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On geometric representations of modular ortholattices

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Abstract. We review results on modular ortholattices related to representations in anisotropic orthogonal geometries. In particular, we establish a 1-1-correspondence between varieties generated by their finite height members and certain classes of geometries.

1. Introduction

We give some historical background and motivation - to be skipped by the non-expert in the first reading. The main body of the paper requires only some basic knowledge of Lattice Theory and Universal Algebra. MOL is a shorthand for 'modular ortholattice'.

Finite height MOLs have been analyzed by Birkhoff and von Neumann [3] in their famous paper marking the beginning of quantum logic: They are isomorphic to finite direct products of irreducibles, and the latter are given by finite dimensional inner product spaces or of height ≤ 3 (this can be reduced to height ≤ 2 is one considers Arguesian MOLs). Projection MOLs of finite von Neumann algebras of operators on Hilbert space were studied by F. J. Murray and J. von Neumann [24] and served as a tool in the classification of these algebras. From there, von Neumann [25] abstracted the concept of continuous geometries, which are just the irreducible complete modular ortholattices, due to the beautiful result of I. Kaplansky [22]. A detailed understanding even of this very special kind of modular ortholattices is still a challenge. Only supposing strong additional structure and axioms, von Neumann [27] was able to relate these back to concrete examples derived from algebras. The motivation behind this note is that it might be easier to study modular ortholattices just in the restricted framework of equational theory and representations within anisotropic orthogonal geometries. There has been some success in this direction cf [4, 28, 14, 15, 16, 13].

In much of the work following von Neumann, the non-modular ortholattice of all closed subspaces of Hilbert space was the primary example which lead to the development of a rich theory of orthomodular lattices [21]. Though, capturing the special features of modularity needs a different approach, having as guideline the Murray-von-Neumann [24] construction of a $*$-regular ring of unbounded operators associated with any finite von Neumann algebra factor. From this one can derive an inner product space, which is an elementary extension of the given Hilbert space, and an embedding of the ortholattice of projections into the ortholattice of

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closed subspaces which is at the same time a lattice embedding into the lattice of all subspaces [13]. In a general approach, inner product spaces are replaced by anisotropic pre-orthogeometries, projective spaces P endowed with an irreflexive symmetric relation $\perp$ between points which is compatible with collinearity. Though, to make use of such, one needs more, namely that mapping p to $p^\perp$, the greatest subspace orthogonal to p, is a polarity from points to hyperplanes, i.e. that $(P, \perp)$ is an orthogeometry. The latter are well known to be closely related to inner product spaces cf. [8].

In the present note, we recall results form the preliminary presentation in [14] and, in simplfied form, from [12] where the more general case of orthogonal geometries and complemented modular lattices with involution has been dealt with. The presentation is intended to be rather elementary and self-containded.

Following the above mentioned example, a representation of a modular ortholattice L in an anisotropic pre-orthogeometry $(P, \perp)$ is a 0-1-lattice embedding into the subspace lattice $L(P)$ which preserves orthogonality, i.e. maps orthogonal elements of L onto orthogonal subspaces of $(P, \perp)$ or, equivalently, yields a simultaneous embedding.

We discuss how representations relate to equational theory. In particular, we show that a variety $\mathcal{V}$ of modular ortholattices is generated by its finite height members if and only if any member of $\mathcal{V}$ has an atomic extension within $\mathcal{V}$ if and only if $\mathcal{V}$ has all members representable within some anisotropic orthogeometry the finite dimensional subspaces of which have ortholattices belonging to $\mathcal{V}$. This can be derived from [12, Thm.2.2]; here we present a simplified proof. The latter classes of geometries are closed under ultraproducts, subgeometries, and orthogonal disjoint unions.

A final result shows that a representation of L within $(P, \perp)$ induces a representation within some (proper projective) subgeometry which is also an (anisotropic) orthogeometry under the restriction of $\perp$, provided that sufficiently many points have $p^\perp$ a hyperplane (for simple L a single p suffices). Though, we cannot provide any serious application.

2. Lattices and Geometries

This section contains introductory material on modular lattices, projective geometries, and the well-known links between the two.

We start with the geometric point of view (cf. [10, Ch.V.5] of [8]). The most elementary Elementary Geometry deals with points, lines (sometimes also planes) and incidence between them.

In an axiomatic approach, there are two disjoint sets P (for points) and G (for lines) and an relation $I \subseteq P \times G$ between them: if $p \in P$ and $l \in G$ with $p\,I\,l$ then we say that p and l are *incident* with each other. If three distinct points p, q, r are incident with the same line then we say that p, q, r are *collinear* or that they form a *collinear triplet.* For a *projective space* one requires the following.

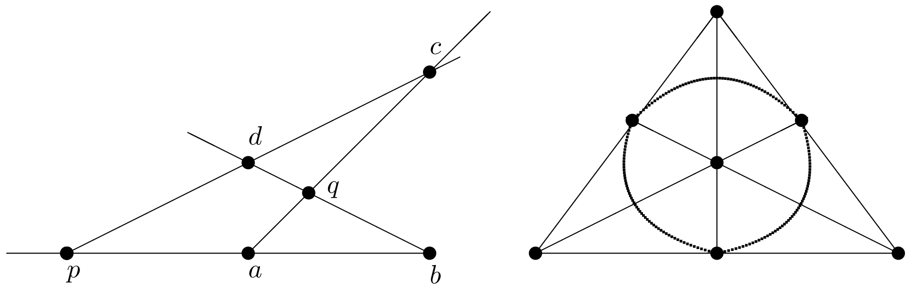

FIGURE 1. Projective property and plane of order 2

(1) Given any two distinct points $p, q \in P$ there is a unique line $l \in G$ incident with them. We write $l = p + q$.
(2) For any line $l \in G$ there are at least two distinct points $p, q \in P$ incident with.
(3) *Projective property*: If p, a, b and p, c, d are both collinear triplets but p, b, c are not collinear, then there is a point q such that a, q, c and b, q, d are collinear triplets, see Figure 1.

In view of (1) and (2) we may identify any line with the set of points incident with it. Most readers will be familiar with projective planes, e.g. the projective plane of order 2 in Figure 1. In a projective plane every pair of distinct lines must intersect in a single point. The projective property is the higher dimensional analogue of this.

Projective Geometry originated from an analysis of perspective drawing: add a point at infinity to each parallel pencil of lines in the familiar (affine) plane or space. In this setting, points are described by 3 respectively. 4 homogeneous coordinates, unique up to a non-zero scalar multiple. More generally, given any vector space V, one obtains a projective space the points and lines of which are the 1- respectively 2-dimensional linear subspaces and where incidence is given by $\subseteq$. Conversely, any projective space can be obtained as a disjoint union of such together with projective lines and planes - where points p, q come from disjoint parts if and only if they are incident with the 2-point line $\{p, q\}$.

A *subspace* of a projective space P is a subset X of P such that the following holds.

> If p, q are distinct points in X then $p + q \subseteq X$, i.e. $r \in X$ for any r such that p, q, r is a collinear triplet.

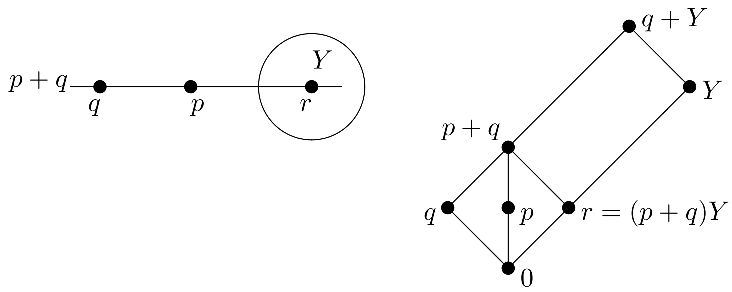

FIGURE 2. Join with a singleton

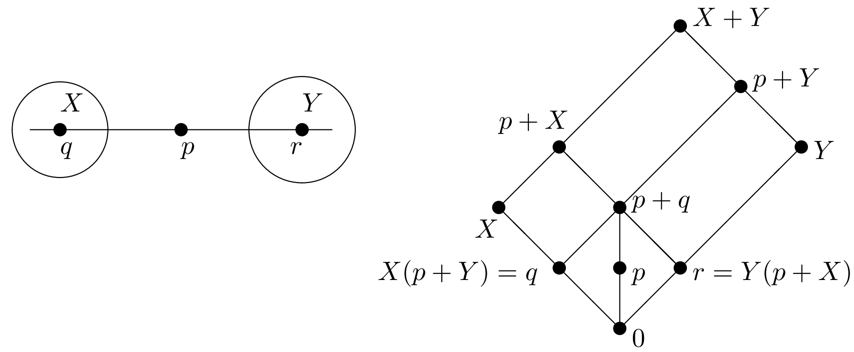

FIGURE 3. Join of disjoint subspaces

Given a subset U of P, its closure X under the collinearity relation, yielding the smallest subspace containing U, can be described iteratively, which is sometimes useful. Let $L(P)$ denote the set of all subspaces of P. $L(P)$ is partially ordered by set inclusion. Moreover, the intersection or *meet* of subspaces is easily seen to be a subspace. Thus, the closure X of U is the intersection of all $Z \in L(P)$ containing given U, i.e. the smallest subspace containing U In particular, this applies to $U = X \cup Y$ with $X, Y \in L(P)$. The closure of $X \cup Y$ is *join* $X + Y$ of X and Y. It is mainly due to the projective property, that the join has a nice description as in Figures 2 and 3. Maximal proper subspaces H of P, will be called *hyperplanes* and are characterized by $p + H = P$ for any point $p \notin H$.

The systems $L(P)$ of subspaces of a projective space with partial order $\subseteq$ and the binary operations of meet and joins are between the fundamental examples which gave rise to the concept of a lattice: a partially ordered set in which infimum (meet) and supremum (join) exist for any pair of elements. We refer the reader to [2], [7], or [10], in particular in regards of the algebraic point of view. We just recall some basic terminology and notation. For lattices pairwise joins will be denoted by $+$ and meets by $\cdot$. We shall often save time and brackets by using juxtaposition for meets and following the convention that meet takes priority over join. When we have occasion to use joins or meets over larger (sometimes infinite) sets we will use the corresponding capitalized symbols. If a lattice L has a smallest element we denote it by 0 and if it has a largest element we denote it by 1. We often treat the bounds 0 and 1 as constants and then we speak of a $0,1$-lattice, and occasionally we just consider the 0 as a constant and speak of a 0-lattice.

If $a, b \in L$, and $a < b$ with no $z \in L$ with $a < z < b$ we say that b covers a and write $a \prec b$. For $a \leq b$ in L the *interval* $[a, b] = \{x \in L \mid a \leq x \leq b\}$ is a sublattice of L. A lattice is *modular* if

$$a \geq c \text{ implies } a(b + c) = ab + c.$$

A useful equivalent characterization is that intervals $[ab,\ a]$ and $b,\ a+b]$ are isomorphic via the mutually inverse isomorphisms $x \mapsto b + x$ and $y \mapsto ay$. For an element a of modular lattice L with 0, if there is a finite maximal chain C in $[0, a]$ then all maximal chains in $[0, a]$ have cardinality $|C|$ and $\dim a = |C| - 1$ is the *height* or *dimension* of a. In particular, $\dim L = \dim 1_L$ and codim $a = \dim[a, 1_L]$ is the *codimension* of a, if these exist. If dimensions exist, the *dimension formula* is valid

$$\dim(a + b) + \dim(ab) = \dim a + \dim b.$$

This applies in the proof of the following well known result.

Lemma 2.1. *(Folklore). Let L and M be modular lattices with* 0 *and all elements of finite height and let $\varphi : L \to M$ be a cover preserving join embedding. Then φ is a lattice embedding.*

Proof. Let $a, b \in L$ with $dim(a) = m$, $dim(b) = n$, $dim(ab) = k$ and, hence $dim(a+b) = m+n-k$. Then $dim(\varphi(a)) = m$, $dim(\varphi(b)) = n$, $dim(\varphi(a) + \varphi(b)) = dim(\varphi(a+b)) = m+n-k$, and hence $dim(\varphi(a)\varphi(b)) = k$. But also $dim(\varphi(ab)) = k$ and, since $\varphi(ab) \leq \varphi(a)\varphi(b)$, $\varphi(ab) = \varphi(a)\varphi(b)$.

□

In a modular lattice, L, the elements of finite dimension form an ideal of L which we will denote by L_{fin}. Dually the elements of finite codimension form a filter L_{cof}. The *atoms* (covers of 0) of a 0-lattice L will be of particular importance to us and we will denote this set by P_L. The elements $h \prec 1$ are *coatoms*, in $L(P)$ these are the hyperplanes.

For any modular lattice, L, the points $p \in P_L$ together with the height 2 elements $p + q$, where $p \neq q$ in P_L, as lines, and the incidence relation $\leq$ form a projective space. In particular, p, q, r is a collinear triplet determining the line l if and only if

$$l = p + q = p + r = q + r \text{ where } p, q, r \in P_L \text{ and } \dim(l) = 2.$$

Only the projective property requires a closer look: We have $p + a = p + b$ and $p + c = p + d$ of height 2 and distinct, whence coatoms of the height 3 interval

$[0,\, p+b+c]$. By the dimension formula we get

$$q = (a+d)(b+c) \in P_L.$$

Conversely, let us recall some lattice properties relevant in the context of lattices $L(P)$. Firstly, a lattice L is *complemented* if it has 0 and 1 and if for all a there is b with $ab = 0$ and $a + b = 1$. If L is complemented and modular, then it is *relatively complemented*, i.e. any of its intervals $[v, u]$ is complemented. A lattice, L, is *atomic* if for any $a > 0$ there is $p \in P_L$ with $p \leq a$; any atomic relatively complemented L is *atomistic*, i.e. any element is, in the partial order of L, a supremum of some $X \subseteq P_L$.

A *geomodular lattice* is an atomic complemented modular complete lattice where the atoms are compact (that is, if an atom is below a join of elements then it is already below the join of a finite subset of these elements). It can been shown that the geomodular lattices are precisely the algebraic complemented modular lattices. The following well-known result sums up the elementary relationship between projective geometries and (geo-)modular lattices.

Theorem 2.2. *If P is a projective space then $L(P)$ is a geomodular lattice. Conversely, if L is a modular lattice then P_L together with the lines $p+q$, $p \neq q$ in P_L, is a projective space. Considering geomodular L, only, these processes define inverse bijections between the two classes.*

If L is geomodular, the isomorphism from L onto $L(P_L)$ is given by

$$a \mapsto \{p \in P_L \mid p \leq a\}.$$

In the case of the projective space P associated with a vector space V_F, this map identifies the lattice $L(V_F)$ of linear subspaces of V_F with the lattice $L(P)$. In the sequel, this map and the above figures will show up in various adaptions. Mostly, we will consider $L(P)$ an abstract geomodular lattice, identify the point p with the singleton subspace p, denote subspaces by lower case letters, and write $v \leq u$ for $v \subseteq u$ and $p \leq u$ for $p \in u$. The above mentioned description of joins in $L(P)$ then reads as follows.

Observation 2.3. *For any atom p of a complemented modular lattice, L, and $a, b \in L$ one has $p \leq a+b$ if and only there are atoms $q \leq a$ and $r \leq b$ such that $p \leq q+r$. If $p \not\leq a$ and $p \not\leq b$ then one can choose $q = a(p+d)$ and $r = d(p+a)$ with a suitable complement d of a. Moreover, in this case p, q, r are collinear.*

In particular, in $L(P)$ one has

$$a + b = \{p \in P \mid p \leq q + r \text{ for some } q \leq a,\ r \leq b\}.$$

Proof. This is trivial if $p \leq a$ or $p \leq b$. Otherwise, let c be a complement of ab in the interval $[0, b]$ of L and d a complement of $a + b$ in the interval $[c, 1]$ of L cf. Figure 4. Then a and d are complements in L. Put $q = a(p + d)$ and $r = d(p + a)$. Then, by modularity, q and r are atoms and $q + r = (p + d)(a + d(p + a)) = (p + d)(p + a)(a + d) \geq p$.

□

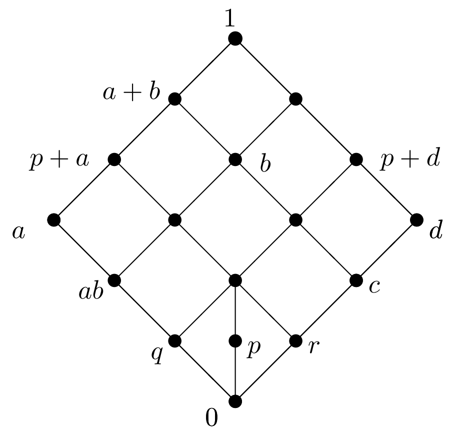

FIGURE 4. Join and atoms

For axiomatic purposes and for the study of subgeometries, projective spaces are more conveniently described in terms of the collinearity relation - which is extended to include trivial cases. (cf. [8]). In this approach we will consider a *projective geometry* or *projective space* to be a set of points P together with a totally symmetric ternary *collinearity* relation, C, on P which satisfies: For all $a, b, c, d, p, q \in P$,

$C(a, b, a)$ for all $a, b \in P$ (any two points are on some a line),

$C(a, b, c)$ and $C(q, b, c)$ and $b \neq c$ imply $C(a, q, b)$ (every line is determined by any two distinct points on it),

and,

$C(p, a, b)$ and $C(p, c, d)$ imply $C(q, a, c)$ and $C(q, b, d)$ for some $q \in P$ (we call this the *projective property*).

The equivalence to the earlier definition is given by

- $C(p, q, r)$ if p, q, r is a collinear triplet or $|\{p, q, r\}| \leq 2$.
- G consists of all at least 2-element subsets l of P such that $p, q \in l$ and $C(p, q, r)$ implies $r \in l$. The incidence relation is $\subseteq$.

Many of the definitions and much of the notation in this paper have been developed semi-independently by different authors in different settings. We have strived to strike a balance between what is already in the literature and what seems appropriate in our particular situation. We have not always found it desirable, or even possible, to follow any one reference exactly, and the reader should consider this when checking other sources.

3. SubGeometries

In this section we discuss the notion of a 'subgeometry' and want the lattice structure of the subspaces of the 'subgeometry' to be reflected accurately in that of the subspaces of the given projective space. There are three basic types of examples which arise as follows where V_F is an F-vector space and K a subfield of F. Firstly, for any F-linear subspace U we have $L(U_F)$ an interval sublattice of $L(V_F)$, Secondly, consider $L(K^n_K)$ (cover preserving embedded in $L(F^n_F)$ via $X \mapsto FX$; the points of $L(K^n_K)$ form a 'Baer subspace' of $L(F^n_F)$ in case F is of degree 2 over K cf. [1, 6]. Finally, Then $L(V_F)$ is a sublattice of $L(V_K)$. It is the combination of the first two whih is behind the concept of 'proper projective subgeometry'. The third will be relevant only later.

Let (P, C) be a projective geometry and let $Q \subseteq P$. In algebraic terms, we may consider Q endowed with a partial binary operation f_Q such that

$$f_Q(p, q) = r \text{ iff } p, q, r \text{ are collinear points in } Q.$$

Thus, (Q, f_Q) is the induced or relative subalgebra of the partial algebra (P, f_P) corresponding to the (induced) *subgeometry* $(Q, Q^3 \cap C)$ of (P, C). This is the terminology common in incidence geometry. In [8] this term occurs in the context of matroids which does not apply if one considers collinearity as the fundamental notion.

We recall some basic facts about closure systems arising in this and similar settings. For any subset X of Q, the smallest subset containing X and closed under f_Q (i.e. the subalgebra generated by X), is the *closure* of X given as

$$C_Q X = \bigcup_{n<\omega} X_n$$

where the X_n are recursively defined by

$$X_0 = X,\ X_{n+1} = X_n \cup \{r \in Q \mid r = f_Q(p, q) \text{ for some } p, q \in X_n\}.$$

See Figure 5 showing the projective plane P of order 2 and $Q = P \setminus \{p_0\}$. It follows

$$(*)\ \ p \in C_Q X \text{ iff } p \in C_Q Y \text{ for some finite } Y \subseteq X$$

The map $X \mapsto C_Q X$ is the *closure operator* associated with the partial algebra (Q, f_Q) and the *closed subsets* $U = C_Q U$ form a complete lattice $L(Q)$ ordered by inclusion with meet $\bigcap_i U_i$ and join $\bigvee^Q_i U_i = C_Q \bigcup_i U_i$. $U \in L(Q)$ is *finitely generated* if $U = C_Q X$ for some finite X. Of course, the above and the following well known observation generalize to any partial algebra with finitary operations. Actually, it only requires $f_Q \subseteq f_P$. i.e. a 'weak subalgebra' Q of P.

Observation 3.1. *Let P and Q as above. Define*

$$\phi : L(Q) \to L(P),\ \ \phi(X) = C_P X$$

1. (1) *$C_P X = C_P C_Q X$ for any $X \subseteq Q$.*
2. (2) *ϕ preserves any joins.*
3. (4) *ϕ is injective iff $p \in C_P X$ implies $p \in C_Q X$ for any finite $X \subseteq Q$ and $p \in Q$.*

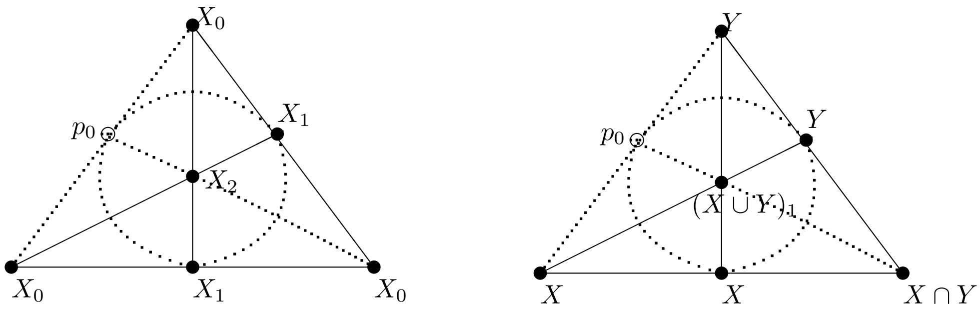

FIGURE 5. Generation process and join in a subgeometry

(4) *$\phi(\bigcap_i U_i) = \bigcap_i \phi(U_i)$ holds for all nonempty families $U_i \in L(Q)$ if it does so for finitely generated $U_i \in L(Q)$.*

(5) *If Q is a projective space and $U \in L(Q)$, then $U \in L(Q)_{\text{fin}}$ if and only if U is finitely generated.*

Proof. (1) From $f_Q \subseteq f_P$ one gets $C_Q X \subseteq C_P X$ whence $C_P X \subseteq C_P C_Q X \subseteq C_P C_P X = C_P X$.

(2) For $U_i \in L(Q)$

$$\phi \bigvee_i^Q U_i = \phi C_Q \bigcup_i U_i = C_P C_Q \bigcup_i U_i = C_P \bigcup_i U_i = \bigvee_i^P C_P U_i = \bigvee_i^P \phi U_i.$$

(3) If ϕ is injective, $X \subseteq Q$ and $p \in Q$ then $p \in C_P X = \phi C_Q X$ implies $\phi C_Q\{p\} \subseteq \phi C_Q X$ whence $p \in C_Q\{p\} \subseteq C_Q X$. Conversely, let $U \subset V$ in $L(Q)$, $U \neq V$. Then there is $p \in V \setminus U \subseteq \phi V$. Due to (*), assuming $p \in \phi U$ would yield $p \in C_P X$ for some finite $X \subseteq U$ whence $p \in C_Q X \subseteq U$, by hypothesis, a contradiction. Thus $\phi U \neq \phi V$.

(4) Let $U_i \in L(Q)$ and $p \in P$, $p \in \bigcap_i \phi U_i$. By (*) there are finite $X_i \leq U_i$ such that

$$p \in C_P X_i = C_P C_Q X_i = \phi C_Q X_i \text{ for all } i$$

in view of (1). Thus by hypothesis

$$p \in \bigcap_i C_P X_i = \phi \bigcap_i C_Q X_i \subseteq \phi \bigcap_i U_i$$

(5) Indeed, by modularity $X \prec Y$ in $L(Q)$ if and only if $Y = p + X$ for some $p \in Y \setminus X$.

$\square$

A subgeometry Q of a projective geometry P is a *projective subgeometry* if it is a projective geometry (under the induced collinearity); then, in a particular, $L(Q)$ is a geomodular lattice. Our main result of this section is the following:

Lemma 3.2. *Let P be a projective space with projective subgeometry Q. Define*

$$\phi : L(Q) \to L(P), \;\; \phi U = \sum\nolimits_P U \;\; \textit{the subspace generated by } U \textit{ in } P\ .$$

Then ϕ is a complete join homomorphism mapping atoms to atoms. Moreover, the following are equivalent.

(i) *For any finite $X \subseteq Q$ and $p \in Q$ one has $p \in \sum_Q X$, the subspace generated by X in the geometry Q, whenever $p \in \phi(\sum_Q X)$;*
(ii) *$\varphi : L(Q)_{\text{fin}} \to LP)_{\text{fin}}$ is a cover preserving join embedding;*
(iii) *$\varphi : L(Q)_{\text{fin}} \to L(P)_{\text{fin}}$ is a 0-lattice embedding;*
(iv) *$\varphi : L(Q) \to L(P)$ is a lattice embedding preserving arbitrary joins and non-empty, but otherwise arbitrary, meets.*

Proof. Singletons in Q are subspaces of Q and P as well; thus ϕ maps atoms to atoms. By observation 3.1 (2), ϕ preserves joins.

Assume (i). Then ϕ is a join embedding by 3.1 (3). If U is a cover of V in $L(Q)_{\text{fin}}$ then there is an atom $p \notin V$ of $L(Q)$ with join $U = V + p$ formed in $L(Q)$. Then $\phi(p) \notin \phi(V)$ by 3.1 (5) and hypothesis (i). Since $\phi(p)$ is an atom of $L(P)$, we conclude by modularity that $\phi U = \phi(V) + \phi(p)$ is a cover of $\phi(V)$ in $L(P)$. Thus (i) implies (ii) and this in turn (iii) by Lemma 2.1.

We assume (iii) and show (i). Identify $L(Q)_{\text{fin}}$ with its image in $L(P)_{\text{fin}}$. Then any computation with joins, meets, and $\leq$ in $L(Q)$ is an instance of the same computation in the geomodular lattice $L(P)$. Given finite $X \subseteq Q$ and $p \leq a = \sum_P X$ with $p \in Q$ we show that p is in the subspace of Q generated by X. We do so by induction on $|X|$ see Figure 6. So let $X = Y \cup \{q\}$ and $b = \sum_P Y$. If $p \leq b$ we are done by induction. Otherwise, $r = b(p+q)$ is a third point on $p+q$ and in Q since both b and $p+q$ belong to the sublattice $L(Q)_{\text{fin}}$. By inductive hypothesis, r is in the subspace of Q generated by Y, whence p is in the subspace of Q generated by X. This proves (i).

Finally, (iii) is contained in (iv). Conversely, (i) and (iii) together imply (iv) in view of 3.1 (4) and (5).

□

Substantial parts of this are known cf. Faure and Frölicher [8]. We follow them calling Q as in the Lemma a *proper projective subgeometry* of P. An example of a projective subgeometry which is not proper is given on page 168 of [8].

If one wants the condition, that Q is a projective subgeometry, to be included only in (i) and still have the equivalences, then one has to replace $L(Q)_{\text{fin}}$ by the union of intervals $[0, U]$, where U ranges over all finitely generated subspaces of the subgeometry Q of P. This might be seen as a lattice theoretic characterization of proper projective subgeometries. The nontrivial step is from (iii) to (i). Consider collinear triplets p, a, b and p, c, d in Q with p, b, c not collinear cf. Figure 1. Then,

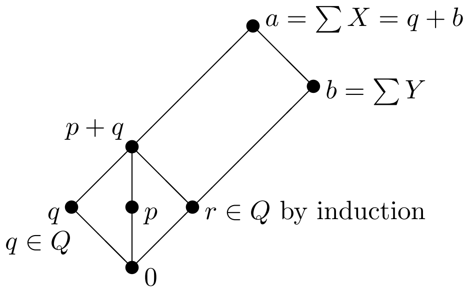

FIGURE 6. Q is proper

by modularity, $q = (a + c)(b + d)$ is an atom in $L(P)$, whence in $L(Q)$, too, i.e. a point of Q. Thus, Q is a projective space.

4. Orthogonality

In Elementary Geometry, orthogonality is defined for lines. Passing to the projective point of view, this means to introduce orthogonality via a scalar product for vectors as done in Linear Algebra. In particular, in an inner product space V over the (skew) field F one has $F\vec{v} \perp F\vec{w}$ if and only $\langle \vec{v} \mid \vec{w} \rangle = 0$. This defines an anisotropic orthogonality on the projective space associated with V and from this orthogonality of subspaces is derived. More generally, one may consider orthogonalities induced by $*$-hermitean forms, which may have isotropic points, a classical subject in finite dimensions and also intensively studied in infinite dimensions cf. [11]. As motivated by these examples, in our abstract approach to Orthogonal Geometry we consider "orthogonality" as a binary relation on points of a projective space. We start with a very general such concept. Though, we will deal always with the anisotropic case.

Let P be a projective space. A *pre-orthogonality* on P is a symmetric binary relation $\perp$ so that, for $p, q, r, s \in P$,

$$p \perp q \text{ and } p \perp r \text{ implies } p \perp s \text{ for all } s \leq p + q.$$

Observe that, if the $\perp_i$, $i \in I$, are pre-orthogonalities on P then so is $\perp$ defined by

$$p \perp q \text{ iff } \ p \perp_i q \ \text{ for all } i \in I.$$

$\perp$ is an *orthogonality* if, in addition, for any $p \neq q$ there is $r \perp p$ collinear with p, q.

A pre-orthogonality which has the property that $p \not\perp p$, for all $p \in P$ is called *anisotropic*. A projective geometry, P, with pre-orthogonality, $\perp$, is formally written $(P, \perp)$. We call such a *pre-orthogeometry* and refer to it as *anisotropic* whenever $\perp$ is. $(P, \perp)$ is an *orthogeometry* if $\perp$ is an orthogonality. For a subgeometry Q of P the *induced* pre-orthogonality is $\perp_Q$ is given by $p \perp_Q q$ if and only if $p, q \in Q$ and

$p \perp q$. An anisotropic pre-orthogonality which is not an orthogonality is discussed in Section 7. We remark, that any anisotropic orthogeometry may be obtained as an orthogonal disjoint union of geometries associated with inner product spaces and such of dimension ≤ 3 cf. [8, Ch.14] and [12, Prop.1.4].

From the definition it is clear that if $X \subseteq P$ then

$$X^\perp = \{p \in P \mid p \perp q \text{ for all } q \in X\}$$

is a subspace of P, the *orthogonal* of X. The map $X \mapsto X^\perp$ is a polarity on the set P and a self-adjoint Galois connection on the ordered set $L(P)$ cf. [2, V.7-8]. In particular, $X \mapsto X^{\perp\perp}$ is a closure operator on $L(P)$ with *closed sets* or *subspaces* of the form $Y = X^\perp$ for some $X \subseteq P$ (or equivalently $Y = Y^{\perp\perp}$). Moreover, intersections of closed sets are closed and the map $X \mapsto X^\perp$ is order reverting. A useful rule to be applied in the sequel is

$$(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp.$$

To see this, let $Z = X_1^\perp \cap X_2^\perp$ and conclude $(X_1 + X_2)^\perp \subseteq Z$ from $X_i \subseteq X_1 + X_2$. On the other hand, from $Z \subseteq X_i^\perp$ we get $Z^\perp \supseteq X_i^{\perp\perp} \supseteq X_i$ whence $Z^\perp \supseteq X_1 + X_2$ and $Z = Z^{\perp\perp} \subseteq (X_1 + X_2)^\perp$. $L(P)$ with the map $X \mapsto X^\perp$ is the structure we have to deal with, primarily. Again, we consider $L(P)$ a geomodular lattice L with pre-orthogonality $\perp$ defined on $P = P_L$ and write $x^\perp$ in place of $X^\perp$.

In order to address the properties of the system of closed subsets, we have to recall some definitions from Lattice Theory cf. [2]. Let L be a $0, 1$ - lattice. An *involution* on L is a dual automorphism, $' : L \to L$, of period two, i.e.. for all $x, y \in L$,

$$(x + y)' = x'y', (xy)' = x' + y', 1' = 0, 0' = 1, \text{ and } x'' = x.$$

Considering $x \mapsto x'$ as an additional unary operation, L is *ortholattice*, OL, iff the involution is an orthocomplementation, i.e.. for all $x \in L$,

$$x + x' = 1, x \cdot x' = 0.$$

Then elements a, b of L are *orthogonal* to each other, denoted by $a \perp b$ if $a \leq b'$, equivalently $b \leq a'$. $\perp$ is the *canonical orthogonality* on L. If L is also modular we have a modular ortholattice, MOL.

As already observed, the closed subspaces form a complete sub-meet semilattice $L_c(P, \perp)$ of $L(P)$. Thus, $L_c(P, \perp)$ is also a lattice with involution $X \mapsto X^\perp$, but of course in general the join operations in the two lattices, $L(P)$ and $L_c(P, \perp)$ are different. Indeed, denoting the join in $L_c(P, \perp)$ by $\vee$, one has

$$a \vee b = (a^\perp b^\perp)^\perp.$$

Indeed, $c = a^\perp b^\perp \in L_c(P, \perp)$ and $c^\perp \geq a, b$. Now, if $x \in L_c(P, \perp)$ and $x \geq a, b$ then $x^\perp \leq c$ and $x = x^{\perp\perp} \geq c^\perp$. If $\perp$ is anisotropic, then $p \notin p^\perp = \{q \in P \mid q \perp p\}$ for all $p \in P$ and $L_c(P, \perp)$ is an ortholattice. This is all well known and the details can be found in [8], [12], [2], for example. Also the following are well known.

Proposition 4.1. *Let $\perp$ be an anisotropic pre-orthogonality on the projective space P. Then the following are equivalent.*

(1) *$\perp$ is an orthogonality on P.*

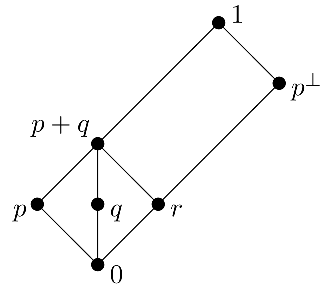

FIGURE 7. $p^\perp$ a hyperplane

(2) *The orthogonal $p^\perp$ of any point is a hyperplane of P.*
(3) $p + p^\perp = 1_{L(P)}$ *for all $p \in P$.*

Proof. Assume $\perp$ to be an orthogonality. Given $p \neq q$, by modularity $r = (p + q)p^\perp$ is a point as required. Thus (1) implies (3). Since $p \cdot p^\perp = 0$ by anisotropicity, (3) implies (2) by modularity. Supposing (2) and given p we have to show $q \leq p+p^\perp$ for any q. This is trivial if $q = p$. Otherwise, by hypothesis there is $r \in p^\perp$ collinear with p, q and it follows $q \leq p + r \leq p + p^\perp$. See Figure 7. Thus (2) implies (1).

□

Only (1), (2), and (3) of the following will be used in the sequel, (2) and (3) not before Sect.6. Otherwise these observations serve for clarification of concepts and as a guideline for the proof of Theorem 11.2.

Proposition 4.2. (1) *If L is an MOL with set P of atoms, then $(P, \perp)$ is an anisotropic orthogeometry with collinearity as in Theorem 2.2 and with $p \perp q$ if and only if $p \leq q'$.*

(2) *In an anisotropic orthogeometry $(P, \perp)$, if $a \in L_c(P, \perp)$ and $a \prec b$ in $L(P)$ then $b^\perp \prec a^\perp$ in $L(P)$ and $b \in L_c(P, \perp)$. Moreover*

$$a \in L_c(P, \perp),\ \operatorname{codim} a^\perp = \dim a,\ \textit{and}\ a + a^\perp = 1 \quad \textit{if}\ \dim a < \omega.$$

(3) *If $(P, \perp)$ is an anisotropic orthogeometry and the lattice $(L(P)$ finite dimensional, then any subspace of P is closed and $L_c(P, \perp)$ is an MOL.*

(4) *Any subspace Q of an anisotropic orthogeometry $(P, \perp)$ is an anisotropic orthogeometry under the induced orthogonality.*

(5) *If $(P, \perp)$ is an anisotropic orthogeometry then $L(P)_{\rm fin} = L_c(P, \perp)_{\rm fin}$ and $\{X^\perp \mid X \in L(P)_{\rm fin}\} = L_c(P, \perp)_{cofin}$ are ideal respectively. filter in both $L(P)$ and $L(P, \perp)$ and their union is a sublattice of $L(P)$ and a sub-OL of $L_c(P, \perp)$. Moreover, $L(P)_{cofin} \subseteq L_c(P, \perp)$ if and only if $(P, \perp)$ is the orthogonal disjoint union of finite dimensional subspaces.*

Proof. (1) Since $x \mapsto x'$ is an involution, we have $p' \prec 1_L$ for each $p \in P$ an, by modularity $r = p'(p+q) \prec p+q$ for $q \in P$, $q \neq p$. Thus $r \in P$, $r \perp p$, and p, q, r are collinear. Now, Proposition 4.1 applies.

(2) We start with an observation. If $a \prec b$ in $L(P)$ then $b = a+q$ for some $q \in P$ and $b^\perp = a^\perp q^\perp$. Now $q \not\leq a$ whence $a^\perp \not\leq q^\perp$ since otherwise $q \leq q^{\perp\perp} \leq a^{\perp\perp} = a$. From $q^\perp \prec 1$ it follows $b^\perp \prec a^\perp$ proving the first claim. Applying this to $b^\perp \prec a^\perp$ it follows $a = a^{\perp\perp} \prec b^{\perp\perp}$ and $a \prec b \leq b^{\perp\perp}$ gives $b = b^{\perp\perp} \in L_c(P, \perp)$.

If $\dim a = n < \omega$, then $\mathrm{codim} a^\perp = n$ and $a \in L_c(P, \perp)$ by n applications of the first claims. From $aa^\perp = 0$ and modularity it follows $\dim[a^\perp, a + a^\perp] = n$ whence $a + a^\perp = 1$.

In (3) observe that $Q \not\leq p^\perp$ for $p \in Q$ by anisotropicity, whence $p^{\perp_Q} = Q \cap p^\perp \prec Q$ by $p \perp P$ and modularity.

(4) follows from (2), immediately. We omit the proof of (5) - a more general positive result can be found in [12, Thm.1.2]. That in an irreducible orthogeometry there are non-closed hyperplanes follows from the fact that in any irreducible projective space the cardinality of the set of hyperplanes exceeds that of the set of points.

□

5. Orthogeometries within anistropic pre-orthogeometries

Let $(P, \perp)$ be a projective space with a not necessarily anisotropic pre-orthogonality. A subset Q of P is a *strong subgeometry* iff

(i) $\sum X = (\sum X)^{\perp\perp}$ for any finite $X \subseteq Q$
(ii) $p, q \in Q$ and $r = r^{\perp\perp} \leq p+q$ imply $r \in Q$

Lemma 5.1. *If Q is a strong subgeometry of $(P, \perp)$ then Q (with the induced collinearity) is a proper projective subgeometry of P.*

Proof. Consider $p, q, r, s \in Q$ such that $x = (p+q)(r+s) \in P$. Then $x^{\perp\perp} \leq (p+q)^{\perp\perp} = p+q$ and $x^{\perp\perp} \leq (r+s)^{\perp\perp} = r+s$ whence $x = x^{\perp\perp}$ and $x \in Q$ by (ii). Thus Q is a projective subgeometry. Now, we show, by induction on $|X|$, for any finite $X \subseteq Q$

$p \leq \sum X$ and $p \in Q$ imply that p is in the subspace of Q generated by X

Assume $p \leq \sum X$, $X \subseteq Q$, $p \in Q$. Then there is $q \in X$ such that $p \leq q + u$ where $u = \sum Y$ with $Y = X \setminus q$. If $p \leq u$ then we are done by induction. Otherwise, $r = u(p+q) \in P$ and $p \leq r+q$ by modularity. By (i), $p+q = (p+q)^{\perp\perp}$ and $u = u^{\perp\perp}$ whence $r^{\perp\perp} \leq u^{\perp\perp}(p+q)^{\perp\perp} = u(p+q) = r$ and $r \in Q$ by (ii). Now, again by induction, r is in the subspace of Q generated by Y, thus p is in the subspace of Q generated by $X = Y \cup q$.

□

Observe that a subgeometry of an anisotropic orthogeometry may fail to be an orthogeometry under the induced pre-orthogonality. Examples can be obtained from inner product spaces F^3 where F is a suitable field with involution and finite prime subfield F_p: Let P and Q be the projective spaces associated with F^3_F and ${F_p^3}_{F_p}$.

Theorem 5.2. *Let $(P, \perp)$ be an anisotropic pre-orthogeometry. Then*

$$Q = \{p \in P \mid p^\perp \prec 1\}$$

is a strong subgeometry of $(P, \perp)$ and an anisotropic orthogeometry under the induced orthogonality relation

Proof. To prove (i) consider finite $X \subseteq Q$ and let $u = \sum X$. We may assume that $|X| = \dim u =: n$. Then $u^\perp = \prod_{p \in X} x^\perp$ whence $\mathsf{codim} u^\perp \leq n$ by modularity. On the other hand, $u^{\perp\perp} u^\perp = 0$ by anisotropicity. By modularity, $[0, u^{\perp\perp}]$ is isomorphic to $[u^\perp, u^{\perp\perp} + u^\perp]$ whence $\dim u^{\perp\perp} \leq n$. From $u \leq u^{\perp\perp}$ it follows that $u = u^{\perp\perp}$.

Now, to prove (ii) consider $p \neq q$ in Q and $r = r^{\perp\perp} \leq p + q =: u$ in P, in particular $r < u$. Then $r^\perp \geq u^\perp = p^\perp q^\perp$ and $\mathsf{codim} u^\perp \leq 2$. Now, $1 > r^\perp$ by anisotropicity; on the other hand, $r^\perp = u^\perp$ would imply $r = r^{\perp\perp} = u^{\perp\perp} = u$, contradiction. Thus, $r^\perp \prec 1$ and $r \in Q$.

Now, to prove that Q is an orthogeometry under the induced orthogonality relation, by Prop.4.1 for any $p \neq q$ in Q we have to provide $r \in Q$ such that $r \perp q$ and $r \leq p + q$. Let $r = q^\perp(p + q)$. Then $r = r^{\perp\perp}$ by (i) and $r \in P$ by modularity (since $r \not\geq q$). Thus, $r \in Q$ by property (ii) of a strong subgeometry.

□

That the situtation of the Theorem can occur in a non-trivial way (e.g. with P the real and Q the rational plane, the latter with the canonical orthogonality) is witnessed by the following.

Proposition 5.3. *Let Q be a proper projective subgeometry of of the projective space P such that P is generated by its subset Q. Let $(Q, \perp_Q)$ be a pre-orthogeometry such that $\sum X$ is closed for any finite $X \subseteq Q$. Then there is $\perp$ on P such that*

1. *$(P, \perp)$ is a pre-orthogeometry inducing $\perp_Q$ on Q*
2. *Q is a strong subgeometry of $(P, \perp)$ and $p \in Q$ iff $p = p^{\perp\perp}$.*
3. *$(P, \perp)$ is anisotropic if $(Q, \perp_Q)$ is so*
4. *$(P, \perp)$ is anisotropic and $Q = \{q \in P \mid q +_{L(P)} q^\perp = 1_P\}$ if $(Q, \perp_Q)$ is an anisotropic orthogeometry*

Proof. Consider $L(Q)_{fin}$ embedded into $L(P)$. For $p \in P$ let $C(p)$ the smallest closed subspace of $(Q, \perp_Q)$ containing p and define

$$p \perp q \text{ iff } C(p) \perp_Q C(q).$$

Equivalently: $p \perp q$ iff $q \perp_Q C(p)^{\perp_Q}$. Thus, $\perp$ is symmetric; moreover $s \perp p, q$ and $r \leq p + q$ imply $C(s)^{\perp_Q} \geq p + q \geq r$, i.e. $s \perp r$. This proves (i).

If $u \in L(Q)_{fin}$ is closed in $(Q, \perp_Q)$ then it is a fortiori closed in $(P, \perp)$. Conversely, let $p = p^{\perp\perp}$ and consider $q \in P$, $q \leq C(p)$. Then for all r with $r \perp p$ one has $C(r) \perp_Q C(p)$ whence $r \perp q$ and it follows $q \leq p^{\perp\perp} = p$. Thus, $p = C(p) \in Q$. This proves (ii).

If $(Q, \perp_Q)$ is anisotropic, then $C(p) \perp_Q C(q)$ implies $C(p) \cap C(q) = 0$ and $p \neq q$. If, in addition, $(Q, \perp_Q)$ is an orthogeometry, then the join of q and $q^{\perp_Q}$ in Q is Q and generates P by hypothesis. Conversely, if $p + p^{\perp} = 1$ in $(P, \perp)$ then $p = p^{\perp\perp}$ whence $p \in Q$. □

We conclude this section with an example illustrating Theorem 11.2. Let V be a unitary space with inner product $\langle \mid \rangle$ and consider self-adjoint endomorpisms ϕ_i, $i \in I$. Let P be the projective space associated with V

$$p \perp q \text{ iff } \quad p = \mathbb{C}v,\ q = \mathbb{C}w,\ \langle v \mid w\rangle = 0, \quad \text{and } \langle \phi_i v \mid w\rangle = 0 \text{ for all } i \in I.$$

Then Q as in Theorem 11.2 consists of all $\mathbb{C}v$ where v is a common eigenvector of all ϕ_i. Here, Q is a disjoint union of subspaces, arising a intersections of eigenspaces.

6. Representations

The following is the basic tool for establishing join preserving maps into lattices $L(P)$, first used by Frink [9]. It follows form Observation 2.3, immediately.

Observation 6.1. *Let P be a projective space and ϕ a map from the lattice L into $L(P)$. Then ϕ preserves finite joins if and only if for any $a, b \in L$ and $p \leq \phi(a+b)$ there are $q, r \in P$ such that $q \leq \phi(a)$, $r \leq \phi(b)$, and $p \leq q + r$.*

Let L be a $0,1$-modular lattice, L. A $0,1$-lattice embedding $\varphi : L \to L(P)$ where P is a projective geometry, is called a *representation of the lattice L in P*.

Let L be a complemented $0,1$-sublattice of the modular $0,1$-lattice M, and let Q be a proper projective subgeometry of P_M. Consider a complementary pair c, d in L and $q \in P_M$. Then $p = c(q + d) \in P_M$ by modularity. We will need that from $q \in Q$ we may derive $p \in Q$.

Q is *L-closed* in M iff for each $p \in Q$ and complementary pairs $a, d \in L$: if $p \not\leq d = 0$ then $q = a(p + d) \in Q$.

Define

$$\eta(a) = \{p \in Q \mid p \leq a\} \quad \text{for } a \in L.$$

Lemma 6.2. *Under the above hypotheses, if Q is L-closed in M then $\eta : L \to L(Q)$ is a homomorphism.*

Proof. This follows immediately from Observation 6.1 and the proof of Observation 2.3.

□

It is well known, that a 0-lattice homomorphism $\phi : L \to M$, where L is complemented modular, is an embedding if and only if $\phi(a) > 0$ for any $a > 0$ in L. Indeed, if $c < d$ in L choose a as a complement of c in $[0, d]$ to conclude $\phi(c) < \phi(d)$. Thus, η from above is an embedding if and only if Q is L-dense in the following sense.

Q is L-*dense* in M if and only if for every $a \in L \setminus \{0\}$ there exists $q \in Q$ with $q \leq a$.

□

Let us now expand the definition of closed and dense to the case where L is mapped homomorphically into M. Let L and M be 0-modular lattices, let $\phi : L \to M$ be a 0-1-lattice homomorphism, and let Q be a proper projective subgeometry of P_M. Then,

Q is (L, ϕ)-*closed* if and only if Q is $\phi(L)$-closed;

Q is (L, ϕ)-*dense* if and only if for any $a > 0$ in L there is $q \in Q$ such that $q \leq \phi(a)$.

Now, $\eta \circ \phi$ is an embedding if and only if Q is (L, ϕ) dense. Since η is a homomorphism by Lemma 6.2, we have the following.

Lemma 6.3. *Let L be a complemented modular lattice, M a $0,1$-modular lattice, $\phi : L \to M$ a $0,1$-lattice homomorphism, and Q an (L, ϕ)-closed and dense proper projective subgeometry of P_M. Then the map $\eta : L \to L(Q)$ given by $\eta(a) = \{q \in Q \mid q \leq \phi(a)\}$, for all $a \in L$, is a representation of the lattice L in the projective space Q.*

Before we turn to representations of MOLs, our main interest, we observe that this change of point of view has no effect on congruences.

Lemma 6.4. *Any lattice congruence on an MOL is also an OL congruence (and, of course, vice versa).*

Proof.[1] Let θ be a lattice congruence on the MOL, L. Suppose $a\theta b$. Then, $ab\theta a + b$ and conversely. Hence we may assume $a \leq b$, so $b' \leq a'$. Now, $a' = a' \cdot 1 = a'(b + b')\theta a'(a + b') = a'a + b' = b'$. The third step because $a\theta b$ and the fourth one by modularity and $b' \leq a'$. Hence, $a'\theta b'$.

□

Corollary 6.5. *If L is a simple MOL, M a modular* 0-1-*lattice, $\phi : L \to M$ a* 0-1-*lattice homomorphism, and if $\emptyset \neq Q \subseteq P_M$ is (L, ϕ)-closed in M, then Q is (L, ϕ)-dense in M.*

Proof. $Q \neq \emptyset$ means $\eta(\phi(1)) > 0$; thus $\eta \circ \phi$ is injective since L is simple. And this means that Q is (L, ϕ) dense.

□

Let $(P, \perp)$ be a pre-orthogeometry and L an MOL. Let η be a representation of L considered as 0-1-lattice L in the projective space P. η is a *representation of the*

---

[1]This result is well known and is actually true for the larger class of OL's called Orthomodular Lattices. These have been studied extensively, see [21] for example, and while we won't discuss them further in this paper, our proof only uses the orthomodular law.

MOL L in the pre-orthogeometry $(P, \perp)$ if in addition

$$\eta(a') \perp \eta(a),$$

for all $a \in L$.

Lemma 6.6. *Let η be a representation of the MOL L in the anisotropic pre-orthogeometry $(P, \perp)$. Then*

$$\eta(a') = \eta(a)^{\perp},$$

for all $a \in L$.

Proof. One direction is trivial. For the other assume $\eta(a') \perp \eta(a)$. Then $\eta(a') \leq \eta(a)^{\perp}$. Also, $1 = \eta(1) = \eta(a) + \eta(a')$ and $0 = \eta(a)\eta(a)^{\perp}$. Hence, $\eta(a') = \eta(a)^{\perp}$ by modularity.

$\square$

Examples of representations of L are obtained from MOL-extensions of L containing sufficiently many atoms.

Proposition 6.7. *Let L be a sub-OL of the MOL, M, and assume P_M to be L dense. Then $\eta(a) = \{p \in P_M \mid p \leq a\}$ is a representation of the MOL L in the anisotropic orthogeometry $(P_M, \perp)$ of Proposition 4.2 (1). In particular, this applies if M is atomic or if $P_M \neq \emptyset$ and L is a simple MOL.*

Proof. By Proposition 4.2 (1) the canonical orthogonality on M induces an anisotropic orthogonality $\perp_P$ on $P = P_M$. Clearly, P_M is L closed in M. Thus ϕ is a representation of L in the projective space P_M by Lemma 6.3. By definition of $\perp_P$, $\eta(a) = \{p \in P \mid p \leq \phi(a)\}$ is also a representation of the MOL L in the orthogeometry $(P, \perp_P)$. Finally, by Corollary 6.5 and Lemma 6.4, P_M is L-dense if L is simple and $P_M \neq \emptyset$.

$\square$

7. Orthoimplications

We now move to the lattice-theoretic point of view. Let L be a 0-lattice. A *pre-orthogonality* on L is a symmetric binary relation, $\perp$, on L with the property that, for all $a, b, c \in L$,

$a \perp b$ and $a \perp c$ imply $a \perp b + c$,

and,

$a \perp c$ and $b \leq c$ imply $a \perp b$.

Analogously, we call $\perp$ *anisotropic* if for all $u \in L$, $u \perp u$ implies that $u = 0$. In particular this occurs if L is an ortholattice with canonical $\perp$. We denote a 0-lattice L with pre-orthogonality $\perp$ by $(L, \perp)$. For any pre-orthogeometry $(P, \perp)$ there is a canonical pre-orthogonality on $L(P)$ given by $X \perp Y$ if and only if $X \subseteq Y^{\perp}$.

Recall from [15] that an orthoimplication in a lattice with with pre-orthogonality is an implication of the form[2]

$$\bigwedge_{i=1}^{n}(x_i \perp y_i) \to f(x_1, ..., x_n, y_1, ..., y_n) = 0,$$

where f is a lattice term. If $(P, \perp)$ is a pre-orthogeometry, then we say that the orthoimplication holds in $(P, \perp)$ if it holds in $(L(P), \perp)$. The first claim in the following Lemma is obvious. The second part is taken directly from the proof of Lemma 2.1 of [15].

Lemma 7.1. *Let L be a lattice with an anisotropic pre-orthogonality. If K is a 0-sublattice of L with the restricted orthogonality then K satisfies the orthoimplications of L. If $(P, \perp)$ is an anisotropic pre-orthogeometry, then $(P, \perp)$ and $(L(P)_{\text{fin}}, \perp)$ satisfy exactly the same orthoimplications. In particular, an orthoimplcation is valid in $(P, \perp)$ if and only if it is in all $(Q, \perp_Q)$ where Q is a finite dimensional subspace of P.*

Lemma 7.2. *For any orthoimplication γ there is a set Σ_γ of first order sentences in the language of pre-orthogeometries such that γ is valid in $(P, \perp)$ is and only if Σ is valid in $(P, \perp)$.*

Proof.

Given finite lists $\bar{p}^i$ of point variables ($i = 1, \dots, n$), for any lattice term f there is a formula ϕ_f such that

$$p \in f(\sum \bar{p}^1, \dots, \sum \bar{p}^n) \;\Leftrightarrow\; \phi_f(p; \bar{p}^1, \dots, \bar{p}^n)$$

holds in any pre-orthogeometry. This is shown by induction: If f is a variable q in $\bar{p}^i$ let ϕ_f the formula $p = q$.

$$\phi_{f_1 f_2} \text{ is } \phi_{f_1} \wedge \phi_{f_2}$$
$$\phi_{f_1+f_2} \text{ is } \exists r_1 \exists r_2.\, C(p, r_1, r_2) \wedge \phi_{f_1}(r_1/p) \wedge \phi_{f_2}(r_2/p).$$

Now, define

$$(p_1, \dots, p_k) \perp (q_1, \dots q_\ell) \;\Leftrightarrow\; \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} p_i \perp q_i.$$

Then Σ_γ for the above orthoimplcation consist of all

$$\bigwedge_{i=1}^{n} \bar{p}_i \perp \bar{q}_i \to \forall p. \neg\phi_f(p; \bar{p}_1, \dots, \bar{p}_n, \bar{q}_1, \dots, \bar{q}_n)$$

where the $\bar{p}_i$ and $\bar{q}_i$ range over all finite lists form a given countable set of variables. □

The relevance of orthoimplications for equational theory is due to the following.

Lemma 7.3. *(Lemma 3.1, [15]) For each MOL identity α there is an orthoimplication α^+ such that, for any MOL L, α holds in L if and only if α^+ holds in L with its canonical orthogonality.*

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[2] $\bigwedge$ refers to logical conjunction here.

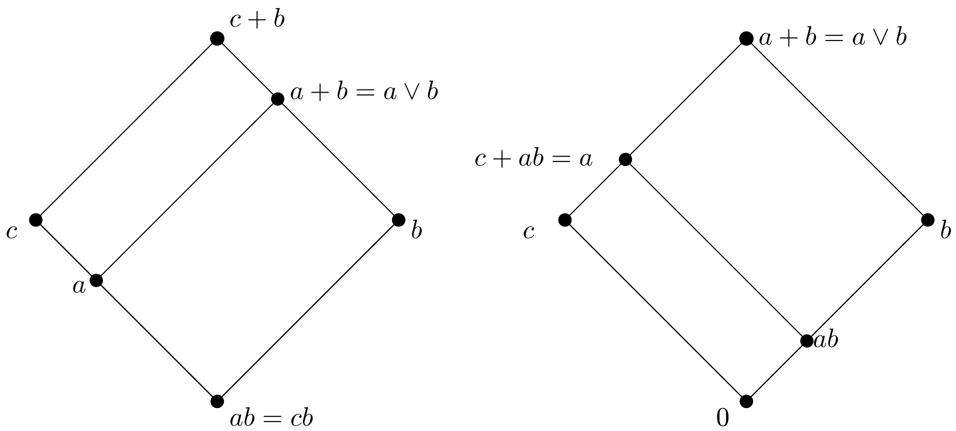

FIGURE 8. Lemma 8.3, part 1

8. Atomic extensions

Corollary 8.1. *Any sub-OL of an atomic MOL, M, has a representation in an anisotropic orthogeometry satisfying the orthoimplications of M, namely $(P_M, \perp)$.*

This follows from Proposition 6.7. Now, we derive the converse, building on ideas in [5]. A more general version has been shown in [12]. Here, we give a simplified proof which is based on Lemma 8.3, below.

Theorem 8.2. *If η is a representation of the MOL L in the anisotropic orthogeometry $(P, \perp)$ then there is an atomic sub-ortholattice M of $L_c(P, \perp)$ which is also a sublattice of $L(P)$ containing all atoms of $L(P)$ such that η is an ortholattice embedding of L into M. In particular, M satisfies the orthoimplications of $(P, \perp)$.*

In the sense of Cor.8.1 and Thm.8.2, representation of an MOL L in an anisotropic orthogeometry and atomic MOL-extension of L are equivalent concepts.

Lemma 8.3. *Let $(P, \perp)$ be an anisotropic orthogeometry and $a, b, c \in L_c(P, \perp)$ such that $a + b = a \vee b$, $a^\perp + b^\perp = a^\perp \vee b^\perp$ and $a \prec c$ or $c \prec a$. Then $c + b = c \vee b$.*

Proof. If $c + b = a + b$ then $b + c \in L_c(P, \perp)$ whence $c + b = c \vee b$. Assume $c + b \neq a + c$. Let $a \prec c$. Then $a + b \prec c + a + b = c + b$ by modularity whence $c + b \in L_c(P, \perp)$ by Proposition 4.2 (2), i.e. $c + b = c \vee b$ see Figure 8, Now, let $c \prec a$. Then $c \geq ab$ (since otherwise $c + b = c + ab + b = a + b$ cf. Figure 8) and $c + b \prec a + b$ by modularity, see Figure 9. By Proposition 4.2 (2), again, $a^\perp \prec c^\perp \leq (ab)^\perp$. Calculating in the ortholattice $L_c(P, \perp)$ and applying the hypothesis, $(ab)^\perp = a^\perp \vee b^\perp = a^\perp + b^\perp$ whence $a^\perp b^\perp \prec c^\perp b^\perp$ by modularity. It follows $c \vee b = (c^\perp b^\perp)^\perp < (ab)^{\perp\perp} = a \vee b = a + b$. Thus $c + b \leq c \vee b < a + b$ and $c \vee b = c + b$ from $c + b \prec a + b$.

□

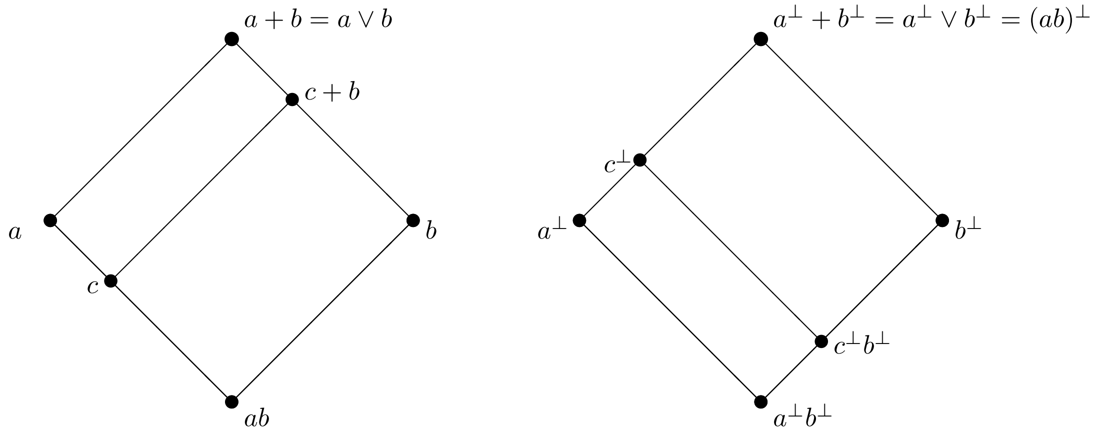

FIGURE 9. Lemma 8.3, part 2

Proof of Theorem 8.2. Identify L with its image under η and observe that L is a sub-ortholattice of $L_c(P,\perp)$ and a sublattice of $L(P)$. In particular, all $x,y\in L$ satisfy

$$(*)\quad x+y=x\vee y \text{ and } x^\perp+y^\perp=x^\perp\vee y^\perp.$$

For $x,y\in L(P)$ define

$$x\theta y \text{ iff } \dim[x,x+y]+\dim[y,x+y]<\omega.$$

This is a well known congruence relation on $L(P)$. Since L is a sublattice of $L(P)$, the union of the congruence classes of members $a\in L$

$$C=\bigcup_{a\in L} a[\theta]$$

is a sublattice of $L(P)$, too. Now, let $M=C\cap L_c(P,\perp)$. Since $L_c(P,\perp)$ is closed under meets in $L(P)$, so is M. By Propositon 4.2 (2) , $x\theta y$ implies $\dim[x^\perp y^\perp,x^\perp]+\dim[x^\perp y^\perp,y^\perp]<\omega$ and this in turn, by modularity $x^\perp\theta y^\perp$, Thus, M is closed under $^\perp$. Finally, any $x\in M$ is connected to some $a\in L$ within M via coverings, e.g. first ascending in $[a,a+x]$, then descending in $[x,a+x]$. Let $d(x)$ denote the minimum distance of $x\in M$ from some $a\in L$ within M and observe that $d(x^\perp)=d(x)$ by Proposition 4.2 (2). By induction on $d(x)+d(y)$ and Lemma 8.3 it follows that all pairs x,y of elements of M satisfy condition (*) and have $x+y=x\vee y=(x^\perp y^\perp)^\perp\in M$.

□

Corollary 8.4. *If a simple MOL, L admits an MOL-extension, M, containing at least one atom, then L admits an atomic extension within the variety of M.*

Proof. This rather serves to illustrate the method. There is also a simple purely algebraic proof. By Proposition 6.7 we have a representation of the MOL L in the anisotropic orthogeometry $(P_M,\perp)$ which satisfies the orthoimplications of M due to Lemma 7.1. Now apply Theorem 8.2 to obtain an atomic MOL $\hat{L}$ which extends L and such that $\hat{L}_{\rm fin} = L(P_M)_{\rm fin}$. By Lemma 7.1 this MOL satisfies the orthoimplications of M and then so do its intervals $[0, u]$ with induced orthocomplementation $x \mapsto ux'$. By Lemma 7.3, these are in the variety of M whence so is $\hat{L}$ by [15].

□

9. Varieties generated by their finite height members

The following is partially in [14] and can be derived from [12, Thm.2.2]. We give a stronger statement and a simplified proof.

Theorem 9.1. *The following are equivalent for a variety $\mathcal{V}$ of MOLs.*

(1) *$\mathcal{V}$ is generated by its finite height members.*
(2) *$\mathcal{V}$ is generated by its atomic members.*
(3) *Any member of $\mathcal{V}$ has an atomic extension in $\mathcal{V}$.*
(4) *Every member of $\mathcal{V}$ has a representation in an anisotropic orthogeometry satisfying the orthoimplications valid in $\mathcal{V}$.*

Proof. The equivalence of (1) and (2) is [16]. The equivalence of (3) and (4) is Prop.8.2. (3) implies (2), obviously. Now, assume (2) and let $\mathcal{C}$ denote the class of all MOLs having an atomic extension in $\mathcal{V}$. Clearly, $\mathcal{C}$ contains all atomic members of $\mathcal{V}$. Thus, to prove that (2) implies (3) we have to show that $\mathcal{C}$ is closed under subalgebras, direct products, and homomorphic images. The first is trivial, in the second, given L_i embedded into atomic $M_i \in \mathcal{V}$ embed the direct product of the L_i into the direct product M of the M_i, canonically. Of course, M is atomic and in $\mathcal{V}$. Finally, closure under homomorphic images follows from Proposition 9.2, below, and the fact that (4) implies (3).

Proposition 9.2. *Let L be an MOL embedded into an atomic MOL M and K a homomorphic image of L. Then the MOL K admits a representation in some anisotropic orthogeometry satisfying the orthoimplications of M.*

The proof of Proposition 9.2 relies on the existence of an extension with some 'saturation property'.

Lemma 9.3. *For any atomic MOL, M_0, there is an ultrapower M such that for any subOL L of M_0. any proper filter of F of L. and any $a \in L$ such that $aF \neq 0$ there is an atom p of M with $p \leq \varepsilon a$ and $p \leq \varepsilon x$ for all $x \in F$ - where ε is the canonical embedding of M_0 into M.*

Proof. Recall the model theoretic proof of the Compactness Theorem. Let I be the set of all finite subsets X of F. For $X \in I$ let

$$X^* = \{Y \in I \mid Y \supseteq X\}.$$

Observe that $X_1^* \cap X_2^* = (X_1 \cup X_2)^*$. Thus, there is an ultrafilter $\mathcal{U}$ on I containing all X^*. Let M the ultrapower $M = M_0^I/\mathcal{U}$. Consider $a \in L$ such that $aF \neq 0$.

Then, for all $X \in I$, $a \prod X > 0$ and by atomicity of M_0 there is an atom $p_{a,X} \leq a \prod X$ of M_0. Observe that for any $X \in I$

$$p_{a,Y} \leq a \prod X \quad \text{for all } Y \in X^*$$

Define

$$p_a = (p_{a,X} \mid X \in I)/\mathcal{U}$$

By Łos' Theorem, p_a is an atom of M and

$$p_a \leq \varepsilon(a \prod X) \quad \text{for any fixed } X \in I.$$

(Here, it suffices to consider singleton X.) □

For the proof of Proposition 9.2 we also need the following Lemma about the " neutral filter" associated with a congruence relation cf. [16]. We write $p \leq F$ if $p \leq x$ for all $x \in F$.

Lemma 9.4. *Let L, M be MOLs, L a subalgebra of M, and θ a congruence of L. Then $F_1 = \{x \in L \mid x\,\theta\,1\}$ is a filter of L. Moreover, $Q = \{p \in P \mid p \leq F_1\}$ is a subspace of P_M and for any $a, b \in L$ and $p \in Q$ with $p \leq a + b$ one has $q, r \in Q$ such that $q \leq a$, $r \leq b$, and $p \leq q + r$.*

Proof. The first claim is well known and checked, straightforwardly. The second is obvious. The last is trivial if $p \leq a$ or $p \leq q$. Referring to the proof of Observation 2.3 we may assume $a + b = 1$. Let $q = a(p + b)$ and $r = b(p + a)$. That q, r are atoms of M and p, q, r collinear is the same elementary calculation as in Lemma 6.2.

Consider $x \in F_1$, i.e. $x\,\theta\,1$ and $p \leq x$. Let

$$y = (a + xb)(b + x) \geq q, \quad z = (b + xa)(a + x) \geq r$$

By modularity, x, y, z coincide or are the atoms of a sublattice of height 2 cf. Figure 10. In particular, all its quotients are in θ whence $1/y \in \theta$ and $y \in F_1$. From $p \leq F_1$ it follows $p \leq y$ and thus $r \leq p+q \leq y$. Hence $r \leq yz \leq x$ and $q \leq x$, symmetrically.

□

Proof of Proposition 9.2. Consider L a subalgebra of the atomic MOL M. W.l.o.g. we may assume M is as in Proposition 9.3. Let θ be a congruence on L with $K \cong L/\theta$ and filter F_1 as in Lemma 9.4. Let a/θ denote the image of a under the canonical projection. Define

$$Q = \{p \in P_M \mid p \leq F_1\}, \quad \eta : L/\theta \to L(Q), \quad \eta(a/\theta) = \{p \in Q \mid p \leq a\}.$$

By Proposition 4.2 (1) and (4), $(P_M, \perp)$ is an anisotropic orthogeometry with subspace $(Q, \perp_Q)$ also an anisotropic orthogeometry, satisfying the orthoimplications of M in view of Lemma 7.1. Obviously, η is meet preserving and $\eta(a/\theta) \perp_Q \eta(a'/\theta)$. If $a/b \in \theta$ then $b = ac$ for some $c \in F_1$ whence $a \geq p \in Q$ implies $p \leq b$; thus, η is well defined. The proof that η preserves joins uses Observation 6.1. Given $a, b \in L$ consider $p \in \eta((a + b)/\theta)$, i.e. $p \in Q$ and $p \leq a + b$. By Lemma 9.4 there are $q, r \in Q$ such that $q \leq a$, $r \leq b$ and $p \leq q + r$ whence $p \in \eta(a/\theta) + \eta(b/\theta)$.

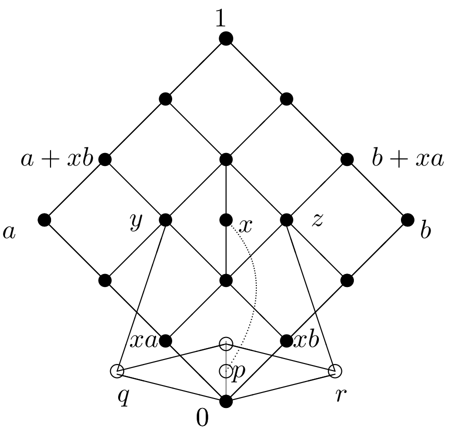

FIGURE 10. Neutral Filter Lemma

Finally, consider $a/\theta \neq 0$ which means $ac > 0$ for all $c \in F_1$. By choice of M there is $p \in P_M$ such that $p \leq x$ for all x in the filter

$$aF_1 = \{x \in L \mid ac \leq x \text{ for some } c \in F_1\}.$$

Thus, $p \in Q$ and $p \leq \eta(a/\theta)$ which proves Q is L-dense in M. Thus, η is a representation by Lemma 6.3.

□

10. U-quasivarieties of anisotropic orthogeometries

A *subgeometry* of an anisotropic orthogeometry $(P, \perp)$ is a proper projective subgeometry Q of P together with the restriction $\perp_Q$. The *orthogonal gisjoint union* of anisotropic orthogeometries $(P_i, \perp_i)$ is the disjoint union P of the P_i together with $p \perp q$ iff $p \perp_i q$ for some i or $p \in P_i$, $q \in P_j$ for some $i \neq j$. Then $X \mapsto (X \cap P_i \mid i \in I)$ yields both an isomorphism of $L(P)$ and $\prod_i L(P_i)$, $L_c(P, \perp)$ and $\prod_i L_c(P_i, \perp_i)$.

For a class $\mathcal{G}$ of anisotropic orthogeometries let $\mathsf{S}_g\mathcal{G}$, $\mathsf{U}\mathcal{G}$, and $\mathsf{P}_u\mathcal{G}$ denote the classes of all subgeometries, orthogonal unions, and ultraproducts of members of $\mathcal{G}$, respectively. Call $\mathcal{G}$ a U-quasivariety if it is closed under these operators.

For any class $\mathcal{G}$ of anisotropic orthogeometries, let $\mathcal{G}_\omega$ denote the class of finite dimensional members of $\mathcal{G}$ and $\mathbb{L}(\mathcal{G})$ the class of all MOLs admitting a representation within some member of $\mathcal{G}$. Let $\mathsf{R}(\mathcal{G})$ denote the class of all finite dimensional anisotropic orthogeometries such that $L_c(P, \perp) \in \mathbb{L}(\mathcal{G})$. Call a U-quasivariety *strong* if it is closed under R.

If $\mathcal{V}$ is a MOL variety, let $\mathbb{G}(\mathcal{V})$ denote the class of all anisotropic orthogeometries $(P, \perp)$ having $L_c(Q, \perp_Q) \in \mathcal{V}$ for all finite dimensional subspaces Q of P.

Theorem 10.1. (1) $\mathbb{L}(\mathcal{G}) \subseteq \mathsf{V}\{L_c(P, \perp) \mid (P, \perp) \in \mathcal{G}_{<\omega}\}$.

(2) $\mathbb{L}(\mathcal{G})$ *is an MOL variety if* $\mathcal{G}$ *is a* U*-quasivariety of anisotropic orthogeometries.*
(3) $\mathbb{G}(\mathcal{V})$ *is a strong* U*-quasivariety of anisotropic orthogeometries, given any MOL variety* $\mathcal{V}$*.*
(4) *(2) and (3) establish a 1-1-correspondence between strong* U*-quasivarieties of anisotropic orthogeometries and MOL varieties generated by their finite height members.* $\mathcal{V}$ *and* $\mathcal{Q}$ *are in correspondence if and only if they satisfy the same orthoimplications.*

Proof. cf. [12, Thm.2.2, Cor.2.3]. (1). Let η be a representation of L in $(P, \perp)$. By Theorem 8.2 L has an atomic M extension satsfying the orthoimplications of $(P, \perp)$. By Lemma 7.1 that are the orthoimplications satisfied by all $(Q, \perp_Q)$, Q finite dimensional subspace of P. Converting orthoimplications into identities by Lemma 7.3 we get that M whence L is in the variety generated by these $L_c(Q, \perp_Q)$.

(2) Closure under subOLs is obvious. If $L = \prod_i L_i$ and L_i represented in $(P_i, \perp_I)$ via ε_i then L is represented in the orthogonal disjoint union by $\varepsilon a = \sum_i \varepsilon_i a$. Concerning homomorphic images, we refer to the proof of Proposition 9.2: Let L be represented in $(P_0, \perp_0)$ and M_0 the atomic MOL-extension of L associated via Thm.8.2, Let M be an ultrapower of M_0 according to Lemma 9.3; then the proof of Prop.9.2 applies to provide for any homomorphic image K of L a representation in some subgeometry $(Q. \perp_Q)$ of the orthogeometry $(P, \perp)$ associated with M. Now, $(P, \perp)$ is an ultrapower of the geometry $(P_0, \perp_0)$ of M_0 - since the geometries are definable in the MOLs. Thus, the $(Q, \perp_Q)$ are in $\mathcal{G}$.

(3) Closure under S_g, U, and R is obvious. If $(P, \perp)$ is ultraproduct of the $(P_i, \perp_i)$ and U is an n-dimensional subspace of P, choose points $p_{ki} \in P_i$ such that the $(p_{ki} \mid i \in I)$, $k = 1, \ldots, n$ determine spanning points $p_1, \ldots, p_n$ of U and let U_i be the subspace of P_i spanned by $p_{i1}, \ldots, p_{in}$. Then U is a subspace of the ultraproduct of the U_i.

(4) For an MOL variety $\mathcal{V}$ let $\mathcal{V}_{<\omega}$ denote its subvariety generated by its finite heigth members. By (1) we have $\mathbb{L}(\mathbb{G}(\mathcal{V})) \subseteq \mathcal{V}_{<\omega}$ Conversely, if $L \in \mathcal{V}$ is of finite height, then $L \cong L_c(P_L, \perp)$ where $\perp$ is the orthogonality, whence $L \in \mathbb{L}(\mathbb{G}(\mathcal{V}))$. It follows $\mathcal{V} = \mathbb{L}(\mathbb{G}(\mathcal{V})$ if and only if $\mathcal{V} = \mathcal{V}_{<\omega}$.

If $(P, \perp) \in \mathcal{G}$, then $L(Q, \perp_Q) \in \mathbb{L}(\mathcal{G})$ for all finite dimensional subspaces Q of P whence $(P, \perp) \in \mathbb{G}(\mathbb{L}(\mathcal{G}))$ by definition. Conversely, assume $(P, \perp) \in \mathbb{G}(\mathbb{L}(\mathcal{G}))$. If Q is a finite dimensional subspace of P, then $L_c(Q, \perp_Q) \in \mathbb{L}(\mathcal{G})$ by definition, whence $(Q, \perp_Q) \in \mathcal{G}$ since $\mathcal{G}$ is closed under R. Now, observe that, by Lemma 10.2 bewlow, $(P, \perp)$ is a subgeometry of an ultraproduct of its finite dimensional subspaces, whence $(P, \perp) \in \mathcal{G}$.

Finally, bserve that by (1) and Lemmas 7.1 and 7.3, the orthoimplications of $\mathcal{G}$ are vaild in $\mathbb{L}(\mathcal{G})$ and the orthoimplcations of $\mathcal{V}$ in $\mathbb{G}(\mathcal{V})$, the latter by the definition of $\mathbb{G}(\mathcal{V})$.

□

Lemma 10.2. *If a first order structure A has substructures A_i such that each finite subset of A is contained in some A_i then A embeds in an ultraproduct of the A_I.*

Proof. This is due to Malcev.

11. One atom might suffice

Lemma 11.1. *In the situation given in Thm.11.2, consider an MOL L and an embedding $\eta : L \to L(P)$ of the 0-1-lattices such that $\eta a \perp \eta a'$ for all $a \in L$. Then Q is L-η-closed.*

Proof. We may consider L a 0-1-sublattice of $L(P)$ and η the identity map. Consider $a \oplus d = 1 = 1_{L(P)}$ in L, $q \not\leq d$, $q \in Q$. Let $p = a(q+d)$. Then, by modularity, $p \in P$ and $q^\perp d' \prec d'$, whence $a' + q^\perp d' \prec 1$. But $p^\perp \geq a'$ from $p \leq a$ and $p^\perp \geq (q+d)^\perp = q^\perp d'$ from $p \leq q+d$; moreover, $p^\perp < 1$ by anisotropictity. Thus, $a' + q^\perp d' \leq p^\perp < 1$ whence $p^\perp \prec 1$ and so $p \in Q$ proving that Q is L-η-closed.

We finish with a result which was intended to establish representations of MOLs: though, no serious application has been found so far.

Theorem 11.2. *Let $(P, \perp)$ be an a anisotropic pre-orthogeometry and Q the set of all points in P having orthogonal $p^\perp$ a hyperplane of P. Then Q is a proper projective subgeometry of P and an orthogeometry under the induced pre-orthogonality.*

Proof. Our proof will be by calculation in the geomodular lattice $L(P)$ endowed with the unary operation $x \mapsto x^\perp$ given by the pre-orthogonality $\perp$. We split the proof up into four Claims and define M as the set of all joins $\sum X$ in $L(P)$, X a finite subset of Q. The first Claim establishes some of the rules of calculation from Proposition 4.2 also in the given context. They will be freely used in proving Claims 2-4.

Claim 1. If $u \in M$ and $dim(u) = n$ then $u = u^{\perp\perp}$, $u + u^\perp = 1$ and $codim(u^\perp) = n$. Furthermore, for $u, v \in M$, $u \leq v$ iff $v^\perp \leq u^\perp$.

Proof of Claim 1. See Figure 11. Let $u \in M$ with $dim(u) = n$ and $u = \Sigma^n p_i$, with $p_i \in Q$, for each i. $u^\perp = \Pi^n p_i^\perp$, and since $p_i \prec 1$ for each i, codim$(u^\perp) \leq n$. Since $\perp$ is anisotropic, $uu^\perp = 0$, and so $dim([u, u+u^\perp]) = n$, by modularity. But $u \leq u + u^\perp \leq 1$ now gives $u + u^\perp = 1$, by modularity, and $codim(u^\perp) = n$. Always $u \leq u^{\perp\perp}$ and, by anisotropicity, $u^\perp u^{\perp\perp} = 0$. Hence, by modularity $u = u^{\perp\perp}$. Finally for this claim, $u \leq v$ in M automatically gives $v^\perp \leq u^\perp$ but now, from above,

$$u = u^{\perp\perp} \leq v^{\perp\perp} = v.$$

This completes the proof of Claim 1.

Claim 2. Q is a projective subgeometry of P.

Proof of Claim 2. It suffices to show that Q has the projective property. To this end suppose (p, a, b) and (p, c, d), where $p, a, b, c, d \in Q$, are collinear triples but that p, b, c are not collinear. Recall that collinearity is that inherited from P, in other words that given by the geomodular lattice $L(P)$. We need to show that there exists $q \in Q$ with (q, a, c) and (q, b, d) both collinear triples. Since P is a projective geometry $q = (a+c)(b+d) \in P$ satisfies the collinearity requirements cf. Figure 1. We need to show $q \in Q$, i.e.. $q^\perp \prec 1$.

Now, (cf. Figure 12) $q^\perp \geq (a+c)^\perp + (b+d)^\perp \geq (a+c)^\perp$, and $codim((a+c)^\perp) = 2$ from Claim 1. If $(a+c)^\perp = (a+c)^\perp + (b+d)^\perp$ then, by the last part of Claim 1,

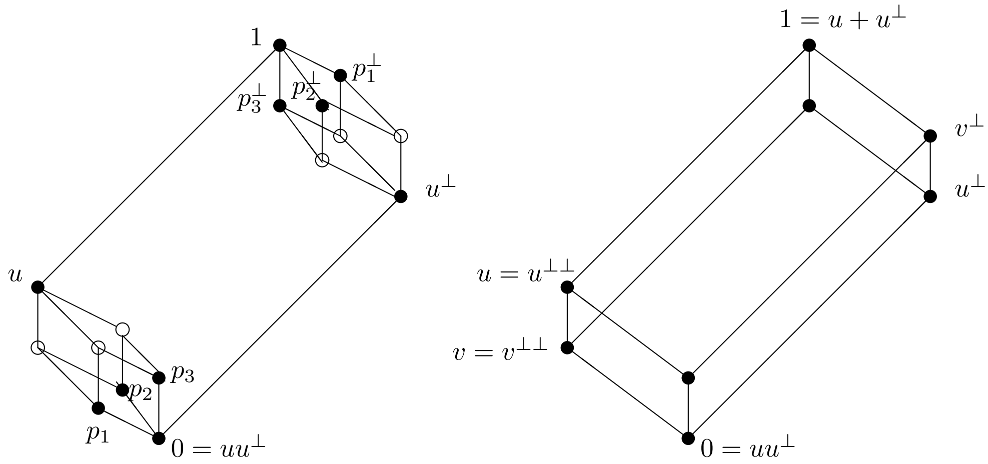

FIGURE 11. Claim 1

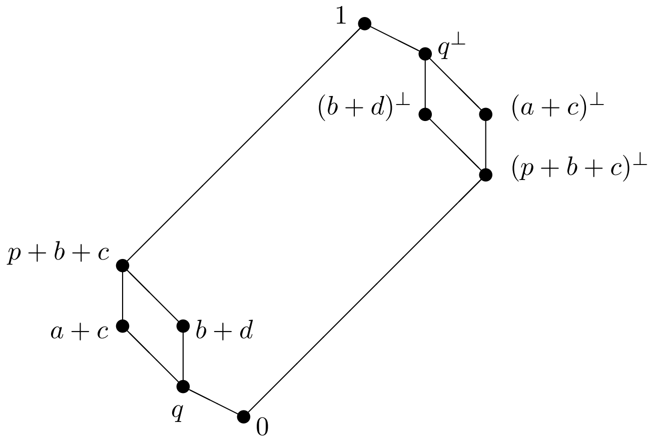

FIGURE 12. Claim 2

$a+c \leq b+d$, which we have assumed does not occur. Also, because $\perp$ is anisotropic, $q^\perp \neq 1$. So now,

$$1 > q^\perp \geq (a+c)^\perp + (b+d)^\perp > (a+c)^\perp,$$

gives, $1 \succ q^\perp = (a+c)^\perp + (b+d)^\perp$, and $q \in Q$.

Claim 3. Q is a proper projective subgeometry of P.

Proof of Claim 3. See Figure 13. We show by induction on $|X|$ that $p \leq u = \sum X$ with $p \in Q$ and finite $X \subseteq Q$ implies that p is in the subspace U of Q generated

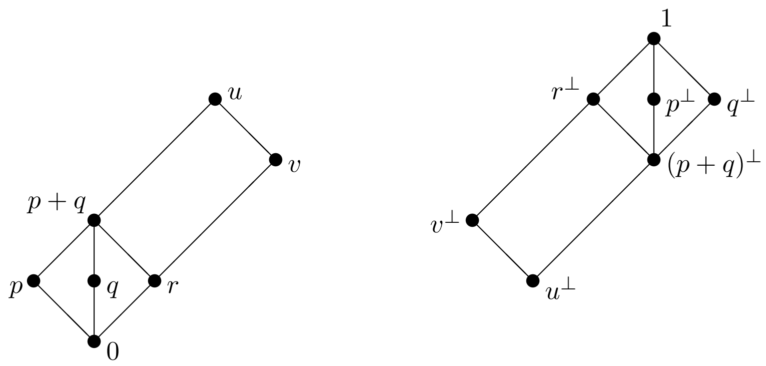

FIGURE 13. Claim 3

by X. So, let $X = Y \cup \{q\}$ with $q \notin Y$ and $v = \sum Y$. If $p \leq v$ we are done by induction. For $p = q$ the claim ist trivial. Otherwise, we have $r = v(p+q) \in P$. Now, in view of Claim 1, $v^\perp \not\leq p^\perp$ since $p \not\leq v$ whence $\dim[v^\perp + p^\perp q^\perp, 1] \leq 1$. On the other hand, $1 > r^\perp \geq v^\perp + (p+q)^\perp \geq v^\perp + p^\perp q^\perp$ whence $r^\perp \prec 1$. Thus $r \in Q$ and so $r \in U$ by induction. Since $q \in U$ and p, q, r are collinear, it follows $p \in U$.

Claim 4. $\perp$ induces an anisotropic orthogonality on Q.

Proof of Claim 4. By hypothesis, $p \not\perp p$ for all $p \in Q$. We are left to show that for $p \neq q$ in Q there is $r \in Q$ such that $r \leq p + q$ and $r \perp p$. See Figure 14. Choose $r = (p+q)p^\perp$. Since $p^\perp \prec 1$ we have $r \in P$. Since p, q and $p+q$ are in M, we already know that they are closed, that $\dim[(p+q)^\perp, 1] = 2$, and $r + p^\perp q^\perp = (p + q + p^\perp q^\perp)p^\perp = (p + q + (p+q)^\perp)p^\perp = 1p^\perp = p^\perp$. From $r \leq p + q$ we have $r^\perp \geq p^\perp q^\perp$. Assuming $r^\perp = p^\perp q^\perp$ it would follow $p = p^{\perp\perp} == (r + p^\perp q^\perp)^\perp = r^\perp (p^\perp q^\perp)^\perp = 0$. Also, $r^\perp = 1$ would contradict anisotropicity. Thus $p^\perp q^\perp \prec r^\perp \prec 1$ since $p^\perp q^\perp = (p+q)^\perp$ has codimension 2, and it follows $r + r^\perp = 1$, i.e. $r \in Q$.

□

Corollary 11.3. *Let ϕ be a representation of the MOL L in the anisotropic orthogeometry $(P, \perp)$ and let $Q = \{p \in P \mid p + p^\perp = 1\}$. Define $\eta : L \to L(Q)$ by*

$$\eta(a) = \{p \in Q \mid p \leq \phi(a)\},$$

for all $a \in L$. If Q is (L, ϕ)-dense (in particular, if L is simple and $Q \neq \emptyset$) then η is a representation of the MOL L in the anisotropic orthogeometry $(Q, \perp_Q)$ which is a proper projective subgeometry of P.

Proof. By Theorem 11.2, Q is a proper projective subgeometry of P and an orthogeometry under the induced orthogonality. Clearly, $\eta(a) \perp_Q \eta(a')$ for $a \in l$. It remains to show that Q is (L, η)-closed; for once this is done we simply apply Lemma 6.3 with $M = L(P)$. For simple L refer to Corollary 6.5.

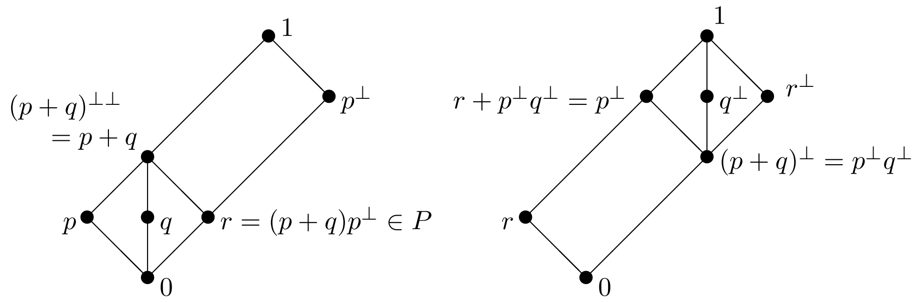

FIGURE 14. Claim 4

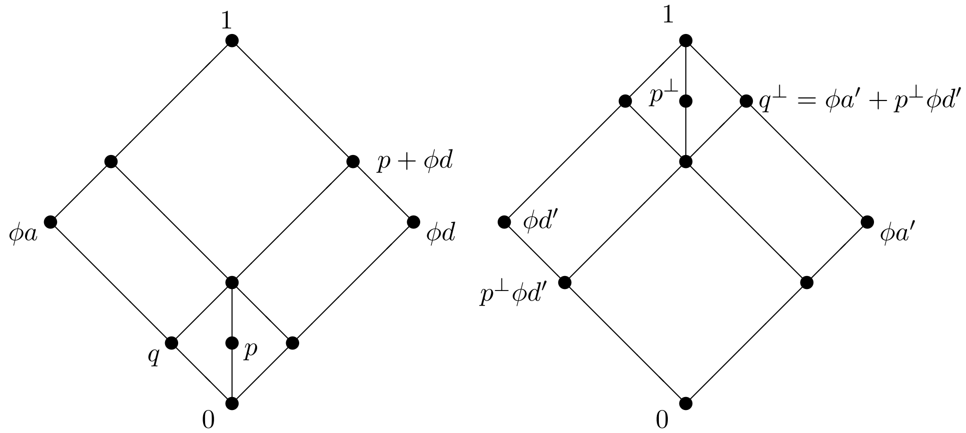

FIGURE 15. Theorem 11.3

Let $a, d \in L$ with $a + d = 1$ and $ad = 0$, whence $\phi(a)$ and $\phi(d)$ are complements of each other in M and, by Lemma 6.6, so are $\phi(a') = \phi(a)^\perp$ and $\phi(d') = \phi(d)^\perp$. See Figure 15. Let $p \in Q$ and $p \not\leq \phi(d)$. Let $q = \phi(a)(p + \phi(d))$. We have $q \in P$, we will show $q^\perp \prec 1$.

Since, $p^\perp \prec 1$, $(\phi(d) + p)^\perp = \phi(d)^\perp p^\perp \prec \phi(d)^\perp$. Now, $q \leq \phi(a)$ and $q \leq p + \phi(d)$ so $q^\perp \geq \phi(a)^\perp + p^\perp \phi(d)^\perp$. But,

$$\phi(a)^\perp + p^\perp \phi(d)^\perp \prec \phi(a)^\perp + \phi(d)^\perp = 1.$$

Since $q^\perp \neq 1$, $q^\perp = \phi(a)^\perp + p^\perp \phi(d)^\perp \prec 1$.

□

Corollary 11.4. *Let M be a modular 0-1-lattice with anisotropic pre-orthogonality $\perp$. Then $P = P_M$ with the restriction of $\perp$ is an anisotropic pre-orthogeometry and $(Q, \perp_Q)$ defined as in Theorem 11.2 satisfies the orthoimplications of $(M, \perp)$.*

Moreover, if L is an MOL and if ϕ is a 0-1- lattice homomorphism of L into M such that $\phi(a) \perp \phi(a')$ for all $a \in L$ and such that, referring to Corollary 11.3, Q is (L, ϕ)-dense (in particular, if L is simple and $Q \neq \emptyset$), then η is a representation of the MOL L in the anisotropic orthogeometry $(Q, \perp_Q)$.

Proof. The first claims follow from Theorem 11.2 and Lemma 7.1. Clearly, P_M is L closed in M and ϕ is a representation of the 0-lattice L in the projective space P_M by Lemma 6.3. By definition of $\perp_P$, $\eta(a) = \{p \in P \mid p \leq \phi(a)\}$ is also a representation of the MOL L in the pre-orthogeometry $(P, \perp_P)$. By Theorem 11.3, η is a representation of the MOL L in the orthogeometry $(Q, \perp_Q)$.

□

12. Discussion

An MOL, L, is *n-distribuitive* if it satisfies the identity

$$x \sum_{i=0}^{n} x_i = \sum_{i=0}^{n} x \sum_{j \neq i} x_i$$

According to [23], this is the case if and only if L is a subdirect product of simple MOLs of height $\leq n$. Due to [15] and the fact that any interval sublattice of L with the induced orthocomplementation is in the variety of L, it follows that for any variety $\mathcal{V}$ of MOLs the following are equivalent

(1) $\mathcal{V}$ contains a simple member of height 3
(2) $\mathcal{V}$ contains an atomic member which is not 2-distributive.

Call $\mathcal{V}$ a *Bruns variety* it $\mathcal{V}$ is if (i) holds for any non-2-distributive subvariety of $\mathcal{V}$. Gunter Bruns [4] had conjectured that any MOL variety is a Bruns variety. The proof he had in mind was to show that any subdirectly irreducible MOL generated by an orthogonal 3-frame would be of height 3. A height 6 conterxample was soon provided by Müller, according to [14] there are counterexamples of arbitrarily large finite as well as infinite height. While the analogous claim for varieties of modular lattices is easily proved (any non-2-distributive lattice admits a 3-frame in a height 3 interval I of its filter lattice, whence I is simple complemented, in particular) all attempts on a similar approach for MOLs failed badly.

At present, no non-trivial example of a Bruns variety of MOLs in knwon - the trivial examples provided by the n-distributive MOL-varieties is Bruns (since its subdirectly irreducible members are of height $\leq n$). Thus, one should discard the Bruns Conjecture as unreasonable and ask the following question.

Problem 12.1. *Is there any Bruns variety of MOLs which is not n-distributive for any n?*

In contrast, there are non-trivial examples of MOL-varieties generated by their finite height members. But these could be verified only on the basis of strong results from Functinal Analysi. (cf. [13, 18]):

> The MOL varieties $\mathcal{V}$ generated by projection lattices of finite Rickart C^*-algebras algebras A_i are generated by either by one or all subspace MOLs $L(\mathbb{C}^n)$ w.r.t. canonical inner product.

Theorem 10.1 establishes a 1-1-correspondence between certain classes $\mathcal{Q}$ of anisotropic orthogeometries and MOL varieties $\mathcal{V}$ generated by their finite height members. From this point of view, the latter appear to be rather special varieties.

The most interesting one is the variety $\mathcal{R}$ generated by the $L(\mathbb{Q}^n)$ $n < \omega$. With [19, Digr.5.22], the fact that $\sqrt{p_0} \notin \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ for distinct primes p_i, and with Jónsson's Lemma it follows that the varieties $L(\mathbb{Q}(\sqrt{p})^3)$, p ranging over primes $\equiv 3 \bmod 4$, form, in the lattice of of subvarieties of $\mathcal{R}$, an independent set of covers of the variety generated by $L(\mathbb{Q}^3)$. In particular, $\mathcal{R}$ has $2^{\aleph_0}$ subvarieties.

Problem 12.2. *Is any subvariety of $\mathcal{R}$ a Bruns variety? Does $\mathcal{R}$ coincide with the variety $\mathcal{N}$ generated by the $L(\mathbb{C}^n)$, $n < \omega$?*

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