On geometric representations of modular ortholattices

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Abstract. A pre-orthogonality on a projective geometry is a symmetric binary relation, $\perp$, such that for each point p, $p^\perp = \{q \mid p \perp q\}$ is a subspace. An orthogonality is a pre-orthogonality such that each $p^\perp$ is a hyperplane. Such $\perp$ is called anisotropic iff it is irreflexive. For projective geometries with an anisotropic pre-orthogonality, we show how to find a (large) projective subgeometry with a natural embedding for the lattices of subspaces and with an orthogonality induced by the given pre-orthogonality. We also discuss (faithful) representations of modular ortholattices within this context and derive a condition which allows one to transform a representation by means of an anisotropic pre-orthogonality into one by means of an anisotropic orthogonality.

1. Introduction

Classical examples of modular ortholattices range from the Boolean Algebras to the continuous geometries of J. von Neumann, [17]. Between these are the modular ortholattices which occur as the ortholattices of projections of type II_1 von Neumann Algebra factors, classified by F. J. Murray and J. von Neumann in [16]. For a recent development related to the ideas in this paper see [10]. Further examples are derived from inner product spaces: they consist of all subspaces of finite dimension and their orthogonals.

Generalizing the last example, in the (informal) conference proceedings [11], we had studied (faithful) representations of modular ortholattices in subspace lattices of projective geometries P endowed with a self-adjoint Galois connection $X \mapsto X^\perp$ (cf. [15]) resp. the induced relation $\perp$ on points; the latter will be called a *pre-orthogonality* in the present note, an *orthogonality* (and $(P, \perp)$ an *orthogeometry*) if, in addition, $p^\perp$ is a hyperplane for any point. In the context of ortholattices, *anisotropicity* ($p \notin p^\perp$) matters and allows one to use representations to derive results in the equational theory of modular ortholattices. (cf. [11, 10]).

In this paper we provide a framework for the possible representation of a general modular ortholattice in an anisotropic orthogeometry. Our main result is a condition under which a representation in a geometry, with an anisotropic pre-orthogonality, can be transformed into a representation in an anisotropic orthogeometry, cf. Theorem 6.6. This is based on a lattice theoretic view on projective subgeometries (cf. Lemma 3.2) and on identifying an (apparently large) sub-orthogeometry within any projective geometry with an anisotropic pre-orthogonality,

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cf. Theorem 5.1. It is our hope that these results and the accompanying framework will pave the way for future work on geometric representations of modular ortholattices.

The accompanying framework has been presented in [11] and in [9], the latter for the more involved case dealing with complemented modular lattices with involution. For convenience, we have restated what is needed, here, and have provided (usually easy) proofs to keep this paper as self-contained as is reasonable. In the same spirit we have made the paper technically complete in order for readers with less experience in the area to check the details fairly easily. The proofs of some results which may seem natural or 'obvious' to the more advanced reader could be skipped on, at least, a first reading.

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2. Lattices and Geometries

This section contains introductory material on modular lattices, projective geometries, and the well-known links between the two. For lattice theory we refer the reader to [1], [3] or [7]. There are many classical references on projective geometry, but the relatively recent [5] is the one we have been using and which we follow for the most part here.

For lattices pairwise joins will be denoted by $+$ and meets by $\cdot$. We shall often save time and brackets by using juxtaposition for meets and following the convention that meet takes priority over join. When we have occasion to use joins or meets over larger (sometimes infinite) sets we will use the corresponding capitalized symbols. If a lattice L has a smallest element we denote it by $0 = 0_L$ and if it has a largest element we denote it by $1 = 1_L$. We sometimes treat the bounds 0 and 1 as constants and then we speak of a $0,1$-lattice, and occasionally we just consider the 0 as a constant and speak of a 0-lattice.

If $a, b \in L$, and $a < b$ with no $z \in L$ with $a < z < b$ we say that b covers a and write $a \prec b$. The *dimension* of L (also called length or height) is the supremum of the non-negative integers $|C| - 1$ where C is a finite chain in L, and we write $\dim(L)$ for this number if it is finite. If $[a, b]$ is an interval in L, then $\dim([a, b])$ is the dimension of $[a, b]$ when treated as a lattice, and if $u \in L$ the *dimension* of u is $\dim(u) = \dim[0, u]$ (this number existing implies that L has a 0). In a $0,1$-lattice the dimension of $[u, 1]$ is called the *codimension* of u, written $\mathrm{codim}(u)$ - this is our excuse for using the old fashioned 'dimension'.

The above serves to establish our notation. Apart from this we expect that the reader is familiar with the rudiments of modular lattice theory. In a modular lattice, L, the elements of finite dimension form a sublattice of L which we will denote by L_{fin}. Dually the elements of finite codimension form a sublattice L_{cof}. If L is a complemented modular lattice then L_{fin} is atomistic, and the complements of an element of L_{fin} are in L_{cof}, and vice-versa. Consequently, $L_{\text{fin}} \cup L_{\text{cof}}$ is a complemented sublattice of L.

We will be particularly interested in certain complemented modular lattices. A *geomodular lattice* is a complete, atomistic, complemented modular lattice where the atoms are compact (that is, if an atom is below a join of elements then it is already below the join of a finite subset of these elements). To put it another way, the geomodular lattices are precisely the algebraic complemented modular lattices.

The atoms (covers of 0) of a 0-lattice L will be of particular importance to us and we will denote the set of these by P_L; the atoms are the points of the corresponding projective geometry. Similarly elements of height two in a complemented modular lattice correspond to the lines of the associated projective geometry. If k is a 'line' and $p, q, r \in P$ are distinct with $k = p + q = q + r = q + r$ we call p, q, r *collinear*. We will return to this topic shortly.

We conclude this part of the section by recalling two well-known results from modular lattice theory which will be used later.

Lemma 2.1. *(cf. page 41 of* [1]*). Let L be a modularlattice and let $u \in L$ with* $\dim(u) = n$*. Then any maximal chain in* $[0, u]$ *is of length n. That is, any maximal chain in* $[0, u]$ *is of the form* $0 \prec c_1 \prec \ldots \prec c_n = u$*. Also, if $a, b \in L$ are of finite dimension then* $\dim(a + b) = \dim(a) + \dim(b) - \dim(ab)$.

Lemma 2.2. *(Folklore). Let L and M be modular lattices where every element has finite height, and let $\varphi\colon L \to M$ be a dimension-preserving join embedding. Then φ is a lattice embedding.*

Proof. Let $a, b \in L$ with $\dim(a) = m$, $\dim(b) = n$, $\dim(ab) = k$ and, hence $\dim(a + b) = m + n - k$. Then $\dim(\varphi(a)) = m$, $\dim(\varphi(b)) = n$, $\dim(\varphi(a) + \varphi(b)) = \dim(\varphi(a+b)) = m+n-k$, and hence $\dim(\varphi(a)\varphi(b)) = k$. But also $\dim(\varphi(ab)) = k$ and, since $\varphi(ab) \le \varphi(a)\varphi(b)$, $\varphi(ab) = \varphi(a)\varphi(b)$. □

We now turn to the geometric point of view. We will consider a projective geometry to be a set of points P together with a totally symmetric ternary *collinearity* relation, C, on P which satisfies (cf. page 26, [5]): For all $a, b, c, d, p, q \in P$,

$C(a, b, a)$ for all $a, b \in P$ (any two points are contained in some line),

$C(a, b, c)$ and $C(q, b, c)$ and $b \neq c$ imply $C(a, q, b)$ (every line is determined by any two distinct points on it),

$C(p, a, b)$ and $C(p, c, d)$ imply $C(q, a, c)$ and $C(q, b, d)$ for some $q \in P$ (we call this the *projective property*).

Most readers will be familiar with projective planes. In a projective plane every pair of distinct lines must intersect in a single point. The projective property is the higher dimensional analogue of this and is the key to the lattice of subspaces of a projective geometry being modular.

A *subspace* of a projective geometry P is a subset Q of P which is closed under the collinearity relation; that is, $r \in Q$ if $C(p, q, r)$ for some $p \neq q$ in Q. One easily sees that the subspaces of a projective geometry ordered by containment form a complete $0, 1$-lattice where meet is intersection and join is closure of the union under the collinearity relation. Hence, if $p, q \in P$ are distinct then $p + q$ is the set of all $r \in P$ with $C(p, q, r)$, ie. the line determined by p and q. We will abuse notation slightly by identifying points with the one-element subspaces they define and usually stating $r \leq p+q$ instead of $r \in p+q$. We call this lattice $L(P)$. The following well-known result sums up the elementary relationship between projective geometries and geomodular lattices

Theorem 2.3. *If P is a projective geometry then $L(P)$ is a geomodular lattice. Conversely, if L is a geomodular lattice then P_L together with the collinearity relation defined earlier in the section is a projective geometry. This establishes a category equivalence between both classes w.r.t. isomorphisms; which can be extended with a suitable concept of morphism for projective geometries on one side, sup-preserving maps on the other, cf.* [4].

Observe that the dimension of a subspace Q, considered as a projective geometry, is $\dim(\sum Q)-1$. Also, recall that for $p_i \in P$ one always has $\dim(\sum_{i=1}^n p_i) \leq n$ and that equality holds iff $p_j \sum_{i\neq j} p_j = 0$ for some/any renumbering of the p_i. We will find one more very easy folklore result useful. Let M be a modular 0-lattice, let P_M be the set of atoms of M and set

$$B = \{x \in M \mid x = \sum_M P_0 \text{ for some } P_0 \subseteq P_M \text{ with } P_0 \text{ finite}\}.$$

Lemma 2.4. *With M, B, and P_M as above: B is an ideal of M, P_M is a projective geometry, and $L_{\text{fin}}(P)$ is isomorphic to B via the map $a \mapsto \sum_M a$, for all $a \in L_{\text{fin}}(P)$.*

Many of the definitions and much of the notation in this paper have been developed semi-independently by different authors in different settings. We have strived to strike a balance between what is already in the literature and what seems appropriate in our particular situation. We have not always found it desirable, or even possible, to follow any one reference exactly, and the reader should consider this when checking other sources.

3. SubGeometries

Let P be a projective geometry and let $Q \subseteq P$. Q is a *projective subgeometry* of P iff Q is itself a projective geometry under the restriction of the collinearity relation on P. Let P be a projective geometry and let Q be a projective subgeometry of P, hence both $L(Q)$ and $L(P)$ are geomodular lattices. Define, $\varphi\colon L(Q) \to L(P)$ by, for all $a \in L(Q)$,

$$\varphi(a) = \sum_{L(P)} a.$$

Observation 3.1. *φ preserves arbitrary joins. In particular, φ is order-preserving.*

By definition, Q is a *Baer subgeometry* of P iff φ is a lattice embedding of $L(Q)$ into $L(P)$ (Baer subspaces are examples of such). Example 7.2.11 of [5] illustrates that not all projective subgeometries of a projective geometry are Baer subgeometries (these are called *proper* in [5]). A second condition that is formally weaker than being a Baer subgeometry is the following: Q *geometrically embeds* in

P iff for all $q \in Q$ and finite dimensional $a \in L(Q)$, $q \leq \varphi(a)$ in $L(P)$ implies $q \leq a$ in $L(Q)$.

The following Lemma is no doubt known by many people but it does not appear to be in the literature in this form. A full proof of the Lemma is tedious but elementary; only relying on elementary compactness arguments, and elementary facts about modular lattices and closure operators. We only use $(i) \Rightarrow (iii)$ and we prove this fully by proving $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$. We leave the rest as an exercise for the reader.

Lemma 3.2. *Let Q be a projective subgeometry of P. The following statements are equivalent:*

(i) *Q geometrically embeds in P.*
(ii) *$\varphi|_{L_{\mathrm{fin}}(Q)} \colon L_{\mathrm{fin}}(Q) \to L_{\mathrm{fin}}(P)$ is a dimension preserving 0-lattice embedding.*
(iii) *(iii) $\varphi \colon L(Q) \to L(P)$ is a 0-lattice embedding (ie. by definition Q is a Baer subgeometry of P).*
(iv) *(iv) $\varphi \colon L(Q) \to L(P)$ is a 0-lattice embedding preserving arbitrary joins and non-empty, but otherwise arbitrary, meets*

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Proof. (partial). $(i) \Rightarrow (ii)$: Let $a \in L_{\mathrm{fin}}(Q)$ with $\dim(a) = n$. Then there exist $q_1, ..., q_n \in Q$ with $a = \sum_{L(P)}^{n} q_i$. Then $\varphi(a) = \sum_{L(P)} q_i$ and so $\dim(\varphi(a)) \leq n$. Suppose $\dim(\varphi(a)) < n$, say $q_n \leq \sum_{L(P)}^{n-1} q_i$. Then, by (i), $q_n \leq \sum_{L(Q)}^{n-1} q_i$, contradicting $\dim(a) = n$. Thus $\varphi|_{L_{\mathrm{fin}(Q)}}$ is dimension preserving. By Observation 3.1 and Lemma 2.2, φ is a 0-lattice embedding.

$(ii) \Rightarrow (iii)$: From Observation 3.1 it suffices to show that φ preserves binary meets. Let $a, b \in L(Q)$ and suppose $p \in P$ with $p \leq \varphi(a)\varphi(b)$. Since p is compact in $L(P)$, there exist $m, n \in \mathbb{N}$, $q_1, ..., q_m \in a$, $r_1, ..., r_n \in b$ with $p \leq \sum_{L(P)}^{m} q_i$ and $p \leq \sum_{L(P)}^{n} r_i$. Let $a_0 = \sum_{L(Q)}^{m} q_i$ and $b_0 = \sum_{L(Q)}^{n} r_i$. Then $p \leq \varphi(a_0) \cdot \varphi(b_0) = \varphi(a_0 b_0)$, by (ii). But now $p \leq \varphi(a_0 b_0) \leq \varphi(ab)$, since φ is order-preserving. □

4. Representations without orthogonality

In this short section we deal with the 'non-ortho' part of the representation process. Let M be a modular 0,1-lattice, let L be a complemented modular lattice, let $\varphi \colon L \to M$ be a 0-lattice embedding, and let Q be a projective subgeometry of P_M.

Q is (L, φ)-*closed* iff $\varphi(a)(p + \varphi(d)) \in Q$ for all $p \in Q$ and $a, d \in L$ such that $p \not\leq \varphi(d)$, $ad = 0$ and $a + d = 1$.

Q is (L, φ)-*dense* iff for each $0 < c \in L$ there is some $p \in Q$ with $p \leq \varphi(c)$.

Define $\eta \colon L \to L(Q)$ by, for all $a \in L$,

$$\eta(a) = \{\, q \in Q \mid q \leq \varphi(a)\}.$$

Lemma 4.1. *((Cf. Lemma 10.2,* [9]*) If Q is (L, φ)-closed then η is a 0-homomorphism.*

Proof. The only part that requires work is closure under joins. Since φ is order preserving one inclusion is trivial. We first establish the reverse when the joinands meet to 0.

Let $a, b \in L$ with $ab = 0$ and let $q \in Q$ with $q \leq \eta(a+b)$. If $q \leq \varphi(a)$ or $q \leq \varphi(b)$ then we are done trivially. So we may assume $q\varphi(a) = 0 = q\varphi(b)$.

Let c be a complement of $a + b$ in L. Set $p = \varphi(a)(q + \varphi(b + c))$ and $r = \varphi(b)(q + \varphi(a + c))$. Easy modularity arguments give that q, a and $b + c$ meet the requirements of the definition of (L, φ)-closed, and hence $p \in Q$. Symmetrically $r \in Q$. By modularity, p, q, r are collinear. Since $p \leq \eta(a)$ and $r \leq \eta(b)$ we have $q \leq \eta(a) + \eta(b)$. It follows $\eta(a + b) = \eta(a) + \eta(b)$ in this case.

If $ab \neq 0$ then let $\tilde{b}$ be a complement of ab in $[0, b]$ and apply the special case to $a, \tilde{b}$ to obtain $\eta(a + b) = \eta(a + \tilde{b}) = \eta(a) + \eta(\tilde{b}) \leq \eta(a) + \eta(b)$. □

Corollary 4.2. *If $Q = P_M$ and $\varphi(1) = 1$ then η is a $0,1$-lattice homomorphism.*

We call $\eta\colon L \to L(P_M)$ the *canonical homomorphism* in this setting.

Corollary 4.3. *If Q is both (L, φ)-closed and (L, φ)-dense then η is a 0-lattice embedding.*

Proof. Let $a < b$ in L and let c be a complement of a in $[0, b]$. Then there is some $q \in Q$ with $q \leq \varphi(c)$. Now $q \leq \eta(b)$ and $q \not\leq \eta(a)$. □

5. Orthogonality

Let P be a projective geometry. A *pre-orthogonality* on P is a symmetric binary relation, $\perp$, on P such that, for all $p, q, r, s \in P$,

$$p \perp q \text{ and } p \perp r \text{ implies } p \perp s \text{ for all } s \leq q + r.$$

A pre-orthogonality which has the property that $p \not\perp p$, for all $p \in P$ is called *anisotropic*. A projective geometry with pre-orthogonality $\perp$ is formally written $(P, \perp)$, although when it is unlikely to cause confusion we just write P. We call $(P, \perp)$ *anisotropic* whenever $\perp$ is.

It is clear that, if $Q \subseteq P$ then $Q^\perp = \{p \in P \mid p \perp q \text{ for all } q \in Q\}$ is a subspace of P. The map $Q \mapsto Q^\perp$ is a self-adjoint Galois connection. The *closed* subspaces of P are those of the form $Q = X^\perp$ for some $X \subseteq P$ (or, equivalently, $Q = Q^{\perp\perp}$). The closed subspaces of P form a sub-ordered set, $L_c(P)$, of $L(P)$ which is closed under arbitrary intersections and hence forms a sub-meet semilattice of $L(P)$. It is thus a complete lattice but, of course, the join operations in $L(P)$ and $L_c(P)$ differ in general.

Thinking of $L(P)$ as as more or less abstract object, we write $u, v, \ldots$ for elements of $L(P)$. Recall the following basic rules of calculation.

$$u \leq v^\perp \text{ iff } v \leq u^\perp, \quad u \leq v \text{ implies } v^\perp \leq u^\perp, \quad u \leq u^{\perp\perp}:$$
$$(u + v)^\perp = u^\perp v^\perp \text{ (de Morgan)}, \quad (uv)^\perp \geq u^\perp + v^\perp \text{ (weak de Morgan)}.$$

Let $(P, \perp)$ be a *pre-orthogeometry*, i.e. a projective geometry with pre-orthogonality. $\perp$ is called an *orthogonality* and $(P, \perp)$ an *orthogeometry* iff $p + p^\perp = 1$, for all $p \in P$. We now come to the main result of this section.

Theorem 5.1. *Let $(P, \perp)$ be an anisotropic pre-orthogeometry. Then,*

$$Q = \{p \in P \mid p + p^\perp = 1\}$$

is a Baer subgeometry of P, and an anisotropic orthogeometry under the induced pre-orthogonality.

Proof. Let M be the set of all finite joins of elements from Q in $L(P)$. If $u \in M$ then $u = \sum_{i=1}^n q_i$ for some minimal $n \in \mathbb{N}$ and some $q_i, ..., q_n \in Q$. It follows that the chain $q_1 \leq q_1 + q_2 \leq .... \leq u$, is of length n and hence $\dim_{L(P)}(u) = n$.

Since $u^\perp = \Pi_{i=1}^n q_i^\perp$ and each $q_i^\perp \prec 1$, $\operatorname{codim}(u) \leq n$. Since $\perp$ is anisotropic, $uu^\perp = 0$ and, by modularity, $\operatorname{codim}(u) = n$. Also, $u^\perp u^{\perp\perp} = 0$ and $u \leq u^{\perp\perp}$, so again by modularity, $u = u^{\perp\perp}$, and $u \in L_c(P)$. It follows that if $u, v \in M$ and $v^\perp \leq u^\perp$ then $u \leq v$ and, hence $u \leq v$ iff $v^\perp \leq u^\perp$. We shall use both these 'opening observations' below.

We next show that Q is a projective subgeometry of P. To do this we show that Q satisfies the projective property, cf. pg 3. Let p, a, b and p, c, d be collinear elements of Q. We need to show that there is some $x \in Q$ with x, a, c and x, b, d both collinear triples. If there are any non-trivial collinearities between the five points p, a, b, c, d other than the two assumed then all five are collinear and x can be chosen to be any one of them. In particular we may assume $a + c \not\leq b + d$.

Since P is a projective geometry, $x = (a + c)(b + d) \in L(P) - \{0\}$ and, since $a+c \not\leq b+d$, $x \in P$. The weak version of de Morgan's law gives $x^\perp \geq (a+c)^\perp + (b+d)^\perp \geq (a + c)^\perp$. From the first of our opening observations, $\operatorname{codim}((a + c)^\perp) = 2$. If $(a + c)^\perp = (a + c)^\perp + (b + d)^\perp$ then $(b + d)^\perp \leq (a + c)^\perp$. From the second of our opening observations $(a + c) \leq (b + d)$, contrary to our assumption. Also, because $\perp$ is anisotropic $x^\perp \neq 1$. But now, $1 > x^\perp \geq (a + c)^\perp + (b + d)^\perp > (a + c)^\perp$ and $\operatorname{codim}((a + c)^\perp) = 2$ and hence $x^\perp \prec 1$, ie. $x \in Q$. Hence Q is a projective geometry, and $L(Q)$ is a geomodular lattice.

Now we show by induction on n that, if $q \leq \sum_{L(P)}^n q_i$, with $q, q_1, ..., q_n \in Q$, then $q \leq \sum_{L(Q)}^n q_i$. This is trivially true for $n = 1$. Let $v = \sum_{L(P)}^{n-1} q_i$ and $u = v + q_n$. If $q \leq v$ then we are done by inductive hypothesis and if $q = q_n$ then we are done trivially; so we assume neither of these occur. Set $r = v(q + q_n)$ which is in P my modularity and since $v \prec u$. Since $q \not\leq v$, $v^\perp \not\leq q^\perp$, again by the second of our opening observations. Since $\operatorname{codim}(q^\perp q_n^\perp) \leq 2$ and $v^\perp \not\leq q^\perp$, $\operatorname{codim}(v^\perp + q^\perp q_n^\perp) \leq 1$. On the other hand, $1 \geq r^\perp \geq v^\perp + q^\perp q_n^\perp$ and, since $\perp$ is anisotropic, $r^\perp \prec 1$ and $r \in Q$. By inductive hypothesis, $r \leq \sum_{L(Q)}^{n-1} q_i$ and $q \leq \sum_{L(Q)}^n q_i$ since $q \leq r + q_n$. By Lemma 3.2, Q is a Baer subgeometry of P.

Finally, we need to show that Q is itself an orthogeometry. Let $q \in Q$. To show that $q + q^\perp = 1$ in $L(Q)$ it suffices to show, that for an arbitrary $r \in Q$ we have $r = q$, $r \perp q$, or p, q, r collinear for some $p \in Q$, $p \perp q$.

As collinearity and $\perp$ on Q are inherited from P, for the final short section of this proof the ordering, the lattice operations, and $^\perp$ are taken in $L(P)$. Assume neither of the first two of the above cases occurs. Put $p = q^\perp(r + q)$. From $q^\perp \prec 1$ it follows $p \in P$ by modularity. Since also $r \in Q$, we have $\operatorname{codim}(q^\perp r^\perp) \leq 2$. From $p \leq r + q$ it follows $q^\perp r^\perp \leq p^\perp < 1$, the latter by anisotropicity of $(P, \perp)$.

Assuming that $p^\perp \prec 1$ fails, one had $p^\perp = q^\perp r^\perp$, in particular $p^\perp \leq q^\perp$, whence $p = p^{\perp\perp} \leq q^{\perp\perp} = q$ (by the 'opening observations') and $p = q$. $\square$

Remark 5.2. Given P and pre-orthogonality $\perp$, call a subset Q of P is a *strong subgeometry* of $(P, \perp)$ iff (i) $\sum X = (\sum X)^{\perp\perp}$ for any finite $X \subseteq Q$ and (ii) $p, q \in Q$ and $r = r^{\perp\perp} \leq p + q$ imply $r \in Q$. Using ideas from the proof of Theorem 5.1 it follows (in a more concise way) that any strong subgeometry of $(P, \perp)$ is a Baer subgeometry of P and that, under the hypotheses of Theorem 5.1, Q is a strong subgeometry of $(P, \perp)$. Moreover, if Q is a Baer subgeometry of P and P its only subspace which contains Q and if, moreover, $\perp$ turns Q into an anisotropic orthogeometry then $\perp$ can be extended to an anisotropic pre-orthogonality so that the situation of Theorem 5.1 takes place. In this sense, the theorem has content.

For an illustration of Theorem 5.1, let V be a unitary space with inner product $\langle \, | \, \rangle$ and consider self-adjoint endomorpisms ϕ_i, $i \in I$. Let P be the projective space associated with V and endowed with

$$p \perp q \text{ iff } \quad p = \mathbb{C}v,\ q = \mathbb{C}w,\ \langle v \mid w \rangle = 0, \quad \text{and } \langle \phi_i v \mid w \rangle = 0 \text{ for all } i \in I.$$

Obviously, $\perp$ is an anisotropic pre-orthogonality. Then Q as in Theorem 5.1 consists of all $\mathbb{C}v$ where v is a common eigenvector of all ϕ_i. Here, Q is a disjoint union of subspaces, arising as intersections of eigenspaces.

6. Ortholattices and representations

We now move to the lattice-theoretic point of view. Let L be a 0-lattice. A *pre-orthogonality* on L is a symmetric binary relation, $\perp$, on L with the property that, for all $a, b, c \in L$,

$$a \perp b \text{ and } a \perp c \text{ imply } a \perp (b + c),$$

and,

$$a \perp c \text{ and } b \leq c \text{ imply } a \perp b.$$

We call $\perp$ *anisotropic* if, in L, $u \perp u$ iff $u = 0$. We formally denote a 0-lattice with pre-orthogonality $\perp$ by $(L, \perp)$. If $(P, \perp)$ is a projective geometry with pre-orthogonality, then there is a canonical pre-orthogonality on $L(P)$ defined by $X \perp Y$ iff $p \perp q$ for all $p \in X$ and $q \in Y$. (For simplicity of notation, given one pre-orthogonality $\perp$, we denote all canonically induced pre-orthogonalities by $\perp$, too.)

Recall from [12] that an orthoimplication in a 0-lattice with pre-orthogonality is an implication of the form

$$(\bigwedge_{i=1}^{n} (x_i \perp y_i)) \Rightarrow f(x_1, ..., x_n, y_1, ..., y_n) = 0,$$

where f is a lattice term and $\bigwedge$ refers to logical conjunction.

Lemma 6.1. *Let L be a lattice with an anisotropic pre-orthogonality. If K is a 0-sublattice of L then, with the restricted pre-orthogonality, K satisfies the orthoimplications that hold in L. If $(P, \perp)$ is a pre-orthogeometry then $L(P)$ and $L_{\mathrm{fin}}(P)$ satisfy the same orthoimplications.*

Proof. The first claim is obvious and that $L_{\text{fin}}(P)$ satisfies the orthoimplications of $L(P)$ follows. In the other direction one can readily adapt the proof of Lemma 2.1 of [12] to show that, if an orthoimplication fails to hold in $L(P)$ then it fails to hold in $L_{\text{fin}}(P)$. $\square$

Let L be a $0,1$ - lattice. An *orthocomplementation* on L is a dual automorphism, $'\colon L \to L$, of period two which is also a complementation, ie. for all $x, y \in L$,

$$(x+y)' = x'y',\ (xy)' = x'+y',\ 1' = 0,\ 0' = 1,\ x'' = x,\ x+x' = 1, \text{ and } x \cdot x' = 0.$$

If we consider the orthocomplementation as an extra unary operation we have an *ortholattice*, abbreviated OL. If an OL, L, is also modular we have a modular ortholattice, MOL. If $(P, \perp)$ is a projective geometry with an anisotropic pre-orthogonality then $L_c(P)$ is an OL (but in general not an MOL) where the orthocomplementation is given by $X \mapsto X^\perp$, for all $X \in L_c(P)$. Any OL carries a canonical anisotropic pre-orthogonality given by $a \perp b$ iff $b \leq a'$.

Lemma 6.2. *(Lemma 3.1,* [12]*) For each MOL identity α there is an orthoimplication α^+ such that, for any MOL L, α holds in L iff α^+ holds in L with its canonical orthogonality.*

Let L be an MOL and let $(P, \perp)$ be a projective geometry with pre-orthogonality.. Then a $0,1$-lattice homomorphism $\psi\colon L \to L(P)$ is called a *representation* of the MOL L in $(P, \perp)$ if

$$\psi(a') = \psi(a)^\perp \text{ for all } a \in L.$$

We observe that, in this situation $\psi(L)$ is simultaneously a sub-$0,1$-lattice of $L(P)$ and, in the context of anisotropic $\perp$ and OL's, a subalgebra of $L_c(P)$. A representation ψ is *faithful* if it is a one-to-one map (in [11, 9] we considered only such.)

Lemma 6.3. *If $(P, \perp)$ is an anisotropic pre-orthogeometry, then a $0,1$-lattice homomorphism $\psi\colon L \to L(P)$ is a representation of L in $(P, \perp)$ iff*

$$\psi(a) \perp \psi(a') \textit{ for all } a \in L.$$

Proof. Let $a \in L$. Since $\varphi(a) \perp \varphi(a')$, $\psi(a') \leq \psi(a)^\perp$. But $\psi(a)\psi(a') = \psi(aa') = 0$ and $\psi(a) + \psi(a') = \psi(a + a') = 1$. Since $\perp$ is anisotropic $\psi(a)\psi(a)^\perp = 0$, and by modularity, $\psi(a') = \psi(a)^\perp$. $\square$

We now return to the setting of Section 4, but now in the presence of 'orthogonality'. That is: let L be an MOL, let M be a modular $0,1$-lattice M equipped with an anisotropic pre-orthogonality, $\perp$, and let $\varphi\colon L \to M$ be a $0,1$-lattice embedding preserving 'orthogonality' in the sense that $\varphi(a) \perp \varphi(a')$ for all $a \in L$. P_M, the set of atoms of M, inherits an anisotropic pre-orthogonality, $\perp$, from M by restriction.

Set $P = P_M$ and let $\psi\colon L \to L(P)$ be the canonical lattice homomorphism, ie. for $a \in L$, $\psi(a) = \{p \in P \mid p \leq \varphi(a) \text{ in M}\}$. By Corollary 4.2 and Lemma 6.3, ψ is a representation of the MOL L in $(P, \perp)$.

Set $Q = \{q \in P \mid q + q^\perp = 1 \text{ in L(P) }\}$

Lemma 6.4. *Q is (L,φ)-closed and is an anisotropic orthogeometry under the induced pre-orthogonality.*

Proof. $(Q,\perp)$ is an anisotropic orthogeometry from Theorem 5.1. We need to show that Q is (L,φ)-closed.

Let $q \in Q$ and $a, d \in L$ satisfy the premise of the definition of (L,φ)-closed, ie. $q \not\leq \varphi(d)$, $ad = 0$ and $a + d = 1$. We need to show that $p = \varphi(a)(q + \varphi(d)) \in Q$. By modularity $p \in P$ whence $p \leq \psi(a)$. Similarly, $r = \varphi(d)(q + \varphi(a)) \in P$ and $r \leq \psi(d)$. Recall that ψ is a representation of L in $(P,\perp)$ and $\psi(1) = 1_{P(L)}$. This is the context of the rest of the proof. Note that $q^\perp \prec 1$ in $L(P)$. It follows $(\psi(d) + q)^\perp = \psi(d)^\perp q^\perp \prec \psi(d)^\perp$. But, $\psi(a)^\perp = \psi(a')$ and $\psi(d)^\perp = \psi(d')$ are complements of each other in $L(P)$ whence by modularity

$$\psi(a)^\perp + q^\perp \psi(d)^\perp \prec \psi(a)^\perp + \psi(d)^\perp = 1.$$

Now, $p \leq \psi(a)$ and $p \leq q + \psi(d)$ imply $p^\perp \geq \psi(a)^\perp + q^\perp \psi(d)^\perp$. Since $p^\perp \neq 1$, it follows $p^\perp = \psi(a)^\perp + q^\perp \psi(d)^\perp \prec 1$. □

Corollary 6.5. *In this setting, the natural map $\eta\colon L \to L(Q)$ given by $\eta(a) = \{\, p \in Q \mid q \leq \varphi(a)\}$ is a representation of the MOL L in $(Q,\perp)$.*

Proof. From the fact that Q is (L,φ)-closed η is a 0-lattice homomorphism by Lemma 4.1. Clearly, $\eta(1) = 1_{L(Q)}$. It follows from Lemma 6.3 that η is a representation. □

For the final results of the paper we fix L, M, φ, $P = P_M$, and Q as above the statement of Lemma 6.4 and η from the statement of Corollary 6.5. Furthermore, we set $Q_0 = \{q \in P \mid q \perp h \text{ for some } h \prec 1 \text{ in } M\}$. In the next few paragraphs we collect earlier material and establish some elementary facts which constitute the proof of our final main result, Theorem 6.6.

From Lemma 2.4 and 6.1, $L_{fin}(P)$ satisfies the orthoimplications of M. And, from Lemma 6.1, $(L_{fin}(P),\perp)$ and $(L(P),\perp)$ satisfy the same orthoimplications as do $(L_{fin}(Q),\perp)$ and $(L(Q),\perp)$, and, using Lemma 6.1 again, all four satisfy the orthoimplications of M.

If Q is (L,φ)-dense then η is a 0-lattice embedding by Corollaries 4.3 and 6.5. Let $u \in L_{fin}(Q)$. Since Q is an anisotropic orthogeometry, cf. Lemma 6.4, the interval $[0,u]$ becomes an MOL with the orthocomplementation $x \mapsto x^\perp u$. Let $\mathcal{V}_{(Q,\perp)}$ be the variety generated by these MOLs. Then, using Lemma 6.2 to pass from MOL identities to orthoimplications and back again, we get $L \in \mathcal{V}_{(Q,\perp)}$.

If L is simple and $Q \neq \emptyset$ then $\eta(1) \neq 0$. Hence, since η is not the trivial homomorphism it must be a lattice embedding.

Finally, let $q \in Q_0$, and let $h \prec 1$ in M with $q \perp h$. As in the final step of the proof of Theorem 5.1 it follows $q \in Q$, and we have shown $Q_0 \subseteq Q$.

Theorem 6.6. *Let L be an MOL, let M be amodular $0,1$-lattice equipped with an anisotropic pre-orthogonality, $\perp$, and let $\varphi : L \to M$ be a $0,1$-lattice embedding such that $\varphi(a) \perp \varphi(a')$, for all $a \in L$. With $Q = \{p \in P_M \mid p + p^\perp = 1 \text{ in } L(P_M)\}$ w.r.t.the induced pre-orthogonality on P_M, $\eta\colon L \to L(Q)$ given by $\eta(a) = \{\, p \in Q \mid q \leq \varphi(a)\}$, and $Q_0 = \{q \in P \mid q \perp h \text{ for some } h \prec 1 \text{ in } M\}$. If $Q \neq \emptyset$ then:*

(a) *$\eta\colon L \to L(Q)$ is a representation of the MOL L in the anisotropic orthogeometry $(Q, \perp)$. Furthermore, $(L(Q), \perp)$ and $(L_{fin}(Q), \perp)$ satisfy the orthoimplications of $(M. \perp)$.*

(b) *If Q is (L, φ)-dense then $\eta\colon L \to L(Q)$ is in addition a faithful representation of L in $(Q, \perp)$. Furthermore, $L \in \mathcal{V}_{(Q,\perp)}$.*

(c) *If L is simple and $Q \neq \emptyset$ then the conclusions of part (b) hold. In particular, they hold if $Q_0 \neq \emptyset$.*

A special case of part (c) is perhaps worth mentioning separately.

Corollary 6.7. *If L is simple and M is an MOL with at least one atom then L is in the variety generated by all finite-dimensional MOLs.*

7. Discussion

Until our Theorem 6.6 has important new applications perhaps the most relevant part of this paper is the framework we have provided for the representation of MOLs. When developing the ideas in this paper we were guilty of focussing our efforts on finding an approach that would give such a representation for an arbitrary MOL. We worked with several different ideas; various ideal/filter and model-theoretic constructions for example. Perhaps the most obvious and natural approach is via the Frink embedding, [6]. We outline this (unsuccessful in general, cf. [14]) approach here because it illustrates many of our ideas well.

Let L be a complemented modular lattice. Then the M of our approach is the $0,1$-lattice of filters of L ordered by reverse inclusion. The map $a \mapsto \{x \in L \,|\, a \leq x\}$ is our $0,1$-lattice embedding $\varphi\colon L \to M$. P is the set of all ultrafilters (maximal proper filters) of L, and $\psi\colon L \to L(P)$ is the map $a \mapsto \{U \subseteq P \mid a \in U\}$. ψ is the classical Frink embedding. Because L and M in this case satisfy the same lattice identities it follows that L has a $0,1$-embedding in an $L(P)$ which satisfies the same lattice identities as L does.

Now if L is an MOL, the orthocomplementation $'\colon L \to L$ induces an anisotropic pre-orthogonality on M given by $F \perp G$ iff there exists $x \in L$ with $x \in F$ and $x' \in G$. According to the material we have developed here, the map $\psi\colon L \to L(P)$ is a faithful representation of L in the anisotropic pre-orthogeometry $L(P, \perp)$. Furthermore, it is easy to see that, in this case, L and $L(P, \perp)$ satisfy exactly the same orthoimplications.

However, in [14], we gave an elementary example of an L where $(P, \perp)$ is not an orthogeometry. Furthermore we suspect that $Q = \emptyset$ in this example also. On the other hand, it is shown that the L of this example generates a variety which is generated by its finite-dimensional members ([8], [14]) and from this it follows that it does admit a faithful representation with respect to some anisotropic orthogeometry (this will be proved in a sequel to this paper).

We are hopeful that Theorem 6.6 might provide representations in situations which we are not as yet aware of.

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