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A second note on the equational theory of modular ortholattices

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Abstract. We show how to alter the material in [4] to prove that every variety of modular ortholattices is generated by its simple members.

1. Introduction

An *ortholattice*, abbreviated OL, is an algebra $(L; \vee, \wedge, {}', 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice (we have decided to adopt the more standard $\vee, \wedge$ notation in place of the $+, \cdot$ notation used in earlier papers; in particular in [4]), and $' : L \to L$ is an *orthocomplementation*, that is, $x \vee x' = 1$, $x \wedge x' = 0$, $x'' = x$, and $x \leq y$ implies $y' \leq x'$, for all $x, y \in L$. Since the last condition, in the presence of the other three, is equivalent to DeMorgan's laws ($(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$), the class of ortholattices forms a variety. An OL L, is a *modular ortholattice*, abbreviated MOL, iff its lattice reduct is modular.

The main result of [4] is the following.

Proposition 1.1 (Proposition 1.2 of [4]). *Every variety of MOLs which is generated by its atomistic members is already generated by its finite dimensional members.*

The two keys to the proof of Proposition 1.1 are Lemma 2.1 in [4] and the idea of replacing ortholattice identities with orthoimplications. In this second note we give a simple application of a result due to B. Jónsson [6], which allows the following modification of Proposition 1.1.

Proposition 1.2. *Every variety of MOLs is generated by its simple members.*

Our proof of this result was outlined in [5]; this manuscript is not widely available.

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2. Alterations for the proof of Proposition 1.2

Recall that the congruences of a MOL are exactly the congruences of its lattice reduct (see, for example, [1], Chapter V, Theorem 3.2).

Let L be a sectionally complemented modular lattice. By definition (following [2], Chapter III, Theorem 20), an ideal I of L is *neutral* iff it is closed under perspectivity, that is, whenever $a, b \in L$ have a common complement in $[0, a \vee b]$ and $a \in I$, then also $b \in I$. As observed in [6], (see also [7], Satz 4.5) the argument for the complemented case (the proof of Theorem 20 of [2] mentioned above) holds to show that $\theta \mapsto I_\theta = \{a \in L \mid a\,\theta\,0\}$ provides an order-isomorphism between the congruences of L and the neutral ideals of L, both ordered by containment. Using this correspondence it follows easily that, if I is a neutral ideal of L, then $(a \vee x) \wedge (a \vee y) \in I$, for all all $a \in I$ and $x, y \in L$ with $x \wedge y = 0$. The elements of finite height in L form a neutral ideal of L.

Lemma 2.1. *Let L be a subdirectly irreducible MOL with minimal non-trivial congruence μ. Then, for any nonzero $c \in L$ with $c\,\mu\,0$, the interval $[0, c]$ is a simple MOL.*

Proof. Let L be the subdirectly irreducible MOL of the statement of the lemma and set $I = I_\mu$, the smallest non-trivial neutral ideal of L. Then I is a sectionally complemented modular lattice and any neutral ideal of I is also a neutral ideal of L. Hence, I is simple, and by Lemma 2.2 of [6], so are all lattices $[0, c]$ with $c\,\mu\,0$ and $c > 0$. These are MOLs under the induced orthocomplementation $x \mapsto x' \wedge c$ whence simple MOLs, too. □

We digress for a moment to make a minor observation. The analog of this result holds for any sectionally complemented modular lattice. Lemma 2.2 of [6] is based on Lemma 1.5 of the same paper, which provides a useful generation process for the neutral ideals of a sectionally complemented modular lattice. Indeed, this generation process immediately yields the congruence extension property, [3], for the ideals of a sectionally complemented modular lattice.

With one minor technical change, detailed below, the proof of Proposition 1.2 can be completed by replacing each occurrence of one of the phrases 'of finite height' or 'is an atom' throughout [4], with the phrase '$\in I$'. The one technical difference is in the proof of Lemma 2.1 of [4]. In this proof we make the following statement.

> One easily computes using modularity that $[0, a_1, a_2, a_1 \vee a_2]$ form an M_3 in $[0, a_1 \vee a_2]$ and, consequently, the a_k, $k = 1, 2$ are atoms of L.

This should be replaced with the following.

Easy calculations using modularity give

$$a_1 \vee a_2 \leq (a \vee e_1) \wedge (a \vee e_2)$$

Also,

$$e_1 \wedge e_2 = 0.$$

Since I is neutral and $a \in I$, for $k = 1, 2$,

$$a_k \leq a_1 \vee a_2 \leq (a \vee e_1) \wedge (a \vee e_2) \in I.$$

Hence, $a_k \in I$, $k = 1, 2$.

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