is closed under $^\perp$ and the first statement in part (1) shows that M is closed under meets. Thus, M is a sub-ortholattice of the ortholattice $L_c(P, \perp)$. Closure under joins in $L(P)$ follows from the second statement of part (1) and from part (4) of Proposition 2.3 as follows.

Let, for $i \in \{1, 2\}$, $x_i \in M$ with associated $c_i \in L$ and $y_i, z_i \in L_c(P)$ as in the definition of M. Then, for $i \in \{1, 2\}$, $c_i \leq z_i$, and $c_i, z_i, c_1 + c_2, c_1^\perp + c_2^\perp$ are all in $L_c(P)$. If follows from part (4) that $z_1 + z_2 \in L_c(P)$. For the other join, $y_1 + y_2$, for $i \in \{1, 2\}$, $c_i^\perp \leq y_i^\perp$, and $c_i^\perp, y_i^\perp, c_1^\perp + c_2^\perp$ and $c_1 + c_2 = c_1^{\perp\perp} + c_2^{\perp\perp}$ are all in $L_c(P)$. Now part (4) gives $y_1 + y_2 = y_1^{\perp\perp} + y_2^{\perp\perp} \in L_c(P)$. Obviously, $y_1 + y_2 \leq x_1 + x_2 \leq z_1 + z_2$ and $y_1 + y_2 \leq c_1 + c_2 \leq z_1 + z_2$, and by part (1), $\dim([y_1 + y_2, z_1 + z_2]) < \infty$.

This shows that M is then also a sublattice of $L(P)$ and hence an MOL. Applying part (2) of Proposition 2.3 with $a = 0$ gives $L_{\text{fin}}(P) \simeq M_{\text{fin}}$ and since M is in the variety generated by its finite-dimensional members, $M \in \mathcal{V}$. This completes the proof of (3) $\Rightarrow$ (2). □

In the following, Lemmas 3.1 and 3.2, we will be dealing with a set F as a subset of different MOLs. The notation $x \leq F$, where x is an element of the relevant MOL, means $x \leq f$, for all $f \in F$.

Lemma 3.1. *Let L, M be MOLs with $L \leq M$ and let F be a neutral filter of L (with corresponding congruence θ). Let $a, b \in L$ with $ab = 0$ and let $p \in P_M$ with $p \leq a + b$ and $p \leq F$. Then $a(p + b), b(p + a) \leq F$.*

Proof of the Lemma 3.1. If $p \leq a$ then $a(b+p) = p$ and $b(a+p) = 0$ and we are done trivially and, symmetrically if $p \leq b$. Otherwise let $q = a(p + b)$ and $r = b(p + a)$. By modularity, p, q, r form a collinear triple in P_M, ie. they are distinct elements of P_M and $p + q = p + r = q + r$.

Let c be a complement of $a + b$ in L and set $\tilde{a} = a + c$. The hypotheses are satisfied if a is replaced by $\tilde{a}$ and the conclusions are formally stronger (although by modularity they are actually the same). So we may assume $a + b = 1$.

Let $x \in F$, ie. $x \,\theta\, 1$. By assumption, $p \leq x$. Let

$$y = (a + xb)(b + x) \geq a(b + p) = q$$

and

$$z = (b + xa)(a + x) \geq b(a + p) = r.$$

Modularity and $ab = 0$ give

$$xy = xz = yz = xa + xb.$$

Similarly, using $a + b = 1$,

$$x + y = x + z = y + z = (a + x)(b + x).$$

Hence, either $x = y = z$ or x, y, z are the atoms of an M_3 as a sublattice of L. In either case, since $x \,\theta\, 1$, we have $x, y, z \in F$. Since $p \leq F$, $p \leq y$. And $q \leq y$ from above. Hence $r \leq p + q \leq y$. From above, $r \leq z$ and now $r \leq yz \leq x$. Symmetrically, $q \leq x$. ☐

Lemma 3.2. *Let K be an atomistic extension of L in $\mathcal{V}$ and let θ be a congruence of L. Then L/θ has a faithful representation in an anisotropic orthogeometry, $(Q, \perp)$ with $[0, u] \in \mathcal{V}$ for all $u \in L_{\text{fin}}(P)$.*

Proof of Lemma 3.2. Extend the language of MOLs to include the elements of L as constants. Let F be the neutral filter of L, $1/\theta$. For each $a \in L$, with $a \not\theta\, 0$, and each $c \in F$, let $\varphi_{a,c}$ be the formula

$$0 < x_a \leq ac$$

with a distinguished variable x_a. For a fixed a, let $\Phi_a = \{\phi_{a,c} \mid c \in F\}$. Consider any finite subset Φ_0 of Φ_a. Since Φ_0 is finite the set $C = \{c \in F \mid \phi_{a,c} \in \Phi_0\}$ is too. Since F is a filter $c_0 = \Pi C \in F$. Set the value of x_a to ac_0. Since $a \not\theta\, 0$ and $c_0 \,\theta\, 1$ we have $ac_0 \not\theta\, 0$. In particular, $0 < ac_0 \leq ac$ for all $c \in C$. Thus Φ_a is finitely satisfiable in K. This holds for all $a \not\theta\, 0$. By the Compactness Theorem of First Order Logic there is an elementary extension M of K so that for each Φ_a there exists an $a^* \in M$ so that $\phi(a^*)$ holds for all $\phi \in \Phi_a$. In particular, $0 < a^* < a$

and $a^* \leq F$. M is atomistic so choose an atom p below a^*. Then $p \in P_M$ with $0 < p \leq a$ and $p \leq F$.

Let $Q = \{p \in P_M \mid p \leq F\}$. Since Q is a subspace of P_M and since $\perp$ is anisotropic, Q with the restriction $\perp_Q$ of $\perp$ is an anisotropic orthogeometry: observe that $p^{\perp_Q} = p^\perp \cap Q \prec Q$ by modularity. We will show that L/θ has a representation in $(Q, \perp_Q)$.

Let $a, b \in L$ with $a\theta b$ we claim that, in M, for all $q \in Q$, $q \leq a$ iff $q \leq b$. Since θ is a congruence is suffices to show this for $a \leq b$ and, in this case that $q \leq b$ implies $q \leq a$. Since $a \leq b$ and $a\theta b$ we have $(a + b')\theta(b + b') = 1$, so $a + b' \in F$. Since $q \leq F$, $q \leq a + b'$ and $q \leq b$ gives $q \leq b(a + b') = a$, by modularity. Now we can unambiguously define $\eta : L/\theta \to L(Q)$ by $\eta(a/\theta) = \{q \in Q \mid q \leq a\}$. It is clear that η preserves meets and $\eta(0) = 0$. Suppose a/θ and b/θ meet to 0 in L/θ. Then, since $(a' + b')\theta 1$ we may replace a with $a(a' + b')$, ie. we may assume $ab = 0$. Let $r \leq a + b$. If $r \leq a$ or $r \leq b$ then trivially $r \in \eta(a/\theta) + \eta(b/)$. Otherwise, $p = a(r + b) \in P_M$ and $q = b(r + a) \in P_M$ and $r \leq p + q$. By Lemma 3.1, $p, q \in Q$ and thus $r \leq \eta(a) + \eta(b)$ in $L(Q)$. It follows from Lemma 2.1 that η is a 0-lattice homomorphism. If $p, q \in Q$ with $p \leq a$ and $q \leq a'$ then $p \perp_Q q$ by definition. Hence $\eta(a/\theta) \perp_Q \eta((a/\theta)')$, for all $a/\theta \in L/\theta$

Above we showed that for every a/θ different from $0/\theta$, $\eta(a/\theta) \neq \emptyset$ and clearly $\eta(1) = 1$. Hence η is a faithful representation of L/θ in the orthogeometry $(Q, \perp_Q)$.

Finally, we need to show that $[0, u] \in \mathcal{V}$ for all $u \in L_{\text{fin}}(Q)$. For any $u \in L_{\text{fin}}(Q)$, $[0, u] \simeq [0, u]_M$, again since Q is a subspace of P_M, and this small claim follows from $M \in \mathcal{V}$. □

Corollary 3.3. *The class of MOLs with an atomistic extension within $\mathcal{V}$ forms a variety.*

Proof of Corollary 3.3. Clearly this class is closed under products and subalgebras. Lemma 3.2 shows any homomorphic image, L/θ, of L has a faithful representation in an anisotropic orthogeometry $(Q, \perp_Q)$ with $[0, u] \in \mathcal{V}$, for all $u \in L_{\text{fin}}(Q)$. Since we have already shown (3) $\Rightarrow$ (2), L/θ has an atomistic extension within $\mathcal{V}$. This gives closure under homorphic images. □

Proof of (1) $\Leftrightarrow$ (3). For $i \in \{1, 2, 3\}$ let $\mathcal{K}_i$ be the class of MOLs satisfying (i). Then, since (2) and (3) are equivalent $\mathcal{K}_2 = \mathcal{K}_3$. But, from (2) $\Rightarrow$ (1), and since every finite-dimensional member of $\mathcal{V}$ is a member of $\mathcal{K}_2$, we have $\mathcal{K}_1$ is the variety of MOLs generated by $\mathcal{K}_2$. But $\mathcal{K}_2 = \mathcal{K}_3$ and $\mathcal{K}_3$ is already a variety by Corollary 3.3, so $\mathcal{K}_1 = \mathcal{K}_3$.

This completes the proof of (1) $\Leftrightarrow$ (3) and hence of Theorem 1.2. □

We close with with a point of clarification. The introduction implicitly (and intentionally) omits something of importance. The statement of Theorem 1.1 is relative to the full variety of MOLs, whereas the statement of Theorem 1.2 refers to an arbitrary variety of MOLs containing the given algebra L. In the preceding discussion, aimed at providing a concise summary of known results, and to provide motivation for the material in this paper, we moved immediately from the full variety of MOLs to, given L, the variety generated by L. This latter is the strongest condition an MOL can satisfy in this regard. And for the results we know of the form 'L is in the variety generated by the finite-dimensional MOLs of some variety'. in principle there are many possibilities; but only a few very special L are known where this variety can be choosen the one generated by L.

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