Computational Complexity of Quantum Satisfiability*

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Abstract. Quantum logic was introduced in 1936 by Garrett Birkhoff and John von Neumann as a framework for capturing the logical peculiarities of quantum observables. It generalizes, and on 1-dimensional Hilbert space coincides with, Boolean propositional logic.
We introduce the weak and strong satisfiability problem for quantum logic terms. It turns out that in dimension two both are also $\mathcal{NP}$–complete.
For higher-dimensional spaces $\mathbb{R}^d$ and $\mathbb{C}^d$ with $d \geq 3$ fixed, on the other hand, we show both problems to be complete for the nondeterministic Blum-Shub-Smale model of real computation. This provides a unified view on both Turing and real BSS complexity theory; and extends the still relatively scarce family of $\mathcal{NP}_{\mathbb{R}}$–complete problems with one perhaps closest in spirit to the classical Cook-Levin Theorem.
Our investigations on the dimensions a term is weakly/strongly satisfiable in lead to satisfiability problems in indefinite finite and finally in infinite dimension. Here, strong satisfiability turns out as polynomial-time equivalent to the feasibility of noncommutative integer polynomial equations over matrix rings.

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This contribution connects both discrete and algebraic complexity theory with the satisfiability problem in certain non-Boolean lattices. It thus aims at particularly broad an audience of pure mathematicians as well as of computer scientists. We have therefore chosen to include some background on computational complexity theory as well as on quantum logic—in particular to the "modular" first version of a "logic of quantum mechanics" due to von Neumann and Birkhoff [BiNe36] based on the finite dimensional Hilbert spaces considered in nowadays Quantum Computation.

This exposition employs, adapts, and tailors several general concepts from various areas of logic for our particular purpose; whereas few dedicated *digressions* offer additional background and deeper connections which may be skipped at first reading.

1 Introduction

s:Intro

Quantum physics is famous for its seemingly paradoxical (yet very real) effects. Shaping them into a mathematically sound physical theory was a big achievement of the last century [Neum32]. There, physical *observables* correspond to linear self-adjoint operators on some Hilbert space $\mathcal{H}$; and *properties* (i.e. observables attaining only (eigen)values 0 or 1) to projection operators—which in turn can be identified with closed subspaces of $\mathcal{H}$. Their logical features (reflecting non-commutativity of operators) have been captured [Mack63,Piro64,BeCa81] as abstract properties of the *ortholattice* of closed subspaces; cmp. also [Rota97]. Let us start with the following

d:qlogic

Definition 1.1 *Fix a $*$-subfield $\mathbb{F}$ of the complex number field $\mathbb{C}$, i.e. a subfield closed under and equipped with complex conjugation $(a+bi)^* = \overline{a+bi} = a - bi = \operatorname{Re}(a+ib) - i\operatorname{Im}(a+ib)$ as involution (which may be ignored if $\mathbb{F} \subseteq \mathbb{R}$). Popular cases are e.g. $\mathbb{F} = \mathbb{C}$ itself, $\mathbb{F} = \mathbb{A}$ algebraic numbers, $\mathbb{F} = \mathbb{R}$ real numbers, $\mathbb{F} = \mathbb{Q}$ rationals, and $\mathbb{F} = \mathbb{A} \cap \mathbb{R}$ algebraic reals. An $\mathbb{F}$-unitary space is an $\mathbb{F}$-vector space $\mathcal{H}$ equipped with a unitary scalar product, i.e. a map $(\vec{x}, \vec{y}) \mapsto \langle \vec{x} \mid \vec{y} \rangle \in \mathbb{F}$, which is $\mathbb{F}$-linear for fixed $\vec{x}$ and satisfies $\langle \vec{y} \mid \vec{x} \rangle = \overline{\langle \vec{x} \mid \vec{y} \rangle}$ and $\langle \vec{x} \mid \vec{x} \rangle \neq 0$ for $\vec{x} \neq \vec{0}$ (cf.* [Halm58, §59-65],[Gelf61, §2-3], [Axle96, Ch.6], *or* [Fare01, Ch.1]*). But observe that we follow the convention of Physics and require linearity in the right hand argument. The basic example is $\mathbb{F}^d$ where $\langle \vec{x} \mid \vec{y} \rangle = \sum_{j=1}^d \overline{x}_i y_i$. The orthogonality relation on $\mathcal{H}$ is then defined by $\vec{x} \perp \vec{y}$ if and only if $\langle \vec{x} \mid \vec{y} \rangle = 0$. A subset U of $\mathcal{H}$ is closed if $U = U^{\perp\perp}$ where $U^\perp = \{\vec{x} \in \mathcal{H} \mid \forall \vec{u} \in U.\ \vec{x} \perp \vec{u}\}$ is the linear subspace orthogonal to U. We write $U \perp V$ if $U \subseteq V^\perp$, equivalently $V \subseteq U^\perp$. The system of quantum logical properties of $\mathcal{H}$, shortly the quantum logic of $\mathcal{H}$, consists of the set $\mathrm{L}(\mathcal{H})$ of all closed subsets of $\mathcal{H}$ equipped with the following connectives:*

$\wedge$: $\mathrm{L}(\mathcal{H}) \times \mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{H})$, $(U,V) \mapsto U \cap V$
$\neg$: $\mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{H})$, $U \mapsto U^\perp$,
$\vee$: $\mathrm{L}(\mathcal{H}) \times \mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{H})$, $(U,V) \mapsto (U \cup V)^{\perp\perp}$

and, in addition, with the constants $\mathbf{0} := \{\vec{0}\} \in \mathrm{L}(\mathcal{H})$ and $\mathbf{1} := \mathcal{H} \in \mathrm{L}(\mathcal{H})$.

Note that any closed subset U is a linear subspace of $\mathcal{H}$ and that $U \vee V = (U+V)^{\perp\perp}$ where $U + V = \{\vec{u} + \vec{v} : \vec{u} \in U, \vec{v} \in V\}$ is the least linear subspace of $\mathcal{H}$ containing $U \cup V$. Moreover, if $\mathcal{H}$ is of finite dimension d, the case we are primarily interested in, then any linear subspace U is closed whence $U \vee V = U + V$. (Indeed, if $\dim U = k$ then $U^\perp$ has dimension $d-k$ being the solution set of the independent linear equations $\langle \vec{u}_i \mid \vec{x} \rangle$, $\vec{u}_1, \ldots, \vec{u}_k$ a basis of U. By the same token, $\dim U^{\perp\perp} = d - (d-k) = k$ and $U = U^{\perp\perp}$ from $U \subseteq U^{\perp\perp}$.) By $\mathrm{L}_k(\mathcal{H})$ we denote the set of k-dimensional subspaces of $\mathcal{H}$.

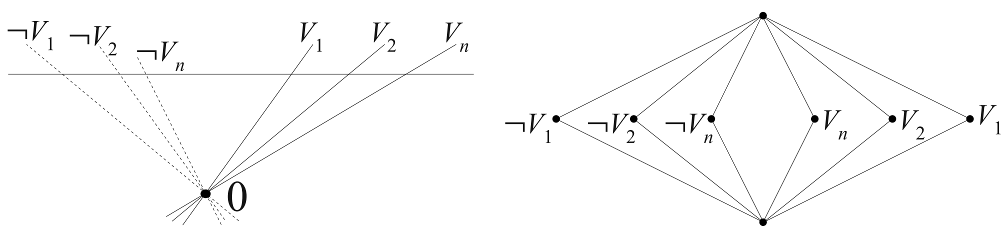

Fig. 1. Prototype $\mathcal{MO}_n$ of a non-distributive modular ortholattice

r:Hilbert

Remark 1.2 *a) In case $d = 1$, $\mathrm{L}(\mathcal{H}) = \{\mathbf{0}, \mathbf{1}\}$ coincides with the set of Boolean truth values. However, starting with dimension 2, the distributive law "$X \vee (Y \wedge Z) = (X \wedge Y) \vee (X \wedge Z)$" generally fails: consider, e.g., $X := \{(t,t) : t \in \mathbb{F}\} \in \mathrm{L}_1(\mathcal{H})$, $Y := \{(t,0) : t \in \mathbb{F}\} \in \mathrm{L}_1(\mathcal{H})$, $Z := \{(0,t) : t \in \mathbb{F}\} \in \mathrm{L}_1(\mathcal{H})$; and compare also Figure 1. This is generally seen as one cause underlying the counter-intuitive effects of quantum physics.*

b) It is well-known that $\big(\mathrm{L}(\mathcal{H}), \wedge, \vee, \mathbf{0}, \mathbf{1}\big)$ constitutes an lattice *a commutative, associative algebraic structure with respect to idempotent operations "$\wedge$" (called* meet*) and "$\vee$" (called* join*) such that $A \wedge (A \vee B) = A = A \vee (A \wedge B)$ and $0 \wedge A = \mathbf{0}$, $\mathbf{1} \wedge A = A$ (alternatively, $A \le B \Leftrightarrow A = A \wedge B$ defines a partial order such that $A \vee B$ is the supremum and $A \wedge B$ infimum of A, B and $\mathbf{0}$ and $\mathbf{1}$ are the smallest and greatest element, respectively, and then $A \le B \Leftrightarrow B = A \vee B$). Moreover, $X \mapsto \neg X$ is an* involution *on L, i.e. $A \le B$ if and only if $\neg B \le \neg A$ and $\neg\neg A = A$. The* de Morgan laws *hold:*

$$\neg(A \vee B) = \neg A \wedge \neg B, \qquad \neg(A \wedge B) = \neg A \vee \neg B. \tag{1}$$

In addition one has $A \wedge \neg A = \mathbf{0}$ and $A \vee \neg A = \mathbf{1}$. All this summarizes into saying that $\big(\mathrm{L}(\mathcal{H}), \wedge, \neg, \vee, \mathbf{0}, \mathbf{1}\big)$ is an ortholattice *and follows easily from the fact that $\perp$ is a symmetric relation such that $\vec{x} \perp \vec{x}$ only for $\vec{x} = \vec{0}$.*

c) The crucial feature of finite dimensional $\mathcal{H}$ is the modular law

$$A \ge C \quad \Rightarrow \quad A \wedge (B \vee C) = (A \wedge B) \vee C \ . \tag{2}$$

Indeed, if $\vec{a} \in A \cap (B + C)$ then $\vec{a} = \vec{b} + \vec{c}$ with $\vec{b} \in B$ and $\vec{c} \in C$ whence $\vec{b} = \vec{a} - \vec{c} \in A \cap B$, proving $\le$, while $\ge$ holds in any lattice. An ortholattice obeying the modular law is a modular ortholattice*, MOL for short. Actually, finite dimension is also necessary for modularity of $\mathrm{L}(\mathcal{H})$ [BiNe36] and we restrict to this case unless stated otherwise; here, $\mathrm{L}(\mathcal{H})$ amounts to the Grassmannian of $\mathcal{H}$ considered as an ordered algebraic structure rather than as a topological space.*

Digression 1.3 *In general, quantum mechanics lives in an* in*finite-dimensional complex Hilbert space $\mathcal{H}$; here, only the special case $B = \neg C$ of the modular law, the* orthomodular law*, $A \ge C \Rightarrow A = (A \wedge \neg C) \vee C$ can be shown. Orthomodular lattices and their generalizations underly most of the work done in quantum logic. Though, in [BiNe36,Neum81]* von Neumann *stipulated certain modular sub-ortholattices of* $\mathrm{L}(\mathcal{H})$ *(related to finite von Neumann algebras of operators) as the proper setting of a 'logic of quantum mechanics' cf. [Duck69,Rede07]. In the context of the present investigation, these can be captured by the finite dimensional case—see Section 7. These modular ortholattices, endowed with a system of von Neumann frames corresponding to a continuous set of matrix units [Neum58] respectively tensor products of von Neumann frames [CzSk10], might provide the framework to take into account newer developments which focus on processes and quantum information and computation [CMW00,Coec10].*

1.1 Motivations from Physics, Logic, and Geometry

We find quantum logic a promising object of study for various reasons and to various communities [Svoz98,CMW00,EGL07].

Physically, it arises from quantum mechanics which our world seems to be based on [Weiz85] yet of which we regularly perceive only macroscopic approximations—which would mean that Boolean logic is strictly speaking 'wrong' and to be replaced with quantum logic [Putn69]: perhaps even in the foundations of mathematics?

Indeed, $\mathrm{L}(\mathcal{H})$ constitutes a logic with semantics $\texttt{false} = \mathbf{0}$ and $\texttt{true} = \mathbf{1}$ plus (in dimensions>1) 'intermediate' values like $V_n := \mathbb{F}\cdot\binom{1}{n} := \{(t, n\cdot t) : t \in \mathbb{F}\}$, $n \in \mathbb{N} = \{1, 2, \ldots\}$. Such many-valuedness resembles probabilistic/fuzzy logic and, more generally, Gödel Logics which 'interpolate' between 0 (impossibility) and 1 (certainty) with values $r \in [0, 1]$ (likelihood). Note, though, that the above V_n are pairwise incomparable; cmp. Figure 1a).

Geometry provides, at least in low dimensions, some important intuition to quantum logic. In fact the incidences among linear subspaces prescribed by some term relate to the field of Convex and Combinatorial Geometry [MnZi93,GoOR04].

1.2 Computational Complexity

is at the core of theoretical computer science [Papa94]. It classifies decision problems (i.e. countable families of yes/no questions) according to the cost inherently incurred for their algorithmic solution: asymptotically with respect to the binary length n of the inputs and expressed in Landau's $\mathcal{O}$-notation. Both algorithms and costs pertain to some model of computation; which usually amounts to the Turing machine capable of reading, storing, processing, and printing a constant number of bits in each step. This leads to classical complexity classes $\mathcal{P}$ (polynomial time), PSPACE (polynomial space). The class $\mathcal{NP}$ (nondeterministic $\mathcal{P}$) on the other hand refers to polynomial-time Turing machines that may 'guess', but have to 'verify' in polynomial time, bits.

x:EvalSat

Example 1.4 *a) The following* Boolean evaluation *problem is in* $\mathcal{P}$*:*

> *Given (the binary encoding* $\langle t\rangle$ *of) a term* t *over* $\wedge, \vee, \neg, \mathbf{0}, \mathbf{1}$ *in variables* $x_1, \ldots, x_n$ *and an assignment within* $\{\mathbf{0}, \mathbf{1}\}$*, does it evaluate to* $\mathbf{1}$ *?*

b) The following Boolean satisfiability *problem,* SAT*, is in* $\mathcal{NP}$*:*

> *Given (an encoding* $\langle t\rangle$ *of) a term* t *over* $\wedge, \vee, \neg$ *and variables* $x_1, \ldots, x_n$*, do they admit an assignment over* $\{\mathbf{0}, \mathbf{1}\}$ *for which* t *evaluates to* $\mathbf{1}$ *?*

c) One (out of many) reasonable way of encoding terms in a) and b) in binary starts by assigning the function symbols $\mathbf{0}, \mathbf{1}, \neg, \vee, \wedge$ *to bit strings* $0, 1, 00, 01, 10$ *and variable symbol* x_k *(*$k \in \mathbb{N}$*) to the number* $k+2$ *in binary. A term in polish notation thus gives rise to a finite sequence of these binary strings. And two such bit strings* $(a_1, \ldots, a_m)$ *and* $(b_1, \ldots, b_\ell)$ *can be 'concatenated' reversibly into a single* $(a_1, 0, a_2, 0, \ldots, a_m, 1, b_1, 0, b_2, 0, \ldots, b_\ell, 1)$*.*

(*Karp* or *many-one*) reduction allows to compare decision problems: $A \preceq B$ ('A is at most as difficult as B') means that instances for A can efficiently be converted into instances for B; formally: there exists a total function f computable in polynomial time such that $\forall x : x \in A \Leftrightarrow f(x) \in B$ holds. Indeed, a (hypothetical) algorithm $\mathcal{M}_B$ deciding membership to B gives rise to one at most polynomially slower for A: Given an instance "$x \in A$?", $\mathcal{M}_A$ calculates $y := f(x)$ and invokes $\mathcal{M}_B$ to decide whether $y \in B$ holds or not. A problem B being $\mathcal{NP}$-*complete* means both that B belongs to $\mathcal{NP}$ (can be solved in nondeterministic polynomial time) and is $\mathcal{NP}$-*hard*: every $A \in \mathcal{NP}$ has $A \preceq B$. In other words, B is a computationally most difficult problem within $\mathcal{NP}$. The Cook-Levin Theorem now states that SAT is $\mathcal{NP}$-complete—and thus not believed to admit a polynomial-time solution by deterministic Turing machines. In fact a vast variety of decision problems naturally arising from various areas have all turned out as polynomial-time equivalent to SAT [GaJo79].

Quantum computers have been suggested as an alternative (and perhaps more powerful) model of computation. They exploit linearity of quantum mechanics, namely, that its evolution extends from states to superpositions. Therefore a physical system realizing some 'computation' on (say polynomially many) so-called qubits also works on linear combinations thereof—simultaneously: quantum parallelism. The difficulty in exploiting this capability algorithmically consists in preparing the superposition and in extracting the output from the resulting state. Quantum logic, on the other hand, as describing operations on observables rather than states, has also been proposed as an approach to computational purposes [Pyka00,Bub07,PaMe07] and to computational concepts [DHMW05,Ying05].

1.3 Blum-Shub-Smale Machines

In algebraic complexity, the BSS machine (also known as *real-RAM*) is a common model [BCSS98] capturing arithmetic on real numbers as entities with unit cost per arithmetic operation. More precisely it can read, store, operate, compare, and output a constant number of reals in each step.

d:BSS

Definition 1.5 *Consider a commutative ring R, possibly ordered. A (deterministic) BSS machine $\mathcal{M}$ over R contains a finite number of constants* $c \in R$, a register array, and three index registers. It receives as input some finite tuple $\bar{x} \in R^n$ together with its length $n =: |\bar{x}| \in \mathbb{N}$. $\mathcal{M}$ can then apply arithmetic operations $+, -, \times$ (and $\div$ in case R is a field, but* no *conjugation $z \mapsto \bar{z}$ even in the complex case**) to these x_j, to its pre-stored constants, or to some array elements accessed via index registers, and store the result. It may furthermore branch based on the test for equality $=, \neq$ (in the ordered case also for $<, \leq, >, \geq$) of two array elements. Each operation/branch is counted for as one step. On a fixed input $\bar{x}$, $\mathcal{M}$ may accept, reject, or loop indefinitely.*

Cf. [BCSS98, Definition 3.1] and compare also, e.g., [Poiz95, §4.A] or [TuZu00, §3]. Note that operations and comparisons are presumed exact.

x:BSS

* Storing the non-recursive constant $\sum_{x \in H} 2^{-n}$ may be exploited by a BSS-machine over $\mathbb{R}$ to decide the Halting problem $H \subseteq \mathbb{N}$ for Turing (!) machines. For machines without constants, refer to Fact 1.12b).

** This is the standard conception [BCSS98, §2.1], noting that otherwise $\mathbb{C}$ would become computationally isomorphic to $\mathbb{R}^2$ and violate the natural differences between algebraic and semi-algebraic geometry [BCSS98, §2.3].

Example 1.6 *Gaussian Elimination for $n \times n$–matrices over a fixed field $\mathbb{F}$ is a typical algorithm for BSS-machines over $\mathbb{F}$ with polynomial running time $\mathcal{O}(n^3)$. Here, exact comparisons are employed during pivot search.*

Furthermore this model commonly underlies algorithms devised, among others, in polynomial system solving [CLOS07] *and in computational geometry* [BKOS97]*.*

BSS machines over the field $\mathbb{Z}_2 = \{0, 1\}$ can be seen equivalent to Turing machines. Definition 1.5 thus extends the traditional, discrete theory of computation; and has led to a rich structural complexity theory [MeMi97]. In particular, a *nondeterministic* BSS machine over R may make and verify guesses from R; and doing so in polynomial time gives rise to the complexity class $\mathcal{NP}_R$, thus naturally translating to this setting the classical question "$\mathcal{P} = \mathcal{NP}$?"—which has turned out as equally inaccessible [FoKo98,FoKo00]. As a matter of fact, both "$\mathcal{P}$ versus $\mathcal{NP}$" *and* "$\mathcal{P}_\mathbb{C}$ versus $\mathcal{NP}_\mathbb{C}$" are propagated as *Third Problem for the Next Century* [Smal98].

x:Feas

Example 1.7 *Fix a commutative**** *ring $R \supseteq \mathbb{Z}$. The following problem* $\mathrm{FEAS}_{R,R}$ *can be decided by a nondeterministic polynomial-time BSS machine over R, i.e. belongs to $\mathcal{NP}_R$:*

> *Given ($n \in \mathbb{N}$ and the list of monomials and coefficients of each of) finitely many polynomials $p_1, \ldots, p_k \in R[X_1, \ldots, X_n]$, do they admit a common root in R, i.e. some $\bar{x} \in R^n$ such that $p_1(\bar{x}) = \cdots = p_k(\bar{x}) = 0$?*

Indeed, such a machine (does not need constants and) will simply 'guess' an assignment $x_1, \ldots, x_n \in R$ and 'verify' by evaluating the polynomials: which is clearly possible in a number steps polynomial in (and noting that n is bounded by) the length of (the descriptions of) the polynomials. $\mathrm{FEAS}_{\mathbb{C},\mathbb{C}}$ is classically characterized by the famous Hilbert's Nullstellensatz in algebraic geometry.

Generalizing the Cook-Levin Theorem, $\mathrm{FEAS}_{R,R}$ has been established complete for $\mathcal{NP}_R$; cf. [BCSS98, Theorem 5.1] and [Good94]. More precisely a Turing (!) machine can, given the description of a nondeterministic polynomial-time BSS machine $\mathcal{M}$ (with symbolic references to its constants from R) and given an input $\vec{y} \in R^m$, output within time polynomial in n some multivariate polynomials over R (with references to $\mathcal{M}$'s constants as above) such that the following holds: $\mathcal{M}$ accepts $\vec{y}$ iff these polynomials admit a common root in R [Cuck93,MeMi97]. The proof employs

f:Constructible

Fact 1.8 *Finite Boolean combinations of polynomial in-/equalities over a field can be expressed as (the feasibility of a system of) polynomial equations:*

a) $p(\bar{x}) = 0 \ \vee\ q(\bar{x}) = 0 \quad \Leftrightarrow \quad \big(p \cdot q\big)(\bar{x}) = 0$
b) $p(\bar{x}) \neq 0 \quad \Leftrightarrow \quad \exists y : y \cdot p(\bar{x}) - 1 = 0$

Note that, other than a BSS machine, even a nondeterministic Turing machine cannot in general guess assignments over R in case the ring is infinite. Hence it is far from clear that $\mathrm{FEAS}_{R,R}$ even be decidable. Not just in view of the Church-Turing Hypothesis the question naturally arises of how the BSS model compares to the Turing model. Of course the latter is not fitted to process inputs containing arbitrary, say, real numbers: Instances of $\mathrm{FEAS}_{R,R}$ consist of both discrete (e.g. a list of monomials with their multi-degrees) and algebraic (e.g. coefficients from R) information, technically being words from a formal language over alphabet $\{0, 1\} \cup R$ while Turing machines work over $\{0, 1\}$.

*** Section 6 will naturally arrive at considering also noncommutative rings.

Example 1.9 *The decision problem* $\{\langle z, \mathrm{bin}(n)\rangle \mid z \in \mathbb{C}, n \in \mathbb{N}, z^n = 1\}$ *for complex roots of unity belongs to* $\mathcal{P}_\mathbb{C}$ *but its instances cannot naturally be presented to a Turing machine.*

This has suggested restricting $\mathrm{FEAS}_{R,R}$, and more generally $\mathcal{NP}_R$, to binary instances:

d:BP

Definition 1.10 *For a ring* $R \supseteq \mathbb{Z}$, *call* $\mathrm{BP}(\mathcal{NP}_R) := \{\mathcal{L} \cap \{0,1\}^* \mid \mathcal{L} \in \mathcal{NP}_R\} = \{\mathcal{L}' \mid \mathcal{L}' \subseteq \{0,1\}^*, \mathcal{L}' \in \mathcal{NP}_R\}$ *the Boolean part of* $\mathcal{NP}_R$*; cmp.* [MeMi97, DEFINITION 3.2].

Note that this complexity class is indeed closed under Turing machine reduction.

x:BP

Example 1.11 *For* $\mathbb{Z} \subseteq R$, *the following problem,* $\mathrm{FEAS}_{\mathbb{Z},R}$, *belongs to* $\mathrm{BP}(\mathcal{NP}_R)$*:*

> *Given (the coefficients, encoded in binary, of) finitely many polynomials* $p_1, \ldots, p_k \in \mathbb{Z}[X_1, \ldots, X_n]$, *do they admit a common root in* R *?*

Although the input polynomials have integer coefficients, their roots may in general 'live' in R: consider for example $x^2 - 2$ or $x^2 + 1$.

f:BSSBP

Fact 1.12 *a)* $\mathrm{FEAS}_{\mathbb{Z},R}$ *is complete for* $\mathrm{BP}(\mathcal{NP}_R)$.

b) Let $\mathcal{NP}^0_R$ *denote the class of languages decidable by a nondeterministic polynomial-time BSS machine without constants. It holds* $\mathrm{BP}(\mathcal{NP}^0_\mathbb{C}) = \mathrm{BP}(\mathcal{NP}_\mathbb{C})$ [BCSS98, PROPOSITION 7.9] *and* $\mathrm{BP}(\mathcal{NP}^0_\mathbb{R}) = \mathrm{BP}(\mathcal{NP}_\mathbb{R})$ [Bürg00, THEOREM 4.1].

c) It holds $\mathcal{NP} \subseteq \mathrm{BP}(\mathcal{NP}_\mathbb{R}) \subseteq \mathsf{PSPACE}$*; cf. e.g.* [Grig88,HRS90,Cann88,MaTo97].

d) Subject to the Generalized Riemann Hypothesis, it holds $\mathrm{BP}(\mathcal{NP}_\mathbb{C}) \subseteq \mathrm{co}\mathcal{RP}^{\mathcal{NP}}$*; cf.* [Koir96].

e) $\mathrm{BP}(\mathcal{NP}_\mathbb{Z})$ *coincides with the class of all binary languages recursively enumerable by a Turing machine; cf.* [Mati70].

f) The decidability of $\mathrm{FEAS}_\mathbb{Q}$ *is a long-standing open question of extending Hilbert's tenth Problem from integers to rationals; cf. e.g.* [Poon09].

g) $\mathrm{BP}(\mathcal{P}^0_\mathbb{R})$ *belongs to the Turing counting hierarchy* [ABKM09].

It remains an open challenge to tighten the relations in Items c)+d)+g).

In particular the class $\mathrm{BP}(\mathcal{NP}_\mathbb{R})$ has turned out to be of interest of its own with several further complete problems [Shor91,CuRo92,Zhan93,Koir99,Rich99,Scha10]. Higher BSS complexity classes arise naturally for problems in algebraic geometry [BüCu06].

We also mention *analytic machines* as variants of BSS machines which may approximate their output with/without error bounds [ChHo99].

1.4 Overview: Related Work, Objectives, Results, and Methods

Recently, there has been new interest in the classical topic of quantum logic, the system of closed subspaces of a Hilbert space and axiomatic investigations of it; see e.g. [Maye07,Megi09]. Finite-dimensional Hilbert spaces are ubiquitous in quantum computation, which motivated HAGGE et al. [DHMW05,Hagg07] to take up the programme of BIRKHOFF and VON NEUMANN [BiNe36] to study the associated subspace ortholattices as particularly tractable objects of quantum logic. They pointed out that TARSKI's famous result applies to obtain decidability of the first order theory of any single such ortholattice. This has been used in [Her10a] to prove decidability of the equational theory of the class comprising all these ortholattices. This theory was also shown to coincide with the equational theory of all projection ortholattices of finite von Neumann algebras.

Our subject, here, is the counterpart of validity, namely satisfiability of equations in a fixed $\mathrm{L}(\mathcal{H})$. As in the Boolean case, satisfiability of a system can be compiled into the (strong) satisfiability of a single equation $t(\bar{x}) = \mathbf{1}$. But, this is no longer the complement of validity of an identity $\forall \bar{x}\ t(\bar{x}) = \mathbf{0}$; this complement $\exists \bar{x}\ t(\bar{x}) \neq \mathbf{0}$ will be called weak satisfiability.

Although computational complexity has become a standard topic of investigation in logic since [Cook71]—cmp. e.g. [BGG01,Marx07]—it seems to have passed on quantum logic. We have taken upon this direction of research in [HeZi11] and arrived at well-known complexity classes $\mathcal{NP}$, $\mathcal{NP}_{\mathbb{R}}$, and $\mathrm{BP}(\mathcal{NP}_{\mathbb{R}})$: For these, the satisfiability problems for real *and* complex quantum propositional terms of appropriate dimensions turn out to be complete. The present work recalls and presents the full proofs in a self-contained way. One-dimensional quantum logic coinciding with the classical Boolean one, Section 2.3 considers the satisfiability problem in 2D, Section 3 the case of dimensions three and higher but fixed, 6 the question of satisfiability in *some* finite dimension, and 7 closes with some aspects of the infinite-dimensional cases.

r:MainResults

Results 1.13 () *a) In fixed dimension, weak and strong satisfiability (in general differ but) are polynomial-time equivalent (Theorem 5.1)*

b) Satisfiability in 2D quantum logic is as hard as its classical Boolean (i.e. 1D) variant: $\mathcal{NP}$-complete, independent of the underlying field $\mathbb{F}$ (Theorem 2.14)

c) Whereas starting with dimension three, (strong) satisfiability over both real and complex *quantum logic is complete for* real *nondeterministic polynomial-time BSS machines (Theorem 3.18)*

d) and remains so even when restricting to terms of the form $\bigwedge\bigvee\bigwedge$ but becomes polynomial-time decidable for $\bigwedge\bigvee$-terms (Theorem 5.5a+e).

e) Another syntactic variant complete for complex *nondeterministic polynomial-time BSS machines is presented in Theorem 5.5c).*

f) Satisfiability over rational 3D quantum logic is equivalent to Hilbert's tenth problem over $\mathbb{Q}$ (Corollary 3.1)

g) and validity over 3D rational quantum logic of a Σ^0_3-formula is undecidable (Corollary 5.2). More generally, quantified quantum logics correspond to the Boolean and the BSS polynomial hierarchy (Theorem 5.3).

h) Weak satisfiability over some *finite real or complex dimension (i.e. asking for both a d and a d-dimensional assignment) is decidable by real nondeterministic polynomial-time BSS machines but not known hard (Theorem 6.2).*

j) Strong satisfiability over some finite dimension is hard for polynomial-time BSS nondeterminism but not known decidable yet polynomial-time equivalent to the feasibility of non*commutative polynomial equations (Theorem 6.14 and Proposition 6.13).*

We find that satisfiability in quantum logic may be even more natural a generalization of the classical Boolean satisfiability problem than the feasibility of a system of ring equations from Fact 1.12a). Moreover these results together provide a unified view on both Turing and real BSS complexity theory. They exhibit some resemblance to those concerning realizability questions for chirotopes [BVSWZ99, §8] and to descriptive complexity theory where complexity classes are captured in appropriate logics. Machine-independent characterizations of some BSS complexity classes have been obtained in [GrMe96,BCNM06].

We also emphasize the broad range of aspects of logic in computer science naturally joining in this contribution: quantum logic, lattice theory, projective geometry, universal algebra, and computational complexity, particularly in the Blum-Shub-Smale Model.

Our methods heavily draw from JOHN VON NEUMANN's legacy. We review, tailor, combine, and add new quantitative asymptotical and computational perspectives to, a variety of techniques known in quantum logic in order to yield the polynomial-time reductions underlying the aforementioned results. For reasons of self-containedness, proofs have been included also of the more technical tools and generally boil down their claims to facts in, e.g., basic linear algebra.

2 Modular Quantum Logic

Quantum Logic studies a wide spectrum of structures, from the concrete $\mathrm{L}(\mathbb{C}^d)$ to abstract orthomodular lattices and more general orthomodular structures (cmp. [EGL07]) — in great variety of methods: from purely axiomatic to analysis of particular structures, with representation results in between. The intermediate concept of *Hilbert lattice* comprises all ortholattices of closed subspaces of generalized Hilbert spaces — the classical case being characterized by SOLÉR [Pres07]. In finite dimension d, Hilbert lattices are just modular ortholattices as introduced by BIRKHOFF and VON NEUMANN who showed that the irreducible ones correspond to d-dimensional inner product spaces $\mathcal{H}$ over division $*$-rings $\mathbb{F}$ (requiring $d \geq 4$ or the Arguesian Law). We deal with the special case where $\mathbb{F}$ is a $*$-subfield of $\mathbb{C}$ and refer to Hilbert lattices $\mathrm{L}(\mathcal{H})$ only in this context. Some basic properties are quite well captured by the abstract concept of a finite dimensional modular ortholattice. We first recall some basic facts about modular lattices and dimension — and give the easy proofs which also shed some new light on proofs in ordinary linear algebra.

f:mod

Fact 2.1 *Let L be a modular lattice with $\mathbf{0}$ and $\mathbf{1}$. In Items c)-e) suppose that L contains a finite maximal* chain*, that is a linearly ordered subset.*

a) For $u, v \in L$, the interval *$[u, v] := \{x : u \leq x \leq v\}$ is a sublattice and, in particular, modular.*
b) Between the intervals $[a \wedge b,\, a]$ and $[b,\, a \vee b]$ one has mutually inverse lattice isomorphisms $x \mapsto b \vee x$ and $y \mapsto a \wedge y$ (this generalizes the canonical isomorphism $A/(A \cap B) \cong (A + B)/B$ for linear subspaces A, B of $\mathcal{H}$).
c) All maximal chains C in L have the same cardinality; $\dim(L) := |C| - 1$ is the dimension *or* height *of L. In particular, $\dim(u) := \dim[\mathbf{0}, u]$ is well defined for any $u \in L$. We call u an* atom *if $\dim(u) = 1$.*
d) For $U \in \mathrm{L}(\mathcal{H})$, $\dim(U)$ coincides with the vector space dimension.
e) One has the dimension formulas

$$\dim(v) = \dim(u) + \dim[u, v] \text{ for } u \leq v \quad \text{and} \quad \dim(a) + dim(b) = \dim(a \wedge b) + \dim(a \vee b).$$

Proof. a) is obvious. b) That the two maps compose to identity, both ways, is a simple application of modularity in each case. Both maps being order preserving, it follows that they are lattice isomorphisms.

c) We use order induction, considering all modular lattices L admitting an $(n+1)$-element maximal chain C. If $n = 1$ then $L = \{\mathbf{0}, \mathbf{1}\}$. Assume $n > 1$. Consider any $a \in C \setminus \{\mathbf{0}, \mathbf{1}\}$, any maximal chain C', and $b \in C' \setminus \{\mathbf{0}, \mathbf{1}\}$. Then the inductive hypothesis applies to any of the intervals $[\mathbf{0},\, a \wedge b]$, $[a \wedge b,\, a]$, $[a,\, a \vee b]$, and $[a \vee b,\, \mathbf{1}]$. Due to the isomorphisms, we have also $\dim[a \wedge b,\, b] = \dim[a,\, a \vee b]$ and $\dim[b,\, a \vee b] = \dim[a \wedge b,\, a]$. This implies $|C'| = |C|$.

d) Given a basis $\vec{v}_1, \ldots, \vec{v}_m$ of U one obtains the $(m+1)$-element maximal chain $U_0 = \{\vec{0}\}$, $U_i = U_{i-1} + \mathbb{F} \cdot \vec{v}_i \ \ (i = 1, \ldots, m)$.

e) The first dimension formula follows since the union of two maximal chains, one in $[\mathbf{0}, u]$, the other in $[u, v]$ is a maximal chain in $[\mathbf{0}, v]$. This, and the earlier argument, also prove the second dimension formula if one chooses for given a, b maximal chains C, C' in L such that $a \wedge b,\ a,\ a \vee b \in C$ and $a \wedge b,\ b,\ a \vee b \in C'$.

Let us call the one-element lattice *trivial*. A homomorphism $\varphi : L \to L'$ between ortholattices is a map having $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ and $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and $\varphi(\neg x) = \neg \varphi(x)$.

f:Foulis

Fact 2.2 *Abbreviate* $C(x, y) := (x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \vee (\neg x \wedge \neg y)$, *the so-called commutator. For* $(L, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ *an ortholattice,* $a, b \in L$ *are said to commute iff* $C(a, b) = \mathbf{1}$ *holds. Since* $C(a, b) = C(b, a)$*, this means that* b, a *commute, too. Now, let* L *be modular.*

a) Principle of Duality: If a statement is true for all modular ortholattices, then so is its dual which arises by interchanging $\leq$ *with* $\geq$, $\wedge$ *with* $\vee$, *and* $\mathbf{0}$ *with* $\mathbf{1}$.

b) Let L *be an MOL and* $u \in L$. *Then* $L_u = [\mathbf{0}, u] \cup [\neg u, 1]$ *is a sub-ortholattice and* $([\mathbf{0}, u], \wedge, \neg_u, \vee, \mathbf{0}, u)$ *is an MOL under the* relative *orthocomplement* $\neg_u x = u \wedge \neg x$ *and a homomorphic image of* L_u *under the map* $\phi(x) = u \wedge x$. *For* $U \in \mathrm{L}(\mathcal{H})$, $[\mathbf{0}, U] \subseteq \mathrm{L}(\mathcal{H})$ *is (isomorphic to) the ortholattice* $\mathrm{L}(U)$ *given by the scalar product induced on* U.

c) For all $a, c \in L$: $C(a, b) = \mathbf{1} \ \Leftrightarrow\ a = (a \wedge b) \vee (a \wedge \neg b) \ \Leftrightarrow\ a = (a \vee b) \wedge (a \vee \neg b) \ \Leftrightarrow\ a \wedge (\neg a \vee b) = a \wedge b \ \Leftrightarrow\ \neg a \wedge (a \vee b) = \neg a \wedge b$. *In particular, comparable* a, b *commute.*

d) If c *commutes with all* $x \in L$ *(i.e. if* c *is central) then one has the isomorphism*

$$x \mapsto (x \wedge c,\ x \wedge \neg c), \quad L \to [\mathbf{0}, c] \times [\mathbf{0}, \neg c]$$

with inverse $(y, z) \mapsto y \vee z$. *In particular,* $L_c \cong [\mathbf{0}, c] \times [\mathbf{0}, \neg c]$ *for any* $c \in L$.

e) If at least one of $a, b, c \in L$ *commutes with the other two, then the distributive laws* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *and* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ *do apply.*

f) If $a_1, \ldots, a_n \in L$ *pairwise commute, they generate a Boolean algebra isomorphic to* $\{\mathbf{0}, \mathbf{1}\}^k$ *for some* $k \leq 2^n$.

g) The sublattice generated by $a, b \in L$ *is isomorphic to* $\{\mathbf{0}, \mathbf{1}\}^k$ *or to* $\{\mathbf{0}, \mathbf{1}\}^k \times \mathcal{MO}_2$ *for some* $k \leq 4$ *where* $\mathcal{MO}_n$ *is depicted in Figure 1b).*

Beran

Remark 2.3 *The first equivalence in c) and e), f), g) also hold for orthomodular lattices — see* [Bera85, §3], [Bera85, Theorem II.3.10], [Bera85, top of p.14 and Theorem II.4.5], *and* [Kalm83, Theorem §3.9].

Although the above claims are folklore in the appropriate community (and thus do not qualify for a lemma), we chose to include some proofs in the simpler case of modular L. These are to serve as illustration of reasoning in this non-Boolean logic, namely employing the modular in place of the distributive law.

Proof. a) Given an ortholattice, the mentioned exchange yields an ortholattice due to the self-dual character of the axioms. The same applies to modularity.

b) $x \leq y$ implies $\neg y \leq \neg x$ whence $\neg_u y \leq \neg_u x$. Also, $\neg_u(\neg_u x) = u \wedge \neg(u \wedge \neg x) = u \wedge (\neg u \vee x) = (u \wedge \neg u) \vee x = x$ by modularity. Thus, $x \mapsto \neg_u x$ is an involution on $[\mathbf{0}, u]$. Finally, $x \vee \neg_u x = u \wedge (x \vee \neg x) = u$ by modularity, i.e. $[\mathbf{0}, u]$ is an ortholattice.

c) We prove the second equivalence, first. Indeed, in one direction, substituting $(a \wedge b) \vee (a \wedge \neg b)$ for a in the third equation one has $a \leq (a \vee b) \wedge (a \vee \neg b) = \big((a \wedge \neg b) \vee b\big) \wedge \big((a \wedge b) \vee \neg b\big) =$

$(a \wedge b) \vee \big(((a \wedge \neg b) \vee b\big) \wedge \neg b\big) = (a \wedge b) \vee (a \wedge \neg b) \vee (b \wedge \neg b) \leq a$ by modularity. The converse direction follows by duality. Now, to prove the first equivalence assume $C(a,b) = \mathbf{1}$. Then, by modularity, $a = a \wedge C(a,b) = (a \wedge b) \vee (a \wedge \neg b) \vee \big(a \wedge \big((\neg a \wedge b) \vee (\neg a \wedge \neg b)\big)\big) = (a \wedge b) \vee (a \wedge \neg b)$. Conversely, if $a = (a \wedge b) \vee (a \wedge \neg b)$ then by de Morgan $\neg a = (\neg a \vee \neg b) \wedge (\neg a \vee b)$ whence, by the second equivalence, $\neg a = (\neg a \wedge b) \vee (\neg a \wedge \neg b)$ and $C(a,b) = a \vee \neg a = \mathbf{1}$. If b, a commute then $b = (b \wedge a)\ vee (b \wedge \neg a)$ whence $a \wedge (b \vee \neg a) = a \wedge \big((b \wedge a) \vee \neg a)\big) = a \wedge b$ by and modularity. Conversely, if $a \wedge (\neg a \vee b) = a \wedge b$ then by modularity $b \leq (a \vee b) \wedge (\neg a \vee b) = b \vee \big(a \wedge (\neg a \vee b)\big) = b \vee (a \wedge b) = b$. The last equivalence is due to the fact that a, b commute if and only if $\neg a, b$ commute. Finally if, say, $a \leq b$ then $\neg a \geq \neg b$ and $C(a,b) \geq \neg a \vee b \geq \mathbf{1}$.

d) Next, we show that if c commutes with x, y then also with $x \vee y$, $x \wedge y$ and $\neg x$. Indeed, $x \vee y \geq \big((x \vee y) \wedge c\big) \vee \big((x \vee y) \wedge \neg c\big) \geq (x \wedge c) \vee (y \wedge c) \vee (x \wedge \neg c) \vee (y \wedge \neg c) \geq x \vee y$ and $(\neg x \wedge c) \vee (\neg x \wedge \neg c) = \neg(x \vee \neg c) \vee \neg(x \vee c) = \neg\big((x \vee \neg c) \wedge (x \vee c)\big) = \neg x$. The claim about $x \wedge y$ follows by duality. These calculations show that the map $x \mapsto (x \wedge c, x \wedge \neg c)$ is an homomorphism — with inverse $(u,v) \mapsto u \vee v$. Indeed, $\big((u \vee v) \wedge c\big) \vee \big((u \vee v) \wedge \neg c\big) = \big((u \vee (v \wedge c)\big) \vee \big((v \vee (u \wedge \neg c)\big) = u \vee v$ by modularity.

e) Assume that c commutes with a and with b. Then $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$ due to that fact that $x \mapsto x \wedge c$ is a lattice homomorphism — the isomorphism in g) followed by projection onto the first direct factor. It follows by c) and modularity $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c) = (a \vee c) \wedge (a \vee \neg c) \wedge \big((b \wedge \neg c) \vee c\big) = \big(b \wedge \neg c \wedge (a \vee c)\big) \vee \big(c \wedge (a \vee \neg c)\big) = (b \wedge \neg c \wedge a) \vee (c \wedge a) \leq (a \wedge b) \vee (a \wedge c)$. The remaining distributive relations follow by symmetry and duality.

f) The calculations in d) also show that if c commutes with all $x \in X$ then it is central in the sub-ortholattice generated by $X \cup \{c\}$ (since c commutes with c). Thus, under the hypothesis of d), any two elements of the sub-ortholattice B generated by the a_i commute with each other whence B is a Boolean algebra by e).

g) We show that a commutes with $C(a,b)$ Indeed, $a \wedge (\neg a \vee (C(a,b)) = a \wedge \big(\neg a \vee (a \wedge b) \vee (a \wedge \neg b)\big) = (a \wedge \neg a) \vee (a \wedge b) \vee (a \wedge \neg b) = (a \wedge b) \vee (a \wedge \neg b) = a \wedge C(a,b)$ by modularity. By symmetry, b commutes with $C(a,b)$, too. Due to d) we have L isomorphic to the direct product $L_1 \times L_2$ where L_i has generators $a_i = \pi_i(a)$, $b_i = \pi_i(b)$ under the homomorphism $\pi_1(x) = x \wedge c$ resp. $\pi_2(x) = x \wedge \neg c$ where $c := C(a,b)$ and π_i denotes the projection onto the i-th component. It follows $C(a_1,b_1) = C(\pi_1(a), \pi_1(b)) = \pi_1(c) = \mathbf{1}_{L_1}$ and $C(a_2,b_2) = C(\pi_2(a), \pi_2(b)) = \pi_2(c) = \mathbf{0}_{L_2}$. By f), L_1 is a Boolean algebra with 2 generators. In L_2 we have $a_2 \wedge b_2 = a_2 \wedge \neg b_2 = \neg a_2 \wedge b_2 = \neg a_2 \wedge \neg b_2 = \mathbf{0}$, and, by de Morgan, all joins $= \mathbf{1}$ which implies that L_2 is either trivial or a copy of $\mathcal{MO}_2$.

In carrying out calculations in modular ortholattices we often write $\neg x = x^\perp$, $x \wedge y = x \cap y$, and $x \vee y = x + y$ (and save brackets according to priority in this order) to improve readability and to appeal at the geometric intuition.

mol

Digression 2.4 *Modular ortholattices of finite height are isomorphic to direct products of irreducibles cf.* [Birk67, Ch. IV]. *The irreducibles in turn can be understood in terms of irreducible projective spaces with anisotropic orthogonality. Irreducibles of height $d \geq 3$ are necessarily infinite [Ball48], for $d \geq 4$ they correspond to inner product spaces over division $*$-rings [BiNe36]. On the other hand, there are examples where the division ring has unsolvable word problem which leads to a finitely presented MOL having unsolvable word problem [Rodd89] and the undecidability of the equational theory of MOLs of height $\leq d$ (where $d \geq 14$ is fixed) [Herr05].*

2.1 Truth, Equivalence, and Satisfiability

The classical Boolean satisfiability problem extends straightforwardly to quantum propositional terms—although truth ("$=\mathbf{1}$") now has to be distinguished from non-falsity ("$\neq \mathbf{0}$"); and we may or may not permit constants (from a fixed MOL L).

d:QL

Definition 2.5 *a) A (ortholattice or quantum logic) term is a syntactically correct expression over certain variables $x_1, \ldots, x_n$ with operations $\wedge$, $\neg$, and $\vee$ and constants $\mathbf{0}$ and $\mathbf{1}$ (the latter rather for formal reasons). We may write $t(\bar{x})$ to emphasize the dependence on $(x_1, \ldots, x_n) =: \bar{x}$.*
The syntactic length *of t is denoted by $|t|$, defined recursively as $|x| = 1$, $|\neg t| = |t| + 1$, and $|s \vee t| = |s| + |t| + 1 = |s \wedge t|$.*

b) For an ortholattice $(L, \wedge, \neg, \vee, \mathbf{0}, \mathbf{1})$ and $\bar{a} = (a_1, \ldots, a_n) \in L^n$, let $t_L(a_1, \ldots, a_n) = t_L(\bar{a}) \in L$ denote the value of t in L when substituting a_i for x_i.

c) A term with constants from L is one where some variables already have been fixed to (i.e. substituted for) elements of L.

d) An n-variate term t is strongly satisfiable in L if there is $\bar{a} \in L^n$ such that $t_L(\bar{a}) = \mathbf{1}$. It is weakly satisfiable in L if there is $\bar{a} \in L^n$ such that $t_L(\bar{a}) \neq \mathbf{0}$.

e) Two n-variate terms s and t are equivalent over L if $s_L(\bar{a}) = t_L(\bar{a})$ for every $\bar{a} \in L^n$.

f) Strong and weak satisfiability over L are the respective decision problems

$$\mathrm{SAT}_L := \big\{\langle t(x_1, \ldots, x_n)\rangle \mid n \in \mathbb{N}, t \textit{ term}, \exists \bar{a} \in L^n : t_L(\bar{a}) = \mathbf{1}\big\} \subseteq \{0,1\}^* \qquad \textit{and}$$
$$\mathrm{sat}_L := \big\{\langle t(x_1, \ldots, x_n)\rangle \;\big|\; n \in \mathbb{N}, t \textit{ term}, \exists \bar{a} \in L^n : t_L(\bar{a}) \neq \mathbf{0}\big\} \subseteq \{0,1\}^*$$

g) More generally, for a class $\mathcal{C}$ of MOLs, consider the question of whether a given term t is strongly/weakly satisfiable over some *$L \in \mathcal{C}$: $\mathrm{SAT}_{\mathcal{C}} := \bigcup_{L \in \mathcal{C}} \mathrm{SAT}_L$, $\mathrm{sat}_{\mathcal{C}} := \bigcup_{L \in \mathcal{C}} \mathrm{sat}_L$.*

h) Returning to single MOLs L, strong satisfiability with constants from *$C \subseteq L$ is*

$$\mathrm{SAT}_{C,L} := \Big\{\big\langle t(x_1, \ldots, x_n, y_1, \ldots, y_m), c_1, \ldots, c_m\big\rangle \;\Big|\; n, m \in \mathbb{N}, t \textit{ term}, c_1, \ldots, c_m \in C, \exists \bar{a} \in L^n : t_L(\bar{a}, \bar{c}) = \mathbf{1}\Big\}$$

and similarly for weak satisfiability with constants. (The encoding $\langle \cdot \rangle$ of terms involving $c_i \in C$ will be specified later...)

Note that weak satisfiability of t means *in*validity of the identify "$t = \mathbf{0}$" in the model-theoretic sense. Moreover, $\mathrm{SAT}_{\{\mathbf{0},\mathbf{1}\}} = \mathrm{sat}_{\{\mathbf{0},\mathbf{1}\}}$ coincides with the classical Boolean satisfiability problem. We also record

o:Product

Observation 2.6 *a) For $U, V \in \mathrm{L}(\mathcal{H})$ with $U \perp V$ it holds that $\omega : \mathrm{L}(U) \times \mathrm{L}(V) \ni (A, B) \mapsto A \vee B \in \mathrm{L}(U \vee V)$ is an embedding of the product $\mathrm{L}(U) \times \mathrm{L}(V)$ of two ortholattices into $\mathrm{L}(U \vee V)$.*

b) For the product $L \times L'$ of ortholattices L and L', it holds $\mathrm{SAT}_{L \times L'} = \mathrm{SAT}_L \cap \mathrm{SAT}_{L'}$ and $\mathrm{sat}_{L \times L'} = \mathrm{sat}_L \cup \mathrm{sat}_{L'}$.

For a) we may assume that $U \vee V = \mathcal{H}$. Then Fact 2.2d) applies. Concerning b) note that, by the very definition of the product, $\mathbf{1}_{L \times L'} = (\mathbf{1}_L, \mathbf{1}_{L'})$ and $\mathbf{0}_{L \times L'} = (\mathbf{0}_L, \mathbf{0}_{L'})$ and $t_{L \times L'}\big((x_1, y_1), \cdots, (x_n, y_n)\big) = \big(t_L(\bar{x}), t_{L'}(\bar{y})\big)$.

r:Boolean

Remark 2.7 *The connective "$\vee$" satisfies the* disjunction property *for weak truth: $x \vee y \neq \mathbf{0}$ holds iff $x \neq \mathbf{0}$ or $y \neq \mathbf{0}$ holds. In dimensions > 1, however, strong truth generally fails this property: $x \vee y = \mathbf{1}$ may well hold with neither $x = \mathbf{1}$ nor $y = \mathbf{1}$. Similarly for the dual connective "$\wedge$". Furthermore, Boolean negation has to be distinguished from complement: $x \neq \mathbf{0} \overset{\Leftarrow}{\not\Rightarrow} \neg x = \mathbf{0}$.*

Whenever this distinction needs emphasis, we denote the traditional propositional connectives as in the `C` programming language: `||` for "*or*", `&&` for "*and*", `!` for "*not*". Lemma 4.20 below explains how to express them within *quantified* quantum logic.

x:QL

Example 2.8 *Fix some MOL L.*

a) $\neg x \vee \neg y$ and $\neg(x \wedge y)$ are terms over variables x, y. According to the de Morgan laws, they are equivalent.
b) Suppose $a \leq b$ and $a \vee \neg b = \mathbf{1}$ holds for some $a, b \in L$. Then $a = b$.
c) Recall the commutator $C(x, y)$ and, for $a, b \in L$, consider the sub-ortholattices $\{\mathbf{0}, \mathbf{1}, a, \neg a\} =: L(a)$ and $L(b)$ they span. Whenever $a \in L(b)$ or $b \in L(a)$, it follows $C_L(a, b) = \mathbf{1}$. In particular, $C(x, y)$ is equivalent to $\mathbf{1}$ on $\mathrm{L}(\mathbb{F}^1)$. Over $\mathrm{L}(\mathbb{F}^2)$, however, $C(\mathbb{F}\binom{1}{r}, \mathbb{F}\binom{1}{s})$ evaluates to $\mathbf{0}$ whenever $r \notin \{s, -1/\bar{s}\}$.
d) Let $t(x, y) := C(x, y) \vee x \vee y$ and $s(x, y, z) := t(x, y) \wedge t(x, z) \wedge t(y, z)$. Then s is equivalent to 1 on $\mathrm{L}(\mathbb{F}^2)$. Over $\mathbb{F}^3$, however, $s(\mathbb{F}\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbb{F}\begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbb{F}\begin{pmatrix}1\\1\\1\end{pmatrix}) = \mathbf{0}$.
e) More generally, $\neg C(x, y) = (x \vee y) \wedge (\neg x \vee y) \wedge (x \vee \neg y) \wedge (\neg x \vee \neg y)$ is strongly satisfiable over each $\mathrm{L}(\mathbb{F}^{2d})$, $d \in \mathbb{N}$; but not over $\mathrm{L}(\mathbb{F}^{2d-1})$.
f) Over any MOL resp. complemented modular lattice it holds:

$$\begin{array}{rl} y \vee \neg(x \vee y) = \mathbf{1} \ \Leftrightarrow x \leq y \Leftrightarrow \exists z: & y \vee z = \mathbf{1} \ \ \texttt{\&\&} \ (x \vee y) \wedge z = \mathbf{0} \ ; \\ (x \wedge y) \vee \neg(x \vee y) = \mathbf{1} \Leftrightarrow x = y \Leftrightarrow \exists z: & (x \wedge y) \vee z = \mathbf{1} \ \ \texttt{\&\&} \ (x \vee y) \wedge z = \mathbf{0} \ . \end{array}$$

Proof. Claim b) is immediate by the orthomodular law. Claim c) is straightforward to verify. Claim d) has been generalized in [Hagg07] to (the complement of) terms equivalent to $\mathbf{1}$ over $\mathbb{F}^{d-1}$, but not over $\mathbb{F}^k$ for every $k \geq d$; cmp. Corollary 4.1 below. For Claim e), consider $a := \mathbb{F}^d \times \{\mathbf{0}\}^d$ and $b := \{(\vec{x}, \vec{x}) : \vec{x} \in \mathbb{F}^d\}$ in the even-dimensional case. Conversely, $A \vee \neg B = \mathbf{1}$ over $\mathbb{F}^D$ implies $D = \dim(A \vee \neg B) \leq \dim(A) + \dim(\neg B) = \dim(A) + D - \dim(B)$; hence $\dim(B) \leq \dim(A)$. And, by symmetry of $C(x, y)$, also $\dim(B) \leq \dim(\neg A)$, $\dim(A) \leq \dim(B)$, and $\dim(A) \leq \dim(\neg B)$: Hence $D = \dim(A) + \dim(\neg A)$ must be even. In Claim f), observe that $y \leq x \vee y$ and that $x \leq y$ if and only if $y = x \vee y$. Thus, the first equivalence follows from Claim b) (actually, it is valid in any orthomodular lattice [Kalm83, §15 THEOREM 3]) and the second by modularity. The same arguing applies to the second line observing that $x \wedge y \leq x \vee y$ and that equality holds if and only if $x = y$. □

Example 2.8f) allows to reduce non-equivalence to weak satisfiability. More generally, we record

f:clear

Fact 2.9 *Let $s_1(\bar{x}), t_1(\bar{x}), \ldots, s_m(\bar{x}), t_m(\bar{x})$ denote n-variate terms and L a MOL.*

a) An assignment $\bar{a}$ in L simultaneously satisfies all equations $s_i(\bar{x}) = t_i(\bar{x})$, $1 \leq i \leq m$, iff it strongly satisfies the single term

$$\bigwedge_{i=1}^{m} \big([s_i(\bar{x}) \wedge t_i(\bar{x})] \vee \neg[s_i(\bar{x}) \vee t_i(\bar{x})] \big) \ .$$

b) *The system* $\{s_i(\bar{x}) = t_i(\bar{x}) \mid i = 1, \ldots m\}$ *of equations is simultaneously satisfiable in* L *iff the following two equations in* $n + m$ *variables* $(\bar{x}, \bar{y})$ *are:*

$$\mathbf{1} = \bigwedge\nolimits_{i=1}^{m} \big([s_i(\bar{x}) \wedge t_i(\bar{x})] \vee y_i\big) \quad \&\& \quad \mathbf{0} = \bigvee\nolimits_{i=1}^{m} \big([s_i(\bar{x}) \vee t_i(\bar{x})] \wedge y_i\big) \ .$$

2.2 $\mathcal{NP}$–Hardness of Strong Satisfiability

p:NPhard

Proposition 2.10 SAT_L *is* $\mathcal{NP}$*–hard for any nontrivial modular ortholattice* L*, uniformly in* L*.*

In particular, for any non-empty class $\mathcal{C}$ of nontrivial MOLs, $\mathrm{SAT}_{\mathcal{C}}$ is $\mathcal{NP}$–hard. Theorem 2.14 below shall extend this to weak satisfiability.

Proof. Convert a given term $t(x_1, \ldots, x_n)$ to the term $s(\bar{x}) := t(\bar{x}) \wedge \bigwedge_{1 \leq i < j \leq n} C(x_i, x_j)$ with the commutator from Fact 2.2: this is clearly computable in polynomial time. Moreover a satisfying Boolean assignment $\bar{b} \in \{\mathbf{0}, \mathbf{1}\}^n$ of t is also one of s in any non-trivial L since $C(\mathbf{0}, \mathbf{0}) = C(\mathbf{0}, \mathbf{1}) = C(\mathbf{1}, \mathbf{0}) = C(\mathbf{1}, \mathbf{1}) = \mathbf{1}$. Conversely a satisfying assignment of g in L consists of pairwise commuting elements $b_1, \ldots, b_n \in L$; hence 'lives' in a Boolean algebra (isomorphic to) $\{\mathbf{0}, \mathbf{1}\}^k$ according to Fact 2.2f): A satisfying Boolean assignment of t is thus obtained by projecting the b_i onto their respective first components.

x:2D

Example 2.11 *Recall the modular ortholattices* $\mathcal{MO}_m$ *from Figure 1b).*

a) *Strengthening Example 2.8f),* $\bigwedge_{1 \leq i < j \leq 2n} (x_i \vee x_j) \wedge (\neg x_i \vee \neg x_j)$ *is weakly/strongly satisfiable in* $\mathcal{MO}_m$ *if and only if* $m \geq n$*.*
b) *For* $a \in \mathcal{MO}_m$ *with* $m \geq 2$*, it holds* $a = \mathbf{1} \Leftrightarrow \exists y, z : \neg C(y, z) \wedge C(a, y) \wedge C(a, z) \wedge a \neq \mathbf{0}$*.*

By a), both concepts of satisfiability thus depend on L at least as far as finite two-dimensional L are concerned. The complexity of SAT_L, however, will turn out to not depend on L as long as it has dimension two.

2.3 Two-Dimensional Satisfiability is in $\mathcal{NP}$

s:2D

Lemma 2.12 *Let* L, L' *be modular ortholattices of dimension* 2*.* l:2D

a) *If* $A \subseteq L$ *then* $A \cup \{\neg a \mid a \in A\} \cup \{\mathbf{0}, \mathbf{1}\}$ *is a subortholattice of* L*. In particular,* $|L| \leq 2n+2$ *if* L *has an* n*-element generating set. Moreover,* L *is isomorphic to* L' *if and only if* $|L| = |L'|$*. And* L *maps isomorphically onto a sub-ortholattice of* L' *if and only if* $|L| \leq |L'|$*.*
b) *A term* $t(x_1, \ldots, x_n)$ *is weakly/strongly satisfiable over* L *iff it is so over* $\mathcal{MO}_m$ *for* $m := \min\{n, \mathrm{Card}(L)/2 - 1\}$ *with the convention that* $\infty/2 - 1 = \infty$*. In particular for infinite* L *we have* $\mathrm{SAT}_L = \mathrm{SAT}_{\mathcal{MO}_\omega}$ *and* $\mathrm{sat}_L = \mathrm{sat}_{\mathcal{MO}_\omega}$*.*

Proof. Indeed, since all maximal chains in a modular ortholattice of dimension 2 are 3-element (Fact 2.1), they are of the form $\mathbf{0} < a < \mathbf{1}$; and $a \vee \neg a = \mathbf{1}$ and $a \wedge \neg a = \mathbf{0}$ require $\neg a \neq a$: hence the set A of atoms of L is a disjoint union $A_0 \cup A_1$ where $a \in A_0$ if and only if $\neg a \in A_1$. This establishes a).

Concerning b), observe that any homomorphism $\varphi : L \to L'$ commutes with the evaluation of a term: $t_{L'}\big(\varphi(a_1), \ldots, \varphi(a_n)\big) = \varphi\big(t_L(a_1, \ldots, a_n)\big)$ for every $a_1, \ldots, a_n \in L$. Now an assignment $a_1, \ldots, a_n \in L$ 'lives' in a sublattice L' of L isomorphic to $\mathcal{MO}_m$ according to a). And, conversely, $\mathcal{MO}_m$ maps isomorphically onto a sublattice L' of L in case $m \leq \mathrm{Card}(L)/2 - 1$.

Every two-dimensional MOL L is thus isomorphic to $\mathcal{MO}_m$ for some cardinal m; and $\mathrm{L}(\mathbb{F}^2)$ is isomorphic to $\mathcal{MO}_{\mathrm{Card}(\mathbb{F})}$: every one-dimensional subspace of $\mathbb{F}^2$ corresponds to an atom. Hence, checking weak/strong satisfiability according to Definition 2.5e) naively involves an infinite choice of possible arguments (=subspaces of $\mathcal{H}$). However from Lemma 2.12 we conclude the following extension of Example 1.4:

p:2DinNP

Proposition 2.13 *a) The following evaluation problem over $\mathcal{M}_\omega$ is decidable in polynomial time:*

Given a term $t(X_1, \ldots, X_n)$ as well as an assignment $x_1, \ldots, x_n \in \mathcal{MO}_\omega$ and value $y \in \mathcal{MO}_\omega$ (with $\mathbf{0}, \mathbf{1}$ encoded as integers $0, 1$ and atoms $a_m, \neg a_m$ as $2m, 2m+1$, say), is $t_{\mathcal{MO}_\omega}(x_1, \ldots, x_n) = y$?

b) For any nonempty class $\mathcal{C}$ of 2-dimensional MOLs, both $\mathrm{SAT}_\mathcal{C}$ and $\mathrm{sat}_\mathcal{C}$ are in $\mathcal{NP}$.

Proof. a) Disregarding parsing details, t can be evaluated by recursion on its subterms. Concerning the recursion bottom observe that $b \vee c = \mathbf{1}$ holds in $\mathcal{MO}_\omega$ for $\mathbf{0} \neq b \neq c \neq \mathbf{0}$; and the other cases are trivial anyway.

b) Consider a nondeterministic Turing machine which, on input of an n-variate term $t(\bar{x})$, calculates from fixed $\max\{\mathrm{Card}(L) : L \in \mathcal{C}\}$ the m according to Lemma 2.12b), and then guesses and verifies an assignment $\bar{b}$ in $(\mathcal{MO}_m)^n$ as in a). □

2-dimensional quantum satisfiability is thus computationally as hard as 1-dimensional (i.e. Boolean) satisfiability:

t:2DNPc

Theorem 2.14. *For any 2-dimensional modular ortholattice L, both SAT_L and sat_L are $\mathcal{NP}$–complete.*

Proof. In view of Propositions 2.10 and 2.13 it suffices to show $\mathcal{NP}$–hardness of sat_L. For $L = \mathcal{MO}_m$ for $m \geq 2$, this follows from Example 2.11b), observing that strong satisfiability $t(\bar{x}) = \mathbf{1}$ here reduces to the weak satisfiability of $\neg C(y, z) \wedge C\big(t(\bar{x}), y\big) \wedge C\big(t(\bar{x}), z\big) \wedge t(\bar{x})$: clearly in polynomial time. For the remaining two-dimensional modular ortholattice $\mathcal{MO}_1 \cong \{\mathbf{0}, \mathbf{1}\}^2$ on the other hand, the claim follows from $\mathrm{sat}_{L \times L'} = \mathrm{sat}_L \cup \mathrm{sat}_{L'}$.

2.4 Fixed-dimensional Satisfiability over Hilbert Lattices is in BSS–$\mathcal{NP}$

ss:UpperComplexity

It has been observed in [DHMW05, Corollary 7]

f:Tarski

Fact 2.15 *Based on Tarski's real (!) quantifier elimination, $\mathrm{sat}_{\mathrm{L}(\mathbb{C}^d)}$ is decidable, uniformly in d.*

Intuitively, a nondeterministic BSS machine can decide satisfiability of a term t by guessing an assignment $\bar{X}$ in $\mathrm{L}(\mathbb{F}^d)$ and evaluating $t_{\mathrm{L}(\mathbb{F}^d)}(\bar{X})$. However for $\mathbb{F} = \mathbb{C}$, the latter requires separate access to real and imaginary parts—which a $\mathbb{C}$-machine does not have (Definition 1.5).

d:ImagField

Digression 2.16 *Recall that we required all $\mathbb{F}$ to be $*$-subfields of $\mathbb{C}$, i.e. subfields closed under conjugation (since conjugation enters into the concept of scalar product).*

a) A mere subfield of $\mathbb{C}$, as $\mathbb{Q}(\sqrt[3]{2}e^{2\pi/3})$, may fail to be $*$-subfield.

b) Observe that $\mathbb{F} \not\subseteq \mathbb{R}$ is a $*$-subfield of $\mathbb{C}$ if and only if $\mathbb{F} = \mathbb{F}' + i\mathbb{F}'$ with subfield $\mathbb{F}' = \mathbb{F} \cap \mathbb{R}$ of $\mathbb{R}$; equivalently: with $\mathbb{F}' = \mathrm{Re}(\mathbb{F}) := \{x \mid x, y \in \mathbb{R}, x + iy \in \mathbb{F}\}$.

c) Strictly speaking, the concept of an $*$-field defines the imaginary unit i and/or the imaginary part only up to the isomorphism $i \mapsto -i$.

Furthermore note that every $U \in \mathrm{L}(\mathbb{F}^d)$ is of the form $U = \operatorname{range} A$ for (many) $A \in \mathbb{F}^{d\times d}$ where $\operatorname{range} A$ denotes the linear subspace of $\mathbb{F}^d$ spanned by the columns of A.

p:UpperComplexity

Proposition 2.17 *Fix $\mathbb{F}$.*

a) Given $d \in \mathbb{N}$ and matrices $A, B \in \mathbb{F}^{d\times d}$
i) a matrix $C \in \mathbb{F}^{d\times d}$ with $\mathrm{range}(C) = \mathrm{range}(A) \vee \mathrm{range}(B)$ and
ii) a matrix $C' \in \mathbb{F}^{d\times d}$ with $\mathrm{range}(C') = \mathrm{range}(A) \wedge \mathrm{range}(B)$
can be calculated by a constant-free BSS-machine over $\mathbb{F}$ in time $\mathcal{O}(d^3)$. Similarly,
iii) a matrix $C'' \in \mathbb{F}^{d\times d}$ with $\mathrm{range}(C'') = \neg\, \mathrm{range}(A)$
can be calculated by a constant-free BSS-machine $\mathcal{M}$ over $\mathbb{F}$ in time $\mathcal{O}(d^3)$: in case $\mathbb{F} \subseteq \mathbb{R}$; otherwise $\mathcal{M}$ in general needs both real part $\mathrm{Re}(A)$ and imaginary part $\mathrm{Im}(A)$.

b) First suppose $\mathbb{F} \subseteq \mathbb{R}$. Then, given an n-variate term t and matrices $A_1, \ldots, A_n \in \mathbb{F}^{d\times d}$ representing $U_j = \mathrm{range}(A_j) \in \mathrm{L}(\mathbb{F}^d)$, a constant-free BSS-machine $\mathcal{M}$ over $\mathbb{F}$ can calculate $C \in \mathbb{F}^{d\times d}$ with $\mathrm{range}(C) = t_{\mathrm{L}(\mathbb{F}^d)}(U_1, \ldots, U_n)$ in time polynomial in both d and $|t|$.
In the general case $\mathbb{F} \subseteq \mathbb{C}$, given matrices $\mathrm{Re}(A_1), \mathrm{Im}(A_1), \ldots, \mathrm{Re}(A_n), \mathrm{Im}(A_n) \in \mathrm{Re}(\mathbb{F})^{d\times d}$, representing $U_j = \mathrm{range}(A_j) \in \mathrm{L}(\mathbb{F}^d)$, a similar machine over $\mathrm{Re}(\mathbb{F})$ can calculate $\mathrm{Re}(C), \mathrm{Im}(C) \in \mathrm{Re}(\mathbb{F})^{d\times d}$ with $\mathrm{range}(C) = t_{\mathrm{L}(\mathbb{F}^d)}(U_1, \ldots, U_n)$.

c) Both weak and strong satisfiability over $\mathrm{L}(\mathbb{F}^d)$ of a given term t can be decided by a nondeterministic constant-free BSS-machine over $\mathrm{Re}\,\mathbb{F}$ in time polynomial in $|t|$ and in d. In particular, it holds

$$\mathrm{sat}_{\mathrm{L}(\mathbb{R}^d)}, \mathrm{SAT}_{\mathrm{L}(\mathbb{R}^d)}, \mathrm{sat}_{\mathrm{L}(\mathbb{A}^d)}, \mathrm{SAT}_{\mathrm{L}(\mathbb{A}^d)}, \mathrm{sat}_{\mathrm{L}(\mathbb{C}^d)}, \mathrm{SAT}_{\mathrm{L}(\mathbb{C}^d)} \;\in\; \mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}}) \;\subseteq\; \mathsf{PSPACE}\;.$$

d) Concerning satisfiability with constants, *we make Definition 2.5h) more precise:*

$$\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d),\mathrm{L}(\mathbb{F}^d)} \;:=\; \Big\{ \big\langle t(x_1, \ldots, x_n, y_1, \ldots, y_m), C_1, \ldots, C_m \big\rangle \;\Big|\; C_1, \ldots, C_m \in \mathbb{F}^{d\times d},$$
$$\exists A_1, \ldots, A_n \in \mathbb{F}^{d\times d} : t_{\mathrm{L}(\mathbb{F}^d)}(\operatorname{range} A_1, \ldots, \operatorname{range} A_n, \operatorname{range} C_1, \ldots, \operatorname{range} C_m) = \mathbf{1} \Big\}$$

where matrices C_j are encoded as d^2-element sequences over $\mathbb{F}$. Then it holds

$$\mathrm{sat}_{\mathrm{L}(\mathbb{F}^d),\mathrm{L}(\mathbb{F}^d)},\; \mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d),\mathrm{L}(\mathbb{F}^d)},\; \mathrm{sat}_{\mathrm{L}(\mathrm{Re}(\mathbb{F})^d),\mathrm{L}(\mathrm{Re}(\mathbb{F})^d)},\; \mathrm{SAT}_{\mathrm{L}(\mathrm{Re}(\mathbb{F})^d),\mathrm{L}(\mathrm{Re}(\mathbb{F})^d)} \;\in \mathcal{NP}^0_{\mathrm{Re}(\mathbb{F})}.$$

Claims b) and c) can be regarded as natural a generalization of Proposition 2.13a) and b) to fixed, higher dimensions.

Proof. a i) Gaussian Elimination (Example 1.6) works for calculation with columns, as well, and allows to turn a matrix into column echelon form. Recall, that Gaussian Elimination leaves the linear span of the columns invariant. Given A and B apply this to the $(d \times 2d)$–matrix $(A|B)$ and put the first d columns (the remaining ones are zero columns) of the echelon form into C to obtain $\mathrm{range}(A) \vee \mathrm{range}(B) = \mathrm{range}(C)$.

a ii) To compute the meet, we use an algorithm due to Zassenhaus [Zass48]. Form the block $2d \times 2d$–matrix $\binom{A\ B}{A\ O}$ of rank $r = \mathrm{rank}(A) + \mathrm{rank}(B)$ (which can be seen using row transformations). Use Gaussian Elimination on columns (according to the upper half of the matrix) to transform this matrix into $\binom{A'\ O}{A''\ C'}$ with the same block structure and rank $r =$

$\mathrm{rank}(A') + \mathrm{rank}(C')$ since $\mathrm{rank}(A'') = \mathrm{rank}(A) \leq \mathrm{rank}(A')$. Then $\mathrm{range}(C') = \mathrm{range}(A) \wedge \mathrm{range}(B)$. Indeed, as observed in i), the columns of A' span $\mathrm{range}(A) \vee \mathrm{range}(B)$. It follows by the dimension formula that $\mathrm{rank}(C') = \dim(\mathrm{range}(A) \wedge \mathrm{range}(B))$ and it remains to show that the columns $\vec{c}$ of C' are in $\mathrm{range}(A) \wedge \mathrm{range}(B)$. But, by construction, the column $\binom{\vec{0}}{\vec{c}}$ is of the form $\binom{\vec{a}}{\vec{a}} + \binom{\vec{b}}{\vec{0}}$ where the first summand is a linear combination of columns of $\binom{A}{A}$, the second of $\binom{B}{O}$. From $\vec{a} + \vec{b} = \vec{0}$ it follows $\vec{c} = \vec{a} \in \mathrm{range}(A) \wedge \mathrm{range}(B)$.

a iii) It is well known that $\ker(A^*) = \neg(\mathrm{range}\, A)$ where A^* is the adjoint of A, i,.e. transposed and complex conjugate; indeed, $A \perp \vec{v}$ (i.e. $\vec{a} \perp \vec{v}$ for all columns $\vec{a}$ of A) if and only if $A^*\vec{v} = 0$ i.e. $\vec{v} \in \ker(A^*)$. Hence let machine $\mathcal{M}$ calculate A^* (using separate access to both $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$); then apply Gaussian Elimination to obtain a basis of its kernel.
We now show that calculating to given $A \in \mathbb{C}^{2\times 2}$ some $C'' \in \mathbb{C}^{2\times 2}$ with $\mathrm{range}(C'') = \neg\,\mathrm{range}(A)$ entails complex conjugation. To this end consider $A := \begin{pmatrix} 0 & z \\ 0 & 1 \end{pmatrix}$. Then $\neg\,\mathrm{range}(A) = \mathbb{C}\binom{-1}{\bar{z}}$ and therefore necessarily $C'' = \begin{pmatrix} -a & -b \\ a\bar{z} & b\bar{z} \end{pmatrix}$ for some $a, b \in \mathbb{C}$ not both 0. A BSS machine over $\mathbb{C}$ can identify a non-zero column of C''; and divide its lower by the upper component to obtain $-\bar{z}$, from which both $\mathrm{Re}(z) = (z + \bar{z})/2$ and $\mathrm{Im}(z) = (z - \bar{z})/2$ are readily obtained.

b) Neglecting parsing details as in Proposition 2.13, recursive application of a) to each subterm of t yields the claim.

c) In the real case, let the nondeterministic BSS-machine 'guess' $A_1, \ldots, A_n \in \mathbb{F}^{d\times d}$; then apply b) to evaluate t thereon and accept iff the result is the identity (SAT) or is not the zero matrix (sat), respectively. In the complex case, similarly 'guess' $\mathrm{Re}(A_1), \mathrm{Im}(A_1), \ldots, \mathrm{Re}(A_n), \mathrm{Im}(A_n) \in \mathrm{Re}(\mathbb{F})^{d\times d}$. The final claim follows from Fact 1.12b)+c).

d) Given $(n+m)$-variate t as well as $C_1, \ldots, C_m \in \mathbb{F}^{d\times d}$, guess $A_1, \ldots, A_n \in \mathbb{F}^{d\times d}$ and then proceed as before to evaluate $t_{\mathrm{L}(\mathbb{F}^d)}(\mathrm{range}\, A_1, \ldots, \mathrm{range}\, A_n, \mathrm{range}\, C_1, \ldots, \mathrm{range}\, C_m)$. □

Observe that the calculations in Proposition 2.17a) can be performed by a BSS-machine over the ring $\mathbb{F}$, i.e. not using inversion. An alternative proof of Proposition 2.17c) replaces Gaussian Elimination in a) by a non-deterministic approach based on Fact 4.6 below.

2.5 Abstract Unitary Spaces and Bases

The results of the preceding subsection have been most conveniently formulated for the $\mathbb{F}$-unitary spaces $\mathbb{F}^d$ and their ortholattices. Transition to abstract spaces $\mathcal{H}$ is possible via bases. A basis $\vec{v}_1, \ldots, \vec{v}_d$ is orthogonal if $\langle \vec{v}_i \mid \vec{v}_j \rangle = 0$ for all $i \neq j$. It is orthonormal if, in addition, $\langle \vec{v}_i \mid \vec{v}_i \rangle = 1$ for all i. An isometry $\omega : \mathcal{H} \to \mathcal{H}'$ between $\mathbb{F}$-unitary spaces $\mathcal{H}$ and $\mathcal{H}'$ is an $\mathbb{F}$-linear isomorphism preserving scalar products: $\langle \omega(\vec{x}) | \omega(\vec{y}) \rangle = \langle \vec{x} | \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathcal{H}$.

Convention 2.18 *Unless stated otherwise, $\mathcal{H}$ is assumed to be a finite dimensional $\mathbb{F}$-unitary space. $\mathbb{F}$ is Pythagorean if in $\mathbb{F} \cap \mathbb{R}$ any sum of squares is a square.*

f:GramSchmidt

Fact 2.19 *a) Any $\mathbb{F}$-unitary space admits an orthogonal basis.*
b) The basis can be chosen orthonormal if $\mathbb{F}$ is Pythagorean.
c) An $\mathbb{F}$-unitary space admitting an orthonormal basis of d vectors is isometric to $\mathbb{F}^d$ (with the basis mapped onto the canonical one).
d) If $\mathcal{H}$ and $\mathcal{H}'$ are isometric, then $\mathrm{L}(\mathcal{H})$ and $\mathrm{L}(\mathcal{H}')$ are isomorphic.

Proof. a) follows from the *Gram-Schmidt* Process, cmp. e.g. [Gelf61, §2-3], [Halm58, §73-74], or [Axle96, 6.19].

b) Having all $(a+bi)(a-bi)=a^2+b^2$ squares in $\mathbb{F}\cap\mathbb{R}$, any $\langle\vec{v}\mid\vec{v}\rangle$ is a square in $\mathbb{F}\cap\mathbb{R}$, too, so that one can normalize.

c) Matching two orthonormal bases extends to an isometry.

d) A linear homomorphism $\omega:\mathcal{H}\to\mathcal{H}'$ satisfies $\omega[A]+\omega[B]=\omega[A+B]$ for all $A,B\in\mathrm{L}(\mathcal{H})$; an injective one also $\omega[A]\cap\omega[B]=\omega[A\cap B]$; and an isometry even $\neg\omega[A]=\omega[\neg A]$. In the latter case, $\mathrm{L}(\mathcal{H})\ni A\mapsto\omega[A]\in\mathrm{L}(\mathcal{H}')$ thus constitutes an isomorphism. □

Fact 2.19d) and its converse can be traced back to [BiNe36], its relevance for Quantum Logic is also pointed out in [Hagg07, line after DEFINITION2]. We weaken Item c) in the following

d:Equinormal

Definition 2.20 *A family $\vec{v}_1,\ldots,\vec{v}_d$ of vectors is equinormal if $\|\vec{v}_i\|^2=\|\vec{v}_j\|^2$ for all i,j.*

For instance, the one-dimensional $\mathbb{Q}$-unitary vector space $\{(x,x):x\in\mathbb{Q}\}$ has the single vector $(1,1)$ as an (trivially equinormal and orthogonal) basis but contains no unit vector.

Digression 2.21 *The two-dimensional $\mathbb{Q}$-unitary vector space $\{(x,x,y):x,y\in\mathbb{Q}\}$ has vectors $(1,1,1)$ and $(1,1,-1)$ constituting an equinormal basis but does not admit an orthogonal equinormal one: Equations $2x^2+y^2=2x'^2+y'^2$ and $2xx'+yy'=0$ have (other than the trivial $x=y=x'=y'=0$) no simultaneous solution over $\mathbb{Z}$ nor, dividing by a common denominator, over $\mathbb{Q}$.*

l:Equinormal

Lemma 2.22 *Let $\mathcal{H}$ and $\mathcal{H}'$ denote finite-dimensional $\mathbb{F}$-unitary spaces.*

a) If both $\mathcal{H}$ and $\mathcal{H}'$ admit equinormal orthogonal bases and have coinciding dimensions, then $\mathrm{L}(\mathcal{H})$ and $\mathrm{L}(\mathcal{H}')$ are isomorphic.

b) If both $\mathcal{H}$ and $\mathcal{H}'$ admit equinormal orthogonal bases and $\dim(\mathcal{H})$ divides $\dim(\mathcal{H}')$, then $\mathrm{L}(\mathcal{H})$ embeds into $\mathrm{L}(\mathcal{H}')$.

Partial converses can be found in Corollary 3.2 below.

Proof. a) Let $\vec{u}_1,\ldots,\vec{u}_d$ be the equinormal orthogonal base for $\mathcal{H}$ and $\vec{v}_1,\ldots,\vec{v}_d$ that for $\mathcal{H}'$ with isomorphism $\omega:\mathcal{H}\to\mathcal{H}'$, $\vec{u}_j\mapsto\vec{v}_j$. In view of the proof of Fact 2.19d) it remains to show that $\vec{x}\perp\vec{y}\Leftrightarrow\omega(\vec{x})\perp\omega(\vec{y})$ holds. Indeed, exploiting orthogonality and equinormality,

$$\Big\langle \sum\nolimits_k \alpha_k\vec{u}_k, \sum\nolimits_\ell \beta_\ell\vec{u}_\ell\Big\rangle \;=\; \sum\nolimits_k\langle\alpha_k\vec{u}_k,\beta_k\vec{u}_k\rangle \;=\; \|\vec{u}\|^2\cdot\sum\nolimits_k\alpha_k\bar{\beta}_k$$

vanishes iff $\|\vec{v}\|^2\cdot\sum_k\alpha_k\bar{\beta}_k=\big\langle\omega\big(\sum_k\alpha_k\vec{u}_k\big),\omega\big(\sum_\ell\beta_\ell\vec{v}_\ell\big)\big\rangle$ does.

b) Let $d=\dim(\mathcal{H})$ and $\dim(\mathcal{H}')=kd$, $k\in\mathbb{N}$. Let $(\vec{v}_{j\ell})_{1\leq j\leq d,1\leq\ell\leq k}$ denote an orthogonal equinormal basis of $\mathcal{H}'$. Then $\mathcal{H}'=V_1\ominus\ldots\ominus V_k$ where $V_j:=\mathbb{F}\vec{v}_{j1}+\cdots+\mathbb{F}\vec{v}_{jd}$. Now by a), there is an isomorphism $\omega_\ell:\mathrm{L}(\mathcal{H})\to\mathrm{L}(V_\ell)$ for each $1\leq\ell\leq k$. On the other hand, from Observation 2.6a) it follows by induction that $\mathrm{L}(V_1)\times\mathrm{L}(V_2)\times\cdots\times\mathrm{L}(V_k)$ embeds into $\mathrm{L}(\mathcal{H}')$. Thus, $\omega(X)=\big(\omega_1(X),\ldots,\omega_k(X)\big)$ defines an embedding of $\mathrm{L}(\mathcal{H})$ into $\mathrm{L}(\mathcal{H}')$. □

Since the standard basis for $\mathbb{F}^d$ is clearly equinormal orthogonal, we conclude from Proposition 2.17c) and Lemma 2.22a)

Corollary 2.1. *Suppose $\mathcal{H}$ is an $\mathbb{F}$-unitary vector space admitting an equinormal orthogonal basis. Then* $\mathrm{sat}_{\mathrm{L}(\mathcal{H})},\mathrm{SAT}_{\mathrm{L}(\mathcal{H})}\in\mathrm{BP}(\mathcal{NP}^0_{\mathrm{Re}(\mathbb{F})})$.

3 Strong Satisfiability is Complete for BSS-$\mathcal{NP}$ in Dimensions ≥ 3

s:Arith2QL

The main result of this section shows that, for fixed $\mathcal{H}$ of dimension $d \geq 3$, strong satisfiability in $\mathrm{L}(\mathcal{H})$ is hard for the complexity class $\mathrm{BP}(\mathcal{NP}^0_{\mathrm{Re}(\mathbb{F})})$. This is shown by recapturing $\mathbb{F}$ within $\mathrm{L}(\mathcal{H})$ using coordinatization methods for modular lattices due to von Neumann [Neum60]. Subsections 3.1 and 3.2 convey the elementary (affine and projective, respectively) geometric intuition behind the quantum logic terms then presented in Subsection 3.3.

3.1 Background from Elementary Affine Geometry

ss:Affine

Let us start with Elementary Geometry in the (affine) plane. Following Hilbert [Hilb03], only the structure matters. The basic objects are points and lines, the basic relation is incidence: a given point is or is not on a given line. A line may be considered as the set of all points incident with it. Following Descartes, to introduce coordinates, we need two lines ℓ_2, ℓ_3 (the choice for these indices will become clear, later) intersecting in a point A_1. Then for any point P and $i \neq j$ in $\{2, 3\}$ we have the intersection P_i of ℓ_i with the parallel to ℓ_j through P. Thus, one obtains a bijective correspondence between points P and their coordinate pairs (P_2, P_3).

In the sequel, let us consider points P on the coordinate line ℓ_2 as scalars and do calculations with them.

o:sub

Observation 3.1 *Subtraction and multiplication of scalars can be defined geometrically:*

Choosing $A_{12} \neq A_1$ on ℓ_2 we fix an orientation on ℓ_2 which allows to understand P as the length of the segment A_1P with sign given by orientation. Then the difference $P \ominus Q$ can be obtained by the geometric construction of Figure 2a): choose $A_{13} \neq A_1$ on ℓ_3 to determine the line ℓ through A_{12} and A_{13}, let Q_{13} the intersection with ℓ_3 of the parallel to ℓ through Q, let S the intersection of ℓ with the parallel to ℓ_2 through Q_{13} and $P \ominus Q$ the intersection of ℓ_2 and the parallel to ℓ_3 through S.

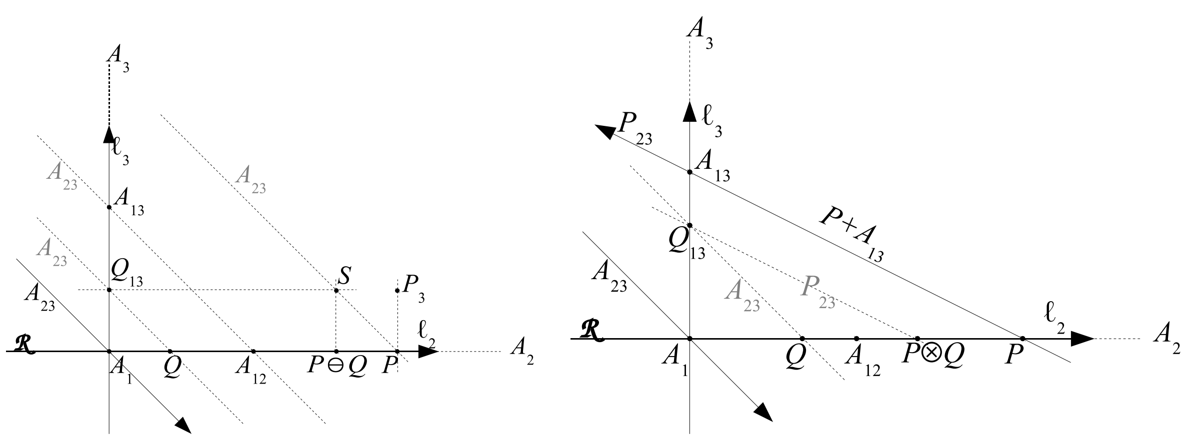

Fig. 2. Geometry of a) subtraction and b) multiplication

Multiplication is based on the *Intercept Theorem*. Having determined Q_{13} as before, let k the line through P and A_{13} and $P \otimes Q$ the intersection of ℓ_2 and the parallel to k through Q_{13}. Observe, that $A_{12} \otimes P = P$.

Notice, that it is not required that ℓ_3 be perpendicular to ℓ_2 or that the distances A_1A_{12} and A_1A_{13} are the same. Operations with scalars on ℓ_3 can be defined, symmetrically, and one can show that the correspondence ω between scalars on ℓ_2 and ℓ_3, respectively, given by ℓ is indeed an isomorphism. Using Desargues' Theorem resp. axiom one can further show that the scalars form a division ring - which is rendered as $\mathbb{R}$ if we have all axioms for incidence and orientation in Elementary Geometry available.

o:coorsy

Observation 3.2 *A coordinate system of the plane is given by the origin A_1 and the coordinate lines ℓ_2, ℓ_3 (resp. their directions A_2, A_3) and unit points $A_{1i} \neq A_i$ on ℓ_i, $i = 2, 3$.*

Suppose that ℓ_2 and ℓ_3 are perpendicular. Consider the lines g and h through A_1 and the points with coordinates (A_{12}, A_{13}) and $(\ominus A_{12}, A_{13})$, respectively. Thus, h is parallel to ℓ.

o:orthogframe

Observation 3.3 *ℓ is perpendicular to g if and only if ω is an isometry, i.e. if the "unit lengths" A_1A_{12} and A_1A_{13} agree, see Figure 3.*

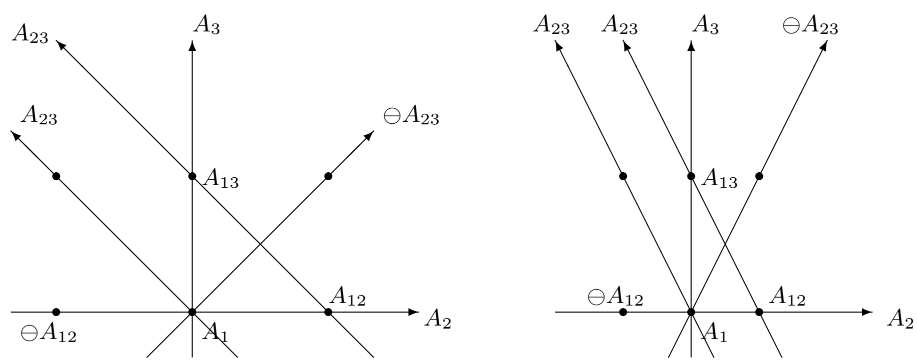

Fig. 3. Orthogonal frames: Orthonormal versus non-orthonormal

3.2 Background from Projective Geometry

ss:Projective

Projective Geometry originated from an analysis of perspective drawing. The basic idea (in the plane) is to add a *point at infinity* for any parallel pencil of lines (and make it incident with each line of the pencil) and to put all these points on a new *line ℓ_∞ at infinity* — just imagine several straight railroads in the plains. Thus, the following applies:

d:pplane

Definition 3.4 *A projective plane $\mathcal{P}$ is an incidence structure such that any two distinct lines meet in exactly one point and any two distinct points P, Q determine a unique line $P \vee Q$ incident with both P, Q.*

The original affine plane $\mathcal{A}$ is recovered within this projective plane $\mathcal{P}$ as $\mathcal{P} \setminus \ell_\infty$ — its lines are given by those lines of $\mathcal{P}$ which are incident with at least two points from $\mathcal{A}$. We still may consider lines just as sets of points. But now there are only two basic operations in $\mathcal{P}$ and these may also be used to recapture $\mathcal{A}$, completely: drawing the parallel g to h through P is replaced by determining the intersection S of h with ℓ_∞ and to join S with P to obtain g. Notice, that choosing any line ℓ of $\mathcal{P}$ we obtain an affine plane $\mathcal{P} \setminus \ell$ for which ℓ is the line at infinity.

In particular, if a line ℓ_∞ at infinity is designated, *directions* are just points on ℓ_∞. Thus, the concept of a coordinate system in Observation 3.2 now translates into the following (cmp. Figure 4b).

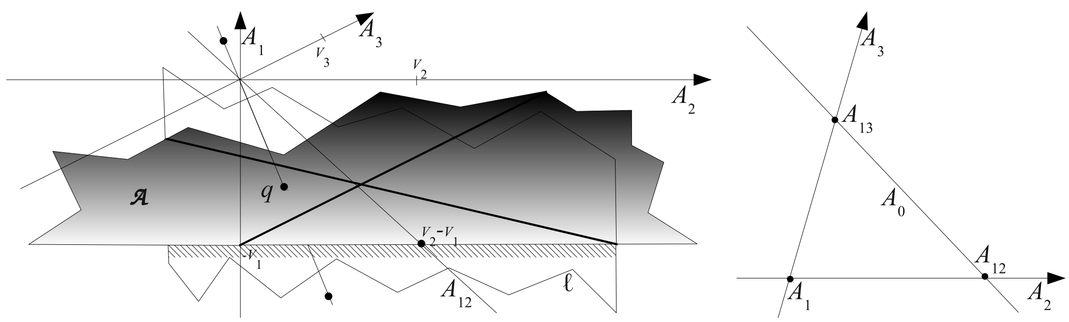

Fig. 4. a) Projective plane and affine space: orthogonal vectors $\vec{v}_i$ induce projective points A_i; the 2-dimensional affine space $\mathcal{A}$ with an affine line $\ell \cap \mathcal{A}$, ℓ a line of projective space, i.e. a homogeneous plane. b) Non-orthogonal coordinate system.

o:coopro

Observation 3.5 *a) A coordinate system is given by three non-collinear* basis points *A_1, A_2, A_3 and* unit points *A_{1i} on $A_1 \vee A_i$ for $i = 2, 3$ such that $A_{1i} \neq A_1, A_i$. Alternatively, we may choose units points $A_{ij} = A_{ji}$ on $A_i \vee A_j$ for $i \leq j$ such that A_{12}, A_{13}, A_{23} are on the same line ℓ and $A_{ij} \neq A_i, A_j$. The latter choice can be also implemented by choosing ℓ.*

b) How does this relate to the lattices $\mathrm{L}(\mathcal{H})$? We first observe, that due to the Dimension Formula $\mathrm{L}_1(\mathcal{H})$ and $\mathrm{L}_2(\mathcal{H})$ as the sets of 'points' and 'lines' with incidence relation $\subseteq$ form a projective plane if $\dim(\mathcal{H}) = 3$. *More generally, $\mathrm{L}_1(\mathcal{H})$ and $\mathrm{L}_2(\mathcal{H})$ form a 'projective space' $\mathcal{P}$ of dimension* $\dim(\mathcal{H}) - 1$, *where the condition that any two lines meet in a some point is required only for lines in a 'subspace' which is a plane.*

c) Of course, with $\mathcal{H}$ we may also associate its affine space *with point set $\mathcal{H}$, lines and planes of the form $\vec{v} + U$ where $U \in \mathrm{L}(\mathcal{H})$ and* $\dim(U) = 1, 2$, *respectively. Choosing* origin $\vec{0}$, *the $\vec{0} + U$ are (homogeneous) line and plane through this origin in the affine space $\mathcal{H}$, points and lines of the projective space $\mathcal{P}$, respectively.*

With respect to a basis $\vec{v}_1, \ldots, \vec{v}_d$, the points of $\mathcal{P}$ are given by their homogeneous coordinates $\bar{x} = (x_1, \ldots, x_d)$, where $\bar{x}$ and $\lambda\bar{x}$ ($\lambda \neq 0$ in $\mathbb{F}$) describe the same point. Then the points with coordinate $x_1 = -1$ (the reason why we do not choose $x_1 = 1$ will become transparent, later) form a $(d-1)$-dimensional affine subspace $\mathcal{A}$ with lines $U \cap \mathcal{A}$, where U is a 2-dimensional linear subspace of $\mathcal{H}$ not parallel to $\mathcal{A}$. (Thus, in case $\mathbb{F}^3$ we may imagine $\mathcal{A}$ as the plane parallel to the x_2–x_3–plane and meeting the x_1–axis at -1; cf. Figure 4a). This space is canonically coordinatized by $\mathbb{F}^{d-1}$. The points with coordinate $x_1 = 0$ form then a $(d-2)$–dimensional projective subspace, of $\mathcal{P}$, the 'hyperplane at infinity".

3.3 Von Neumann Frames for Coordinatizing Lattices

ss:vonNeumann

Frames were introduced by von Neumann to recapture linear algebraic structure within certain modular lattices, including all Hilbert lattices $\mathrm{L}(\mathcal{H})$. The intuition came from Projective Geometry but the concept can be explained within Linear Algebra, too. A basis $\vec{v}_1, \ldots, \vec{v}_d$ of $\mathcal{H}$ is turned into certain elements of $\mathrm{L}(\mathcal{H})$ satisfying relations which, conversely, guarantee that they have been induced by a basis—sufficiently unique to construct an isomorphism of $\mathcal{H}$ onto $\mathbb{F}^d$ via an isomorphism from $\mathrm{L}(\mathcal{H})$ onto $\mathrm{L}(\mathbb{F}^d)$.

From the $A_i = \mathbb{F}\vec{v}_i$ we may recover the $\vec{v}_i$ up to scalar multiple. Moreover, relations

$$A_1 \vee \ldots \vee A_d = \mathbf{1} \quad \text{and} \quad A_i \wedge \bigvee\nolimits_{j \neq i} A_j = \mathbf{0} \quad (1 \leq i \leq d)$$

make sure that $\mathcal{H}$ is a direct sum $\mathcal{H} = A_1 \oplus \ldots \oplus A_d$.

In particular $\dim(A_i) = 1$ since $\dim(\mathcal{H}) = d$; and one recovers a basis of $\mathcal{H}$ by choosing non-zero $\vec{v}_i \in A_i$ $(1 \leq i \leq d)$. What has to be added is a simultaneous scaling of the $\vec{v}_i$ in terms of the lattice. The basis vectors $\vec{v}_i$ are naturally related via the isomorphisms $\eta_{ij} : A_i \to A_j$ such that $\eta_{ij}(\vec{v}_i) = \vec{v}_j$. Clearly, $\eta_{jk} \circ \eta_{ij} = \eta_{ik}$.

To capture these isomorphisms within the lattice, one might associate with each η_{ij} its graph $\mathbb{F}(\vec{v}_i + \vec{v}_j) = \{\vec{x} + \eta_{ij}(\vec{x}) \mid \vec{x} \in A_i\}$. Though, as e.g. the graphs of η_{12}, η_{13} and η_{23} give a direct decomposition of $\mathbb{R}^3$, there is no meaningful lattice relation on these graphs. Von Neumann realized, that there are meaningful relations if one uses *negative graphs* $A_{ij} = A_{ji} = \mathbb{F}(\vec{v}_i - \vec{v}_j)$. The relevant relations are

$$A_{ij} \wedge A_i = \mathbf{0}, \quad A_{ij} \vee A_i = \mathbf{1}, \quad A_{ik} = (A_i \vee A_k) \wedge (A_{ij} \vee A_{jk}) \qquad (i \neq j \neq k \neq i)$$

(cmp. Observation 3.5a). The first two express the fact that A_{ij} defines an isomorphism from A_i onto A_j, the third guarantees the above composition laws. Observe that the linear automorphisms leaving the A_i and A_{ij} invariant are exactly the multiplications with a scalar. Following [Neum60] this leads to the

d:frame

Definition 3.6 *A system $\bar{a} = (a_{ij} \mid 1 \leq i, j \leq d)$ of elements of a modular lattice, L, is a d-frame if it satisfies (abbreviating $a_i =: a_{ii}$)*

$$\begin{array}{lll}
\mathbf{1} & = a_1 \vee \ldots \vee a_d, & \\
\mathbf{0} & = a_i \wedge \bigvee_{j \neq i} a_j & \textit{for all } i, j = 1, \ldots, d,\ i \neq j, \\
\mathbf{0} & = a_i \wedge a_{ij} & \textit{for all } i, j = 1, \ldots, d,\ i \neq j, \\
a_i \vee a_j & = a_i \vee a_{ij} & \textit{for all } i, j = 1, \ldots, d,\ i \neq j, \\
a_{ij} & = a_{ji} & \textit{for all } i, j = 1, \ldots, d,\ i \neq j, \\
a_{ik} & = (a_i \vee a_k) \wedge (a_{ij} \vee a_{jk}) & \textit{for all } i, j, k = 1, \ldots, d,\ i \neq j \neq k \neq i.
\end{array}$$

The frame relations and Fact 2.1 imply the following.

f:frame

Fact 3.7 *Let $\bar{a}$ be a d-frame in the modular lattice L.*

a) $\dim a_i = \dim a_j = \dim a_{ij}$ *for all* $i, j > 0$.
b) $\dim \bigvee_{i \in I} a_i = |I| \cdot \dim a_1$ *for any* $I \subseteq \{1, \ldots, d\}$.
c) $\dim(L) = d \cdot \dim(a_1)$.
d) *Let* $\dim(\mathcal{H}) = d$. *Then* $\bar{A} = (A_{ij} \mid 1 \leq i, j \leq d)$ *is a d-frame of* $\mathrm{L}(\mathcal{H})$ *if and only if there exists a basis* $\vec{v}_1, \ldots, \vec{v}_d$ *of* $\mathcal{H}$ *such that* $A_i = A_{ii} = \mathbb{F}\vec{v}_i$ *and* $A_{ij} = \mathbb{F}(\vec{v}_i - \vec{v}_j)$ *for* $i \neq j$.

e) In particular, every d-dimensional $\mathrm{L}(\mathcal{H})$ *admits a d-frame.*

Proof. a) to c) are immediate by Fact 2.1. Concerning d), assume that $\bar{A} = (A_{ij} \mid 1 \leq i, j \leq d)$ is a d-frame in $L = \mathrm{L}(\mathcal{H})$. Since $\mathcal{H} = A_1 \oplus \ldots \oplus A_d$ with $\dim(A_i) = 1$, we may choose a basis $\vec{v}_1, \ldots, \vec{v}_d$ of $\mathcal{H}$ such that $A_i = \mathbb{F}\vec{v}_i$. Since $\dim A_{1i} = 1$ for $i > 1$, we may choose $\vec{w}_{1i}$ such that $A_{1i} = \mathbb{F}\vec{w}_{1i}$. From $A_{1i} \subseteq A_1 + A_i$ we get $\vec{w}_{1i} = r_i\vec{v}_1 + s_i\vec{v}_i$ and $r_i \neq 0$ since $A_{1i} + A_1 = A_1 + A_i \neq A_i$. Thus, by scaling, i.e. replacing $\vec{v}_i$ by $r_i^{-1}s_i\vec{v}_i$, we may assume that $\vec{w}_{1i} = \vec{v}_1 - \vec{v}_i$ for $i > 1$. Then, it follows $A_i = \mathbb{F}\vec{v}_i$, $A_{ij} = \mathbb{F}(\vec{v}_i - \vec{v}_j)$ for all $i \neq j$. Indeed, by the sixth line of equations in Definition 3.6, for $1 < i, j$ and $i \neq j$, $\vec{0} \neq \vec{v}_i - \vec{v}_j = \vec{v}_1 - \vec{v}_j - (\vec{v}_1 - \vec{v}_i) \in (A_i + A_j) \cap (A_{1i} + A_{1j}) = A_{ij}$.

3.4 Recovering the Ground Field in Hilbert Lattices of Dimension ≥ 3

ss:IntField

The next step is to recover the field $\mathbb{F}$ within $\mathrm{L}(\mathcal{H})$ where $d = \dim(\mathcal{H}) \geq 3$. Let a d-frame $\bar{A}$ be given as above. Observe that with each of the isomorphisms η_{ij} from above and each scalar $r \in \mathbb{F}$ we also have the linear map $r\eta_{ij} : A_i \to A_j$ (which is an isomorphism if $r \neq 0$). For simplicity of notation we just consider the $r\eta_{12}$; but the discussion applies to any other choice as well. Since we have seen above that the η_{ij} should be captured by their negative graphs, associate with each scalar $r \in \mathbb{F}$ a 1D subspace via

$$\Theta_{\bar{A}} : \mathbb{F} \ni r \mapsto \mathbb{F}(\vec{v}_1 - r\vec{v}_2) \;\in\; \mathcal{R}_{\bar{A}} \;:=\; \{X \in \mathrm{L}(\mathcal{H}) \mid X \cap A_2 = \mathbf{0},\; X + A_2 = A_1 + A_2\}\;. \quad (3)$$

Conversely, if $X \in \mathcal{R}_{\bar{A}}$ then $\dim(X) = 1$ (since $X \notin \{\mathbf{0}, A_1 + A_2\}$ and $\dim(A_1 + A_2) = 2$) whence $X = \mathbb{F}(x_1\vec{v}_1 + x_2\vec{v}_2)$ for some $x_1, x_2 \in \mathbb{F}$. Assuming $x_1 = 0$ it would follow $X \subseteq A_2$ and $A_1 \subseteq A_2$. Thus $x_1 \neq 0$ and $X = \Theta(r)$ with $r = -x_1^{-1}x_2$. This demonstrates that $\Theta_{\bar{A}} : \mathbb{F} \to \mathcal{R}_{\bar{A}} \subseteq \mathrm{L}_1(\mathcal{H})$ is a bijection. Observe that $\Theta_{\bar{A}}(0) = A_1$ and $\Theta_{\bar{A}}(1) = A_{12}$.

Comment 3.8 *Imagining* $\mathcal{A}$ *and its projective extension* $\mathcal{P}$ *represented within* $\mathcal{H}$ *as in Observation 3.5c), we may consider* Θ *a bijection from* $\mathbb{F}$ *onto the coordinate line* ℓ_2 *of* $\mathcal{A}$*: mapping* r *onto the point with coordinate* r *on this line (recall that the point with coordinate* 1*, the unit point on this line, is* $A_{12} = \Theta(1)$*). Indeed, the (homogeneous) line* $\Theta(r)$ *of* $\mathcal{H}$ *meets the affine subspace* $\mathcal{A}$ *in this point. For the following definition compare Observation 3.1.*

Now define binary operations $\ominus$ and $\otimes$ on R in terms of $\bar{A}$ and lattice operations such that Θ becomes an isomorphism of $(\mathbb{F}, 0, 1, -, \cdot)$ onto $(R, A_1, A_{12}, \ominus, \otimes)$:

$$\begin{aligned} P \ominus Q &= \big([(Q_{13} + A_2) \cap (P + A_{23})] + A_3\big) \cap (A_1 + A_2), \qquad & Q_{13} &:= (Q + A_{23}) \cap (A_1 + A_3) \\ P \otimes Q &= (Q_{13} + P_{32}) \cap (A_1 + A_2), & P_{32} &:= (P + A_{13}) \cap (A_2 + A_3) \end{aligned}$$

Indeed, referring to the above basis one has

$$A_i + A_j = \{x_i\vec{v}_i + x_j\vec{v}_j \mid x_i, x_j \in \mathbb{F}\}, \quad P = \mathbb{F}(\vec{v}_1 - p\vec{v}_2), \quad Q = \mathbb{F}(\vec{v}_1 - q\vec{v}_2)$$

for unique $p, q \in \mathbb{F}$ and calculates

$$\begin{aligned} Q_{13} &= \{x\vec{v}_1 - xq\vec{v}_2 + y\vec{v}_2 - y\vec{v}_3 \mid x, y \in \mathbb{F}\} \cap (A_1 + A_3) = \mathbb{F}(\vec{v}_1 - q\vec{v}_3) \\ P_{32} &= \{x\vec{v}_1 - xp\vec{v}_2 + y\vec{v}_1 - y\vec{v}_3 \mid x, y \in \mathbb{F}\} \cap (A_2 + A_3) = \mathbb{F}(-p\vec{v}_2 + \vec{v}_3) \end{aligned}$$

since $y - xq = 0$ respectively $x + y = 0$. It follows for $S := (Q_{13} + A_2) \cap (P + A_{23})$ and $P \ominus Q = (S + A_3) \cap (A_1 + A_2)$ that

$$\begin{aligned} S \quad &= \{x\vec{v}_1 + y\vec{v}_2 - x\vec{q}x_3 \mid x, y \in \mathbb{F}\} \cap \{r\vec{v}_1 - (rp - s)\vec{v}_2 - s\vec{v}_3 \mid r, s \in \mathbb{F}\} \\ &= \{r\vec{v}_1 - r(p - q)\vec{v}_2 - rq\vec{v}_3 \mid r \in \mathbb{F}\} \\ P \ominus Q &= \{t\vec{v}_1 - t(p - q)\vec{v}_2 \mid t \in \mathbb{F}\} = \Theta(p - q) \end{aligned}$$

since $r = x$ and $s = xq = rq$. Finally, observing $y = xq$, one gets

$$P \otimes Q = \{x\vec{v}_1 - yp\vec{v}_2 - (xq - y)\vec{v}_3)\} \cap (A_1 + A_2) = \{x\vec{v}_1 - xqp\vec{v}_2 \mid x \in \mathbb{F}\} = \Theta_{\bar{A}}(p \cdot q).$$

Comment 3.9 *The term for multiplication has been defined in order to generalize in Subsection 6.2 to matrices resp. endomorphisms when writing applications of maps on the left. Doing so, one should write scalars to the right of vectors, i.e consider right vector spaces, if commutativity of $\mathbb{F}$ is not to be presumed.*

We define $P \oplus Q = P \ominus (A_1 \ominus Q)$ and summarize the above.

f:IntField

Fact 3.10 *For $\mathcal{H}$ of $\dim(\mathcal{H}) = d \geq 3$ with orthogonal basis $\vec{v}_1, \ldots, \vec{v}_d$ and $\bar{A}$ a d-frame of $\mathrm{L}(\mathcal{H})$, $\Theta_{\bar{A}}(r) = \mathbb{F}(\vec{v}_1 - r\vec{v}_2)$ constitutes an isomorphism of the field $\mathbb{F}$ with operations $+, -, 0, \cdot, 1$ onto $\mathcal{R}_{\bar{A}}$ with operations $\oplus_{\bar{A}}, \ominus_{\bar{A}}, A_1, \otimes_{\bar{A}}, A_{12}$ defined as above in terms of $\bar{A}$.*

There is of course nothing particular about the indices 123. So Fact 3.10 holds also for any triple ijk of pairwise distinct indices between 1 and d.

Comment 3.11 *We will refer, informally, to results as in Fact 3.10 as* interpretations *- established by maps $\Theta_{\bar{c}}$ (with or without parameters $\bar{c}$) and formulas relating the two structures (cmp. Subsection 4.2).*

3.5 Strong Satisfiability is Hard for BSS–$\mathcal{NP}$ in Dimension ≥ 3

ss:realBSSNPc

Having Fact 3.10, a well-known method applies to translate information about the ring $\mathbb{F}$ into such about $\mathrm{L}(\mathcal{H})$, given a d-frame $\bar{A}$ ($d = \dim \mathcal{H} \geq 3$); and we observe that both the translation and the conditions for a d-frame can be encoded in polynomial time:

p:IntField

Proposition 3.12 *Let $\mathcal{H}$ denote a d-dimensional $\mathbb{F}$-unitary vector space, $d \geq 3$.*

a) $\mathrm{SAT}_{\mathrm{L}(\mathcal{H})}$ *is* $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{F}})$*–hard*
b) $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d),\mathrm{L}(\mathbb{F}^d)}$ *is* $\mathcal{NP}_{\mathbb{F}}$*–hard.*
c) Claim a) holds uniformly in the following sense: Given $3 \leq d \in \mathbb{N}$ and a finite system $\bar{p} = 0$ of of polynomial equations $p_j(X_1, \ldots, X_n) = 0$ with $p_j \in \mathbb{Z}[X_1, \ldots, X_n]$, a Turing machine can produce within time polynomial in d and in the length of the binary encoding of the p_j's coefficients and monomials, an orthologic term $t_{\bar{p},d}$ such that, for every field $\mathbb{F} \subseteq \mathbb{C}$ and every d-dimensional $\mathbb{F}$-unitary vector space $\mathcal{H}$, the p_j admit a common root in $\mathbb{F}$ iff $t_{\bar{p},d}$ admits a strongly satisfying assignment in $\mathrm{L}(\mathcal{H})$.

Proof. Claim c) asserts $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{F}})$–complete $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}}$ to be polynomial-time reducible to $\mathrm{SAT}_{\mathrm{L}(\mathcal{H})}$, hence a) follows from c). Concerning c) itself, consider the following algorithm:

i) Replace each integer coefficient c by an expression over $(1, +, -, \cdot)$ according to Observation 3.13 below, thus obtaining n-variate terms p'_j in the language of rings equivalent to p_j over $\mathbb{F}$.

ii) In these terms, replace each constant/operation $+, -, 0, \cdot, 1$ by the corresponding term $\oplus_{\bar{A}}, \ominus_{\bar{A}}, A_1, \otimes_{\bar{A}}, A_{12}$ according to Fact 3.10, thus obtaining terms $t_j(X_1, \ldots, X_n; \bar{A})$ in the language of lattices.
iii) Now take the lattice equations $t_j(X_1, \ldots, X_n; \bar{A}) = A_1$ and
iv) for each variable X_i add conditions $X_i \wedge A_2 = \mathbf{0}$ and $X_i \vee A_2 = A_1 \vee A_2$ from Equation (3)
v) as well as those from Definition 3.6 for $\bar{A}$ to constitute a d-frame
vi) and combine the resulting system of lattice equations into a single ortholattice equation "$t_{\bar{p}} = \mathbf{1}$" according to Fact 2.9a).

First, a Turing machine can indeed perform each of the above steps in time polynomial in d and in the joint binary length of the p_j.

Secondly, a d-frame $\bar{A}$ exists according to Fact 3.7e) and hence complies with (v). Structural induction on (ii) yields for any $b_1, \ldots, b_n$

$$t_j\big(\Theta_{\bar{A}}(b_1), \ldots \Theta_{\bar{A}}(b_n)\big) \;=\; \Theta_{\bar{A}}\big(p'_j(b_1, \ldots b_n)\big) \stackrel{\text{(i)}}{=} \Theta_{\bar{A}}\big(p_j(b_1, \ldots b_n)\big) \;. \tag{4}$$

Thus, a common root $b_1, \ldots, b_n \in \mathbb{F}$ of all p_j gives rise via $B_i := \Theta_{\bar{A}}(b_i)$ to an assignment $B_1, \ldots, B_n$ in $\mathrm{L}(\mathcal{H})$ satisfying the requirements of (iv) as well as (iii) and

$$t_j\big(\Theta_{\bar{A}}(b_1), \ldots \Theta_{\bar{A}}(b_n)\big) \;=\; \Theta_{\bar{A}}(0) \;=\; A_1 \;. \tag{5}$$

leading to a strongly satisfying assignment $\bar{B}, \bar{A}$ over $\mathrm{L}(\mathcal{H})$ of $t_{\bar{p}}$.

Conversely, a strongly satisfying assignment over $\mathrm{L}(\mathcal{H})$ of $t_{\bar{p}}$ consists by (v) of a d-frame $\bar{A}$ and, due to (iv), of $B_1, \ldots, B_n \in \mathcal{R}_{\bar{A}}$, that is $B_i = \Theta_{\bar{A}}(b_i)$ for certain $b_1, \ldots, b_n \in \mathbb{F}$. Moreover, $t_j(\bar{B}) = A_1$ due to (iii) implies $p_j(\bar{b}) = 0$ by reading Equations (4) and (5) backwards: $b_1, \ldots, b_n$ a common root of all p_j.

Concerning the case (b) with constants, let p_j denote polynomials over $\mathbb{F}$. Now, instead of (v), fix $\bar{A}$ to denote the d-frame corresponding to the standard basis $\vec{v}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ of $\mathbb{F}^d$ according to Fact 3.7d); and, instead of (i), replace the polynomials' coefficients $c \in \mathbb{F}$ by constants $C := \Theta_{\bar{A}}(c) = \mathbb{F}(0, \ldots, c, \ldots, -c, \ldots, 0) \in \mathrm{L}_1(\mathbb{F}^d)$. □

o:MultChain

Observation 3.13 *To each $c \in \mathbb{N}$ there exists a term t_c over $(1, +, \cdot)$ of length $|t_c| \leq \mathcal{O}(\log c)$ evaluating to c over each ring containing $\mathbb{N}$. Moreover, such t_c can be computed from c in time polynomial in the binary length of c.*

Note that an *addition chain* 'expresses' c as unique solution to a system of equations of logarithmic size; however we are interested in a single term and thus employ also multiplication.

Proof. By induction, claiming $|t_c| \leq 2 + 7\log_2(c)$: Indeed $2c = (1+1) \cdot t_c =: t_{2c}$ and $2c + 1 = (1+1) \cdot t_c + 1 =: t_{2c+1}$ both have length at most $7 + |t_c| \leq 7 + 2 + 7\log_2(c) = 2 + 7\log_2(2c)$ by induction hypothesis.

c:realBSSNPc

Corollary 3.1. *In case $\mathbb{F} \subseteq \mathbb{R}$ and for every $d \geq 3$, $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d)}$ is $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{F}})$–complete; and $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d),\mathrm{L}(\mathbb{F}^d)}$ is $\mathcal{NP}_{\mathbb{F}}$–complete.*

In particular, the decidability of $\mathrm{SAT}_{\mathrm{L}(\mathbb{Q}^3)}$ is as open as that of $\mathrm{FEAS}_{\mathbb{Q}}$, recall Fact 1.12f). And $\mathrm{SAT}_{\mathrm{L}(\mathbb{R}^d)}$ is complete for $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$ and perhaps closest in spirit to the Boolean satisfiability problem complete for $\mathcal{NP}$.

Note however the gap between Proposition 2.17c) and Proposition 3.12 in case $\mathbb{F} \not\subseteq \mathbb{R}$: to be closed in the sequel.

3.6 Orthonormal Frames

ss:OrthonFrame

In order to recapture $\mathbb{F} \not\subseteq \mathbb{R}$ as a $*$-ring within $\mathrm{L}(\mathbb{F}^d)$, i.e. taking into account the involution of $\mathbb{F}$, we have to include orthogonality in our concept of frames. The first step is again due to VON NEUMANN.

d:OrthogFrame

Definition 3.14 *For L a modular ortholattice, call a d-frame $\bar{a}$ of L orthogonal if $a_i \leq \neg a_j$ holds for all $i \neq j$ (which makes the second line of conditions in Definition 3.6 redundant).*

In the situation of Fact 3.7d), orthogonality of $\bar{A}$ corresponds to orthogonality of the basis $\vec{v}_i$. What we want, to make things a simple as possible, is a concept of frames which corresponds to equinormal orthogonal bases. So, assume that an orthogonal basis $\vec{v}_1, \ldots, \vec{v}_d$ is given with associated orthogonal d-frame $\bar{A}$. We refer to Fact 3.10 and claim that $\|\vec{v}_1\|^2 = \|\vec{v}_2\|^2 \quad \Leftrightarrow$ $\Theta_{\bar{A}}(1) = A_{12} \ \perp\ \Theta_{\bar{A}}(-1) = \ominus_{\bar{A}} A_{12}$: Recall Observation 3.3 for the geometric intuition or calculate, with $\langle \vec{v}_1 \mid \vec{v}_2 \rangle = 0$ in mind,

$$0 = \langle \vec{v}_1 - \vec{v}_2 \mid \vec{v}_1 + \vec{v}_2 \rangle = \langle \vec{v}_1 \mid \vec{v}_1 \rangle + \langle \vec{v}_1 \mid \vec{v}_2 \rangle - \langle \vec{v}_2 \mid \vec{v}_1 \rangle - \langle \vec{v}_2 \mid \vec{v}_2 \rangle = \langle \vec{v}_1 \mid \vec{v}_1 \rangle - \langle \vec{v}_2 \mid \vec{v}_2 \rangle.$$

By Fact 3.10, it holds $\ominus_{\bar{A}} A_{12} = A_1 \ominus A_{12} = \big([(A_{13} + A_2) \cap (A_1 + A_{23})] + A_3\big) \cap (A_1 + A_2)$. In view of the last paragraph of Subsection 3.4, similarly to $\ominus_{\bar{A}12} \stackrel{\text{def}}{=} \ominus_{\bar{A}}$ we may consider $\ominus_{\bar{A}1k} A_{1k}$ for all $2 < k \leq d$. This motivates defining the following terms $\ominus_{\bar{y}1k} y_{1k}$ in variables $\bar{y} = (y_{ij} \mid 1 \leq i, j \leq d)$ where index $1 < \ell \leq d$ is distinct from $1, k$ but otherwise arbitrary:

$$\ominus_{\bar{y}1k} y_{1k} := \big([(y_{1\ell} \vee y_k) \wedge (y_1 \vee y_{k\ell})] \vee y_\ell\big) \wedge (y_1 \vee y_k).$$

d:OrthonFrame

Definition 3.15 *An orthogonal d-frame $\bar{a}$ of a modular ortholattice is orthonormal if $\ominus_{\bar{a}1k} a_{1k} \leq \neg a_{1k}$ for all $k \geq 2$.*

Fact 3.7d) and the above calculation prove

l:OrthonFrame

Lemma 3.16 *For $d \in \mathbb{N}$, a system $\bar{A} = (A_{ij} \mid 1 \leq i, j \leq d)$ is an orthonormal d-frame of $\mathrm{L}(\mathcal{H})$ if and only if there is an equinormal orthogonal basis $\vec{v}_1, \ldots, \vec{v}_d$ of $\mathcal{H}$ such that $A_{ii} = \mathbb{F}\vec{v}_i$ and $A_{ij} = \mathbb{F}(\vec{v}_i - \vec{v}_j)$ for $i \neq j$, $i, j = 1, \ldots, d$.*

For $\mathcal{H} = \mathbb{F}^d$ with the canonical basis this frame is the canonical orthonormal d-frame of $\mathrm{L}(\mathbb{F}^d)$.

Now, the image of an orthonormal d-frame under an isomorphism of $\mathrm{L}(\mathbb{F}^d)$ onto $\mathrm{L}(\mathcal{H})$ is again an orthonormal d-frame; which yields an equinormal orthogonal basis for $\mathcal{H}$ according to Lemma 3.16. We have thus concluded the promised partial converse to Lemma 2.22a):

c:Equinormal

Corollary 3.2. *a) Let $\mathcal{H}$ be a d-dimensional unitary $\mathbb{F}$-space and $\mathrm{L}(\mathcal{H})$ isomorphic to $\mathrm{L}(\mathbb{F}^d)$. Then $\mathcal{H}$ admits an equinormal orthogonal base (i.e. is a 'scaled' isometric copy of $\mathbb{F}^d$).*

b) Suppose $\mathcal{H}$ and $\mathcal{H}'$ are finite-dimensional unitary $\mathbb{F}$-spaces such that $\mathrm{L}(\mathcal{H})$ embeds into $\mathrm{L}(\mathcal{H}')$. Then $\dim(\mathcal{H})$ divides $\dim(\mathcal{H}')$.

In a) we even have $\mathcal{H}$ and $\mathbb{F}$ 'isometric up to scaling'. Concerning Claim b), take a d-frame $\bar{a}$ of $\mathrm{L}(\mathcal{H})$, $d := \dim(\mathcal{H})$. Its image $\bar{b}$ in $\mathrm{L}(\mathcal{H}')$ under the embedding thus also constitutes a d-frame; hence $\dim(\mathcal{H}') = d \cdot \dim(b_1)$ according to Fact 3.7c). □

3.7 Recovering the $*$-Field in Dimension ≥ 3

ss:IntStarField

Based on the above preparation, the interpretation of $\mathbb{F}$ in $\mathrm{L}(\mathcal{H})$ from Subsection 3.4 can be extended to respect also involution. In the more general and abstract setting of projective spaces with an orthocomplementation on the subspace lattice, this is due to Birkhoff and von Neumann [BiNe36]. Yet, in our setting there is no need for abstract calculations. Recall Fact 3.10.

l:IntStarField

Lemma 3.17 *Let $d \geq 3$. For any $\mathbb{F}$-unitary space $\mathcal{H}$ of dimension d admitting an equinormal orthogonal basis, there is some orthonormal d-frame $\bar{A}$ of $\mathrm{L}(\mathcal{H})$. Given such, there is a bijection $\Theta_{\bar{A}} : \mathbb{F} \to \mathcal{R}_{\bar{A}}$ such that the following hold*

a) $\Theta_{\bar{A}}$ is an isomorphism of the ring $(\mathbb{F}, +, -, 0, \cdot, 1)$ onto $(\mathcal{R}_{\bar{A}}, \oplus_{\bar{A}}, \ominus_{\bar{A}}, A_1, \otimes_{\bar{A}}, A_{12})$.
b) For $P = \Theta_{\bar{A}}(p)$ and $Q = \Theta_{\bar{A}}(q)$ with $p, q \neq 0$ in $\mathbb{F}$ one has $q = \bar{p}$, the conjugate of p, if and only if there is $R \in \mathcal{R}_{\bar{A}}$ such that $Q \otimes_{\bar{A}} R = \ominus_{\bar{A}} A_{12}$ and $P \perp R$ or, equivalently, $Q \otimes_{\bar{A}} ((A_1 \vee A_2) \wedge \neg P) = \ominus_{\bar{A}} A_{12}$.

Proof. The existence of $\bar{A}$ follows from Lemma 3.16; and conversely it is no loss of generality to assume that $\bar{A}$ is induced by an equinormal orthogonal basis. Now a) is just a special case of Fact 3.10. In b) we first show that $q = \bar{p}$ if and only there is $R \in \mathcal{R}_{\bar{A}}$ such that $Q \otimes_{\bar{A}} R = \ominus_{\bar{A}} A_{12}$ and $P \perp R$. Namely, for $R = \Theta_{\bar{A}}(r)$ we have $Q \otimes R = \ominus A_{12}$ if and only if $qr = -1$, i.e. $R = \mathbb{F}(q\vec{v}_1 + \vec{v}_2)$. Moreover, $P \perp R$ amounts to

$$0 = \langle \vec{v}_1 - p\vec{v}_2 \mid q\vec{v}_1 + \vec{v}_2 \rangle = \langle \vec{v}_1 \mid q\vec{v}_1 \rangle + \langle \vec{v}_1 \mid \vec{v}_2 \rangle - \langle p\vec{v}_2 \mid q\vec{v}_1 \rangle - \langle p\vec{v}_2 \mid \vec{v}_2 \rangle = q\|v_1\|^2 - \bar{p}\|v_2\|^2$$

i.e. to $q = \bar{p}$. Conversely, if $q = \bar{p}$ choose $R = \Theta_{\bar{A}}(-q^{-1})$. Now, observe that here R is uniquely determined as the inverse of $\ominus_{\bar{A}} Q$ and that $P \perp R$ means $R = (A_1 \vee A_2) \wedge \neg P$.

We can now extend Proposition 3.12 to match with Proposition 2.17c) also in the complex case. Here, a $*$-polynomial is a term in the language of (commutative) $*$-rings, i.e. may involve involution as in $p(X, Y) = (X + Y^\dagger)^2/4 + (X - X^\dagger)^2/2 + (Y + Y^\dagger)^2/4 + (Y - Y^\dagger)^2/2$. Considering substitutions in the $*$-field $\mathbb{F}$, it is convenient to define such polynomials as members of the polynomial ring $\mathbb{F}[X_1, X_1^\dagger, \ldots, X_n, X_n^\dagger]$ where $X^\dagger$ is a variable to be interpreted in $\mathbb{F}$ as the conjugate a^* if X is interpreted as a.

t:IntStarField

Theorem 3.18. *a) Given $d \geq 3$ and finitely many $*$-polynomials $p_j \in \mathbb{Z}[X_1, X_1^\dagger, \ldots, X_n, X_n^\dagger]$, a Turing machine can within time polynomial in d and in the length of the binary encoding of the p_j's coefficients and monomials produce an orthologic term $t_{\bar{p},d}$ such that, for every $*$-field $\mathbb{F} \subseteq \mathbb{C}$ and every d-dimensional $\mathbb{F}$-unitary vector space $\mathcal{H}$ admitting an equinormal orthogonal basis, the p_j have a common root in $\mathbb{F}$ iff $t_{\bar{p},d}$ admits a strongly satisfying assignment in $\mathrm{L}(\mathcal{H})$.*
b) For every $\mathbb{F}$-unitary vector space $\mathcal{H}$ of dimension $d \geq 3$ admitting an equinormal orthogonal basis, $\mathrm{SAT}_{\mathrm{L}(\mathcal{H})}$ is $\mathrm{BP}(\mathcal{NP}^0_{\mathrm{Re}\,\mathbb{F}})$–complete.
c) For $d \geq 3$, satisfiability with constants $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d),\mathrm{L}(\mathbb{F}^d)}$ is $\mathcal{NP}_{\mathrm{Re}\,\mathbb{F}}$–complete.

Proof. a) In view of Lemma 3.17 extending Fact 3.10, modify the algorithm in the proof of Proposition 3.12 as follows:

v') Add equations from Definitions 3.6, 3.14, and 3.15 for $\bar{A}$ to constitute an orthonormal d-frame.

iv') For each variable X_i add conditions $X_i \wedge A_2 = \mathbf{0}$ and $X_i \vee A_2 = A_1 \vee A_2$. Do similarly for each of the newly introduced variables $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ and add equations $Y_k \otimes_{\bar{A}} Z_k = \ominus_{\bar{A}} A_{12}$ and $X_k = \neg Z_k \wedge (A_1 \vee A_2)$. Finally replace any starred occurrence $X_i^\dagger$ of X_i with Y_i.

b) by reduction from $\mathrm{FEAS}_{\mathbb{Z},\mathrm{Re}\,\mathbb{F}}$: To the given polynomials $p_j(X_1, \ldots, X_n)$ over $\mathbb{Z}$ add $*$-polynomial equations $X_i = X_i^\dagger$ $(1 \leq i \leq n)$, thus restricting solutions from $\mathbb{F}$ to $\mathrm{Re}\,\mathbb{F}$. Now apply a) to conclude $\mathrm{BP}(\mathcal{NP}^0_{\mathrm{Re}\,\mathbb{F}})$–hardness and combine with Proposition 2.17c).

c) Combine b) with the (proof of) Proposition 3.12b), i.e. use the constants permitted in the term to encode the fixed orthonormal standard frame $\bar{A}$ as well as the polynomials' coefficients' $c \in \mathbb{F}$ by constants $C := \Theta_{\bar{A}}(c)$: this proves $\mathcal{NP}_{\mathrm{Re}\,\mathbb{F}}$–hardness, for the upper bound recall Proposition 2.17d). □

c:complexBSSNPc

Corollary 3.3. $\mathrm{SAT}_{\mathrm{L}(\mathbb{C}^d)}$ *is* $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$*–complete; and* $\mathrm{SAT}_{\mathrm{L}(\mathbb{C}^d),\mathrm{L}(\mathbb{C}^d)}$ *is* $\mathcal{NP}^0_{\mathbb{R}}$*–complete. For any fixed* $\mathbb{F}$ *and* $k, d \geq 3$, $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d)}$ *is polynomial-time equivalent to* $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^k)}$.

d:ax

Digression 3.19 *a) From Theorem 3.18 one can derive, for any $d \geq 4$, an axiomatization of the first order theory of $\mathrm{L}(\mathbb{R}^d)$ — namely as that of modular ortholattice of height d admitting an orthonormal d-frame such that the coordinate $*$-ring $\mathcal{R}$ is a real closed field with identity involution. This follows as a special case of the coordinatization of projective spaces and the result of [BiNe36] cmp.* [Her10b, Thm.13.2]. *For $\mathrm{L}(\mathbb{C}^d)$ the coordinate $*$-ring $\mathcal{R}$ is required to be a field having real closed subfield $\mathcal{R}_0 = \{x \in \mathcal{R} \mid x = x^*\}$ and an element i such that $i^2 = -1$, $\mathcal{R} = \mathcal{R}_0 + i\mathcal{R}_0$, and $(a + ib)^* = a = -ib$. Of course, there is no finite axiomatization.*

b) Concerning axiomatization, for $d = 3$ the modular law has to be replaced by the stronger Arguesian law. The easiest cases are $d = 1$ (all height 1 ortholattices) and $d = 2$ (all infinite height 2 ortholattices).

c) The question of axiomatizating the universal or equational theory in fixed dimension $d \geq 3$ leads to open problems cmp. Digression 5.12. In contrast, the equational theory of $\{\mathrm{L}(\mathbb{F}^d) \mid d < \infty\}$ can be recursively axiomatized (whence decided) for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ cmp. Fact 2.15, Section 7, and [HRSZ11]. On the other hand, the universal (Horn) theory of this class is not recursively axiomatizable (cmp. [Her10a, Corollary 7.2]*).*

4 More Background from Linear Algebra, Logic, and Modular Orthologic

s:Facts

We recall some definitions and results from Linear Algebra (Subsection 4.1) and Logic (Subsection 4.3) to be referred to in the sequel. Also, we discuss (Subsection 4.2) a particularly simple instance of the concept of interpretation, relational interpretation, which captures the idea of evaluating a term by keeping track of the intermediate results (and hint upon the general concept and the fact that many of our results may be seen as instances). The main contribution is adapting (in Subsection 4.4) a variant of the concept of d-frames, Huhn's d-diamonds, to capture (in Subsection 4.5) the quantifier free part of the first order theory of d-dimensional MOLs by equations, quantified existentially or, by choice, universally.

4.1 Endomorphism Rings, Adjoints, and Matrix Units

ss:endo

Matrices form a non-commutative ring naturally equipped with transposition (and possibly complex conjugation). For any $\mathbb{F}$-vector spaces U, V let $\mathrm{Hom}(U, V)$ denote the set of all $\mathbb{F}$-linear maps $\phi : U \to V$. In particular, $\mathrm{End}(U) = \mathrm{Hom}(U, U)$ is the endomorphim ring of U.

Given a basis $\vec{v}_1, \ldots, \vec{v}_d$ of $\mathcal{H}$, there is a canonical isomorphism of $\mathrm{End}(\mathcal{H})$ onto the matrix ring $\mathbb{F}^{d \times d}$, the matrix A of ϕ has as its columns the coordinate vectors of the $\phi(\vec{v}_j)$.

Recall that (cmp. [Rowe88, §2.13], [Fare01, §5.1]) a $*$-ring is a ring R equipped with an involution $r \mapsto r^*$, i.e. a map such that $(r^*)^* = r$, $(r+s)^* = r^* + s^*$, and $(rs)^* = s^* r^*$. Thus, the ring $\mathbb{F}^{d \times d}$ of d-by-d-matrices over the $*$-subfield $\mathbb{F}$ of $\mathbb{C}$ becomes a $*$-ring with the involution $A \mapsto B = A^*$, the conjugate transpose of A, i.e. $b_{ij} = \bar{a}_{ji}$. Occasionally, we write $A^\dagger = A^*$. In the setting of $\mathbb{F}$-unitary spaces, the involution is given as adjunction.

d:adjoints

Definition 4.1 *(Cf.* [Gelf61, §11-13]*,* [Halm58, §73]*,* [Axle96, Ch.7]*,* [Fare01, 1.20]*).*

a) Given subspaces U, V of $\mathcal{H}$, linear maps $\phi : U \to V$ and $\psi : V \to U$ are adjoint to each other if $\langle \vec{u} \mid \phi(\vec{u}) \rangle = \langle \psi(\vec{v}) \mid \vec{v} \rangle$ for all $\vec{u} \in U$, $\vec{v} \in V$.
b) $\phi : U \to V$ is an isometry if $\langle \vec{x} \mid \vec{y} \rangle = \langle \phi(\vec{x}) \mid \phi(\vec{y}) \rangle$ for all $\vec{x}$, $\vec{y} \in U$.
c) An idempotent in a ring is an element e such that $e^2 = e$.
d) Dealing with $*$-rings, call r selfadjoint if $r = r^*$.
e) An idempotent e is a projection if it is also selfadjoint.

f:adjointa

Fact 4.2 *Consider $U, V, W \in \mathrm{L}(\mathcal{H})$, $\mathcal{H}$ a finite-dimensional $\mathbb{F}$-unitary space.*

a) If $\phi \in \mathrm{Hom}(U, V)$ has an adjoint ψ then ψ is unique and denoted by ϕ^*.
b) If $\phi \in \mathrm{Hom}(U, V)$ and $\psi \in \mathrm{Hom}(V, W)$ have adjoints, then so does $\psi \circ \phi \in Hom(U, W)$ and it holds $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
c) Any $\phi \in \mathrm{End}(\mathcal{H})$ has an adjoint ϕ^*: If ϕ has matrix A w.r.t. the orthogonal basis $\vec{v}_1, \ldots, \vec{v}_d$ then ϕ^* has matrix $D^{-1} \cdot A^\dagger \cdot D$, D diagonal with diagonal entries $\langle \vec{v}_i \mid \vec{v}_i \rangle$.
d) $\mathrm{End}(\mathcal{H})$ is a $*$-ring with adjunction as involution.
e) Any equinormal orthogonal basis $\vec{v}_1, \ldots, \vec{v}_d$ of $\mathcal{H}$ establishes an isomorphism of $*$-rings $\mathrm{End}(\mathcal{H}) \ni \phi \mapsto A \in \mathbb{F}^{d \times d}$ such that A is the matrix of ϕ with respect to this basis.
f) $\psi \in \mathrm{Hom}(U, V)$ is an isometry if and only if it is a linear isomorphism and $\phi^{-1} = \phi^*$.
g) For any subspace U of $\mathrm{L}(\mathcal{H})$ there is a unique projection π such that $U = \mathrm{range}(\pi)$; moreover $\neg U = \mathrm{range}(\mathrm{id} - \pi)$ and, for any projection ϕ, $U \subseteq \mathrm{range}\, \phi$ if and only if $\phi | U = \mathrm{id}_U$ if and only if $\pi \circ \phi = \pi$ if and only if $\phi \circ \pi = \pi$.

Proof. a) to g) are standard Linear Algebra.

Matrix units provide a well known method to establish an isomorphism of a ring or $*$-ring onto a suitable matrix ring.

x:matunit

Example 4.3 *For $1 \leq i, j \leq d$ let E_{ij} denote the $d \times d$-matrix mapping the i-th canonical basis vector onto the j-th and the others to $\vec{0}$, i.e.*

$$E_{ij} = (\delta_{i\ell} \cdot \delta_{jk})_{k\ell} \quad \textit{where } \delta_{i\ell} = \begin{cases} 1 & \textit{if } i = \ell \\ 0 & \textit{if } i \neq \ell \end{cases}$$

Note the deliberate transposition implicit in this example to make it comply with

d:matunit

Definition 4.4 *In a fixed (not necessarily commutative) ring R, elements $(\varepsilon_{ij} \mid 1 \leq i, j \leq d)$ form a d–system $\bar{\varepsilon}$ of matrix units if*

$$\sum\nolimits_{i=1}^{d} \varepsilon_{ii} = 1, \quad \varepsilon_{k\ell} \cdot \varepsilon_{ij} = \delta_{jk} \varepsilon_{i\ell} \tag{6}$$

In particular, the ε_{ii} are idempotents. For R a $*$-ring, $\bar{\varepsilon}$ is a d-system of $*$-matrix units if, in addition, $\varepsilon_{ij}^* = \varepsilon_{ji}$ for all i, j; in particular, the ε_{ii} are projections.

Following common (though not reasonable) use to write maps $\phi : X \to Y$ as $\phi(x) = y$ and composition as $\psi\big(\phi(x)\big) = \big(\psi \circ \phi\big)(x)$, on indexed domains X_i this naturally leads to $\phi_{ij} : X_i \to X_j$ and composition $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$.

f:matunit

Fact 4.5 *a) Given an idempotent (projection) $\varepsilon \in \mathrm{End}(\mathcal{H})$ one has a direct (orthogonal) decomposition $\mathcal{H} = \mathrm{range}(\varepsilon) \oplus \mathrm{range}(\mathrm{id} - \varepsilon)$ and obtains a ring ($*$-ring) $\varepsilon \circ \mathrm{End}(\mathcal{H}) \circ \varepsilon := \{\varepsilon \circ \phi \circ \varepsilon \mid \phi \in \mathrm{End}(\mathcal{H})\}$, with unit ε while all other structure is inherited from $\mathrm{End}(\mathcal{H})$, which is isomorphic to $\mathrm{End}(\mathrm{range}\,\varepsilon)$.*

b) Given a d-system $(\varepsilon_{ij} \mid 1 \leq i, j \leq d)$ of matrix ($*$-matrix) units in $\mathrm{End}(\mathcal{H})$ one has $\mathcal{H}$ a (orthogonal) direct sum of the $U_i = \mathrm{range}(\varepsilon_{ii})$, and the restrictions $\eta_{ij} := \varepsilon_{ij}|U_j$ isomorphisms (isometries) $\eta_{ij} : U_i \to U_j$ such that $\eta_{jk} \circ \eta_{ij} = \eta_{ik}$. In particular $\eta_{ii} = \mathrm{id}_{U_i}$ and $\eta_{ji} = \eta_{ij}^{-1}$ $(= \eta_{ij}^*)$ and $\dim(U_i)$ does not depend on i. Conversely, given such U_i and η_{ij}, one obtains a d-system of matrix ($*$-matrix) units via $\varepsilon_{ij} := \eta_{ij} \circ \pi_i$ where $\pi_i(\vec{v}) = \vec{u}_i$ for $\vec{v} = \sum_j \vec{u}_j$ with $\vec{u}_j in U_j$.

c) Any (orthogonal equinormal) basis $\vec{v}_{ik}$ $(1 \leq i \leq d; 1 \leq k \leq n)$ of $\mathcal{H}$ determines a d-system of matrix ($*$-matrix) units where $U_i = \sum_{k=1}^n \mathbb{F}\vec{v}_{ik}$ and $\varepsilon_{ij}(\vec{v}_{ik}) = \vec{v}_{jk}$. If $d = \dim \mathcal{H}$, every d-system of matrix ($*$-matrix) units arises in this way.

d) Given a d-system $\bar{\varepsilon}$ of matrix ($*$-matrix) units in $\mathrm{End}(\mathcal{H})$ where $\dim(\mathcal{H}) = m < \infty$, one has unique ℓ with $m = \ell d$ and a ring ($*$-ring) isomorphism of $\mathbb{F}^{\ell \times \ell}$ onto $\mathrm{End}(\mathrm{range}\,\varepsilon_{11})$.

e) If ε is a d-system of matrix ($*$-matrix) units in $\mathrm{End}(\mathcal{H})$ and $\dim(\mathcal{H}) = d$ then $a \mapsto a\,\mathrm{id}$ is an ring ($*$-ring) isomorphism of $\mathbb{F}$ onto $\{\phi \in \mathrm{End}(\mathcal{H}) \mid \phi \circ \varepsilon_{ij} = \varepsilon_{ij} \circ \phi$ for all $i, j\}$.

Proof. a) Clearly, $\mathrm{id} - \varepsilon$ is an idempotent (projection) , too, and yields the decomposition (which is orthogonal if ε is a projection). Moreover, $\varepsilon \circ \mathrm{End}(\mathcal{H}) \circ \varepsilon$ is a ($*$-) subring (except unit) of $\mathrm{End}(\mathcal{H})$. Mutually inverse isomorphisms between this and $\mathrm{End}(U)$, $U := \mathrm{range}(\varepsilon)$, are given by

$$\varepsilon \circ \phi \circ \varepsilon \mapsto (\varepsilon \circ \phi \circ \varepsilon)|U, \quad \psi \mapsto \hat{\psi} \quad \text{where } \hat{\psi}(\vec{u} + \vec{v}) := \psi(\vec{u}) \text{ for } \vec{u} \in U,\ \vec{v} \in \mathrm{range}(\mathrm{id} - \varepsilon)$$

b) Given a d-system of ($*$-) matrix units, the first condition yields $\vec{x} = \sum_{i=1}^d \varepsilon_{ii}(\vec{x})$ whence $\mathcal{H} = \sum_i U_i$ and this sum is direct (orthogonal) by the second condition. Moreover, given $*$-matrix units, for $i \neq j$ we have $\langle \varepsilon_{ii}(\vec{x}) \big| \varepsilon_{jj}(\vec{y}) \rangle = \langle \big(\varepsilon_{ji} \circ \varepsilon_{jj} \circ \varepsilon_{ij}\big)(\vec{x}) \big| \varepsilon_{jj}(\vec{y}) \rangle = \langle \big(\varepsilon_{jj} \circ \varepsilon_{ij}\big)(\vec{x}) \big| \big(\varepsilon_{ji}^* \circ \varepsilon_{jj}\big)(\vec{y}) \rangle = 0$ since $\varepsilon_{ji}^* = \varepsilon_{ij}$ and $\varepsilon_{ij} \circ \varepsilon_{jj} = 0$: $U_i \perp U_j$. Also, $\eta_{ij}^* = \eta_{ji}$ since $\varepsilon_{ij}^* = \varepsilon_{ji}$. The remaining claims follow from similar calculations. c) and d) are then obvious. In e) choose a basis according to c) thus turning ε_{ij} into E_{ij}.

f:lat

Fact 4.6 *Let $\mathbb{F}$ be a $*$-subfield of $\mathbb{C}$.*

a) Any subspace U of $\mathbb{F}^d$ is the range of some $A \in \mathbb{F}^{d \times d}$.

b) For any $A, B \in \mathbb{F}^{d \times d}$ one has $\exists X \in \mathbb{F}^{d \times d}$, $A = BX \;\Rightarrow\; \mathrm{range}(A) \subseteq \mathrm{range}(B)$.

c) For $A, B, C \in \mathbb{F}^{d \times d}$ it holds: $\mathrm{range}(C) = \mathrm{range}(A) \vee \mathrm{range}(B) \quad \Leftrightarrow$

$$\exists W, X, Y, Z \in \mathbb{F}^{d \times d} : \; C = A \cdot X + B \cdot Y \;\;\&\&\;\; A = C \cdot W \;\;\&\&\;\; B = C \cdot Z \;\;.$$

d) For $A, C \in \mathbb{F}^{d \times d}$ it holds

$$\mathrm{range}(C) = \neg\,\mathrm{range}(A) \quad \Leftrightarrow \quad \exists X, Y \in \mathbb{F}^{d \times d} : \; A^\dagger \cdot C = O \;\;\&\&\;\; A \cdot X + C \cdot Y = I \;\;.$$

Similarly, for finite dimensional $\mathbb{F}$-unitary spaces $\mathcal{H}$ and their endomorphism $*$-rings $\mathrm{End}(\mathcal{H})$.

Proof. In a) choose $\mathbb{F}^d = U \oplus V$ and let $A(\vec{u} + \vec{v}) = \vec{u}$ for $\vec{u} \in U$ and $\vec{v} \in V$.
b) This writes the generators of range(A) as linear combinations of those of range(B). In c), the first condition in the formula is equivalent to the columns of C being linear combinations of those of $(A|B)$, i.e. to range$(C) \subseteq$ range$(A) \vee$ range(B); while in view of b) the converse inclusion is equivalent to the conjunction of the last two conditions. The condition in d) tells range(C) orthogonal to range(A) and joining with it to $\mathbb{F}^d$. This characterizes the orthocomplement of range(A). The reasoning carries over to abstract spaces and their endomorphism $*$-rings; in a)–c) we may use any basis, then in d) there is no more need to refer to a basis.

4.2 Relational Interpretations

ss:Interpret

A particularly useful kind of reduction is implicit in the proof of the Cook-Levin-Theorem and in its adaptation to the Blum-Shub-Smale Model [Cuck93, p.403]: here the evaluation of a term is decomposed into a polynomial-size, existentially quantified system of equations together with a simple term providing the final value:

x:RingInterp

Example 4.7 *Fix some commutative ring R with unity.*
The evaluation of a k-variate polynomial $p \in R[x_0, x_{-1}, \ldots, x_{-k+1}]$ decomposes into a series of basic binary operations $+$, $\times$, and constants (i.e. 0-ary) $c \in R$. More precisely, a straight-line program Γ of length N over R calculating $R^k \ni \bar{r} \to p(\bar{r}) \in R$ consists of a sequence of assignments "$x_n := f_n(x_{n_1}, \ldots, x_{n_{k_f}})$" ($n = 1, \ldots, N$) each applying a function f_n of arity $k_{f_n} =: k_n$ from R's signature to previous intermediate results x_{n_i} ($-k < n_i < n$) such that $x_N = p(\bar{x})$ yields the final result. That is, for each choice of $r_0, r_{-1}, \ldots, r_{-k+1}, y \in R$, the following system of equations in variables $x_1, \ldots, x_N$ is satisfiable over R (and uniquely so) iff $p(\bar{r}) = y$ holds:

$$y = x_N \quad \&\& \quad x_n = f_n(x_{n_1}, \ldots, x_{n_{k_n}}), \quad n = 1, \ldots, N \ . \tag{7}$$

Conversely if R is ordered, any such system can be combined into a single equation: $\&\&_{n=1}^N p_n(\bar{x}) = q_n(\bar{x})$ for polynomials p_n, q_n is equivalent to $0 = \sum_{n=1}^N \big(p_n(\bar{x}) - q_n(\bar{x})\big)^2$. In the case of a $*$-subfield $\mathbb{F} \not\subseteq \mathbb{R}$ of $\mathbb{C}$ use $0 = \sum_{n=1}^N \big(p_n(\bar{x}) - q_n(\bar{x})\big)^* \big(p_n(\bar{x}) - q_n(\bar{x})\big)$.

This motivates the following.

d:RelInterp

Definition 4.8 *(*[Hodg93, §2.6.1]*) The **relational interpretation** Ψ_σ for a signature σ associates with any term $t = t(\bar{x})$ a conjunction $\Psi_\sigma(t)$ of equations with **output variable** X_t according to the following recursive definition: $\Psi_\sigma(x)$ is $X_x = x$ for any variable x; for any k-ary operation symbol f and $t = f(t_1, \ldots, t_k)$ the formula $\Psi_\sigma(t)$ is the conjunction of $X_t = f(z_1, \ldots, z_k)$ and the $\Psi_\sigma(t_i)(z_i/X_{t_i})$ with new auxiliary variables z_i substituted for the X_{t_i}. The relational interpretation $\Psi_\sigma(s = t)$ of an equation $s = t$ is then the conjunction of $\Psi_\sigma(s)$, $\Psi_\sigma(t)$ and $X_s = X_t$.*

For example, the relational interpretation of $t = \big(x_1 \wedge (x_2 \vee \neg x_1)\big) \vee (x_2 \vee \neg x_1)$ is

$$X_t = z_1 \vee z_2 \ \&\& \ z_1 = x_1 \wedge z_3 \ \&\& \ z_2 = x_2 \vee z_4 \ \&\& \ z_3 = x_2 \vee z_5 \ \&\& \ z_4 = \neg x_1 \ \&\& \ z_5 = \neg x_1.$$

o:RelInterp

Observation 4.9 *The relational interpretation Ψ_σ captures the term $t = t(\bar{x})$ within the class of all structures of signature σ: for all a_0 and $\bar{a}$ in $\mathcal{A}$, $a_0 = t_{\mathcal{A}}(\bar{a})$ if and only if $\mathcal{A} \models \exists\, \Psi_\sigma(t)(a_0; \bar{a})$ where quantification is over all auxiliary variables. Given (a suitable encoding* *of) a system of equations $s_i(\bar{x})=t_i(\bar{x})$ $(1\leq i\leq I)$ with terms s_i,t_i of some signature σ, their image under the relational interpretation Ψ_σ can be calculated by a polynomial-time Turing machine to produce a system $y_j=f_j(\bar{y})$ $(1\leq j\leq J)$ of equations of signature σ such that*

- *the original system is satisfiable in a structure $\mathcal{S}$ of signature σ iff the new system is*
- *the new system is basic in the sense that each equation has a variable symbol on the left and one function symbol on the right.*

In Example 4.7, observing that the p_n in Equation (7) are just pairwise distinct variables and the q_n are either linear of quadratic polynomials (w.l.o.g. with coefficients $0,\pm1$ according to Observation 3.13) it follows the well-known

f:RingInterp

Fact 4.10 *For any ordered field $\mathbb{F}$, $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}}$ is polynomial-time equivalent to the question of whether a list of quadratic integer polynomials (w.l.o.g. with coefficients $0,\pm1$) has a joint root; which in turn is polynomial-time equivalent to that of a single quartic integer polynomial $P(X_1,\ldots,X_N)$ (with coefficients $0,\pm1,\ldots,\pm N$) having a root.*

Indeed, the squared polynomials $(p_n-q_n)^2$ $(1\leq n\leq N)$ arising at the end of Example 4.7 involve in expanded form only coefficients $0,\pm1,\pm2$; which in $Q=\sum_{n\leq N}(p_n-q_n)^2$ cannot add up beyond $\pm N$.

Similar reformulations in the structure of ortholattices will be employed extensively in Theorem 5.5 below.

r:Interp

Remark 4.11 *Reductions respecting truth had been employed in logic at least since [TMR53] and have become ubiquitous in complexity theory with [Cook71,Karp72]. For counting problems, they have been generalized to parsimonious reductions [Vali79]. And for more general (i.e. not necessarily integer-valued) function problems $f:S\to T$, e.g. in algebraic complexity theory* [Bürg96, Remark 4.4], *morphisms give rise to a notion of reduction—provided that S and T share the same signature.*

Subsection 3.4 however 'reduces' $(\mathbb{R},+,-,0,\cdot,1)$ to $\big(\mathrm{L}(\mathbb{R}^3),\vee,\wedge,\neg,\mathbf{0},\mathbf{1}\big)$: clearly structures of very different signatures. Moreover the mapping $\Theta_{\bar{A}}:\mathbb{R}\to\mathrm{L}(\mathbb{R}^3)$ depends on the frame $\bar{A}$. This can be seen as an instance of the quite general concept *interpretation* regularly employed in Model Theory [BuSa81, §5.5] and [Hodg93, §5.3]. We have relied in Theorem 3.18 on the famous *coordinatization* due to von Neumann, one of many results in Geometry and Algebra which can be understood as interpretations. Another such result is the isomorphism between $\mathrm{L}(\mathbb{F}^d)$ and the ortholattice of (principal) right ideals of the matrix $*$-ring $\mathbb{F}^{d\times d}$ — related to the use of Fact 4.6 to prove Proposition 2.17 by reduction to $\mathrm{FEAS}_{\mathbb{Z},\mathrm{Re}\,\mathbb{F}}$.

d:Interp

Digression 4.12 *Such interpretations also provide translations from sentences in one language (e.g. fields) into another (e.g. lattices) which often are actually polynomial time reductions. E.g. the case of fixed $d\geq3$ in Theorem 5.3 may be recovered from interpretations yielding polynomial time equivalence between any prenex sentences in the language of $*$-rings w.r.t validity in $\mathbb{F}$ to such in the language of ortholattices w.r.t validity in $\mathrm{L}(\mathbb{F}^d)$ — preserving alternation type of quantifications and uniform in $\mathbb{F}$. Here, in view of Lemma 4.20 we may assume that ortholattice sentences ϕ have matrix of the form $t(\bar{x})=\mathbf{1}$. Such ϕ is translated to a prenex sentence in $*$-ring language – refering to $\mathbb{F}^{d\times d}$, first, and then proceeding to $\mathbb{F}$. Depending on the innermost quantifier of ϕ one has to use the existentially respectively the universally quantified variant of relational interpretation (cmp.* [Hodg93, §2.5.1]*), the first combined with* *the interpretation given by Fact 4.6, the second with the interpretation of* $\mathrm{L}(\mathbb{F}^d)$ *as the ortholattice of projections in* $\mathbb{F}^{d\times d}$ *(cmp. Section 7). In the converse direction, given a sentence* ψ *in* $*$*-ring language proceed as in the proof of Propositiom 3.12, using in addition the term* $x^{\dagger_{\bar{y}}}$ *of Lemma 6.12 capturing conjugation, to translate atomic subformulas whence matrix of* ψ *into the matrix of* ψ' *in ortholattice language. Then, as in the proof of Theorem 5.3. introduce the orthonormal d-frame* $\bar{y}$ *by bounded existential or universal quantification put in front of* ψ' *and convert quantifiers of* ψ' *into bounded ones to encode the conditions* $x \in \mathcal{R}_{\bar{y}}$*; pass from there to prenex form by Lemma 4.13 or, more efficiently, avoid bounded quantifications using terms based on [MaRo87], evaluating in* $\mathrm{L}(\mathbb{F}^d)$ *to an orthonormal d-frame or to* $\mathbf{0}$*, and an ortholattice term* $r(x,\bar{y})$ *such that* $r(x,\bar{a}) \in \mathcal{R}_{\bar{a}}$ *and* $x = r(x,\bar{a}) \Leftrightarrow x \in \mathcal{R}_{\bar{a}}$ *for all orthogonal d-frames. (cmp. [HRSZ11]).*

Finally, we mention the particularly useful interpretations of the field $\mathrm{Re}\,\mathbb{F}$ *in the* $*$*-subfield* $\mathbb{F}$ *of* $\mathbb{C}$ *and vice versa. In order to comply with the BSS-machine concept, we derive most results for real fields, first, and then use this equivalence to read them as results about real parts. But, working with interpretations more directly, one could deal with* $*$*-fields* $\mathbb{F}$ *(including the real case), primarily, and then derive the results involving real parts* $\mathrm{Re}\,\mathbb{F}$*.*

4.3 Some Tools from Logic and Orthologic

ss:tool

Recall the bounded quantifiers

$$(\exists \bar{x}\alpha(\bar{x})).\ \phi(\bar{x},\bar{y})\ :\Leftrightarrow\ \exists \bar{x}.\ \alpha(\bar{x}) \,\&\&\, \phi(\bar{x},\bar{y}) \quad \text{and} \quad (\forall \bar{x}\alpha(\bar{x})).\ \phi(\bar{x},\bar{y})\ :\Leftrightarrow\ \forall \bar{x}.\ \alpha(\bar{x}) \supset \phi(\bar{x},\bar{y}).$$

We use the old fashioned $\phi \supset \psi$ as a shorthand notation for logical implication $!\,\phi \,||\, \psi$.

l:bdquant

Lemma 4.13 *Let* $\phi(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y})$ *be a first order formula (all listed variables pairwise distinct) and* ψ *given as*

$$\big(\mathrm{Q}_1\,\bar{x}^1\alpha_1(\bar{x}^1)\big)\,\big(\mathrm{Q}_2\,\bar{x}^2\alpha_2(\bar{x}^1,\bar{x}^2)\big)\,\big(\mathrm{Q}_3\,\bar{x}^3\alpha_3(\bar{x}^1,\bar{x}^2,\bar{x}^3)\big)\ldots\big(\mathrm{Q}_\ell\,\bar{x}^\ell\alpha_l(\bar{x}^1,\ldots,\bar{x}^\ell)\big).\,\phi(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y})$$

where $\mathrm{Q}_i \in \{\forall,\exists\}$*. Then there is a Boolean combination* β *(to be determined in polynomial time) of the* α_i *and* ϕ *such that* ψ *is logically equivalent to*

$$\mathrm{Q}_1\,\bar{x}^1\,\mathrm{Q}_2\,\bar{x}^2\ldots\mathrm{Q}_\ell\,x^\ell.\ \beta(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y}).$$

Proof. Recall the logical equivalences

$$\alpha \,\&\&\, \mathrm{Q}\,\bar{x}\beta \Leftrightarrow \mathrm{Q}\,\bar{x}\,(\alpha \,\&\&\, \beta), \quad \alpha \supset \mathrm{Q}\,\bar{x}\beta \Leftrightarrow \mathrm{Q}\,\bar{x}\,(\alpha \supset \beta)$$

where $\mathrm{Q} \in \{\forall,\ \exists\}$ and no variable form $\bar{x}$ does occur freely in α - for the easy proof one may assume that the $\bar{x}$ are the only variables occuring freely. Proceeding by induction on ℓ we have

$$\big(\mathrm{Q}_2\,\bar{x}^2\alpha_2(\bar{x}^1,\bar{x}^2)\big)\,\big(\mathrm{Q}_3\,\bar{x}^3\alpha_3(\bar{x}^1,\bar{x}^2,\bar{x}^3)\big)\ldots\big(\mathrm{Q}_\ell\,\bar{x}^\ell\alpha_\ell(\bar{x}^1,\ldots,\bar{x}^\ell)\big).\,\phi(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y})$$

logically equivalent to

$$\mathrm{Q}_2\,\bar{x}^2\,\mathrm{Q}_3\,\bar{x}^3\ldots\mathrm{Q}_\ell\,\bar{x}^\ell.\ \chi(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y})$$

where χ is a boolean combination of ϕ and the $\alpha_2,\ldots,\alpha_\ell$. Thus, depending on its first quantifier, ψ is logically equivalent to one of

$$\begin{array}{lcl}\exists\bar{x}^1.\ \alpha_1(\bar{x}^1) & \&\& & \mathrm{Q}_2\,\bar{x}^2\,\mathrm{Q}_3\,\bar{x}^3\ldots\mathrm{Q}_\ell\,\bar{x}^\ell.\ \chi(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y}) \\ \forall\bar{x}^1.\ \alpha_1(\bar{x}^1) & \supset & \mathrm{Q}_2\,\bar{x}^2\,\mathrm{Q}_3\,\bar{x}^3\ldots\mathrm{Q}_\ell\,\bar{x}^\ell.\ \chi(\bar{x}^1,\ldots,\bar{x}^\ell,\bar{y})\end{array}$$

and these in turn by the above equivalence rules to

$$\begin{array}{lcl} \exists\bar{x}^1\, \mathrm{Q}_2\, \bar{x}^2\, \mathrm{Q}_3\, \bar{x}^3 \ldots \mathrm{Q}_\ell\, \bar{x}^\ell .\ \alpha_1(\bar{x}^1) & \texttt{\&\&} & \chi(\bar{x}^1, \ldots, \bar{x}^\ell, \bar{y}) \\ \forall\bar{x}^1\, \mathrm{Q}_2\, \bar{x}^2\, \mathrm{Q}_3\, \bar{x}^3 \ldots \mathrm{Q}_\ell\, \bar{x}^\ell .\ \alpha_1(\bar{x}^1) & \supset & \chi(\bar{x}^1, \ldots, \bar{x}^\ell, \bar{y}) \end{array}$$

having a form as required.

The following extends Lemma 2.12b) and Observation 2.6b): l:2Dx

Lemma 4.14 *a) For a first-order sentence ϕ in the language of ortholattices with n (bound) variables, the following are equivalent:*

i) $L \models \phi$ for some/any infinite MOL L of dimension 2 *ii) $\mathcal{MO}_n \models \phi$*

b) For a term $t(x_1, \ldots, x_n)$ in the language of ortholattices and MOLs L, L', the sentences ϕ and χ given as $\mathrm{Q}_1\, x_1\, \mathrm{Q}_2\, x_2 \ldots \mathrm{Q}_n\, x_n : t(\bar{x}) = \mathbf{1}$ and $\mathrm{Q}_1\, x_1\, \mathrm{Q}_2\, x_2 \ldots \mathrm{Q}_n\, x_n : t(\bar{x}) \neq \mathbf{0}$, respectively, have

$$L \models \phi \ \textit{ and } \ L' \models \phi \ \Leftrightarrow \ L \times L' \models \phi \qquad \textit{and} \qquad L \models \chi \ \textit{ or } \ L' \models \chi \ \Leftrightarrow \ L \times L' \models \chi$$

Proof. a) Fix an infinite MOL L of dimension 2 and add all its elements a as constants $\underline{a}$ to the language. For a sentence ϕ in this language let n_ϕ be total number of variables and new constants occurring in ϕ. Given ϕ, consider the system $\mathcal{S}_\phi$ of all sub-ortholattices S of L of size $|S| = 2n_\phi + 2$ and such that S contains all a such that $\underline{a}$ occurs in ϕ. We consider $S \in \mathcal{S}_\phi$ in an extended signature where constants $\underline{a}$ occurring in ϕ are interpreted as $a \in S$. We show by induction on the length of the formula ϕ (with logical connectives $\texttt{!}$, $\texttt{\&\&}$, and $\exists x$) that the following are equivalent

$L \models \phi$, $S \models \phi$ for some $S \in \mathcal{S}_\phi$, $S \models \phi$ for all $S \in \mathcal{S}_\phi$.

The last equivalence is immediate by the fact that all $S \in \mathcal{S}_\phi$ are isomorphic. This follows from (the proof of) Lemma 2.12); as does the remaining equivalence if ϕ is an atomic sentence. In the inductive step consider ϕ given as $\phi_1 \,\texttt{\&\&}\, \phi_2$ and choose $S \in \mathcal{S}_{\phi_1} \cap \mathcal{S}_{\phi_2} = \mathcal{S}_\phi$; then the claim follows, readily. Even more so ϕ being $\texttt{!}\ \psi$ where $\mathcal{S}_\phi = \mathcal{S}_\psi$. Finally, let ϕ be given as $\exists x.\, \psi(x)$. If $L \models \phi$ then $L \models \psi(\underline{a})$ for some $a \in L$. Choose $S \in \mathcal{S}_{\psi(\underline{a})}$ and apply the inductive hypothesis to conclude $S \models \psi(\underline{a})$ whence $S \models \phi$. Conversely, assume that $S \models \phi$ for some $S \in \mathcal{S}_\phi$; then $S \models \psi(\underline{a})$ for some $a \in S$; now $S \in \mathcal{S}_{\psi(\underline{a})}$ whence $L \models \psi(\underline{a})$ by inductive hypothesis

b) Again, it is convenient to add constants to the language: for each $(a, b) \in L \times L'$ let $\underline{(a, b)}$ a constant interpreted as (a, b) in $L \times L'$, as a in L and as b in L'. Observe, that $L \times L'$ is the direct product of L and L' also in the extended signature. We prove the first equivalence for sentences ϕ involving such constants by induction on the number n of quantifiers. For $n = 0$ the definition of direct product applies. Let $n > 0$ and $\phi = \mathrm{Q}_1\, \psi(x_1)$. Then, using induction, $L \times L' \models \phi$ iff $L \times L' \models \psi(\underline{(a, b)})$ (for some respectively all (a, b)) iff $L \models \psi(\underline{(a, b)})$ and $L' \models \psi(\underline{(a, b)})$ iff $L \models \phi$ and $L' \models \phi$. The second equivalence follows by contraposition and converting $\texttt{!}\ \phi$ into prenex form, having replaced equation $t(\bar{x}) = \mathbf{1}$ by $\neg t(\bar{x}) = \mathbf{1}$. □

4.4 Huhn's Diamonds

ss:Huhn

An equivalent concept of frame (more convenient for equational questions) has been established by Huhn [Huhn72]. The idea is to consider instead of the A_{ij} just the single $A_0 := A_{12} + \ldots + A_{1d}$.

From the A_i on can recover A_{1i} as $A_0 \wedge (A_1 \vee A_i)$, $i > 0$: If $\vec{x} \in A_0 \cap (A_1 + A_i)$ then $\vec{x} = x_1 \vec{v}_1 + x_i \vec{v}_i = \sum_{j=2}^{d} y_j (\vec{v}_1 - \vec{v}_j)$ requires $y_j = 0$ for each $j \notin \{1, i\}$. And from A_{1i} and

A_{1j} one obtains A_{ij} as $(A_i + A_j) \cap (A_{1i} + A_{1j})$. The relevant relations between A_0 and the A_i $(i > 0)$ are

$$A_0 \vee A_i = \mathbf{1}, \quad A_0 \wedge A_i = \mathbf{0} \quad \text{for } i = 1, \ldots, d \tag{8}$$

Indeed observe that $\vec{v}_1 \in A_0 + A_i$ whence $\vec{v}_j \in A_0 + A_i$ for all j; and that for $\vec{x} = y_i(\vec{v}_1 - \vec{v}_i) \in A_k$, where $k \in \{1, i\}$, it follows $y_i = 0$. This motivates the following

Definition 4.15 *An orthogonal d-diamond in a MOL, L, is a system $\bar{a} = (a_i \mid 0 \leq i \leq d)$ of elements such that*

$$\begin{array}{ll} \mathbf{1} = a_1 \vee \ldots \vee a_d & \\ a_i \leq \neg a_j & \textit{for all } i, j = 1, \ldots, d,\ i \neq j, \\ \mathbf{0} = a_0 \wedge a_i & \textit{for all } i = 1, \ldots, d, \\ \mathbf{1} = a_0 \vee a_i & \textit{for all } i = 1, \ldots, d. \end{array}$$

Actually, Huhn calls this a $(d-1)$–diamond and requires independence of the $a_1, \ldots, a_d$ in place of orthogonality.

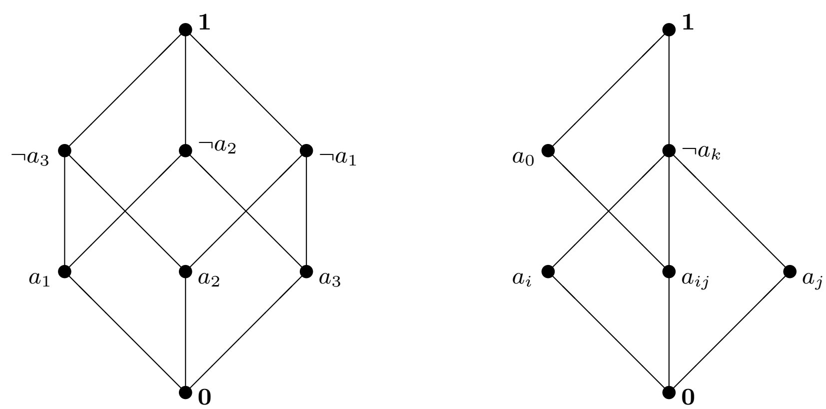

Fig. 5. Relations of a 3-diamond: Order diagrams of relevant sublattices

x:huhn

Example 4.16 *a) For a d-dimensional $\mathbb{F}$-unitary space $\mathcal{H}$, the $(A_0, A_1, \ldots, A_d)$ as above form an orthogonal d-diamond if the basis $\vec{v}_1, \ldots, \vec{v}_d$ is chosen orthogonal (Fact 2.19a).*

b) In every 2-dimensional MOL L, pairwise distinct atoms $a_0, a_1, \neg a_1 =: a_2$ give rise to an orthogonal 2-diamond.

f:Huhn

Fact 4.17 $\dim[a_0, \mathbf{1}] = \dim a_i$ *for all $i > 0$. The claims b) and c) of Fact 3.7 remain valid.*

Consider the following ortholattice terms where $\bar{y} = (y_0, y_1, \ldots, y_d)$.

$$\begin{aligned} h_d(\bar{y}) &= \big(\textstyle\bigvee_{i=1}^{d} (y_i \wedge \bigwedge_{i \neq j > 0} \neg y_j)\big) \wedge \bigwedge_{i=1}^{d} \big((y_0 \vee y_i) \wedge (\neg y_0 \vee \neg y_i)\big), \\ \tilde{h}_d(x, \bar{y}) &= h_d(\bar{y}) \wedge \hat{h}_d \ \text{ where } \hat{h}_d = \textstyle\bigvee_{i,j>0} \Big(\big((x \vee \neg y_i) \wedge y_i\big) \vee y_0\big) \wedge y_j\Big), \\ g_d(\bar{y}) &= \neg y_0 \wedge \textstyle\bigwedge_{i=1}^{d} \big(y_0 \vee (y_i \wedge \bigwedge_{i \neq j > 0} \neg y_j)\big), \\ \tilde{g}_d(x, \bar{y}) &= g_d(\bar{y}) \wedge \textstyle\bigwedge_{i=1}^{d} \big(y_0 \vee (x \wedge y_i)\big) \end{aligned}$$

l:Huhn

Lemma 4.18 *Let L be any MOL, $1 \leq \dim(L) \leq d$; In a), b) and d) let $\dim(L) = d$.*

a) $\bar{a} \in L^{d+1}$ is an orthogonal d-diamond of L if and only if $h_d(\bar{a}) = \mathbf{1}$.

b) *Given an orthogonal d-diamond $\bar{a}$ and b in L, one has $b > \mathbf{0}$ if and only if $\hat{h}_d(b,\bar{a}) = \mathbf{1}$. In particular, if L admits an orthogonal d-diamond, then $b > \mathbf{0}$ if and only if $\tilde{h}_d(b,\bar{y})$ is strongly satisfiable in L.*
c) *$\bar{a} \in L^{d+1}$ is an orthogonal d-diamond of L and $\dim(L) = d$ if and only if $g_d(\bar{a}) > \mathbf{0}$.*
d) *Given an orthogonal d-diamond $\bar{a}$ and b in L, one has $b = \mathbf{1}$ if and only if $\tilde{g}_d(b,\bar{a}) > \mathbf{0}$. In particular, if L admits an orthogonal d-diamond, then $b = \mathbf{1}$ if and only if $\tilde{g}_d(b,\bar{y})$ is weakly satisfiable in L.*

[DHMW05, Theorem 1] and [Hagg07] have presented terms t_d equivalent to 0 over $\mathrm{L}(\mathbb{C}^{d-1})$ but not over $\mathrm{L}(\mathbb{C}^d)$. A careful analysis of their construction reveals the length $|t_d|$ to be exponential in d. Based on Huhn's diamonds, we improve that in the following

c:Hagge

Corollary 4.1. *The term g_d has length quadratic in d. It is equivalent to $\mathbf{0}$ over $\mathrm{L}(\mathbb{F}^{d-1})$ but not over $\mathrm{L}(\mathbb{F}^d)$.*

Indeed, a lattice of height $< d$ does not admit an orthogonal d-diamond whereas, by Example 4.16a), $\mathrm{L}(\mathbb{F}^d)$ does.

Proof (of Lemma 4.18).

a) Observe that for any $i > 0$ one has $a_i \leq \neg a_j$ for all $0 < j \neq i$ if and only if $a_i = a_i \wedge \bigwedge_{0<j\neq i} \neg a_j$.

b) Given an orthogonal d-diamond $\bar{a}$ we have $h_d(\bar{a}) = \mathbf{1}$. Consider $b > \mathbf{0}$. Then $b \not\leq a_i^\perp$ for some $i > 0$ (since $\bigcap_{i=1}^n a_i^\perp = \mathbf{0}$) and $b + a_i^\perp = \mathbf{1}$ since $\dim[a_i^\perp, \mathbf{1}] = 1$. It follows $(b+a_i^\perp) \cap a_i + a_0 = a_i + a_0 = \mathbf{1}$ whence $((b+a_i^\perp) \cap a_i + a_0) \cap a_j = a_j$ for all $j > 0$ and $\tilde{h}_d(b,\bar{a}) = \mathbf{1}$. Conversely, assume $\tilde{h}_d(b,\bar{a}) = \mathbf{1}$. Then $h_d(\bar{a}) = \mathbf{1}$ and $\bar{a}$ is an orthogonal d-diamond. Assuming $b = \mathbf{0}$ one has $(b + a_i^\perp) \cap a_i = a_i^\perp \cap a_i = \mathbf{0}$ and $((a + a_i^\perp) \cap a_i + a_0) \cap a_j = a_0 \cap a_j = \mathbf{0}$ whence $\tilde{h}_d(b,\bar{a}) = \mathbf{0}$, as contradiction.

c) Assume $g_d(\bar{a}) > \mathbf{0}$ and put $b_i := a_i \cap \bigcap_{i\neq j>0} a_j^\perp$ for $i > 0$. Assuming $b_i = \mathbf{0}$ we get $g_d(\bar{a}) \leq a_0^\perp \cap (a_0 + b_i) = a_0 \cap a_0^\perp = \mathbf{0}$, a contradiction. Thus $b_i > \mathbf{0}$. Also, by definition, $b_i \leq a_i$ and $b_i \leq a_j^\perp \leq b_j^\perp$ for $i \neq j > 0$, i.e. the b_i, are pairwise orthogonal. Thus $d \leq \sum_{i=1}^d \dim b_i = \dim \sum_{i=1}^d b_i \leq d$ whence $\dim b_i = 1$, $\dim(L) = d$, and $\sum_{i=1}^d b_i = \mathbf{1}$. From $\mathbf{0} < a_i \leq b_i$ it follows $b_i = a_i$ for $i > 0$. Assuming $a_i \leq a_0$ for some $i > 0$ gives, $g_d(\bar{a}) \leq a_0^\perp \cap (a_0 + a_i) = a_0 \cap a_0^\perp = \mathbf{0}$, a contradiction. Thus, $a_0 \cap a_i = \mathbf{0}$ and $\dim[a_0, a_0 + a_i] = 1$ for all $i > 0$. Assuming $a_0 + a_i \neq a_0 + a_j$ for some $i \neq j$, $i,j > 0$ gives $a_0 = (a_0 + a_i) \cap (a_0 + a_j)$ whence $g_d(\bar{a}) \leq a_0^\perp \cap (a_0 + a_i) \cap (a_0 + a_j) = \mathbf{0}$, a contradiction. It follows $a_0 + a_i = a_0 + a_j$ for all $i \neq j$, $i,j > 0$ whence $a_0 + a_i \geq \sum_{j=1}^d a_j = \mathbf{1}$.
Conversely, given an orthogonal d-diamond $\bar{a}$ one calculates from the relations that $g_d(\bar{a}) = a_0^\perp$. Assuming $a_0^\perp = \mathbf{0}$ would imply $a_0 = \mathbf{1}$, $a_i = \mathbf{1}$ for all $i > 0$, and $\mathbf{1} = \mathbf{0}$: contradiction.

d) If $b = \mathbf{1}$ substitute into $\bar{y}$ an orthogonal d-diamond $\bar{a}$. Conversely, assume $g_d(\bar{a}) \cap \bigcap_{i=1}^d (a_0 + b \cap a_i) > \mathbf{0}$ for some $\bar{a}$. Then $g_d(\bar{a}) > \mathbf{0}$ and $\bar{a}$ is an orthogonal d-diamond by c). Assume $b < \mathbf{1}$. Then $b \not\geq a_i$ for some $i > 0$, whence $b \cap a_i = \mathbf{0}$ and it follows

$$g_d(\bar{a}) \cap \bigcap\nolimits_{i=1}^d (a_0 + b \cap a_i) \leq a_0^\perp \cap (a_0 + b \cap a_i) = a_0^\perp \cap a_0 = \mathbf{0}$$

a contradiction. Thus, $b = \mathbf{1}$. $\square$

4.5 Expressing Boolean within First-Order Quantum Logic

ss:Quantifiers

The following results are used in Subsection 5.2, only, but are of some interest by themselves. Recall that a quantifier free formula $\phi(\bar{x})$ in the first order language of ortholattices is a Boolean combination of equations $s_i(\bar{x}) = t_i(\bar{x})$ of equations between ortholattice terms $s_i(\bar{x}), t_i\bar{x})$. Recall, that to form Boolean combinations within Logic we use connectives `!`, `&&`, `||` and $\supset$ for 'not, 'and', 'or' and 'implies'. We have to carefully distinguish these from ortholattice terms built from $\neg$, $\wedge$, and $\vee$.

The following Lemma shows that within fixed dimension d and in the presence of orthogonal d-frames any such formula $\phi(\bar{x})$ can be equivalently expressed in the form $\mathrm{Q}\,\bar{y}.t(\bar{x},\bar{y}) = 1$ with an ortholattice term $t(\bar{x},\bar{y})$ and quantifier Q which may be chosen both as $\exists$ and as $\forall$. In particular, considering atomic formulae of the form $x = \mathbf{1}$ and $x \neq \mathbf{1}$ (i.e. $\mathtt{!}(x = \mathbf{1})$) this shows that the Boolean (propositional) theory of strong truth can be expressed within quantified modular Quantum Logic of fixed dimension.

Following [Kalm83, §15 Theorem 3] we define the ortholattice terms

$$x \to y := y \vee \neg(x \vee y), \quad x \leftrightarrow y := (x \wedge y) \vee \neg(x \vee y)$$

and recall (Example 2.8f) that in any MOL

$$a \to b = \mathbf{1} \;\Leftrightarrow\; a \leq b, \quad a \leftrightarrow b = \mathbf{1} \;\Leftrightarrow\; a = b.$$

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Lemma 4.19 *For any $d \geq 2$ there is an ortholattice term $f_d(\bar{y})$, $\bar{y} = (y_0, \ldots, y_d)$, such that for any MOL L of $\dim(L) = d$ and any $\bar{a}$ in L*

a) $f_d(\bar{a}) = \mathbf{1}$ if and only if $\bar{a}$ is an orthogonal d-diamond; $f_d(\bar{a}) = \mathbf{0}$, otherwise.
b) $f_d(\bar{a}) = \mathbf{1} \Rightarrow x = \mathbf{1}$ if and only if $f_d(\bar{a}) \to x = \mathbf{1}$.

Proof. Define

$$f_d(\bar{y}) = \bigvee_{i=1}^{d} \big((g_d(\bar{y}) \vee \neg y_i) \wedge y_i\big).$$

If $\bar{a}$ is an orthogonal d-diamond, then $g_d(\bar{a}) = \neg a_0$ and $\neg a_0 \vee \neg a_i = \mathbf{1}$ for $i > 0$ since $a_0 \wedge a_i = \mathbf{0}$. It follows $f_d(\bar{a}) = \bigvee_{i=1}^{d} a_i = \mathbf{1}$. Conversely, assume that $\bar{a}$ is not an orthogonal d-diamond. Then $g_d(\bar{a}) = \mathbf{0}$ from Lemma 4.18c) and its follows $f_d(\bar{a}) = \bigvee_{i=1}^{d} \neg a_i \wedge a_i = \mathbf{0}$. This proves a) and b) follows immediately.

l:Boolean2

Lemma 4.20 *Fix $d \geq 2$. For every quantifier free formula $\phi(\bar{x})$ in the first order language of ortholattices there are ortholattice terms $t' = t'_{d,\phi}(\bar{x},\bar{y})$ and $t'' = t''_{d,\phi}(\bar{x},\bar{y})$ computable in polynomial time from ϕ such that all modular ortholattices L, of $\dim(L) = d$ and admitting an orthogonal d-diamond, and for all for all $\bar{b}$ in L*

$$L \models \exists \bar{y} :\; t'_{d\phi}(\bar{b},\bar{y}) = \mathbf{1} \;\Leftrightarrow\; L \models \phi(\bar{b}) \;\Leftrightarrow\; L \models \forall \bar{y} :\; t''_{d\phi}(\bar{b},\bar{y}) = \mathbf{1}.$$

More precisely, we have $\bar{y} = (y_0, \ldots, y_d)$ and ortholattice terms $t_{d\phi}(\bar{x},\bar{y})$ such that

$$L \models \exists \bar{y} :\; f_d(\bar{y}) \wedge t_{d\phi}(\bar{b},\bar{y}) = \mathbf{1} \;\Leftrightarrow\; L \models \phi(\bar{b}) \;\Leftrightarrow\; L \models \forall \bar{y} :\; f_d(\bar{y}) \to t_{d\phi}(\bar{b},\bar{y}) = \mathbf{1}$$

for all $\bar{b}$ in L – where f_d is the term from Lemma 4.19. Moreover

1. $t_{d\phi}(\bar{x},\bar{y}) = s_1(\bar{x}) \leftrightarrow t_1(\bar{x})$ for ϕ an equation $s_1(\bar{x}) = t_1(\bar{x})$.
2. $t_{d\phi}(\bar{x},\bar{y}) = t_{d\phi_1}(\bar{x},\bar{y}) \wedge t_{d\phi_2}(\bar{x},\bar{y})$ for ϕ given as ϕ_1 `&&` ϕ_2.

3. $t_{d\phi}(\bar{x},\bar{y}) = \hat{h}_d(\neg t_{d\psi}(\bar{x},\bar{y}),\bar{y})$ *for* $\phi = \,!\,\psi$ *and term* $\hat{h}_d(x,\bar{y})$ *from Lemma 4.18.*

Proof. For atomic formulas ϕ, i.e. equations $s_1 = t_1$ we refer to Example 2.8f). Now, it suffices to proceed with structural induction for formulas built with propositional connectives && and ! and to verify that (2) and (3) do the job for these.

Considering (2), assume that $L \models \phi(\bar{b})$, i.e. that $L \models \phi_i(\bar{b})$ for $i = 1, 2$. Due to inductive hypothesis on the second equivalence we have $L \models t_{d\phi_i}(\bar{b},\bar{a}) = \mathbf{1}$ for any orthogonal d-diamond $\bar{a}$, whence $L \models t_{d\phi}(\bar{b},\bar{a}) = \mathbf{1}$ and $L \models \forall\bar{y}.\ f_d(\bar{y}) \to t_{d\phi}(\bar{b},\bar{a}) = \mathbf{1}$. Due to inductive hypothesis on the first equivalence, there $\bar{a}_i$ such that $L \models f_d(\bar{a}_i) \wedge t_{d\phi_i}(\bar{b},\bar{a}_i) = \mathbf{1}$; in particular, both $\bar{a}_i$ are orthogonal d-diamonds. As just observed, also $L \models t_{d\phi_2}(\bar{b},\bar{a}_1) = \mathbf{1}$ whence $L \models \exists\bar{y} : f_d(\bar{y}) \wedge t_{d\phi}(\bar{b},\bar{y}) = \mathbf{1}$.

Conversely, assume $L \models \exists\bar{y}.\ f_d(\bar{y}) \wedge t_{d\phi}(\bar{b},\bar{y}) = \mathbf{1}$. Choose a witnessing (orthogonal d-diamond) $\bar{a}$, conclude $L \models t_{d\phi_i}(\bar{b},\bar{y}) = \mathbf{1}$ and, by induction, $L \models \phi_i(\bar{b})$. Now, assume $L \models \forall\bar{y}.\ f_d(\bar{y}) \to t_{d\phi}(\bar{b},\bar{y}) = \mathbf{1}$. Choose any orthogonal d-diamond $\bar{a}$ (we supposed that such exist) and conclude $L \models t_{d\phi_i}(\bar{b},\bar{a})$ and, by induction, $L \models \phi_i(\bar{b})$ for $i = 1, 2$.

Coming to (3) let $\phi = \,!\,\psi$. Observe that, by Lemma 4.18, for any orthogonal d-diamond $\bar{a}$ one has

$$t_{d\psi}(\bar{b},\bar{a}) \neq \mathbf{1} \ \Leftrightarrow\ \neg t_{d\psi}(\bar{b},\bar{a}) \neq \mathbf{0} \ \Leftrightarrow\ \hat{h}_d(\neg t_{d\phi}(\bar{b},\bar{a}),\bar{a}) = \mathbf{1}.$$

Now, assume that $L \models \phi(\bar{b})$, i.e. that $L \not\models \psi(\bar{b})$. By inductive hypothesis, for all orthogonal d-diamonds (and there are such) one has $t_{d\psi}(\bar{b},\bar{a}) \neq \mathbf{1}$. With the above observation we derive the other two sentences in (3) to be valid in L. Conversely if for some, in particular if for all, orthogonal d-diamonds $\bar{a}$ one has $\hat{h}_d(\neg t_{d\phi}(\bar{b},\bar{a}),\bar{a}) = \mathbf{1}$ then $t_{d\psi}(\bar{b},\bar{a}) \neq \mathbf{1}$ and $L \not\models \phi(\bar{b})$ by induction.

d:Boolean

Digression 4.21 *Adding the constants of an orthogonal d-diamond to a height d MOL, L, gives rise to a discriminator term on L, see [Herr05]. This is the general reason behind Lemma 4.20. For any $d \geq 3$, Lemmas 3.17 and 4.20 associate with any quantifier free formula $\psi(\bar{x})$ in the language of $*$-rings an ortholattice term $s_{d\psi}(\bar{x},\,\bar{y})$ such that $\psi(\bar{r})$ holds in $\mathbb{F}$ if and only if $s_{d\psi}(\Theta_{\bar{a}}r_1, \ldots \Theta_{\bar{a}}r_n,\,\bar{a})$ holds in $\mathrm{L}(\mathbb{F}^d)$ where $\bar{a}$ is any orthonormal d-frame of $\mathrm{L}(\mathbb{F}^d)$. Cmp. [Mnëv88] for related results.*

5 Variations of the Quantum Satisfiability Problem

s:WeakStrong

This section establishes (Theorem 5.1) the polynomial-time equivalence of sat_L and SAT_L for any fixed modular ortholattice L of finite height. We investigate the complexity of satisfiability for terms with a limited number of $\bigwedge\bigvee$-alternations and for negation-free terms (Subsection 5.3). The effect of changing the ground field and the dimension is explored in Subsections 5.4 and 5.5, respectively. Generalizing satisfiability, Subsection 5.2 determines the complexity of deciding the truth of quantified formulas.

5.1 Weak versus Strong Satisfiability

ss:WeakStrong

Based on Lemma 4.18 and Example 4.16 we can now conclude

t:WeakStrong

Theorem 5.1. *Fix $d \in \mathbb{N}$. Then for any modular ortholattice L of height d admitting an orthogonal d–diamond and for any ortholattice term $t(\bar{x})$, the following hold:*

a) $t(\bar{x})$ *is weakly satisfiable in* L *if and only if* $\tilde{h}_d\big(t(\bar{x}),\bar{y}\big)$ *is strongly satisfiable in* L.

b) $t(\bar{x})$ is strongly satisfiable in L if and only if $\tilde{g}_d\big(t(\bar{x}),\bar{y}\big)$ is weakly satisfiable in L.

In particular, weak and strong satisfiability over Hilbert quantum logics are mutually polynomial-time reducible: For every finite-dimensional unitary $\mathbb{F}$-vector space $\mathcal{H}$ it holds $\mathrm{sat}_{\mathrm{L}(\mathcal{H})} \preceq \mathrm{SAT}_{\mathrm{L}(\mathcal{H})} \preceq \mathrm{sat}_{\mathrm{L}(\mathcal{H})}$.

Indeed, the translations indicated in a) and b) can easily be computed in time polynomial in the length of t (and in d, otherwise independent of L). Theorem 3.18b) thus extends to

c:WeakStrong

Corollary 5.1. *For every $\mathbb{F}$-unitary vector space $\mathcal{H}$ of dimension $d \geq 3$ admitting an equinormal orthogonal basis, $\mathrm{sat}_{\mathrm{L}(\mathcal{H})}$ is $\mathrm{BP}(\mathcal{NP}^0_{\mathrm{Re}\,\mathbb{F}})$–complete.*

Comment 5.2 *a) One can prove Theorem 5.1a) using the transitivity of the orthogonal group: map t to $\bigvee_{i=1}^d t_i$ where t_i arises from t, replacing each variable x in t by a variable x_i occurring only in t_i. Also, in Theorem 5.1b) one can use terms provided by Hagge [Hagg07] and an iterative construction.*

b) According to Comment 2.4, any MOL of height d is isomorphic to a direct product of irreducibles of heights $d_1,\ldots,d_k$ such that $d = d_1 + \ldots + d_k$. And an MOL of height d_i is irreducible if and only if it admits an orthogonal d_i-diamond. Since in Lemma 4.18 the only requirement was the existence of such, the equivalence stated in Theorem 5.1 can be established for any given MOL, L, of finite height $d \geq 1$ via $\bigvee_i \phi_{d_i}$ and $\bigwedge_i \psi_{d_i}$. Though, there are L where both problems are undecidable and the same holds if one considers satisfiability within some (indefinite) MOL of height $\leq d$ (resp. irreducible of height d) where $d \geq 14$ is fixed (cmp. Digression 2.4).

5.2 Quantified Quantum Propositions

ss:PolyHierarchy

Stockmeyer's polynomial hierarchy for Turing machines is based on, and extends, the complexity classes $\mathcal{P}$ and $\mathcal{NP}$ as well as the class $\mathrm{co}\mathcal{NP}$ of complements of $\mathcal{NP}$-problems. More specifically, starting with $\Sigma_0^{\mathcal{P}} = \mathcal{P} = \Pi_0^{\mathcal{P}}$ and $\Sigma_1^{\mathcal{P}} = \mathcal{NP}$ and $\Pi_1^{\mathcal{P}} = \mathrm{co}\mathcal{NP}$, higher classes $\Sigma_\ell^{\mathcal{P}}$ and $\Pi_\ell^{\mathcal{P}}$ ($\ell \geq 2$) can be characterized both syntactically and semantically: For the latter, $\Sigma_\ell^{\mathcal{P}}$ contains precisely those decision problems accepted by nondeterministic polynomial-time Turing machines with oracle access to some $V \in \Sigma_{\ell-1}^{\mathcal{P}}$ (equivalently: to some $V' \in \Pi_{\ell-1}^{\mathcal{P}}$); and $\Pi_\ell^{\mathcal{P}}$ consists of the complements of members from $\Sigma_\ell^{\mathcal{P}}$ [Papa94]. For the former characterization, $\Sigma_\ell^{\mathcal{P}}$ and $\Pi_\ell^{\mathcal{P}}$ contain all decision problems of the form

$$\begin{aligned}
&\big\{\bar{z} \in \{\mathbf{0},\mathbf{1}\}^n \;\big|\; n \in \mathbb{N},\ \exists \bar{y}^{(1)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)}\ \forall \bar{y}^{(2)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)}\ \exists \bar{y}^{(3)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)}\ \forall \bar{y}^{(4)} \ldots \\
&\qquad\qquad \ldots \mathrm{Q}_\ell\, \bar{y}^{(\ell)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)} :\ \langle \bar{z}, \bar{y}^{(1)}, \ldots \bar{y}^{(\ell)}\rangle \in V\big\} \\
&\big\{\bar{z} \in \{\mathbf{0},\mathbf{1}\}^n \;\big|\; n \in \mathbb{N},\ \forall \bar{y}^{(1)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)}\ \exists \bar{y}^{(2)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)}\ \forall \bar{y}^{(3)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)}\ \exists \bar{y}^{(4)} \ldots \\
&\qquad\qquad \ldots \mathrm{Q}'_\ell\, \bar{y}^{(\ell)} \in \{\mathbf{0},\mathbf{1}\}^{p(n)} :\ \langle \bar{z}, \bar{y}^{(1)}, \ldots \bar{y}^{(\ell)}\rangle \in V'\big\}
\end{aligned} \tag{9}$$

respectively, with $V, V' \in \mathcal{P}$ and $p \in \mathbb{N}[N]$ a polynomial. Here Q_ℓ denotes the existential quantifier when ℓ is odd and otherwise the universal one; vice versa for Q'.

Generalizing the Cook-Levin Theorem, a natural problem complete for $\Sigma_\ell^{\mathcal{P}}$ asks for the truth of a given Boolean formula with ℓ blocks of alternating quantifiers, starting with the existential one:

$$\begin{aligned}
\mathrm{SAT}^\ell \;=\; &\big\{\langle t(\bar{x}^{(1)}, \bar{x}^{(2)}, \ldots \bar{x}^{(\ell)})\rangle \;\big|\; n_1, \ldots n_\ell \in \mathbb{N},\ \exists \bar{a}^{(1)} \in \{0,1\}^{n_1}\ \forall \bar{a}^{(2)} \in \{0,1\}^{n_2}\ \exists \bar{a}^{(3)} \\
&\qquad \ldots \mathrm{Q}_\ell\, \bar{a}^{(\ell)} \in \{0,1\}^{n_\ell} :\ t(\bar{a}^{(1)}, \bar{a}^{(2)}, \ldots \bar{a}^{(\ell)}) = 1\big\}\ .
\end{aligned}$$

So binarily encoded terms $\langle t(\bar{x}^{(1)}, \bar{x}^{(2)}, \ldots, \bar{x}^{(\ell)})\rangle$ replace $\bar{z}$ in Equation (9). $\overline{\mathrm{SAT}}^{\ell}$ is defined similarly but starting with the universal quantifier—and complete for $\Pi_\ell^{\mathcal{P}}$. Moreover, the following problem QSAT is complete for PSPACE:

$$\big\{\langle t(x_1,\ldots x_n)\rangle \;\big|\; n\in\mathbb{N}\;\exists a_1\in\{0,1\}\;\forall a_2\in\{0,1\}\;\exists a_3\ldots \mathrm{Q}_n\, a_n\in\{0,1\}:\; t(a_1,\ldots a_n)=1\big\}$$

More generally, sequential polynomial space corresponds to parallel polynomial time [Boro77]; as well as to parallel alternating time [CKS81] hence PSPACE is sometimes also denoted as $\mathsf{PAR}=\mathsf{PAT}$.

Both the polynomial hierarchy and its two characterizations translate (although with notably different proofs) to the BSS setting [Cuck93, §4], [BCSS98, §21.4]: $\Sigma_{\ell,\mathbb{F}}^{\mathcal{P}}$ contains precisely those decision problems accepted by nondeterministic polynomial-time BSS machines over $\mathbb{F}$ with oracle access to some $\mathbb{V}\in\Sigma_{\ell-1,\mathbb{F}}^{\mathcal{P}}$ (equivalently: to some $\mathbb{V}'\in\Pi_{\ell-1,\mathbb{F}}^{\mathcal{P}}$); and $\Pi_{\ell,\mathbb{F}}^{\mathcal{P}}$ consists of the complements of members from $\Sigma_{\ell,\mathbb{F}}^{\mathcal{P}}$. For the former characterization, $\Sigma_{\ell,\mathbb{F}}^{\mathcal{P}}$ and $\Pi_{\ell,\mathbb{F}}^{\mathcal{P}}$ consist of all sets of the form

$$\big\{\bar{z}\in\mathbb{F}^n \;\big|\; n\in\mathbb{N},\;\exists\bar{y}^{(1)}\in\mathbb{F}^{p(n)}\;\forall\bar{y}^{(2)}\in\mathbb{F}^{p(n)}\;\exists\bar{y}^{(3)}\ldots \mathrm{Q}_\ell\,\bar{y}^{(\ell)}\in\mathbb{F}^{p(n)}:\langle\bar{z},\bar{y}^{(1)},\ldots\bar{y}^{(\ell)}\rangle\in\mathbb{V}\big\}$$
$$\big\{\bar{z}\in\mathbb{F}^n \;\big|\; n\in\mathbb{N},\;\forall\bar{y}^{(1)}\in\mathbb{F}^{p(n)}\;\exists\bar{y}^{(2)}\in\mathbb{F}^{p(n)}\;\forall\bar{y}^{(3)}\ldots \mathrm{Q}'_\ell\,\bar{y}^{(\ell)}\in\mathbb{F}^{p(n)}:\langle\bar{z},\bar{y}^{(1)},\ldots\bar{y}^{(\ell)}\rangle\in\mathbb{V}'\big\}$$

respectively, with $\mathbb{V},\mathbb{V}'\in\mathcal{P}_{\mathbb{F}}$ and $p\in\mathbb{N}[N]$; cmp. also [BCNM06]. And

$$\begin{aligned}
\mathrm{FEAS}^{2\ell-1}_{\mathbb{F},\mathbb{F}} =& \big\{\langle p_1,\ldots p_k\rangle \;\big|\; k,n_1,\ldots\in\mathbb{N},\; p_j\in\mathbb{F}[\bar{X}^{(1)},\ldots\bar{X}^{(2\ell-1)}],\;\exists\bar{y}^{(1)}\in\mathbb{F}^{n_1}\;\forall\bar{y}^{(2)}\in\mathbb{F}^{n_2}\\
&\exists\bar{y}^{(3)}\;\ldots\exists\bar{y}^{(2\ell-1)}\in\mathbb{F}^{n_{2\ell-1}}: p_1(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell-1)})=0\;\texttt{\&\&}\cdots\texttt{\&\&}\;p_k(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell-1)})=0\big\}\\
\mathrm{FEAS}^{2\ell}_{\mathbb{F},\mathbb{F}} =& \big\{\langle p_1,\ldots p_k\rangle \;\big|\; k,n_1,\ldots\in\mathbb{N},\; p_j\in\mathbb{F}[\bar{X}^{(1)},\ldots\bar{X}^{(2\ell)}],\;\exists\bar{y}^{(1)}\in\mathbb{F}^{n_1}\;\forall\bar{y}^{(2)}\in\mathbb{F}^{n_2}\\
&\exists\bar{y}^{(3)}\;\ldots\forall\bar{y}^{(2\ell)}\in\mathbb{F}^{n_{2\ell}}: p_1(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell)})\neq 0\;\;\texttt{||}\cdots\texttt{||}\;\;p_k(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell)})\neq 0\big\}\\
\mathrm{FEAS}^{2\ell-1}_{\mathbb{Z},\mathbb{F}} =& \big\{\langle p_1,\ldots p_k\rangle \;\big|\; k,n_1,\ldots\in\mathbb{N},\; p_j\in\mathbb{Z}[\bar{X}^{(1)},\ldots\bar{X}^{(2\ell-1)}],\;\exists\bar{y}^{(1)}\in\mathbb{F}^{n_1}\;\forall\bar{y}^{(2)}\in\mathbb{F}^{n_2}\\
&\exists\bar{y}^{(3)}\ldots\exists\bar{y}^{(2\ell-1)}\in\mathbb{F}^{n_{2\ell-1}}: p_1(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell-1)})=0\;\texttt{\&\&}\cdots\texttt{\&\&}\;p_k(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell-1)})=0\big\}\\
\mathrm{FEAS}^{2\ell}_{\mathbb{Z},\mathbb{F}} =& \big\{\langle p_1,\ldots p_k\rangle \;\big|\; k,n_1,\ldots\in\mathbb{N},\; p_j\in\mathbb{Z}[\bar{X}^{(1)},\ldots\bar{X}^{(2\ell)}],\;\exists\bar{y}^{(1)}\in\mathbb{F}^{n_1}\;\forall\bar{y}^{(2)}\in\mathbb{F}^{n_2}\\
&\exists\bar{y}^{(3)}\ldots\forall\bar{y}^{(2\ell)}\in\mathbb{F}^{n_{2\ell}}: p_1(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell)})\neq 0\;\;\texttt{||}\cdots\texttt{||}\;\;p_k(\bar{y}^{(1)},\ldots\bar{y}^{(2\ell)})\neq 0\big\}
\end{aligned}$$

are complete for $\Sigma_{2\ell-1,\mathbb{F}}^{\mathcal{P}}$, $\Sigma_{2\ell,\mathbb{F}}^{\mathcal{P}}$, $\mathrm{BP}(\Sigma_{2\ell-1,\mathbb{F}}^{\mathcal{P}})$, and $\mathrm{BP}(\Sigma_{2\ell,\mathbb{F}}^{\mathcal{P}})$, respectively. Note that (Fact 1.8) the matrix form defining $\mathrm{FEAS}^{\ell}_{\mathbb{F},\mathbb{F}}$ and $\mathrm{FEAS}^{\ell}_{\mathbb{Z},\mathbb{F}}$ can be relaxed to arbitrary finite Boolean combinations of, and in the case admitting an order (bottom of Example 4.7) also restricted to one single, polynomial equality or inequality: depending on ℓ's parity!

Space complexity does not translate as nicely to the BSS setting [Mich89,CuBr07]; however $\mathsf{PAT}_{\mathbb{F}}$ is a natural counterpart to $\mathsf{PAT}=\mathsf{PSPACE}$ and $\mathrm{QSAT}_{\mathbb{F}}$ complete for it, where $\mathsf{PAT}_{\mathbb{F}}$ consists of all subsets of $\mathbb{F}^*$ of the form

$$\begin{aligned}
&\big\{\bar{z}\in\mathbb{F}^n \;\big|\; n\in\mathbb{N},\;\exists y_1\in\mathbb{F}\;\forall y_2\in\mathbb{F}\;\exists y_3\in\mathbb{F}\;\forall y_4\ldots\;\ldots\mathrm{Q}_n\,y_n\in\mathbb{F}:\langle\bar{z},\bar{y}\rangle\in\mathbb{V}\big\} \qquad \text{and}\\
&\mathrm{QSAT}_{\mathbb{F}} \;:=\; \big\{\langle p_1,\ldots,p_k\rangle \;\big|\; k,n\in\mathbb{N},\; p_j\in\mathbb{F}[X_1,\ldots,X_n],\\
&\qquad\qquad \exists y_1\in\mathbb{F}\;\forall y_2\in\mathbb{F}\;\exists y_3\in\mathbb{F}\;\forall y_4\ldots\ldots\mathrm{Q}_n\,y_n\in\mathbb{F}: p_1(\bar{y})=\cdots=p_k(\bar{y})=0\big\}
\end{aligned}$$

with $\mathbb{V}$ running through $\mathcal{P}_{\mathbb{F}}$ [Cuck93, Theorem 4.1]; similarly for $\mathrm{QSAT}^0_{\mathbb{F}}$ complete for $\mathrm{BP}(\mathsf{PAT}_{\mathbb{F}})$. Natural problems in $\mathsf{PAR}_{\mathbb{C}}\subset\mathsf{PAT}_{\mathbb{C}}$ and $\mathsf{PAR}_{\mathbb{R}}\subset\mathsf{PAT}_{\mathbb{R}}$ traditionally arise in semi-/algebraic geometry [Cann88,GiHe91,Lece00,JKSS04,BüSc09,BaZe10,Sche12].

In view of our generalization of Boolean satisfiability to ortholattices L (Definition 2.5f) this suggests to consider first-order quantified quantum (i.e. predicate) logic and to define

$$\begin{aligned}
\mathrm{SAT}^\ell_L &:= \big\{\langle t(\bar x^{(1)},\ldots \bar x^{(\ell)})\rangle \;\big|\; n_1,\ldots n_\ell\in\mathbb{N},\; \exists \bar a^{(1)}\in L^{n_1}\;\forall\bar a^{(2)}\in L^{n_2}\;\exists\bar a^{(3)}\ldots \\
&\qquad\qquad \ldots \mathrm{Q}_\ell\, \bar a^{(\ell)}\in L^{n_\ell}:\; t_L(\bar a^{(1)},\bar a^{(2)},\ldots \bar a^{(\ell)})=\mathbf{1}\big\}\;, \\
\overline{\mathrm{SAT}}^\ell_L &:= \big\{\langle t(\bar x^{(1)},\ldots \bar x^{(\ell)})\rangle \;\big|\; n_1,\ldots n_\ell\in\mathbb{N},\; \forall \bar a^{(1)}\in L^{n_1}\;\exists\bar a^{(2)}\in L^{n_2}\;\forall\bar a^{(3)}\ldots \\
&\qquad\qquad \ldots \mathrm{Q}'_\ell\, \bar a^{(\ell)}\in L^{n_\ell}:\; t_L(\bar a^{(1)},\bar a^{(2)},\ldots \bar a^{(\ell)})\neq\mathbf{1}\big\}\;, \\
\mathrm{sat}^\ell_L &:= \big\{\langle t(\bar x^{(1)},\ldots \bar x^{(\ell)})\rangle \;\big|\; n_1,\ldots n_\ell\in\mathbb{N},\; \exists \bar a^{(1)}\in L^{n_1}\;\forall\bar a^{(2)}\in L^{n_2}\;\exists\bar a^{(3)}\ldots \\
&\qquad\qquad \ldots \mathrm{Q}_\ell\, \bar a^{(\ell)}\in L^{n_\ell}:\; t_L(\bar a^{(1)},\bar a^{(2)},\ldots \bar a^{(\ell)})\neq\mathbf{0}\big\}\;, \\
\overline{\mathrm{sat}}^\ell_L &:= \big\{\langle t(\bar x^{(1)},\ldots \bar x^{(\ell)})\rangle \;\big|\; n_1,\ldots n_\ell\in\mathbb{N},\; \forall \bar a^{(1)}\in L^{n_1}\;\exists\bar a^{(2)}\in L^{n_2}\;\forall\bar a^{(3)}\ldots \\
&\qquad\qquad \ldots \mathrm{Q}'_\ell\, \bar a^{(\ell)}\in L^{n_\ell}:\; t_L(\bar a^{(1)},\bar a^{(2)},\ldots \bar a^{(\ell)})=\mathbf{0}\big\}\;, \\
\mathrm{QSAT}_L &:= \big\{\langle t(x_1,\ldots x_n)\rangle \;\big|\; \exists a_1\in L\;\forall a_2\in L\;\exists a_3\in L\;\forall a_4\;\ldots \mathrm{Q}_n\, a_n\in L: t_L(\bar a)=\mathbf{1}\big\}\;, \\
\mathrm{qsat}_L &:= \big\{\langle t(x_1,\ldots x_n)\rangle \;\big|\; \exists a_1\in L\;\forall a_2\in L\;\exists a_3\in L\;\forall a_4\;\ldots \mathrm{Q}_n\, a_n\in L:\; t_L(\bar a)\neq\mathbf{0}\big\}
\end{aligned}$$

and analogously for terms *with constants*; cmp. also [Roma06]. Observe that in all of $\mathrm{QSAT}_{\mathbb{F}}$, QSAT_L, and qsat_L dummy variables are admitted; in the latter two such variable x can be camouflaged by meeting with $(x\vee\neg x)$.

We emphasize that the definitions of $\mathrm{SAT}^\ell_L, \mathrm{sat}^\ell_L$ do *not* depend on the parity of ℓ.

t:PolyHierarchy

Theorem 5.3. *Fix* $\ell\in\mathbb{N}$.

a) For any infinite two-dimensional MOL L, *both* SAT^ℓ_L *and* sat^ℓ_L *are* $\Sigma^{\mathcal{P}}_\ell$*–complete; and* $\overline{\mathrm{SAT}}^\ell_L, \overline{\mathrm{sat}}^\ell_L$ *are* $\Pi^{\mathcal{P}}_\ell$*–complete; while* QSAT_L *and* qsat_L *are* PSPACE*–complete.*

b) For $\mathbb{F}\subseteq\mathbb{R}$ *and* $d\geq 3$, *both* $\mathrm{SAT}^\ell_{\mathrm{L}(\mathbb{F}^d)}$ *and* $\mathrm{sat}^\ell_{\mathrm{L}(\mathbb{F}^d)}$ *are* $\mathrm{BP}(\Sigma^{\mathcal{P}}_{\ell,\mathbb{F}})$*–complete; and* $\overline{\mathrm{SAT}}^\ell_{\mathrm{L}(\mathbb{F}^d)}$ *and* $\overline{\mathrm{sat}}^\ell_{\mathrm{L}(\mathbb{F}^d)}$ *are* $\mathrm{BP}(\Pi^{\mathcal{P}}_{\ell,\mathbb{F}})$*–complete; while* $\mathrm{QSAT}_{\mathrm{L}(\mathbb{F}^d)}$ *and* $\mathrm{qsat}_{\mathrm{L}(\mathbb{F}^d)}$ *are* $\mathrm{BP}(\mathsf{PAT}_{\mathbb{F}})$*–complete.*

c) For every $\mathbb{F}$*-unitary vector space* $\mathcal{H}$ *of dimension* $d\geq 3$ *admitting an equinormal orthogonal basis, both* $\mathrm{SAT}^\ell_{\mathrm{L}(\mathcal{H})}$ *and* $\mathrm{sat}^\ell_{\mathrm{L}(\mathcal{H})}$ *are* $\mathrm{BP}(\Sigma^{\mathcal{P}}_{\ell,\mathrm{Re}\,\mathbb{F}})$*–complete; and* $\overline{\mathrm{SAT}}^\ell_{\mathrm{L}(\mathcal{H})}$ *and* $\overline{\mathrm{sat}}^\ell_{\mathrm{L}(\mathcal{H})}$ *are* $\mathrm{BP}(\Pi^{\mathcal{P}}_{\ell,\mathrm{Re}\,\mathbb{F}})$*–complete; while* $\mathrm{QSAT}_{\mathrm{L}(\mathcal{H})}$ *and* $\mathrm{qsat}_{\mathrm{L}(\mathcal{H})}$ *are* $\mathrm{BP}(\mathsf{PAT}_{\mathrm{Re}\,\mathbb{F}})$*–complete.*

d) Similarly, for $\mathcal{H}=\mathbb{F}^d$ *with* $d\geq 3$, *the above problems* with constants $\mathrm{SAT}^\ell_{\mathrm{L}(\mathcal{H}),\mathrm{L}(\mathcal{H})}$ *and* $\mathrm{sat}^\ell_{\mathrm{L}(\mathcal{H}),\mathrm{L}(\mathcal{H})}$ *are* $\Sigma^{\mathcal{P}}_{\ell,\mathrm{Re}\,\mathbb{F}}$*–complete; and* $\overline{\mathrm{SAT}}^\ell_{\mathrm{L}(\mathcal{H}),\mathrm{L}(\mathcal{H})}$ *and* $\overline{\mathrm{sat}}^\ell_{\mathrm{L}(\mathcal{H}),\mathrm{L}(\mathcal{H})}$ *are* $\Pi^{\mathcal{P}}_{\ell,\mathrm{Re}\,\mathbb{F}}$*–complete; while* $\mathrm{QSAT}_{\mathrm{L}(\mathcal{H}),\mathrm{L}(\mathcal{H})}$ *and* $\mathrm{qsat}_{\mathrm{L}(\mathcal{H}),\mathrm{L}(\mathcal{H})}$ *are* $\mathsf{PAT}_{\mathrm{Re}\,\mathbb{F}}$*–complete. Here, lattice constants are considered encoded as (ranges of) matrices as in Theorem 3.18c)*

Since [Poon09] proved $\mathrm{FEAS}^3_{\mathbb{Q}}$ undecidable, we conclude

c:Poonen

Corollary 5.2. $\mathrm{SAT}^3_{\mathrm{L}(\mathbb{Q}^3)}$ *and* $\mathrm{sat}^3_{\mathrm{L}(\mathbb{Q}^3)}$ *are undecidable (to a Turing machine***).*

It should be emphasized that (to the best of our knowledge) Theorem 5.3 cannot be just deduced from the satisfiability case $\ell=1$ but requires additional considerations such as Lemmas 4.20 and 4.14.

Proof (of Theorem 5.3). We first prove SAT^ℓ_L polynomial-time equivalent to sat^ℓ_L: $\exists\bar a^{(1)}\forall\bar a^{(2)}\ldots:$ $t_L(\bar a^{(1)},\bar a^{(2)},\ldots)\neq\mathbf{0}$ is equivalent within L to $\exists\big(\bar b,\bar a^{(1)}\big),\forall\bar a^{(2)}\ldots:\tilde h_d\big(t_L(\bar a^{(1)},\bar a^{(2)},\ldots),\bar b\big)=\mathbf{1}$ according to Lemma 4.18b) and Example 4.16, thus yielding a polynomial-time reduction from sat^ℓ_L to SAT^ℓ_L. The converse reduction similarly follows from Lemma 4.18d). QSAT_L and qsat_L are seen polynomial-time equivalent analogously.

a) In view of the above preconsiderations it suffices to treat the cases SAT_L^ℓ and QSAT_L. Concerning the first, the case $\ell=1$ amounts to satisfiability and was already shown complete for $\mathcal{NP}=\Sigma_1^{\mathcal{P}}$ in Theorem 2.14. To locate

$$\mathrm{SAT}_L^2 \;=\; \big\{\langle t(x_1,\ldots,x_k,y_1,\ldots,y_m)\rangle \;\big|\; \exists\bar a\in L^k\;\forall\bar b\in L^m:\; t_L(\bar a,\bar b)=\mathbf{1}\big\}$$

in $\Sigma_2^{\mathcal{P}}$ observe that, for a given $t(\bar x,\bar y)$ in $k+m$ variables and according to Lemma 4.14a), it suffices to consider the case $L=\mathcal{MO}_{k+m}$. Now elements $z\in\mathcal{MO}_{k+m}$ can be encoded as in Proposition 2.13a) as integers in $\{0,1,\ldots,2k+2m\}$ which, in unary, correspond to $\bar z\in\{0,1\}^{2(k+m)}$. Tuples $(a_1,\ldots,a_k)\in\mathcal{MO}_{k+m}^k$ and $(b_1,\ldots,b_m)\in\mathcal{MO}_{k+m}^m$ thus can be encoded as $(\bar a_1,\ldots,\bar a_k)\in\{0,1\}^{p(k+m)}$ and $(\bar b_1,\ldots,\bar b_m)\in\{0,1\}^{p(k+m)}$ of size $p(n):=4n^2$ polynomial in the binary length $n\geq k+m$ of the $(k+m)$-variate input. This rephrases SAT_L^2 in the form of Equation (9) with

$$\begin{aligned} V \;:=\; \big\{\langle t(x_1,\ldots,x_k,y_1,\ldots,y_m),(\bar a_1,\ldots,\bar a_k),(\bar b_1,\ldots,\bar b_m)\rangle \;\big|\; \\ k,m\in\mathbb{N},\; t_{\mathcal{MO}_{k+m}}(a_1,\ldots,a_k,b_1,\ldots,b_m)=\mathbf{1}\big\}\end{aligned}$$

a problem in $\mathcal{P}$ according to Proposition 2.13a). More generally regarding SAT_L^ℓ, each k_j-tuple $\bar a^{(j)}$ in $\mathcal{MO}_n$ with $n\geq\sum_{j=1}^\ell k_j$ can be encoded as an $\leq(k_j\cdot n)$-tuple in $\{0,1\}$; similarly for QSAT_L.
Concerning $\Sigma_\ell^{\mathcal{P}}$–hardness of SAT_L^ℓ, we reduce from SAT^ℓ and recall (Fact 2.2f) that pairwise commuting elements 'live' in a Boolean subalgebra of L, i.e. in $\{0,1\}^2$. Let us therefore abbreviate

$$\bar C(x_1,\ldots x_n):=\bigwedge_{i<j} C(x_i,x_j) \quad\text{and}\quad \bar C\big((x_1,\ldots x_n),(y_1,\ldots y_m)\big) \;:=\; \bigwedge_{i\leq n,j\leq m} C(x_i,y_j)$$

expressing in terms of strong truth that all $\bar x$ to commute and commute with all $\bar y$, respectively. Observe that $\bar C(\bar x,\bar y\bar z)=\bar C(\bar x,\bar y)\wedge\bar C(\bar x,\bar z)$ and $\bar C(\bar x\bar y)=\bar C(\bar x)\wedge\bar C(\bar y)\wedge\bar C(\bar x,\bar y)$. We now adapt the proof of Proposition 2.10 and want to require all quantifiers to range over pairwise commuting elements. In the existential case $\exists a_1,\ldots,a_n: t(\bar a)=\mathbf{1}$ this was accomplished by proceeding to $t'(\bar a):=t(\bar a)\wedge\bar C(\bar a,\bar a)$. More generally, SAT^ℓ–instance

$$\exists\bar a^{(1)}\;\forall\bar a^{(2)}\;\exists\bar a^{(3)}\;\ldots\mathrm{Q}_\ell\,\bar a^{(\ell)}:\; t\big(\bar a^{(1)},\bar a^{(2)},\ldots\bar a^{(\ell)}\big)=1 \quad ?$$

over $\{\mathbf{0},\mathbf{1}\}$ is equivalent to the following formula with bounded quantifiers over L:

$$\begin{aligned}\big(\exists\bar b^{(1)}.\bar C(\bar b^{(1)})=\mathbf{1}\big)\;\big(\forall\bar b^{(2)}.\bar C(\bar b^{(2)},\bar b^{(1)}\bar b^{(2)})=\mathbf{1}\big)\;\big(\exists\bar b^{(3)}.\bar C(\bar b^{(3)},\bar b^{(1)}\bar b^{(2)}\bar b^{(3)})=\mathbf{1}\big)\;\ldots \\ \ldots\big(\,\mathrm{Q}_\ell\,\bar b^{(\ell)}.\bar C(\bar b^{(\ell)},\bar b^{(1)}\bar b^{(2)}\cdots\bar b^{(\ell)})=\mathbf{1}\big):\; t\big(\bar b^{(1)},\bar b^{(2)},\ldots\bar b^{(\ell)}\big)=\mathbf{1}\end{aligned} \tag{10}$$

Indeed, all quantification in (10) is bounded to mutually commuting elements and thus 'live' in $\{\mathbf{0},\mathbf{1}\}^2$; hence Lemma 4.14b) applies. Now according to Lemma 4.13, (10) is in turn equivalent over L to the unboundedly quantified

$$\begin{aligned}\exists\bar b^{(1)}\;\forall\bar b^{(2)}\;\exists\bar b^{(3)}\;\ldots\mathrm{Q}_\ell\,\bar b^{(\ell)}:\;\beta\Big(t'\big(\bar b^{(1)},\bar b^{(2)},\ldots\bar b^{(\ell)}\big)=\mathbf{1},\bar C(\bar b^{(1)})=\mathbf{1},\bar C(\bar b^{(2)},\bar b^{(1)}\bar b^{(2)})=\mathbf{1}, \\ \bar C(\bar b^{(3)},\bar b^{(1)}\bar b^{(2)}\bar b^{(3)})=\mathbf{1},\ldots\bar C(\bar b^{(\ell)},\bar b^{(1)}\cdots\bar b^{(\ell)})=\mathbf{1}\Big)\end{aligned}$$

with Boolean propositional formula β polynomial time computable; which finally can be converted in similar time into the SAT^ℓ_L–instance

$$\exists\bar b^{(1)}\,\forall\bar b^{(2)}\,\exists\bar b^{(3)}\,\ldots\mathrm{Q}_\ell\,\big(\bar b^{(\ell)},\bar x\big):\ t''\big(\bar b^{(1)},\bar b^{(2)},\ldots,\bar a^{(4)},\bar x\big)=\mathbf{1}\quad ?$$

with t'' according to Lemma 4.20. The reduction from QSAT to QSAT_L proceeds analogously.

b) Again, we first consider the case $\ell=2$: Encode $a_j\in\mathrm{L}(\mathbb{F}^d)$ as in Proposition 2.17 as any matrix $A_j\in\mathbb{F}^{d\times d}$ with $a_j=\mathrm{range}(A_j)$. Then

$$\begin{aligned}\langle t(x_1,\ldots,x_k,y_1,\ldots,y_m)\rangle\in\mathrm{SAT}^2_{\mathrm{L}(\mathbb{F}^d)}\quad&\Leftrightarrow\quad\exists\bar A\in\mathbb{F}^{kd^2}\,\forall\bar B\in\mathbb{F}^{md^2}:\ \langle t(\bar x,\bar y),\bar A,\bar B\rangle\in\mathbb{V},\\ \mathbb{V}\ :=\ \big\{\langle t(x_1,\ldots,x_k,y_1,\ldots,y_m),\bar A,\bar B\rangle\ \big|\ &A_i,B_j\in\mathbb{F}^{d\times d},t_{\mathrm{L}(\mathbb{F}^d)}(\mathrm{range}\,\bar A,\mathrm{range}\,\bar B)=\mathbf{1}\big\}\end{aligned}$$

with $\mathbb{V}\in\mathcal{P}_\mathbb{F}$ according to Proposition 2.17: hence indeed $\mathrm{SAT}^2_{\mathrm{L}(\mathbb{F}^d)}\in\mathrm{BP}(\Sigma^\mathcal{P}_{2,\mathbb{F}})$. The case of general ℓ proceeds similarly; and so does $\mathrm{QSAT}_{\mathrm{L}(\mathbb{F}^d)}\in\mathrm{BP}(\mathsf{PAT}_\mathbb{F})$.
In order to show $\mathrm{SAT}^\ell_{\mathrm{L}(\mathbb{F}^d)}$ hard for $\mathrm{BP}(\Sigma^\mathcal{P}_{\ell,\mathbb{F}})$ we reduce from $\mathrm{FEAS}^\ell_{\mathbb{Z},\mathbb{F}}$. To this end recall the proof of Proposition 3.12, based on the isomorphisms $\Theta_{\bar A}:\mathbb{F}\to\mathcal{R}_{\bar A}\subseteq\mathrm{L}(\mathbb{F}^d)$ induced by any d-frame $\bar A$ (and the fact that such exist). Recall step (v), the term $t(\bar z)$ such that $t(\bar A)=\mathbf{1}$ if and only if $\bar A$ constitutes a d-frame, as well as steps i) to iii) translating polynomials $p_j\in\mathbb{Z}[X_1,\ldots,X_n]$ into orthoterms $t_j(\bar A;x_1,\ldots,x_n)$ such that $t_j(\bar A;\Theta_{\bar A}(r_1),\ldots,\Theta_{\bar a}(r_n))=\mathbf{1}$ is equivalent to $p_j(r_1,\ldots,r_n)=0$ for any substitution of $r_i\in\mathbb{F}$. Recall restriction (iv) for all variables x_i to range over the coordinate ring $\mathcal{R}_{\bar A}=\{U\in\mathrm{L}(\mathbb{F}^d):\rho_{\bar A}(U)\}$. Now, due to the isomorphisms $\Theta_{\bar A}$, a $\mathrm{FEAS}^\ell_{\mathbb{Z},\mathbb{F}}$–instance

$$\exists\bar y^{(1)}\,\forall\bar y^{(2)}\,\exists\bar y^{(3)}\,\ldots\mathrm{Q}_\ell\,\bar y^{(\ell)}:\ \alpha\big(p_1(\bar y^{(1)},\ldots,\bar y^{(\ell)})=0,\cdots,p_k(\bar y^{(1)},\ldots,\bar y^{(\ell)})\big)\quad ?$$

with arbitrary Boolean propositional formula α is equivalent to

$$\begin{aligned}&(\exists\bar z)\,(\exists\bar y^{(1)}\in\mathcal{R}_{\bar z})\,(\forall\bar y^{(2)}\in\mathcal{R}_{\bar z})\,(\exists\bar y^{(3)}\in\mathcal{R}_{\bar z})\,\ldots(\mathrm{Q}_\ell\,\bar y^{(\ell)}\in\mathcal{R}_{\bar z}):\\ &\qquad t(\bar z)=\mathbf{1}\ \&\&\ \alpha\Big(t_{p_1}\big(\bar z,\bar y^{(1)},\ldots\bar y^{(\ell)}\big)=\mathbf{1},\cdots,t_{p_k}\big(\bar z,\bar y^{(1)},\ldots\bar y^{(\ell)}\big)=\mathbf{1}\Big)\quad ?\end{aligned}$$

with quantifiers $\mathrm{Q}_i\,\bar y^{(i)}$ restricted by $\bar y^{(i)}\in\mathcal{R}_{\bar z}$, the latter defined as $\&\&_h\,\rho_{\bar z}(y^{(i)}_h)$ where $\rho_{\bar z}(y)$ is the formula $(y\wedge z_{22}=\mathbf{0}\ \&\&\ y\vee z_{22}=z_{11}\vee z_{22})$ and equivalent to certain $t_\rho(\bar z,y)=\mathbf{1}$ (by Fact 2.9). As in a), these restrictions to the quantifiers can be moved, according to Lemma 4.13, to the matrix, which thus becomes a Boolean combination α' of $t(\bar z)=\mathbf{1}$, $t_{p_j}\big(\bar z,\bar y^{(1)},\ldots\bar y^{(\ell)}\big)=\mathbf{1}$, and $\bigwedge_h t_\rho(\bar z,y^{(i)}_h)=\mathbf{1}$; and, based on the Q_ℓ-version of Lemma 4.20, the whole sentence can be transformed furtheron to a form representing a $\mathrm{SAT}^\ell_{\mathrm{L}(\mathbb{F}^d)}$–instance. The reduction from $\mathrm{BP}(\mathsf{PAT}_\mathbb{F})$ to $\mathrm{QSAT}_{\mathrm{L}(\mathbb{F}^d)}$ proceeds analogously.

c) Similarly to b), now with

$$\mathbb{V}\ :=\ \big\{\langle t(x_1,\ldots x_k,y_1,\ldots y_m),\mathrm{Re}\,\bar A,\mathrm{Im}\,\bar A,\mathrm{Re}\,\bar B,\mathrm{Im}\,\bar B\rangle\ \big|\ t_{\mathrm{L}(\mathbb{F}^d)}(\mathrm{range}\,\bar A,\mathrm{range}\,\bar B)=\mathbf{1}\big\}$$

$\in\mathcal{P}_{\mathrm{Re}\,\mathbb{F}}$ according to Proposition 2.17 to show $\in\mathrm{BP}(\Sigma^\mathcal{P}_{2,\mathrm{Re}\,\mathbb{F}})$; and, concerning hardness, with modifications as in the proof of Theorem 3.18a): conditions added for $\bar A$ to constitute an orthonormal d-frame and quantifiers to range over the real subring $\mathcal{R}_0:=\{y\in\mathcal{R}:y=y^*\}$ expressible as $y\otimes_{\bar z}((z_{11}\vee a_{22})\wedge\neg y)=\ominus_{\bar z}z_{12}$ by the second condition in Lemma 3.17b); the latter is translated into certain $t'_\rho(\bar z,y)=\mathbf{1}$ via relational interpretation (Definition 4.8) of this equation and Fact 2.9.

d) Similarly to c), now with $\bar{A}$ fixed to the orthonormal standard frame and polynomial coefficients $c \in \mathbb{F}$ encoded as lattice constants $C := \Theta_{\bar{A}}(c)$; conversely, lattice constants are encoded by constant matrices, i.e. lists of numbers in $\mathbb{F}$; recall the proof of Theorem 3.18c). □

5.3 Syntactically Restricted Terms

ss:Syntax

It is well-known that the Boolean satisfiability problem becomes no more simple when restricted to terms in conjunctive form ($\bigwedge\bigvee$-terms) and even with at most three literals per clause a problem known a 3SAT—whereas 2SAT can be decided in polynomial time; cmp. e.g. [Papa94, §9.2]. [Scha78] has succeeded in closely delineating syntactically the border between $\mathcal{P}$ and $\mathcal{NP}$-completeness for Boolean satisfiability problems; see [Chen09] for a modern presentation from the general perspective of *constraint satisfaction*. Regarding quantum logic, however, conjunctive form is semantically a proper restriction:

x:Conjunctive

Example 5.4 $(X \wedge Y) \vee (X \wedge \neg Y)$ *is a term in disjunctive form not equivalent over* $\mathrm{L}(\mathbb{F}^2)$ *to any term in conjunctive form.*

Thus more than two alternations of $\bigvee$ and $\bigwedge$ are required to obtain reasonable syntactical restrictions of the satisfiability problem; and Theorem 5.5 below explores the boundary between $\mathcal{P}$ and BSS-$\mathcal{NP}$ in terms of the number of these alternations.

A different kind of syntactic restriction, consider terms without negation: the operation only a real but no complex BSS machine can compute on $\mathrm{L}(\mathbb{C}^d)$, recall Proposition 2.17a iii). Now (so-called *lattice*) terms t over $\vee$ and $\wedge$ only are monotone (Lemma 5.8e); hence the question of whether a given lattice term admits an assignment making it evaluate to $\mathbf{1}$ (or $\mathbf{0}$) is easy to decide: by checking the assignment $(\mathbf{1}, \ldots, \mathbf{1})$ (or $(\mathbf{0}, \ldots, \mathbf{0})$). The alternation classes $\mathcal{Q}$ of lattice terms are defined, inductively: variables are in $\bigwedge$ as well as $\bigvee$; if t_1, t_2 are in $\bigwedge \mathcal{Q}$, then so is $t_1 \wedge t_2$ but $t_1 \vee t_2$ is in $\bigvee\bigwedge \mathcal{Q}$. Dually, for $\bigvee \mathcal{Q}$. An ortholattice term is in the alternation class $\mathcal{Q}$ if by de Morgan's rules it reduces to a lattice term (in variables and negated variables) in the class $\mathcal{Q}$ (cmp. Lemma 5.8d).

t:Syntax

Theorem 5.5. *a) Strong satisfiability over* $\mathrm{L}(\mathbb{F}^d)$, $d \geq 2$, *of* $\bigwedge\bigvee$*-terms is (independent of* $\mathbb{F}$ *and depending only on d's parity and in this sense uniformly) polynomial-time decidable.*

b) Strong satisfiability over L of general orthoterms is polynomial-time reducible to the strong satisfiability over L of $\bigwedge\bigvee\bigwedge\bigvee$*-terms, uniformly in L.*

c) For every $\mathbb{F}$ *and* $d \geq 3$*, the language*

$$\big\{\langle t(x_1, \ldots, x_n), s(x_1, \ldots, x_n)\rangle \mid \exists a_1, \ldots, a_n \in \mathrm{L}(\mathbb{F}^d) : t(a_1, \ldots, a_n) = \mathbf{1} \ \&\& \ s(a_1, \ldots, a_n) = \mathbf{0}\big\} \ \subseteq \ \{0,1\}^* \ ,$$

that is the question of whether two given lattice terms t and s admit a joint *assignment over* $\mathrm{L}(\mathbb{F}^d)$ *making t evaluate to* $\mathbf{1}$ and *s to* $\mathbf{0}$*, is complete for* $\mathrm{BP}(\mathcal{NP}_{\mathbb{F}})$.
The analogous question for terms (t, s) with *constants is* $\mathcal{NP}^0_{\mathbb{F}}$*–complete.*

d) The languages from c) remain $\mathrm{BP}(\mathcal{NP}_{\mathbb{F}})$*–complete and* $\mathcal{NP}^0_{\mathbb{F}}$*–complete, respectively, when restricting to* $t \in \bigwedge\bigvee\bigwedge$ *and* $s \in \bigvee\bigwedge\bigvee$.

e) For $\mathbb{F}$ *and* $d \geq 3$*, strong satisfiability over* $\mathrm{L}(\mathbb{F}^d)$ *of* $\bigwedge\bigvee\bigwedge$*-terms is* $\mathrm{BP}(\mathcal{NP}_{\mathrm{Re}\,\mathbb{F}})$*–complete; for* $d \leq 2$ *it is* $\mathcal{NP}$*–complete.*

In view of the gap between a), b), and e) we ask q:Syntax

Question 5.6 *What is the computational complexity of the strong satisfiability problem for $\bigvee\bigwedge\bigvee$-terms? Does it depend on dimension or the ground field?*

The proof of Theorem 5.5 is deferred until later in this subsection as it relies on some tools. Recall that the atoms $a_1, \ldots, a_n$ of $\mathcal{MO}_n$ satisfy

$$\mathbf{1} \;=\; a_k \vee a_\ell \;=\; a_k \vee \neg a_\ell \;=\; \neg a_k \vee a_\ell \;=\; \neg a_k \vee \neg a_\ell \quad (1 \leq k < \ell \leq n)\ . \tag{11}$$

Lemma 2.12 yields such elements also in $\mathrm{L}(\mathbb{F}^2)$ and, by Observation 2.6a), also in $\mathrm{L}(\mathbb{F}^{2d})$ for every $\mathbb{F}$ and d. But Equation (11) cannot be satisfied in odd dimensions. Instead, consider the following

x:IndepOddD

Example 5.7 *For every $d \geq 2$ and $\mathbb{F}$ and $n \in \mathbb{N}$, there exist $a_1, \ldots, a_n \in \mathrm{L}(\mathbb{F}^d)$ with*

$$\mathbf{1} \;=\; a_k \vee \neg a_\ell \;=\; \neg a_k \vee \neg a_\ell \;=\; a_k \vee a_\ell \vee a_j \quad (k \neq \ell \neq j \neq k)\ .$$

Indeed, the case of even d has been treated above. Whereas the $(3{+}2d)$-dimensional case follows from Observation 2.6a) by combining a 3D instance $(a_1, \ldots, a_n)$ with an even-dimensional one $(b_1, \ldots, b_n)$ to $(a_1 \oplus b_1, \ldots, a_n \oplus b_n)$. In the remaining case $d = 3$ observe that any three distinct vectors $\vec{v}_k := (1, k, k^2) \in \mathbb{F}^3$ form a Vandermonde matrix and are thus linearly independent; hence $a_k := \mathbb{F}\vec{v}_k \in \mathrm{L}_1(\mathbb{F}^3)$ satisfy $a_k \vee a_\ell \vee a_j = \mathbf{1}$ for every $k \neq \ell \neq j \neq k$. Moreover $\langle \vec{v}_k, \vec{v}_\ell \rangle = 1 + k\ell + k^2\ell^2 > 0$ shows $\vec{v}_k \not\perp \vec{v}_\ell$, hence $\dim(a_k \vee \neg a_\ell) = \dim(a_k) + 3 - \dim(a_\ell) - \dim(a_k \wedge \neg a_\ell) = 1 + 3 - 1$ by the dimension formula (Fact 2.1e). Similarly, $\neg a_k \neq \neg a_\ell$ and $\dim(\neg a_k) = 2 = \dim(\neg a_\ell)$ imply $\dim(\neg a_k \vee \neg a_\ell) \geq 3$.

l:Conjunctive

Lemma 5.8 *a) Let $t(x_1, \ldots, x_n)$ denote a $\bigwedge\bigvee$-term with at least two different literals in each clause. Then t is strongly satisfiable over $\mathrm{L}(\mathbb{F}^{2d})$ for every $\mathbb{F}$ and every d.*

b) Let $t(x_1, \ldots, x_n)$ denote a $\bigwedge\bigvee$-term with at least three different literals in each clause. Then t is strongly satisfiable over $\mathrm{L}(\mathbb{F}^{2d+1})$ for every $\mathbb{F}$ and every $d \geq 1$.
More precisely for $a_1, \ldots, a_n \in \mathrm{L}_d(\mathbb{F}^{2d+1})$ according to Example 5.7, every choice of $b_j \in \{a_j, \neg a_j\}$ gives rise to a strongly satisfying assignment $(b_1, \ldots, b_n)$ of t.

c) For a $\bigwedge\bigvee$-term $t(x_1, \ldots, x_n)$ with exactly two different literals in each clause, the following are equivalent:
  - *i) t is strongly satisfiable over $\mathrm{L}(\mathbb{F}^d)$ for some $\mathbb{F}$ and some odd d.*
  - *ii) t is strongly satisfiable over $\{\mathbf{0}, \mathbf{1}\}$.*
  - *iii) For $a_1, \ldots, a_n \in \mathrm{L}_d(\mathbb{F}^{2d+1})$ from Example 5.7, there exist $b_j \in \{a_j, \neg a_j\}$ such that $\bar{b}$ constitutes a strongly satisfying assignment of t.*
  - *iv) t is strongly satisfiable over $\mathrm{L}(\mathbb{F}^d)$ for every $\mathbb{F}$ and every odd d.*

d) For any ortholattice term $t(\bar{x})$ in n variables, there exists a lattice term $\hat{t}(\bar{x}, \bar{y})$ in $2n$ variables such that $L \models \hat{t}(x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n) = t(x_1, \ldots, x_n)$ holds over any ortholattice L. Moreover, $\hat{t}$ can be calculated from t by a Turing machine in polynomial time.

e) For any lattice term t and lattice L, $a_i \leq b_i$ $(1 \leq i \leq n)$ implies $t_L(\bar{a}) \leq t_L(\bar{b})$.

An orthoterm $\hat{t}(x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n)$ as in d) is called in *negation normal form*, since it has negations only in front of variables.

Proof. a) As argued before Example 5.7, $\mathrm{L}(\mathbb{F}^{2d})$ contains $a_1, \ldots, a_n$ satisfying Equation 11, i.e. rendering `true` every clause $u \vee v$ with distinct $u, v \in \{a_1, \neg a_1, \ldots, a_n, \neg a_n\}$.

b) Similarly observe that, for any three distinct $u, v, w \in \{a_j, \neg a_j, a_k, \neg a_k, a_\ell, \neg a_\ell\}$, $u \vee v \vee w$ evaluates to $\mathbf{1}$.

c i⇒ii) Let $\bar{y}$ denote a strongly satisfying assignment over $\mathrm{L}(\mathbb{F}^{2d+1})$. We claim that the derived assignment $b_j := \mathbf{1}$ for $\dim(y_j) \geq d+1$ and $b_j := \mathbf{0}$ for $\dim(\neg y_j) \geq d+1$ is also a satisfying one. To this end consider a arbitrary clause $u \vee v$ of t with literals $u, v \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$. By hypothesis it evaluates to $\mathbf{1}$ when plugging in $\bar{y}$ for $\bar{x}$, requiring $2d+1 \leq \dim(u[\bar{y}]) + \dim(v[\bar{y}])$ and thus that at least one of u, v had been assigned a subspace of dimension $\geq d+1$: which in the derived assignment becomes 1 and keeps the clause `true`.

c ii⇒iii) Let $\bar{y}$ denote a satisfying assignment over $\{\mathbf{0}, \mathbf{1}\}$. We claim that the derived assignment $b_j := \neg a_j$ for $y_j = \mathbf{1}$ and $b_j := a_j$ for $y_j = \mathbf{0}$ is a satisfying one. According to Example 5.7, this makes all clauses $u \vee v$, $(u \neq v)$, evaluate to `true` for which at least one of the literals u, v are assigned to the *complement* of some a_j. On the other hand, since $u[\bar{y}] \vee v[\bar{y}] = \mathbf{1}$, by construction also at least one of $u[\bar{b}]$ and $v[\bar{b}]$ is of the form $\neg a_j$.

c iii⇒iv) follows from Example 5.7. And c iv⇒i) is a tautology.

d) Recursive application of de Morgan's Laws as in Example 2.8a).

e) is straightforward by term induction since $\vee$ and $\wedge$ are sup and inf, respectively. □

Our next tool is designed to deal with Item c) of Theorem 5.5. We consider formulae $\phi(\bar{x})$ of the form

$$\exists \bar{z}.\ \texttt{\&\&}_{i=1}^{k}\, t_i(\bar{x}, \bar{z}) = \mathbf{1}\ \texttt{\&\&}\ \texttt{\&\&}_{j=1}^{\ell}\, s_j(\bar{x}, \bar{z}) = \mathbf{0}$$

with lattice terms $t_i(\bar{x}, \bar{z}) \in \bigwedge\bigvee$ and $s_j(\bar{x}, \bar{z}) \in \bigvee\bigwedge$. Let Λ denote the set of all such formulae (that is, certain *pp-formulae* in the sense of Model Theory). Recall that a complemented modular lattice is a modular lattice L with $\mathbf{0}$ and $\mathbf{1}$ such that for any $a \in L$ there is $b \in L$ such that $a \wedge b = \mathbf{0}$ and $a \vee b = \mathbf{1}$. Examples are the lattices of all subspaces of vector spaces.

p:collin

Proposition 5.9 *a) The Boolean conjunction of two formulae in Λ is logically equivalent to a formula in Λ.*

b) For any $\phi(\bar{x})$ in Λ there exist lattice terms $t(\bar{x}, \bar{z}) \in \bigwedge\bigvee\bigwedge$ and $s(\bar{x}, \bar{z}) \in \bigvee\bigwedge\bigvee$ such that, within any lattice, $\phi(\bar{x})$ is equivalent to $\exists \bar{z}.\ t(\bar{x}, \bar{z}) = \mathbf{1}$ `&&` $s(\bar{x}, \bar{z}) = \mathbf{0}$.

c) Given lattice terms $t(\bar{x}, u_1, \ldots u_m)$ and $t_i(\bar{x}, u_i)$ $(i = 1, \ldots, m)$ such that, within the lattice L, the formulae $v = t(\bar{x}, u_1, \ldots, u_m)$ and $u_i = t_i(\bar{x})$ are equivalent to $\phi(v, \bar{x}, u_1, \ldots, u_m)$ and $\phi_i(u_i, \bar{x})$, respectively, then

$$\exists u_1 \ldots \exists u_m.\ \phi(v, \bar{x}, u_1, \ldots u_m)\ \texttt{\&\&}\ \phi_1(u_1, \bar{x})\ \texttt{\&\&} \ldots \texttt{\&\&}\ \phi_m(u_m, \bar{x})$$

is equivalent within L to $v = t\big(\bar{x}, t_1(\bar{x}), \ldots, t_m(\bar{x})\big)$.

d) To any of the formulae $x \leq y$, $x = y$, $x \leq y \vee z$, $x \vee y = x \vee z$, there exists a formula in Λ equivalent to it within complemented modular lattices.

e) To every $d \in \mathbb{N}$ there is $\chi_d(\bar{y})$ in Λ such that in any complemented modular lattice L one has $L \models \chi_d(\bar{a})$ if and only if $\bar{a}$ is a d-frame of L.

f) To every $d \geq 3$ there is $\rho_d(x, \bar{y})$ in Λ such that for any a d-frame $\bar{A}$ in $\mathrm{L}(\mathcal{H})$, $\dim(\mathcal{H}) = \lceil$, one has $P \in \mathcal{R}_{\bar{A}}$ if and only $\mathrm{L}(\mathcal{H}) \models \rho_d(P, \bar{A})$.

g) For any variable u, and any given $d \geq 3$ there are $\phi_{d,\ominus}(u, x_1, x_2, \bar{y})$ and $\phi_{d,\otimes}(u, x_1, x_2, \bar{y})$ in Λ such that for any d-frame $\bar{A}$ in $\mathrm{L}(\mathcal{H})$, $\dim(\mathcal{H}) = \lceil \geq \ni$, and R, P, Q in $\mathcal{R}_{\bar{A}}$ one has $R = P \ominus_{\bar{A}} Q$ if and only $\mathrm{L}(\mathcal{H}) \models \phi_{d,\ominus}(R, \bar{P}, \bar{A})$ and $R = P \otimes_{\bar{A}} Q$ if and only $\mathrm{L}(\mathcal{H}) \models \phi_{d,\otimes}(R, \bar{P}, \bar{A})$.

h) In any of the above, the claimed formulae can be obtained in polynomial time from the data.

Proof. Item a) follows from the basic rules of logic dealing with $\exists$ and &&, In b) let $t = \bigwedge_i t_i$ and $s = \bigvee_j s_j$. In c) generalize the reasoning of Observation 4.9). The first two cases in Item d) are dealt with in Example 2.8f). Now, by modularity and existence of complements, $x \leq y \vee z$ is equivalent to $\exists u.\ y \vee z \vee u = \mathbf{1}$ && $(x \vee y \vee z) \wedge u = \mathbf{0}$ and, in any lattice, $x \vee y = x \vee z$ to the conjunction of $y \leq x \vee z$ and $z \leq x \vee y$. e) and f) follow immediately — only the last condition in the definition of a frame needs a closer look: it is equivalent to $a_{ik} \leq a_{ij} \vee a_{jk}$; indeed, given this $a_{ik} = (a_i \vee a_k) \wedge (a_{ij} \vee a_{jk})$ follows by modularity.

In g), considering the subterms of $P \ominus Q$ and $P \otimes Q$ and the calculations in Subsection 3.4 (resp. their geometric motivations in Subsection 3.1) we derive the following, which proves g) in view of d), c), and a),

$$\begin{array}{lll}
u_1 = Q_{13} & \Leftrightarrow u_1 \leq Q + A_{13} & \&\&\ u_1 \leq A_1 + A_3 \\
u_2 = P_{32} & \Leftrightarrow u_2 \leq P + A_{13} & \&\&\ u_2 \leq A_2 + A_3 \\
u_3 = P \otimes Q & \Leftrightarrow u_3 \leq Q_{13} + P_{32} & \&\&\ u_3 \leq A_1 + A_2 \\
u_4 = S & \Leftrightarrow u_4 \leq Q_{13} + A_2 & \&\&\ u_4 \leq P + A_{23} \\
u_5 = P \ominus Q & \Leftrightarrow: u_5 \leq S + A_3 & \&\&\ u_5 \leq A_1 + A_2.
\end{array}$$

h) is clear from the above proofs.

Proof (of Theorem 5.5).

a) Given a $\bigwedge\bigvee$-term $t(x_1, \ldots, x_n)$, first eliminate all clauses with only one literal by substituting it with $\mathbf{1}$: This simplification can obviously be performed in polynomial time and maintains t's $\bigwedge\bigvee$-form as well as strong satisfiability. If it fails (like for instance in $\neg x \wedge \neg y \wedge (x \vee y)$), reject t. Otherwise, in the even-dimensional case, accept. In odd dimensions, collect all clauses with precisely two (remaining) literals and report whether this instance of 2SAT is satisfiable: as mentioned above, in polynomial time. It remains to assert the correctness of this algorithm. Regarding the case of even dimensions, this holds due to Lemma 5.8a). Over $\mathrm{L}(\mathbb{F}^{2d+1})$, any satisfying assignment of t must in particular make all its two-literal clauses evaluate to `true`; which requires their conjunction to be a positive instance for 2SAT according to Lemma 5.8c i$\Rightarrow$ii). Conversely, if the two-literal clauses are satisfiable over $\{0,1\}$ then both, they *and* the clauses with at least three literals, admit a joint satisfying assignment from $\{a_1, \neg a_1, \ldots, a_n, \neg a_n\} \subseteq \mathrm{L}(\mathbb{F}^{2d+1})$ according to Lemma 5.8b) and c ii$\Rightarrow$iii).

b) By Lemma 5.8d) we may w.l.o.g. presume the input $t(\bar{x})$ to be in negation normal form, i.e. equal to $\hat{t}(\bar{x}, \neg\bar{x})$ with $\hat{t}(\bar{x}, \bar{y})$ over $\vee$ and $\wedge$ only. The relational interpretation (Observation 4.9) produces from that in polynomial time a system of basic lattice equations, i.e. each of one of the forms $y_j = y_k \wedge y_\ell$ and $y_j = y_k \vee y_\ell$ in (possibly negated) variables y_j jointly satisfiable iff t admits a strongly satisfying assignment. Now according to Example 2.8f), these equations are in turn equivalent to

$$\mathbf{1} \;=\; (y_j \wedge y_k \wedge y_\ell) \vee \big(\neg y_j \wedge (\neg y_k \vee \neg y_\ell)\big) \quad \text{and} \quad \mathbf{1} \;=\; \big(y_j \wedge (y_k \vee y_\ell)\big) \vee (\neg y_j \wedge \neg y_k \wedge \neg y_\ell).$$

Their right hand sides are of the form $\bigvee\bigwedge\bigvee$; hence combining these equations according to Fact 2.9a) yields a single $\bigwedge\bigvee\bigwedge\bigvee$-term $s(\bar{y})$ as desired.

c) To see the problem in $\mathrm{BP}(\mathcal{NP}_{\mathbb{F}})$ (and not just in $\mathrm{BP}(\mathcal{NP}_{\mathrm{Re}\,\mathbb{F}})$), recall that the evaluation in Proposition 2.17b) needs access to real and imaginary parts only for terms with

negation when invoking Proposition 2.17a iii); whereas for s and t as in this case, Proposition 2.17a i) and a ii) suffices.
Regarding completeness, observe that steps i) to v) in the proof of Proposition 3.12 produce a finite list of lattice equations: terms $\otimes_{\bar{A}}$ and $\ominus_{\bar{A}}$ do not invoke negation in ii) and iii), nor does iv) or v) according to Definition 3.6. Replacing in Step vi) Fact 2.9a) with Fact 2.9b) thus yields two lattice terms s and t with the claimed properties.

d) Regarding completeness as in the proof of c), observe that, due to Proposition 5.9, Steps i) to v) in the proof of Proposition 3.12 produce a finite list of formulae in Λ: in v) use e) and in iv) use f). In ii) observe that any polynomial p with integer coefficients is equivalent within (commutative) rings to a term $q(\bar{x})$ in constants $0, 1$ and binary operation symbols $-, \cdot$. Replace these according to Proposition 5.9g) by y_{11}, y_{12}, $\phi_{d,\ominus}$, and $\phi_{d,\otimes}$, respectively, and apply, iteratively, c) and a) of Proposition 5.9 to obtain $\phi_{d,p}(u, \bar{x}, \bar{y})$ in Λ such that for any d-frame $\bar{A}$ in $\mathrm{L}(\mathcal{H})$, $\dim(\mathcal{H}) = d \geq 3$ and any $R, \bar{P}$ in $\mathcal{R}_{\bar{A}}$ one has $R = p_{\mathcal{R}_{\bar{A}}}(\bar{P})$ if and only if $\mathrm{L}(\mathcal{H}) \models \phi_{d,p}(R, \bar{P}, \bar{A})$. This deals with Step iii). Replacing in Step vi) Fact 2.9b) with Proposition 5.9b) thus yields two lattice terms s and t with the claimed properties — and all this in time polynomial in the input length of p.

e) Let $d \geq 3$. Equation $s = \mathbf{0}$ with $s \in \bigvee\bigwedge\bigvee$ is equivalent to $\neg s = \mathbf{1}$ and $\neg s$ is equivalent to a term in $\bigwedge\bigvee\bigwedge$ by de Morgan's Laws. In the case $\mathbb{F} \not\subseteq \mathbb{R}$ we also have to take in account Steps (v') and (iv') of the proof of Theorem 3.18. Here, the additional conditions $A_i \leq \neg A_j$ and $X_k \leq \neg Z_k$ are dealt with by Example 2.8f), again. In the case $d \leq 2$, for hardness we refer to the proof of Proposition 2.10 and the fact that the commutator is in $\bigvee\bigwedge$; membership in $\mathcal{NP}$ follows from Proposition 2.13. □

5.4 Varying the Ground Field

ss:Fields

In dimension two, weak/strong satisfiability did not depend on the underlying field under consideration: recall Lemma 2.12b). This becomes different starting with dimension three.

f:Fields

Fact 5.10 *a) If $\mathbb{F}$ is a $*$-subfield of $\mathbb{E}$ then $\mathrm{L}(\mathbb{F}^d)$ embeds into $\mathrm{L}(\mathbb{E}^d)$.*
b) $\mathrm{L}\left(\mathbb{F}(i)^d\right)$ embeds into $\mathrm{L}(\mathbb{F}^{2d})$.
c) For an irreducible monic polynomial $p(X) \in \mathbb{F}[X]$ of degree k and $\mathbb{F}$-vector space V, there is an endomorphism ϕ of V such that $p(\phi) = 0$ if and only if $k \leq \dim(V)$.

Proof. In a) map U to $\mathbb{E}U$. In b) consider $\mathbb{F}(i)^d$ as an $\mathbb{F}$-vector space with scalar product the real part of that in $\mathbb{F}(i)^d$. For c) See [Jaco53, Ch.III.4].

Recall that two algebraic structures are elementarily equivalent if they satisfy the same first-order sentences. In particular, all real closed fields $\mathbb{F}$ (such as $\mathbb{R}$ and $\mathbb{A} \cap \mathbb{R}$) are pairwise elementarily equivalent (cf. [Shoe67, §5.5]); and so are all the associated $*$-fields $\mathbb{F}(i)$.

p:Fields

Proposition 5.11 *a) For elementarily equivalent $\mathbb{F}_1, \mathbb{F}_2$ and $\mathbb{F}_i$-unitary spaces $\mathcal{H}_i$ of the same dimension d admitting equinormal orthogonal bases, every orthologic term t is weakly (resp. strongly) satisfiable in $\mathrm{L}(\mathcal{H}_1)$ if and only if it is so in $\mathrm{L}(\mathcal{H}_2)$.*
b) A term is weakly (resp. strongly) satisfiable either in both or in none of $\mathrm{L}(\mathbb{C}^d)$ and $\mathrm{L}(\mathbb{A}^d)$; similarly for $\mathrm{L}(\mathbb{R}^d)$ and $\mathrm{L}\left((\mathbb{R} \cap \mathbb{A})^d\right)$.
c) If $\mathbb{F}$ is a $*$-subfield of $\mathbb{E}$ then any term weakly resp. strongly satisfiable in $\mathrm{L}(\mathbb{F}^d)$ is so in $\mathrm{L}(\mathbb{E}^d)$.
d) Any term weakly resp. strongly satisfiable in $\mathrm{L}\left(\mathbb{F}(i)^d\right)$ is so in $\mathrm{L}(\mathbb{F}^{2d})$.

e) Let polynomial $p(X) \in \mathbb{Z}[X]$ have a root in $\mathbb{E}$ but not in $\mathbb{F}$ and $d \geq 3$. Then the term $t_{p,d}$ according to Proposition 3.12c) is strongly satisfiable in $\mathrm{L}(\mathbb{E}^d)$ *but not in* $\mathrm{L}(\mathbb{F}^d)$.

f) For any $d \geq 3$ and any $k \geq 2$ there exists a term t such that t is strongly satisfiable in $\mathrm{L}(\mathbb{R}^d)$ *and in* $\mathrm{L}(\mathbb{Q}^{kd})$ *but not in any* $\mathrm{L}(\mathbb{Q}^n)$ *with $n < kd$.*

So Item e) says that for instance $t_{X^2-2,d}$ is strongly satisfiable over $\mathrm{L}(\mathbb{R}^d)$ but not over $\mathrm{L}(\mathbb{Q}^d)$; and $t_{X^2+1,d}$ is strongly satisfiable over $\mathrm{L}(\mathbb{C}^d)$ but not over $\mathrm{L}(\mathbb{R}^d)$. Proposition 5.11e) and f) has analoga for weak satisfiability according to Theorem 5.1b).

Proof (of Proposition 5.11).

a) As observed, e.g., in [DHMW05, Theorem 6], weak/strong satisfiability over $\mathrm{L}(\mathbb{F}^d)$ amounts to a first-order sentence to be valid in $\mathbb{F}$. In case $\mathcal{H}$ admits an equinormal orthogonal basis, invoke Lemma 2.22a).

b) follows from a); c) follows from Fact 5.10a); d) follows from Fact 5.10b).

e) is immediate by Proposition 3.12c).

f) Consider a real algebraic integer α of degree k over $\mathbb{Q}$ like, for instance, $\alpha := 2^{1/k}$; cmp. [Lang99, Ch. VI Thm.9.1]. Its associated minimal polynomial $p(X) \in \mathbb{Z}[X]$ is monic and irreducible — over $\mathbb{Q}[X]$ by Gauss' Lemma. Fact 6.9c) below yields a term $t_{p,d}$ strongly satisfiable over $\mathrm{L}(\mathbb{F}^n)$ iff there is an endomorphism ϕ of an m-dimensional subspace W of $\mathbb{F}^n$ where $n = dm$. In view of Fact 5.10c) this happens for $\mathbb{F} = \mathbb{Q}$ if and only if $k \leq m$. On the other hand, if $\mathbb{A} \cap \mathbb{R} \subseteq \mathbb{F}$ then $p(X)$ factors in $\mathbb{F}[X]$ into linear factors and one may choose $m = 1$. □

d:uclass

Digression 5.12 *For $*$-subfields $\mathbb{F} \subseteq \mathbb{E}$ of $\mathbb{C}$, if $\mathbb{E}$ is a purely transcendental extension of $\mathbb{F}$ then one has an embedding of* $\mathrm{L}(\mathbb{E}^d)$ *into some ultrapower of* $\mathrm{L}(\mathbb{F}^d)$*. If $\mathbb{E}$ is an algebraic extension of $\mathbb{F}$ then* $\mathrm{L}(\mathbb{E}^d)$ *is the direct union of the* $\mathrm{L}(\mathbb{E}_i^d)$ *with $\mathbb{E}_i$ ranging over the finite algebraic extensions of $\mathbb{F}$. Thus, the interesting case is $\mathbb{E} = \mathbb{F}(\alpha)$ where α is a root of an irreducible polynomial $p(X)$ in $\mathbb{F}[X]$.*

Recall that $\mathrm{L}(\mathbb{E}^d)$ *embeds into* $\mathrm{L}(\mathbb{F}^n)$ *if there is a $*$-ring embedding of $\mathbb{E}^{d\times d}$ into $\mathbb{F}^{n\times n}$ (whence, in particular, $n = kd$ for some k) and that for $d \geq 3$ this is also necessary (this can be derived from the proofs of the coordinatization results of [Neum60]). For $\mathbb{E} = \mathbb{F}(i)$ there is a canonical such embedding with $k = 2$: consider $\mathbb{F}(i) \cong \left\{\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \mid a, b \in \mathbb{F}\right\}$ and convert the entries of a matrix in $\mathbb{F}(i)^{d\times d}$ into blocks of a matrix over $\mathbb{F}$, accordingly (cmp. Fact 5.10b). In view of Digression 2.16 this leaves to consider the case $\mathbb{E} \subseteq \mathbb{R}$. Here, existence of a $*$-ring embedding amounts to existence of a symmetric $A \in \mathbb{F}^{k\times k}$ such that $p(A) = 0$. For $p(X) = X^2 - c$ and $k = 2$ such A are of those of the form $A = \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$ with $c = a^2 + b^2$. Such $A \in \mathbb{Q}^{2\times 2}$ exist for $c = 2$ and primes $c \equiv 1 \bmod 4$, but not for integers $c \equiv 3 \bmod 4$.*

Now, for a prime $p \equiv 1 \bmod 4$ we have integers a, b such that $p = a^2 + b^2 + 2$. Consider $\mathbb{F} = \mathbb{Q}(\sqrt{2}, i)$ and the selfadjoint matrix $A = \left(\begin{smallmatrix} \sqrt{2} & c \\ c^* & -\sqrt{2} \end{smallmatrix}\right)$ where $c = a + bi$. Then $A^2 = pI$ and we conclude that $\mathrm{L}(\mathbb{F}(\sqrt{p})^d)$ *embeds into* $\mathrm{L}(\mathbb{F}^{2d})$ *whence in* $\mathrm{L}(\mathbb{Q}^8)$.

To us, all other cases are open. As is the question whether there is an ortholattice identity valid in all $\mathrm{L}(\mathbb{Q}^n)$ *but not in* $\mathrm{L}(\mathbb{R}^3)$*. Though, such identity cannot be a mere lattice identity.*

5.5 Dimensions of Satisfiability

ss:Dimensions

o:Dimensions **Observation 5.13** *a) A term weakly satisfiable in* $\mathrm{L}(\mathcal{H})$ *is also weakly satisfiable in* $\mathrm{L}(\mathcal{H}')$ *for every subspace $\mathcal{H} \subseteq \mathcal{H}'$.*

b) A term strongly satisfiable in both $\mathrm{L}(\mathcal{H})$ *and in* $\mathrm{L}(\mathcal{H}')$ *is also strongly satisfiable in* $\mathrm{L}(\mathcal{H} \oplus \mathcal{H}')$.

Put differently, for each term t, the set $\dim_{\mathbb{F}}(t) \subseteq \mathbb{N}$ is an ideal in $(\mathbb{N}, +)$ and $\mathrm{DIM}_{\mathbb{F}}(t) \subseteq \mathbb{N}$ a sub-semigroup, where

$$\begin{aligned} \dim_{\mathbb{F}}(t) &:= \big\{ d \in \mathbb{N} \mid t \text{ is weakly satisfiable over } \mathrm{L}(\mathbb{F}^d) \big\} \\ \mathrm{DIM}_{\mathbb{F}}(t) &:= \big\{ d \in \mathbb{N} \mid t \text{ is strongly satisfiable over } \mathrm{L}(\mathbb{F}^d) \big\} \end{aligned} \tag{12}$$

Indeed, a) follows from Fact 2.2b) and the fact that weak satisfiability of t is the complement of validity of the equation $t = \mathbf{0}$, the latter being preserved in substructures and homomorphic images. b) follows from Observation 2.6a).

x:Dimensions

Example 5.14 *a) For any* d*, the term* h_d *from Lemma 4.18a) is (independent of* $\mathbb{F}$ *and) strongly satisfiable in* $\mathrm{L}(\mathcal{H})$ *if and only if* d *divides* $\dim(\mathcal{H})$.
b) For any k*, there exists a term* t_k *of length* $\mathcal{O}(k)$ *strongly satisfiable over* $\mathrm{L}(\mathcal{H})$ *for* $\dim(\mathcal{H}) = 2^k$ *but not for* $\dim(\mathcal{H}) < 2^k$.
c) For any d*, the term* g_d *from Lemma 4.18c) is (independent of* $\mathbb{F}$ *and) weakly satisfiable in* $\mathrm{L}(\mathcal{H})$ *if and only if* $d \leq \dim(\mathcal{H})$.

Proof. a) Suppose $\dim(\mathcal{H}) = d \cdot k$. Then $\mathcal{H} = V_1 \oplus \cdots \oplus V_k$ for appropriate V_j pairwise orthogonal. By Example 4.16a), there exists an orthogonal d-diamond $\bar{A}_j$ in $\mathrm{L}(V_j)$, $1 \leq j \leq d$; and thus one in $\mathrm{L}(\mathcal{H})$ by Observation 2.6a): strongly satisfying h_d by Lemma 4.18a).
Conversely, any strongly satisfying assignment $\bar{A}$ of h_d over $\mathcal{H}$ constitutes by Lemma 4.18a) an orthogonal d-diamond; hence d divides $\dim(\mathcal{H})$ according to Fact 4.17c).

b) Consider the $(2n+1)$-variate term

$$x_{n+1} \ \wedge \ \bigwedge\nolimits_{i=1}^{n} \Big((x_i \vee \neg y_i) \ \wedge \ (y_i \vee \neg x_i) \ \wedge \ (\neg x_i \vee \neg y_i) \\ \wedge \ \Big(\big(x_{i+1} \wedge (x_i \vee y_i) \big) \vee \big(\neg x_{i+1} \wedge \neg (x_i \vee y_i) \big) \Big) \Big)$$

Note that $x_i \vee \neg y_i = \mathbf{1}$ in $\mathrm{L}(\mathbb{F}^d)$ implies $\dim(x_i) + \dim(\neg y_i) \geq d$; hence the first two terms in the big conjunction require $\dim(x_i) = \dim(y_i)$. The fourth term amounts to condition $x_{i+1} = x_i \vee y_i$ according to Example 2.8f); hence $\dim(x_{i+1}) = \dim(x_i) + \dim(y_i)$ because of $x_i \wedge y_i = \mathbf{0}$ (third term). Concluding, any satisfying assignment $(a_1, b_1, \ldots, a_n, b_n, a_{n+1})$ has $\dim(a_{i+1}) = 2 \cdot \dim(a_i)$. Therefore $2^n \cdot \dim(a_1) = \dim(a_{n+1}) = d$ by the very first term. Conversely, the following is easily verified to constitute a satisfying assignment:

$$a_1 := \mathbb{F} \times \{0\}, \ \ b_1 := \{(x,x) : x \in \mathbb{F}\}, \quad a_2 := \mathbb{F}^2 \times \{0\}^2, \ \ b_2 := \{(\vec{x},\vec{x}) : \vec{x} \in \mathbb{F}^2\}, \\ \ldots, \quad a_{i+1} := \mathbb{F}^{2^i} \times \{0\}^{2^i}, \ \ b_{i+1} := \{(\vec{x},\vec{x}) : \vec{x} \in \mathbb{F}^{2^i}\} \quad \ldots, \quad a_{n+1} = \mathbb{F}^{2^n}$$

(all understood embedded into $\mathbb{F}^{2^n}$ by appending zeros).

c) As pointed out in Corollary 4.1, g_d is not weakly satisfiable over $\mathrm{L}(V)$ for $\dim(V) < d$ but weakly satisfiable for $\dim(V) = d$. In case $\dim(\mathcal{H}) > d$, there exists $V \in \mathrm{L}_d(\mathcal{H})$; hence by Observation 5.13a), g_d is weakly satisfiable also over $\mathrm{L}(\mathcal{H})$. □

Digression 5.15 *An alternative term (and historically the perhaps first, although of length cubic in d) weakly satisfiable in $\mathrm{L}(\mathcal{H})$ if and only if $d \leq \dim(\mathcal{H})$ is also due to* Huhn *[Huhn72]: He has shown that the "$(d-1)$-distributive law"*

$$x \vee \bigwedge_{j=1}^{d} y_j = \bigwedge_{j=1}^{d} \big(x \vee \bigwedge_{j\neq i=1}^{d} y_i\big) \tag{13}$$

fails in a modular lattice L if and only if L contains $v < u$ and a d-diamond in $[v,u]$ — where the condition $a_i \leq \neg a_j$ for $i \neq j$ is replaced by the independence requirement $a_i \wedge \bigvee_{j=1,\,j\neq i}^{d} = \mathbf{0}$ for $i \geq 1$. As in Fact 4.17 this implies $\dim[v,u] \geq d$. Now consider $r \wedge \neg\ell$ and observe that $\ell \leq r$ holds in any lattice, where ℓ and r denote left and right hand side of Equation (13), respectively.

A partial converse to Example 5.14c) appears in Item b) of the next

l:Dimensions

Lemma 5.16 *Recall from Lemma 5.8d)+e) that a lattice term is one over $\vee$ and $\wedge$ only, i.e. contains no negations.*

a) For any lattice term t and $\vec{v} \in t_{\mathrm{L}(\mathcal{H})}(U_1,\ldots,U_n)$ there are $V_i \subseteq U_i$ in $\mathrm{L}(\mathcal{H})$ such that $\sum_{i=1}^n \dim V_i \leq |t|$ and $\vec{v} \in t_{\mathrm{L}(\mathcal{H})}(V_1,\ldots,V_n)$.
b) A term t is weakly satisfiable in $\mathrm{L}(\mathcal{H})$ if and only if there exists $W \in \mathrm{L}(\mathcal{H})$ with $\dim W \leq |t|$ such that t is weakly satisfiable in $\mathrm{L}(W)$.

Proof. a) by term induction: Let $t = t_1 \vee t_2$ and $m_j := |t_j|$ whence $|t| \geq m_1 + m_2$. Then $\vec{v} \in t_{1\,\mathrm{L}(\mathcal{H})}(U_1,\ldots,U_n) + t_{2\,\mathrm{L}(\mathcal{H})}(U_1,\ldots,U_n)$, i.e. there are $\vec{v}_j \in t_{j,\mathrm{L}(\mathcal{H})}(U_1,\ldots,U_n)$ such that $\vec{v} = \vec{v}_1 + \vec{v}_2$ and by inductive hypothesis there are $V_{ji} \subseteq U_i$ for $j = 1,2$ such that $\sum_{i=1}^n \dim V_{ji} \leq m_j$ and $\vec{v}_j \in t_{j,\mathrm{L}(\mathcal{H})}(V_1,\ldots,V_n)$. So put $V_i := V_{1i} + V_{2i}$.
For $t = t_1 \wedge t_2$ the same arguing applies with $\vec{v} = \vec{v}_1 = \vec{v}_2$.

b) In view of Lemma 5.8d) we may assume $t(\bar{x}) = \hat{t}(\bar{x}, \neg\bar{x})$ with a lattice term $\hat{t}(\bar{x},\bar{y})$. Now, assume $\vec{v} \in t_{\mathrm{L}(\mathcal{H})}(U_1,\ldots,U_n)$. Then $\vec{0} \neq \vec{v} \in \tilde{t}_{\mathrm{L}(\mathcal{H})}(U_1,\ldots,U_n,U_1^\perp,\ldots,U_n^\perp)$. According to a) there are $V_i \subseteq U_i$ and $V_i' \subseteq U_i^\perp$ such that $\sum_{i=1}^n \dim V_i + \dim V_i' \leq m = |t|$ and $\vec{v} \in \tilde{t}_{\mathrm{L}(\mathcal{H})}(V_1,\ldots,V_n,V_1',\ldots,V_n')$. Put $W := \sum_{i=1}^n V_i + V_i'$ and $W_i := U_i \cap W$. Then $\dim W \leq m$, $V_i \subseteq W_i \subseteq U_i$, and $V_i' \subseteq W \cap W_i^\perp$ since $V_i' \subseteq U_i^\perp \subseteq W_i^\perp$. By Lemma 5.8e) it follows $\vec{v} \in \tilde{t}_{\mathrm{L}(\mathcal{H})}(W_1,\ldots,W_n,W\cap W_1^\perp,\ldots,W\cap W_n^\perp) = t_{\mathrm{L}(W)}(W_1,\ldots,W_n)$. □

6 Satisfiability in Indefinite (yet Finite) Dimension

s:Indef

In an aim to approach the infinite-dimensional case, we now consider satisfiability questions quantifying existentially not just over assignments but also over the (finite) dimension the assignment lives in.

d:Indef

Definition 6.1 *a) Call a term weakly respectively strongly satisfiable in $\mathrm{L}(\mathbb{F}^*)$ if and only if it is so in $\mathrm{L}(\mathbb{F}^d)$ for some $d \in \mathbb{N}$.*
b) The corresponding decision problems are abbreviated as $\mathrm{sat}_{\mathrm{L}(\mathbb{F}^*)} := \mathrm{sat}_{\{\mathrm{L}(\mathbb{F}^d):d\in\mathbb{N}\}} = \bigcup_d \mathrm{sat}_{\mathrm{L}(\mathbb{F}^d)}$ and $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^*)} := \mathrm{SAT}_{\{\mathrm{L}(\mathbb{F}^d):d\in\mathbb{N}\}} = \bigcup_d \mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d)}$, respectively.

t:WeakIndef

Theorem 6.2. *a) For Pythagorean $\mathbb{F}$ it holds $\mathrm{sat}_{\mathrm{L}(\mathbb{F}^*)} \in \mathrm{BP}(\mathcal{NP}_{\mathrm{Re}\,\mathbb{F}})$. In particular, $\mathrm{sat}_{\mathrm{L}(\mathbb{R}^*)}, \mathrm{sat}_{\mathrm{L}(\mathbb{C}^*)} \in \mathrm{BP}(\mathcal{NP}_{\mathbb{R}})$.*

b) A term is weakly (resp. strongly) satisfiable either in both or in none of $\mathrm{L}(\mathbb{F}^*)$ *and* $\mathrm{L}(\mathrm{Re}\,\mathbb{F}^*)$*, i.e. it holds* $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}*)} = \mathrm{SAT}_{\mathrm{L}(\mathrm{Re}\,\mathbb{F}^*)}$ *and* $\mathrm{sat}_{\mathrm{L}(\mathbb{F}*)} = \mathrm{sat}_{\mathrm{L}(\mathrm{Re}\,\mathbb{F}^*)}$*.*

That is, weak satisfiability over $\mathrm{L}(\mathbb{F}^*)$ is decidable by a nondeterministic polynomial-time BSS-machine over $\mathrm{Re}\,\mathbb{F}$—and thus no more hard than in the fixed-dimensional case. A qualitative precursor to this result is the decidability of the equational theory of the class $\{\mathrm{L}(\mathbb{C}^d) \mid d < \omega\}$ in [Her10a].

Proof. a) According to Lemma 5.16d), t is weakly satisfiable over $\mathrm{L}(\mathbb{F}^*)$ iff it is so over $\mathrm{L}(W)$ for some d-dimensional $\mathbb{F}$-unitary space W for $d := |t|$; iff it is over $\mathrm{L}(\mathbb{F}^d)$, according to Fact 2.19. Now recall (Proposition 2.17c) that satisfiability over $\mathrm{L}(\mathbb{F}^d)$ can be decided by a nondeterministic BSS-machine over $\mathrm{Re}\,\mathbb{F}$ in time polynomial in $|t|$ *and* d.

b) follows from Proposition 5.11c+d) in view of $\mathbb{F} = \big(\mathrm{Re}\,\mathbb{F}\big)(i)$ for $*$-fields $\mathbb{F} \not\subseteq \mathbb{R}$. □

q:WeakIndef

Question 6.3 *Is* $\mathrm{sat}_{\mathrm{L}(\mathbb{R}^*)}$ *hard for* $\mathcal{NP}$ *or even for* $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$*?*

The rest of this section explores a strong counterpart to Theorem 6.2a) and Question 6.3, namely the complexity (and computability) of $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^*)}$. To this end, the next subsections extend Subsections 3.3, 3.4, and 3.6 from interpreting into quantum logic not just the ring $\mathbb{F}$ of scalars but the ring $\mathbb{F}^{m\times m}$ of matrices, uniformly in m, and similarly for $*$-rings. We first discuss the feasibility problems for those.

6.1 Feasibility in Matrix Rings

ss:intermat

Dealing with non-commutative rings, the classical concept corresponding to the notion of a term amounts to that of polynomials $p \in \mathbb{Z}\langle X_1,\ldots,X_n\rangle$ with integer coefficients in non-commuting variables $\bar{X}$. Similarly for non-commutative $*$-rings, a term $t(X_1,\ldots,X_n)$ is (equivalent to) a polynomial $p \in \mathbb{Z}\langle X_1,\ldots,X_n,X_1^\dagger,\ldots,X_n^\dagger\rangle$ – again with the convention that $X^\dagger$ has to be interpreted as $A^* = A^\dagger$ if X is interpreted as A. And the ring anti-automorphism interchanging X_i and $X_i^\dagger$ yields a polynomial equivalent to the term t^* within $*$-rings. These naturally suggest to generalize the family of problems $\mathrm{FEAS}_{\mathbb{Z},R}$ from Example 1.11 and the feasibility of commutative $*$-polynomials from Theorem 3.18.

d:Feas2

Definition 6.4 *a) For a not necessarily commutative ring* $R \supseteq \mathbb{Z}$ *let* $\mathrm{FEAS}_{\mathbb{Z},R}$ *denote the following decision problem:*

Given polynomials $p_1,\ldots,p_k \in \mathbb{Z}\langle X_1,\ldots,X_n\rangle$*, do they admit a common root in* R*, i.e. some assignment* $\bar{x} \in R^n$ *such that* $p_1(\bar{x}) = \cdots = p_k(\bar{x}) = 0$ *?*

b) $\mathrm{QUAD}_{1,R}$ *is the similar problem but restricted to quadratic polynomial systems with coefficients from* $\{0,\pm1\}$*.*

c) For a $*$-ring R, $\mathrm{FEAS}^\dagger_{\mathbb{Z},R}$ *is the question of whether non-commutative $*$-polynomials* $p_j \in \mathbb{Z}\langle X_1,X_1^\dagger,\ldots,X_n,X_n^\dagger\rangle$ *admit a common root in* R*.*

d) $\mathrm{QUART}^\dagger_{n\mapsto 2n,R}$ *is the similar problem but restricted to one single quartic $*$-polynomial* $p(X_1,X_1^\dagger,\ldots,X_n,X_n^\dagger)$ *with coefficients from* $\{0,\pm1,\pm2,\ldots,\pm2n\}$*.*

e) For $\mathcal{R}$ *a class of rings,* $\mathrm{FEAS}_{\mathbb{Z},\mathcal{R}} := \bigcup_{R\in\mathcal{R}} \mathrm{FEAS}_{\mathbb{Z},\mathcal{R}}$ *asks for a common root to* $p_1,\ldots,p_k$ *over* some $R \in \mathcal{R}$*. Similarly for* $\mathrm{QUAD}_{1,\mathcal{R}}$*.*

f) For a class $\mathcal{R}$ *of $*$-rings, write* $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathcal{R}} := \bigcup_{R\in\mathcal{R}} \mathrm{FEAS}^\dagger_{\mathbb{Z},\mathcal{R}}$ *and similarly for* $\mathrm{QUART}^\dagger_{n\mapsto 2n,\mathcal{R}}$*.*

We point out that (the complements of) these computational problems arise in practice [BDDK03], hence their complexity is worthwhile investigating. Fact 4.10 carries over to the case of noncommutative ($*$-) rings:

l:RingInterp2

Lemma 6.5 *a) For R a (not necessarily commutative) ring,* $\mathrm{FEAS}_{\mathbb{Z},R}$ *and* $\mathrm{QUAD}_{1,R}$ *are polynomial-time equivalent, independently of R.*

b) The matrix $*$-ring $\mathbb{F}^{d\times d}$ is formally real *in the sense that $\sum_{j=1}^{J} B_j^\dagger \cdot B_j = 0$ implies $B_j = 0$ $(1 \leq j \leq J)$.*

c) For R a formally real $*$-ring, $\mathrm{FEAS}^\dagger_{\mathbb{Z},R}$ *and* $\mathrm{QUART}^\dagger_{n\mapsto 2n,R}$ *are polynomial-time equivalent, independently of R.*

Proof. a) follows straightforwardly using the relational interpretation Example 4.7 in connection with Observation 3.13.

b) Suppose there is some k and some $\vec{x} \in \mathbb{F}^d$ such that $B_k\vec{x} \neq 0$. Then $0 = \langle \vec{x} \,|\, \sum_j B_j^\dagger \cdot B_j\vec{x}\rangle = \sum_j \langle B_j\vec{x}|B_j\vec{x}\rangle \geq \|B_j\vec{x}\|^2 > 0$: contradiction.

c) Similarly to a), Observation 4.9 yields in polynomial time a system of quadratic $*$-polynomials q_n $(1 \leq n \leq N)$ whose joint feasibility is equivalent to that of the original ones. From these, proceed to the single $*$-polynomial $\sum_{n=1}^{N} q_n^* \cdot q_n$. Similarly to Fact 4.10, the latter can be achieved to have coefficients bounded by the number of 'literals' X_j and $X_j^\dagger$. □

p:feas

Proposition 6.6 *Abbreviate* $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}^*} := \mathrm{FEAS}_{\mathbb{Z},\{\mathbb{F}^{d\times d}:d\in\mathbb{N}\}}$, $\mathrm{QUAD}_{1,\mathbb{F}^*} := \mathrm{QUAD}_{1,\{\mathbb{F}^{d\times d}:d\in\mathbb{N}\}}$, $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}^*} := \mathrm{FEAS}^\dagger_{\mathbb{Z},\{\mathbb{F}^{d\times d}:d\in\mathbb{N}\}}$, *and* $\mathrm{QUART}^\dagger_{n\mapsto 2n,\mathbb{F}^*} := \mathrm{QUART}^\dagger_{n\mapsto 2n,\{\mathbb{F}^{d\times d}:d\in\mathbb{N}\}}$.

a) For any fixed field $\mathbb{F}$ and $d \geq 1$, the three problems $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}}$ *and* $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}^{d\times d}}$ *and* $\mathrm{QUAD}_{1,\mathbb{F}^{d\times d}}$ *are mutually polynomial-time equivalent.*

b) $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}^*}$ *is polynomial-time equivalent to* $\mathrm{QUAD}_{1,\mathbb{F}^*}$.

c) For algebraically closed $\mathbb{F}$ (say, $\mathbb{F} = \mathbb{C}$), $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}}$ *reduces polynomially to* $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}^*}$.

d) For an $*$-subfield $\mathbb{F}$ of $\mathbb{C}$ and $d \geq 1$, the following are polynomial time equivalent:
$\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}}$, $\mathrm{FEAS}_{\mathbb{Z},\mathrm{Re}\,\mathbb{F}}$, $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}^{d\times d}}$, $\mathrm{QUART}^\dagger_{n\mapsto 2n,\mathbb{F}^{d\times d}}$.

e) For an $*$-field $\mathbb{F}$, $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}^*}$ *is polynomial-time equivalent to* $\mathrm{QUART}^\dagger_{n\mapsto 2n,\mathbb{F}^*}$.

f) If $\mathbb{F}$ is real or algebraically closed (e.g. $\mathbb{F} = \mathbb{R},\mathbb{C}$) then there is a polynomial time reduction from $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}}$ *to* $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}^*}$.

Proof. a) In view of Lemma 6.5a), it remains to show the equivalence of $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}}$ and $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}^{d\times d}}$. Concerning one direction, a polynomial-time nondeterministic BSS machine over $\mathbb{F}$ can guess the entries of a matrix assignment over $\mathbb{F}^{d\times d}$ and verify them by evaluating the polynomials using operations from $\mathbb{F}$. This demonstrates $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}^{d\times d}} \in \mathrm{BP}(\mathcal{NP}_\mathbb{F})$. Since $\mathrm{FEAS}_{\mathbb{Z},\mathbb{F}}$ is complete for $\mathrm{BP}(\mathcal{NP}_\mathbb{F})$, the first reduction follows.
More directly, the relational interpretation (Observation 4.9) expresses matrix calculations as a system of equations over $\mathbb{F}$.
For the converse reduction, apply Fact 4.5 requiring, in addition to the given equations the existence of a d-system of matrix units commuting with all variables.

b) Apply Lemma 6.5a) to $R = \mathbb{F}^{d\times d}$, uniformly in d.

c) Given equations $p_1(\bar{X}) = \ldots = p_k(\bar{X}) = 0$ add all commutativity conditions $X_iX_j - X_jX_i$. Clearly, if $a_1, \ldots a_n$ is a solution in $\mathbb{F}$ for the old system, then $a_1I, \ldots a_nI$ is a solution in $\mathbb{F}^{d\times d}$ for any d. Conversely, if $A_1, \ldots, A_n$ is a solution of the extended system in some $\mathbb{F}^{d\times d}$, then the $A_1, \ldots, A_n$ commute with each other whence have a common eigenvector $\vec{v}$ (cmp. [Gelf61, p.109], [Jaco53, Ch.IV.9]) to respective eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$; which constitute a solution in $\mathbb{F}$.

d) To reduce $\mathrm{FEAS}^{\dagger}_{\mathbb{Z},\mathbb{F}}$ to $\mathrm{FEAS}_{\mathbb{Z},\mathrm{Re}\,\mathbb{F}}$, use the relational interpretation and the description of operations in $\mathbb{F} = (\mathrm{Re}\,\mathbb{F})(i)$ in terms of the Cartesian representation $x = x_1 + ix_2$ e.g. replacing $X = YZ$ by $X_1 = Y_1Z_1 - Z_2Z_2 = 0$ and $X_2 = Y_1Y_2 + Y_2Z_1 = 0$. For the converse, add the condition $X_j - X_j^{\dagger} = 0$ for each variable. The remaining equivalences follows as in a) using $*$-matrix units.

e) Apply Lemma 6.5b+c) to the $*$-ring $\mathbb{F}^{d\times d}$, uniformly in d.

f) Add the commutativity conditions and also conditions $X_j = X_j^{\dagger}$, that is, require the X_j to be selfadjoint. Having a solution $A_1, \dots A_n$, it holds also in the real case that each invariant subspace U of A_i contains an eigenvector of A_i (since the endomorphism defined by A_i is selfadjoint, also when restricted to U). Thus, by the same proof as for $\mathbb{F} = \mathbb{C}$ in c), the A_i have some common eigenvector and the proof proceeds as before (cmp. [Jaco53, Ch.VI. Thm.7]. □

6.2 Interpreting the Matrix Rings $\mathbb{F}^{m\times m}$ into $\mathrm{L}(\mathbb{F}^{3m})$, Uniformly in m

ss:MatinGrH

The next step is to recover the ring $\mathbb{F}^{n\times m}$ in $\mathrm{L}(\mathbb{F}^{3m})$, a subject dealt with exhaustively by von Neumann, who also derived existence and uniqueness of coordinatizing $*$-rings in much more general context. Again, we first use general frames in order to interpret the matrix ring with*out* involution; then (Subsection 6.3) take also the latter into account based on orthonormal frames. The intuitions and motivations from Section 3 still apply, though under a more abstract view.

In the first step, we consider the $\mathrm{Hom}(U, V)$ as $\mathbb{F}$-vector spaces and may allow infinite dimension. For subspaces U, V of a vector space $\mathcal{H}$ define

$$\mathcal{R}_{U,V} := \{X \in \mathrm{L}(\mathcal{H}) \mid X \cap V = \mathbf{0},\ X + V = U + V\} \quad \text{and} \quad \Gamma_{U,V}(\phi) := \{\vec{u} - \phi(\vec{u}) \mid \vec{u} \in U\}\ ,$$

for maps $\phi : U \to V$, that is, the graph of $-\phi$.

endoring

Fact 6.7 *Consider $U \cap V = \mathbf{0}$ in $\mathrm{L}(\mathcal{H})$.*

a) $\Gamma_{U,V}$ is a bijection of $\mathrm{Hom}(U, V)$ onto $\mathcal{R}_{U,V}$.
b) $\phi \in \mathrm{Hom}(U, V)$ is an isomorphism if and only if $\Gamma_{U,V}(\phi) \in \mathcal{R}_{V,U}$. Moreover, $\Gamma_{V,U}(\phi^{-1}) = \Gamma_{U,V}(\phi)$ in this case.
c) For a direct sum $U \oplus V \oplus W$ of subspaces of $\mathcal{H}$ and $\phi \in \mathrm{Hom}(U, V)$ and $\psi \in \mathrm{Hom}(V, W)$

$$\Gamma_{U,W}(\psi \circ \phi) = (U + W) \cap \big(\Gamma_{U,V}(\phi) + \Gamma_{V,W}(\psi)\big)$$

d) $\bar{A} = (A_{ij} \mid 1 \le i, j \le d)$ is a d-frame of $\mathrm{L}(\mathcal{H})$ if and only if there is a d-system $\bar{\varepsilon}$ of matrix units such that $A_i = A_{ii} = \mathrm{range}(\varepsilon_{ii})$ and for all pairwise distinct i, j

$$A_{ij} = \{x - \varepsilon_{ij}(x) \mid x \in A_i\}.$$

In particular, one has the maps η_{ij} of Fact 4.5b).

Proof. a)-c) are well known exercises in Linear Algebra. By hypothesis, $U + V = U \oplus V$ is a direct sum, i.e. the $\vec{x} \in U + V$ have unique representation $\vec{x} = \vec{u} + \vec{v}$ with $\vec{u} \in U$ and $\vec{v} \in V$.

a) Consider $\phi \in \mathrm{Hom}(U, V)$ and $X = \Gamma_{U,V}(\phi)$. By linearity, $X \in \mathrm{L}(\mathcal{H})$. Moreover, if $\vec{u} - \phi(\vec{u}) \in V$ then $\vec{u} = \vec{0}$ whence $\phi(\vec{u}) = \vec{0}$ by linearity. This proves $X \cap V = 0$. On the other hand, $\vec{u} = \vec{x} + \vec{v}$ with $\vec{x} = \vec{u} - \phi(\vec{u}) \in X$ and $\vec{v} = \phi(\vec{u})$ whence $U \subseteq X + V$ and so

$U + V = X + V$.

Conversely, consider $X \in \mathcal{R}_{U,V}$. Then $X + V = X \oplus V$ is also a direct sum. Since $U \subseteq X + V$, for each $\vec{u} \in U$ there are unique $\vec{x} \in X$ and $\vec{v} \in V$ such hat $\vec{u} = \vec{x} + \vec{v}$. Define $\phi(\vec{u}) = \vec{v}$. Then $X = \Gamma_{U,V}(\phi)$ by definition. Linearity of ϕ follows from the assumption that X is a linear subspace of $\mathcal{H}$.

b) If the inverse $\phi^{-1} \in \mathrm{Hom}(V, U)$ exists, apply a) to it and observe that $\vec{u} - \phi(\vec{u}) = -(\vec{v} - \phi^{-1}(\vec{v}))$ where $\vec{v} = \phi(\vec{u})$ resp. $\vec{u} = \phi^{-1}(\vec{v})$. Assuming $X = \Gamma_{U,V}(\phi) \in \mathcal{R}_{V,U}$ apply a) to conclude that $X = \Gamma_{V,U}(\psi)$ for some $\psi \in \mathrm{Hom}(V, U)$ and use the preceding argument to derive $\psi = \phi^{-1}$.

c)
$$\begin{array}{lll} \Gamma_{U,V}(\phi) + \Gamma_{V,W}(\psi) & = \{\vec{u} - \phi(\vec{u}) + \vec{v} - \psi(\vec{v}) \mid \vec{u} \in U, \vec{v} \in V & \} \\ U + W & = \{\vec{x} \qquad\qquad\qquad - \vec{w} \mid \vec{x} \in U, \qquad \vec{w} \in W & \} \end{array}$$

whence, by uniqueness of summands $\vec{u} \in U$, $\vec{v} \in V$, and $\vec{w} \in W$ in $\vec{x} = \vec{u} + \vec{v} + \vec{w}$, the intersection is obtained as $\{\vec{u} - \psi(\vec{v}) \mid \vec{u} \in U, \vec{v} \in V, \vec{v} = \phi(\vec{u})\} = \Gamma_{U,W}(\psi \circ \phi)$.

d) That $\mathcal{H}$ is the direct sum of the A_i is encoded into the first two conditions in Definition 3.6 . The remaining ones and a)-c) give the η_{ij} as in Fact 4.5b). □

Guided by the development in Subsection 3.4 and following von Neumann [Neum60], one is led to

d:MatinGrH

Definition 6.8 *For any $d \geq 3$ and variables x, y, and $\bar{y} = (y_{ij} \mid 1 \leq i, j \leq d)$, define the following lattice formulae and terms.*

$$\begin{array}{ll} \rho_{\bar{y}}(x) & :\Leftrightarrow x \wedge y_2 = \mathbf{0} \;\&\&\; x \vee y_2 = y_1 \vee y_2 \\ \pi^{ij}_{\bar{y}ik}(x) & := (y_i \vee y_k) \wedge (x \vee y_{jk}) \\ \pi^{ij}_{\bar{y}kj}(x) & := (y_k \vee y_j) \wedge (x \vee y_{ik}) \\ x \ominus_{\bar{y}} z & := \big([(\pi^{12}_{\bar{y}13}(z) \vee y_2) \wedge (x \vee y_{23})] \vee y_3\big) \wedge (y_1 \vee y_2) \\ \ominus_{\bar{y}} z & := y_1 \ominus_{\bar{y}} z \\ x \oplus_{\bar{y}} z & := x \ominus_{\bar{y}} (\ominus_{\bar{y}} z) \\ x \otimes_{\bar{y}} z & := (\pi^{12}_{\bar{y}13}(z) \vee \pi^{12}_{\bar{y}32}(x)) \wedge (y_1 \vee y_2) \end{array}$$

For d-frame $\bar{A}$ of $\mathrm{L}(\mathcal{H})$ and $\phi : A_i \to A_j$ abbreviate $\mathcal{R}_{\bar{A}ij} := \mathcal{R}_{A_i,A_j}$, $\Gamma_{\bar{A}ij}(\phi) := \Gamma_{A_i,A_j}(\phi)$.

f:MatinGrH

Fact 6.9 *a) Fix a d-frame $\bar{A}$ in $\mathrm{L}(\mathcal{H})$ with η_{ij} as in Fact 6.7d). For any pairwise distinct $i, j, k \leq 3$, the polynomials $\pi^{ij}_{\bar{A}ik}$ and $\pi^{ij}_{\bar{A}kj}$ of Definition 6.8 provide bijections from $\mathcal{R}_{\bar{A}ij}$ onto $\mathcal{R}_{\bar{\mathcal{A}}\rangle\|}$ and $\mathcal{R}_{\bar{A}kj}$, respectively. Moreover, for $\phi \in \mathrm{Hom}(A_i, A_j)$,*

$$\pi^{ij}_{\bar{A}ik}\big(\Gamma_{\bar{A}ij}(\phi)\big) = \Gamma_{\bar{A}ik}(\eta_{jk} \circ \phi), \quad \pi^{ij}_{\bar{A}kj}\big(\Gamma_{\bar{A}ij}(\phi)\big) = \Gamma_{\bar{A}kj}(\phi \circ \eta_{ki})$$

b) The map $\Theta_{\bar{A}}$, $\phi \mapsto \Gamma_{\bar{A}12}(\eta_{12} \circ \phi)$, is an isomorphism from the ring $\mathrm{End}(A_1)$ with operations $+, -, 0, \circ, \mathrm{id}$ onto $\mathcal{R}_{\bar{A}12}$ with operations $\oplus_{\bar{A}}$, $\ominus_{\bar{A}}$, A_1, $\otimes_{\bar{A}}$, A_{12}.

c) To every $d \geq 3$ and every finite family $\bar{p}$ of integer polynomials $p_j \in \mathbb{Z}\langle X_1, \ldots, X_n\rangle$ $(1 \leq j \leq J)$ in n non-commuting variables, there exists a term $t_{\bar{p},d}$ in the language of ortholattices such that the following holds:
For every field $\mathbb{F}$ and every $\mathbb{F}$-unitary vector space $\mathcal{H}$, $t_{\bar{p},d}$ is strongly satisfiable over $\mathrm{L}(\mathcal{H})$ iff there exists $m \in \mathbb{N}$ and $W \in \mathrm{L}_m(\mathcal{H})$ and $\phi_1, \ldots, \phi_n \in \mathrm{End}(W)$ such that $\dim(\mathcal{H}) = md$ and $p_j(\phi_1, \ldots, \phi_n) = 0$ for every $1 \leq j \leq J$.

Recall that Claim c) has already been employed in the proof of Proposition 5.11f).

Proof. a) and bijectivity of $\Theta_{\bar{A}}$ follow immediately from Fact 6.7.

b) Consider $\phi, \psi \in \mathrm{End}(A_1)$. By a) we have

$$\pi^{12}_{\bar{A}32}(\Theta_{\bar{A}}(\phi)) = \Gamma_{\bar{A}32}(\eta_{12} \circ \phi \circ \eta_{31}), \quad \pi^{12}_{\bar{A}13}(\Theta_{\bar{A}}(\psi)) = \Gamma_{\bar{A}12}(\eta_{23} \circ \eta_{12} \circ \psi) = \Gamma_{\bar{A}12}(\eta_{13} \circ \psi)$$

and with c) of Fact 6.7 it follows $\Theta_{\bar{A}}(\phi \circ \psi) = \Theta_{\bar{A}}(\phi) \otimes_{\bar{A}} \Theta_{\bar{A}}(\psi)$. Moreover

$$\begin{aligned} \Theta_{\bar{A}}(\phi) + A_{23} &= \{\vec{x} + \vec{y} - \eta_{12}(\phi(\vec{x})) - \eta_{23}(\vec{y}) \mid \vec{x} \in A_1, \vec{y} \in A_2\} \\ \Theta_{\bar{A}}(\psi) + A_2 &= \{\vec{u} + \vec{v} - \eta_{13}(\psi(\vec{u})) \mid \vec{u} \in A_1, \vec{v} \in A_2\} \end{aligned}$$

whence for $S := (\Theta_{\bar{A}}(\psi) + A_2) \cap (\Theta_{\bar{A}}(\phi) + A_{23})$ one obtains

$$S = \{\vec{x} + \vec{y} - \eta_{12}(\phi(\vec{x})) - \eta_{23}(\vec{y}) \mid \vec{x} \in A_1,\ \vec{y} \in A_2,\ \eta_{23}(\vec{y}) = \eta_{13}(\psi(\vec{x}))\}$$

Now, applying η_{32} we see that $\eta_{23}(\vec{y}) = \eta_{13}(\psi(\vec{x}))$ is equivalent to $\vec{y} = \eta_{12}(\psi(\vec{x}))$ and

$$\begin{aligned} S &= \{\vec{x} + \eta_{12}(\psi(\vec{x})) - \eta_{12}(\phi(\vec{x})) - \eta_{23}(\vec{y}) \mid \vec{x} \in A_1,\ \vec{y} \in A_2\} \\ &= \{\vec{x} - \eta_{12}((\phi - \psi)(\vec{x})) - \eta_{23}(\vec{y}) \mid \vec{x} \in A_1,\ \vec{y} \in A_2\} \end{aligned}$$

whence $\Theta_{\bar{A}}(\phi) \ominus_{\bar{A}} \Theta_{\bar{A}}(\psi) = (S + A_3) \cap (A_1 + A_2) = \Theta_{\bar{A}}(\phi - \psi)$.

c) Similar to the proof of Proposition 3.12c), let $t_{\bar{p},d}(\bar{y}, \bar{x})$ be the conjunction of conditions requiring $\bar{y}$ to evaluate to a d-frame $\bar{A}$ (Definition 3.6), all variables x_j to belong to the induced ring $\mathcal{R}_{\bar{A}12}$, integer constants replaced by terms (Observation 3.13), and operations $+, -, 0, \circ, \mathrm{id}$ by $\oplus_{\bar{A}}, \ominus_{\bar{A}}, A_1, \otimes_{\bar{A}}, A_{12}$, respectively. Note that, according to Fact 6.7, a d-frame exists precisely in dimensions an integer multiple of d. □

s:MatinGrH

Scholium 6.10 *The term $t_{\bar{p},d}$ from Fact 6.9c) can be computed from $\bar{p}$ and d by a Turing machine in time polynomial in d and in the binary encoding length of $\bar{p}$.*

6.3 Uniformly Interpreting the $*$-Rings $\mathbb{F}^{m\times m}$ into $\mathrm{L}(\mathbb{F}^{3m})$

ss:adjoint

In order to include adjunction into the above result, we have to relate adjoint pairs of linear maps between subspaces of $\mathcal{H}$ to the ortholattice $\mathcal{H}$.

f:adjoint

Fact 6.11 *Consider $U, V \in \mathrm{L}(\mathcal{H})$, $\mathcal{H}$ a finite-dimensional $\mathbb{F}$-unitary space.*

a) If $U \perp V$ then ϕ and ψ are adjoint to each other if and only if $\Gamma_{U,V}(\phi) \perp \Gamma_{V,U}(-\psi)$.
b) $\bar{A} = (A_{ij} \mid 1 \le i, j \le d)$ is an orthonormal d-frame of $\mathrm{L}(\mathcal{H})$ if and if and only if the η_{ij} in Fact 6.7d) are isometries, in other words, $\eta_{ij}^{-1} = \eta_{ji} = \eta_{ij}^*$.
c) If d divides $\dim(\mathcal{H})$ and $\mathbb{F}$ is Pythagorean, there is an orthonormal d-frame of $\mathrm{L}(\mathcal{H})$.

Put differently, b) establishes a 1-1-correspondence between orthonormal d-frames of $\mathrm{L}(\mathcal{H})$ and d-systems of matrix units of the $*$-ring $\mathrm{End}(\mathcal{H})$.

Proof. Concerning a), $U \perp V$ implies

$$\langle \vec{u} - \phi(\vec{u}) \mid \vec{v} + \psi(\vec{v}) \rangle = \langle \vec{u} \mid \psi(\vec{v}) \rangle - \langle \phi(\vec{u}) \mid \vec{v} \rangle \quad \text{for all } \vec{u} \in U, \vec{v} \in V$$

proving the claim. For b) observe that by Fact 6.7e) $A_{12} = \Gamma_{\bar{A}12}(\mathrm{id})$ and $\ominus_{\bar{A}12} A_{12} = \Gamma_{\bar{A}12}(-\mathrm{id})$. These being orthogonal to each other means $\langle \vec{x} \mid \vec{y} \rangle = \langle \eta_{12}(\vec{x}) \mid \eta_{12}(\vec{y}) \rangle$ since, in view of $A_1 \perp A_2$, for $\eta = \eta_{12}$ and all $\vec{x}, \vec{y} \in A_1$

$$\langle \vec{x} - \eta(\vec{x}) \mid \vec{y} + \eta(\vec{y}) \rangle = \langle \vec{x} \mid \vec{y} \rangle + \langle \vec{x} \mid \eta(\vec{y}) \rangle - \langle \eta(\vec{x}) \mid \vec{y} \rangle - \langle \vec{x} \mid \eta(\vec{y}) \rangle = \langle \vec{x} \mid \vec{y} \rangle - \langle \eta(\vec{x}) \mid \eta(\vec{y}) \rangle.$$

Symmetry applies to the η_{1k}. The claim for the η_{ij} follows from $\eta_{ij} = \eta_{1j} \circ \eta_{1i}^{-1}$. c)) By virtue of Fact 2.19b) choose an orthonormal basis $\vec{v}_{ik}$, $i = 1, \ldots, d$, $k = 1, \ldots, n$ and put $A_i = \sum_{k=1}^n \mathbb{F}\vec{v}_{ik}$, $A_{ij} = \sum_{k=1}^n \mathbb{F}(\vec{v}_{ik} - \vec{v}_{jk})$.

Now, suppose that an orthonormal d-frame $\bar{A}$ of $\mathrm{L}(\mathcal{H})$ is given as in Fact 6.11b) for finite dimensional $\mathcal{H}$. For $\phi \in \mathrm{End}(A_1)$ we want to recover ϕ^* in terms of $\mathrm{L}(\mathcal{H})$. Write $\Gamma_{\bar{A}ij} = \Gamma_{A_1,A_j}$ and consider $\psi \in \mathrm{End}(A_1)$. We claim that

$$\psi = \phi^* \text{ if and only if } \Gamma_{\bar{A}21}(\psi \circ \eta_{21}) = (A_1 + A_2) \cap (\Gamma_{\bar{A}12}(\eta_{12} \circ \phi))^{\perp}.$$

Indeed, by Fact 4.2b+f) it holds that

$$\psi = \phi^* \text{ if and only if } \psi \circ \eta_{21} = \phi^* \circ \eta_{21} = \phi^* \circ \eta_{12}^* = (\eta_{12} \circ \phi)^*.$$

By Fact 6.11b) the latter amounts to $\Gamma_{\bar{A}21}(\psi \circ \eta_{21}) \subseteq (A_1 + A_2) \cap \big(\Gamma_{\bar{A}12}(-\eta_{12} \circ \phi)\big)^{\perp}$. Here, equality holds since both sides are complements of A_1 in $[\mathbf{0}, A_1 + A_2]$ — with $X = \Gamma_{\bar{A}12}(\eta_{12} \circ -\phi)$ one derives $A_1 + X^{\perp} = A_1^{\perp\perp} + X^{\perp} = (A_1^{\perp} \cap X)^{\perp} = (\sum_{i=2}^{d} A_i \cap X)^{\perp} = \mathbf{0}^{\perp} = \mathbf{1}$ whence $A_1 + \big((A_1 + A_2) \cap X^{\perp}\big) = A_1 + A_2$.

Thus, it remains to recover $\Gamma_{\bar{A}12}(\eta_{12} \circ \psi)$ from $\Gamma_{\bar{A}21}(\psi \circ \eta_{21})$ by means of lattice operations. This is easily done using Fact 6.9a):

$$\eta_{12} \circ \psi = \eta_{12} \circ \psi \circ \eta_{11} = \eta_{12} \circ \psi \circ \eta_{21} \circ \eta_{32} \circ \eta_{13} = \big(\eta_{12} \circ ((\psi \circ \eta_{21}) \circ \eta_{32})\big) \circ \eta_{13}$$

implies $\Gamma_{\bar{A}12}(\eta_{12} \circ \psi) = t_{\bar{A}}\big(\Gamma_{\bar{A}21}(\psi \circ \eta_{21})\big)$ where $t_{\bar{y}}(x)$ is defined as

$$t_{\bar{y}}(x) \;:=\; \pi^{32}_{\bar{y}12}\big(\pi^{31}_{\bar{y}32}[\pi^{21}_{\bar{y}31}(x)]\big) \qquad \text{and} \qquad x^{\dagger_{\bar{y}}} \;:=\; t_{\bar{y}}\big((y_1 \vee y_2) \wedge \neg(\ominus_{\bar{y}} x)\big) \tag{14}$$

captures adjunction. So, together with Fact 6.9b) and Scholium 6.10 we have proved

l:MatinGrH

Lemma 6.12 *a) Let $\mathbb{F}$ be Pythagorean, $\mathcal{H}$ a finite dimensional $\mathbb{F}$-unitary space, and $d = \dim \mathcal{H} \geq 3$. Given an orthonormal d-frame $\bar{A}$ of $\mathrm{L}(\mathcal{H})$, there is an isomorphism from the $*$-ring* $\mathrm{End}(A_1)$ *with operations* $+$, $-$, 0, $\circ$, id, * *onto $\mathcal{R}_{\bar{A}12} = \{U \in \mathrm{L}(\mathcal{H}) \mid \rho_{\bar{A}}(U)\}$ with operations defined by the terms $x \oplus_{\bar{A}} z$, $\ominus_{\bar{A}} z$, A_1, $x \otimes_{\bar{A}} z$, A_{12}, $x^{\dagger_{\bar{A}}}$ with constants $\bar{A} = (A_{ij} \mid 1 \leq i, j \leq 3)$ from $\mathrm{L}(\mathcal{H})$, as given by Definition 6.8 and Equation (14).*

b) Given $d \geq 3$ and a finite family $\bar{p}$ of $*$-polynomials $p_j \in \mathbb{Z}\langle X_1, X_1^{\dagger}, \ldots, X_n, X_n^{\dagger}\rangle$ $(1 \leq j \leq J)$ in n noncommuting variables, a polynomial-time Turing machine can produce a term $t_{\bar{p},d}$ in the language of ortholattices such that the following holds:
For every Pythagorean $*$-field $\mathbb{F}$, and every $\mathbb{F}$-unitary vector space $\mathcal{H}$, $t_{\bar{p},d}$ is strongly satisfiable over $\mathrm{L}(\mathcal{H})$ iff $\dim(\mathcal{H}) = dm$ and there exist $\phi_1, \ldots, \phi_n \in \mathrm{End}(\mathbb{F}^m)$ such that $\dim(\mathcal{H}) = md$ and $p_j(\phi_1, \phi_1^*, \ldots, \phi_n, \phi_n^*) = 0$ for every $1 \leq j \leq J$.

Item b) thus generalizes Theorem 3.18a) to the noncommutative case, uniformly in m.

6.4 Strong Satisfiability in Indefinite Finite Dimension

ss:StrongIndef

Picking up on Theorem 6.2a), and since Theorem 5.1 depends on the dimension being fixed, we now consider the complexity and computability of *strong* satisfiability over indefinite finite dimensions — and approach the boundary between complexity and mere computability.

p:StrongIndef

Proposition 6.13 *For a fixed $*$-subfield $\mathbb{F}$ of of $\mathbb{C}$ there are the following polynomial time reductions:*

a) SAT *to* $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^*)}$ *to* $\mathrm{FEAS}^{\dagger}_{\mathbb{Z},\mathbb{F}^*}$.

b) For Pythagorean $\mathbb{F}$, $\mathrm{FEAS}^{\dagger}_{\mathbb{Z},\mathbb{F}^*}$ *to* $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^*)}$.

c) For real resp. algebraically closed $\mathbb{F}$, $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}}$ *to* $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^*)}$.

Proof. a) The first reduction is Proposition 2.10. According to Proposition 2.17c), $\mathrm{SAT}_{\mathrm{L}(\mathbb{F}^d)}$ reduces in polynomial time to $\mathrm{FEAS}_{\mathbb{Z},\mathrm{Re}\,\mathbb{F}}$ and, by Fact 1.12a), furtheron to any problem in $\mathrm{BP}(\mathcal{NP}_{\mathrm{Re}\,\mathbb{F}})$ – such as $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{F}^{d\times d}}$. However both reductions depend on d while Fact 4.6 provides a short-cut uniformly in d. b) follows from Lemma 6.12b). For c), combine b) with Proposition 6.6f). □

t:StrongIndef

Theorem 6.14. $\mathrm{SAT}_{\mathrm{L}(\mathbb{R}^*)} = \mathrm{SAT}_{\mathrm{L}(\mathbb{C}^*)}$ *is* $\mathrm{BP}(\mathcal{NP}_\mathbb{R})$*–hard. Moreover, the following are equivalent:*

i) $\mathrm{SAT}_{\mathrm{L}(\mathbb{C}^*)}$ *is decidable*
ii) $\mathrm{FEAS}^\dagger_{\mathbb{Z},\mathbb{C}^*}$ *is decidable*
iii) $\mathrm{QUART}^\dagger_{n\mapsto 2n,\mathbb{R}^*}$ *is decidable*
iv) There exists a total recursive function $\delta:\mathbb{N}\to\mathbb{N}$ *such that the following holds:*

$$\begin{array}{l}\textit{Whenever a term } t \textit{ is strongly satisfiable over } \mathrm{L}(\mathbb{C}^*),\\ \textit{it is so over } \mathrm{L}(\mathbb{C}^d) \textit{ for some } d \le \delta(|t|).\end{array} \tag{15}$$

v) There exists a total recursive function $\delta':\mathbb{N}\to\mathbb{N}$ *such that the following holds:*

$$\begin{array}{l}\textit{Whenever a quartic } p \in \{0,\pm1,\pm2,\dots,\pm2n\}\langle X_1,X_1^\dagger,\dots,X_n,X_n^\dagger\rangle \textit{ admits}\\ \textit{a root over } \mathbb{R}^{d\times d} \textit{ for some } d, \textit{ it also does so over } \mathbb{R}^{d'\times d'} \textit{ for some } d' \le \delta'(n).\end{array} \tag{16}$$

Note that the following functions δ,δ' do satisfy Equations (15) and (16), respectively:

$$\begin{aligned}\delta : n \mapsto \max\big\{d \;\big|\; &\forall \text{ ortholattice terms } t : \\ &\big(|t| = n \;\wedge\; \exists d' \in \mathbb{N}\;\exists A_1',\dots,A_n' \in \mathrm{L}(\mathbb{R}^{d'}) : t_{\mathrm{L}(\mathbb{R}^d)}(A_1',\dots,A_n') = \mathbf{1}\big) \\ &\qquad \Rightarrow\; \exists d' \le d\;\exists A_1,\dots,A_n \in \mathrm{L}(\mathbb{R}^{d'}) : t_{\mathrm{L}(\mathbb{R}^{d'})}(A_1,\dots,A_n) = \mathbf{1}\big\}\end{aligned} \tag{17}$$

$$\begin{aligned}\delta' : n \mapsto \max\big\{d' \;\big|\; &\forall p \in \{0,\pm1,\pm2,\dots,\pm2n\}\langle X_1,X_1^\dagger,\dots,X_n,X_n^\dagger\rangle : \\ &\big(\exists d \in \mathbb{N}\;\exists A_1,\dots,A_n \in \mathbb{R}^{d\times d} : p(A_1,A_1^\dagger,\dots,A_n,A_n^\dagger) = 0\big) \\ &\qquad \Rightarrow\; \exists d \le d'\;\exists A_1',\dots,A_n' \in \mathbb{R}^{d\times d} : p(A_1',A_1'^\dagger,\dots,A_n',A_n'^\dagger) = 0\big\}\;.\end{aligned} \tag{18}$$

Indeed, the number of terms of given length is finite; hence the maximum exists. Similarly, a quartic n-variate polynomial consists of $\mathcal{O}(n^4)$ monomials; hence there can be no more than $|K|^{\mathcal{O}(n^4)}$ of them with coefficients in $K := \{0,\pm1,\pm2,\dots,\pm n\}$.

q:StrongIndef

Question 6.15 *Is the function* δ' *(unlike the* busy beaver*) recursively bounded?*

According to Example 5.14b), such a bound has to be at least exponential.

Proof (of Theorem 6.14). $\mathrm{SAT}_{\mathrm{L}(\mathbb{C}^*)} = \mathrm{SAT}_{\mathrm{L}(\mathbb{R}^*)}$ holds due to Fact 5.10a+b); for hardness invoke Proposition 6.13c).

i)⇔ii) Proposition 6.13a+b).

i)⇔iii) $\mathrm{SAT}_{\mathrm{L}(\mathbb{C}^*)} = \mathrm{SAT}_{\mathrm{L}(\mathbb{R}^*)}$ is equivalent $\mathrm{FEAS}_{\mathbb{Z},\mathbb{R}^*}$ according to Proposition 6.13a+b); which by Proposition 6.6e) is in turn equivalent to $\mathrm{QUART}^\dagger_{n\mapsto 2n,\mathbb{R}^*}$.

i)⇒iv) Based on an algorithm deciding $\mathrm{SAT}_{\mathrm{L}(\mathbb{C}^*)}$, the function δ from Equation (18) can be computed as follows: Given n enumerate all (the finitely many, up to renaming variables) terms t of length n strongly satisfiable over $\mathrm{L}(\mathbb{C}^*)$; for each one search (Fact 2.15) for the first dimension t is strongly satisfiable in and return the maximum.

iv)⇒i) Given t, calculate $d := \delta(|t|)$ and decide satisfiability of t over $\mathrm{L}(\mathbb{C}^1)$, $\mathrm{L}(\mathbb{C}^2)$, ..., $\mathrm{L}(\mathbb{C}^d)$: if none succeeds then t is not satisfiable over $\mathrm{L}(\mathbb{C}^*)$ either.

ii)⇒v) Similarly to i)⇒iv), enumerate all (the finitely many) n-variate quartic $*$-polynomials over $\{0, \pm 1, \pm 2, \dots, \pm 2n\}$ in $\mathrm{QUART}^{\dagger}_{n\mapsto 2n,\mathbb{R}^*}$; for each search for the first d such that it admits a root in $\mathbb{R}^{d\times d}$ and return the maximum.

v)⇒ii) Similarly to iv)⇒i), given $p \in \{0, \pm 1, \pm 2, \dots, \pm 2n\}\langle X_1, \dots, X_n, X_1^{\dagger}, \dots, X_n^{\dagger}\rangle$ calculate $d' := \delta'(n)$ and invoke Proposition 6.6d) to decide whether p admits a root in $\mathbb{R}^{1\times 1}$, $\mathbb{R}^{2\times 2}$, ..., $\mathbb{R}^{d'\times d'}$: if none succeeds then $p \notin \mathrm{FEAS}^{\dagger}_{\mathbb{Z},\mathbb{R}^*}$. □

Digression 6.16 *For Pythagorean $\mathbb{F}$, the following problem allows a polynomial time reduction to strong satisfiability in $\mathrm{L}(\mathbb{F}^*)$. Namely, to decide for any finite semigroup resp. group presentation (the latter are considered as semigroup presentations with a generator symbol g' and relations $gg' = e = g'g$ added for each group generator symbol g) whether it admits a finite dimensional $\mathbb{F}$-algebra model $\mathcal{A}$. The proof is by interpreting the relations of the presentation as in Lemma 6.12 (with d = 3), Then, the ortholattice relations are combined into strong satisfiability of t via Fact 2.9a). If there is a model $\mathcal{A}$ as required, then t will be strongly satisfied in $\mathrm{L}(\mathcal{H})$ where $\mathcal{H}$ is the 3-fold orthogonal sum of $\mathbb{F}[S]$ considered as a right vector space with canonical scalar product and where any generator symbol g is interpreted as negative graph of the linear map $x \mapsto gx$. Conversely, if t is strongly satisfied, then the endomorphisms of U_1 corresponding to generator symbols generate a finite dimensional model $\mathcal{A}$.*

7 Towards the Infinite-Dimensional Case

s:Neumann

Recall that, for an infinite dimensional Hilbert space $\mathcal{H}$, $\mathrm{L}(\mathcal{H})$ consists of all *closed* linear subspaces of $\mathcal{H}$. We also define $\mathrm{L}_{\mathrm{l}}(\mathcal{H})$ as the modular lattice of all linear subspaces of the $\mathbb{F}$-vector space $\mathcal{H}$; and $\mathrm{L}_*(\mathcal{H})$ as the subset of all U and $U^{\perp}$ where $U \in \mathrm{L}_{\mathrm{l}}(\mathcal{H})$ is finite dimensional (and in particular closed).

Fact 7.1 *$\mathrm{L}_*(\mathcal{H})$ is a sub-ortholattice of $\mathrm{L}(\mathcal{H})$ and a sublattice of $\mathrm{L}_l(\mathcal{H})$, in particular a modular ortholattice. All claims from Lemma 5.16 remain valid when applied to $\mathrm{L}_*(\mathcal{H})$. The analogous results hold for abstract atomic MOLs cf. [Her10a].*

Though, one has to be aware of the fact that $\mathrm{L}_{\mathrm{l}}(\mathcal{H})$ is much more special than $\mathrm{L}(\mathcal{H})$ even if $\mathcal{H}$ is a separable Hilbert space. In particular, nothing is known about satisfiability in $\mathrm{L}(\mathcal{H})$. Neither is it known whether the equational theory of $\mathrm{L}(\mathcal{H})$ is decidable.

Recall that a von Neumann algebra $\mathbf{A}$ (of bounded operators on a Hilbert space) is finite if $AA^* = I$ implies $A^*A = I$ for all $A \in \mathbf{A}$. It is a factor if its center is $\mathbb{C}$. Its projections $P = P^* = P^2$ are partially ordered by $P \le Q \Leftrightarrow QP = P$. The following is due to Murray and von Neumann cmp. [Take79].

Fact 7.2 *The projections of a finite von Neumann algebra $\mathbf{A}$ form a continuous modular ortholattice $\mathrm{P}(\mathbf{A})$. If $\mathbf{A}$ is a factor then either $\mathrm{P}(\mathbf{A}) \cong \mathrm{L}(\mathbb{C}^d)$ for some $d < \infty$ or $\mathrm{P}(\mathbf{A})$ is simple with no atoms; the latter are called of* type II_1 *and contain all $\mathrm{L}(\mathbb{C}^d)$ as sub-ortholattices.*

Proof. Most of this is in [MuNe36]. The key to modularity is that $\mathbf{A}$ extends to a $*$-regular ring having the same projections (§14.1). In [Murray and von Neumann 1943, Thm. XIII] it is shown that a type II_1-algebra has a subalgebra $\mathbb{C}^{d\times d}$ for any d. From this it follows that $\mathrm{L}(\mathbb{C}^{d\times d})$ embeds into $\mathrm{P}(\mathbf{A})$ – cmp. [Her10a, Cor.3.2]

The $\mathrm{P}(\mathbf{A})$ with $\mathbf{A}$ of type II_1 are examples of *continuous geometries*, i.e. continuous modular ortholattices L, admitting a dimension function mapping L onto the unit interval. In [Neum81] von Neumann introduced additional structure and axioms on the latter, motivated by his view on what a 'quantum logic' should be, to recover these abstract continuous geometries as projection MOLs of algebras of operators, cf. [Rede07].

Theorem 7.3 ([Her10a]). *Given a finite von Neumann algebra factor* $\mathbf{A}$ *of type* II_1*, an ortholattice identity is valid in* $\mathrm{P}(\mathbf{A})$ *if and only if it is valid in* $\mathrm{L}(\mathbb{C}^d)$ *for all resp. infinitely many* d*.*

Corollary 7.1. *a) The (indefinite) weak satisfiability problem for any class of of projection ortholattices of finite von Neumann algebra factors is in* $\mathrm{BP}(\mathcal{NP}_{\mathbb{R}})$*.*

b) The decision problem for the equational theory of any class of projection ortholattices of finite von Neumann algebra factors is in $\mathrm{BP}(\mathrm{co}\mathcal{NP}_{\mathbb{R}})$*.*

Digression 7.4 *The concept of* $*$-regular rings*,* $*$*-rings such that for any* a *there is* x *such that* $a = axa$ *and* $aa^* = 0$ *only for* $a = 0$*, has been introduced by* von Neumann *as an abstract approach to certain algebras of unbounded operators. He showed that* $*$*-regular rings have MOLs of projections and that any MOL admitting an orthogonal d-frame* $\bar{a}$*,* $d \geq 4$*, is isomorphic to the lattice of projections of some* $*$*-regular ring (cf. [Neum60]), actually the matrix ring* $\mathcal{R}_{\bar{a}}^{d\times d}$ *with involution mimicking Fact f:adjointc). This has been extended to 'large partial orthogonal d-frames' by* Jónsson *[Jóns60], including the case* $d = 3$ *under the stronger Arguesian Law (cf. [Her10b]) which is valid in all MOL of projections. In particular, this applies to all simple Arguesian MOLs of height* ≥ 3*.*

References

[ABKM09] E. Allender, P. Bürgisser, J. Kjeldgaard-Pedersen, P. Miltersen: "On the Complexity of Numerical Analysis", pp.1987–2006 in *SIAM J. Comput.* vol.**38:5** (2009).

[Axle96] Sh. Axler: "*Linear Algebra Done Right*", Springer, New York (1996).

[Ball48] R. W. Ball: "Dualities of finite projective planes", pp.929–940 in *Duke Math. J.* vol.**15:4** (1948)

[BaZe10] S. Basu, T. Zell: "Polynomial hierarchy, Betti Numbers and a Real Analogue of Toda's Theorem", pp.429–454 in *Foundations of Computational Mathematics* vol.**10** (2010).

[BeCa81] E.G. Beltrametti, G. Cassinelli: "*The Logic of Quantum Mechanics*", vol.**15** in (G.-C. Rota, Editor) the Encyclopedia of Mathematics and Its Applications, Addison-Wesley (1981).

[BDDK03] F. Benanti, J. Demmel, V. Drensky, P. Koev: "Computational Approach to Polynomial Identities of Matrices—a Survey", pp.141–178 in *Polynomial Identities and Combinatorial Methods* (A. Giambruno, A. Regev, M. Zaicev, Edts), Dekker Series of Lecture Notes in Pure and Applied Mathematics vol.**234** (2003).

[Bera85] L. Beran: "*Orthomodular Lattices*", Springer (1985).

[BKOS97] M. de Berg, M. van Kreveld, M. Overmars, O. Schwarzkopf: "*Computational Geometry, Algorithms and Applications*", Springer (1997).

[BiNe36] G. Birkhoff, J. Von Neumann: "The Logic of Quantum Mechanics", pp.823–843 in *Annals of Mathematics* vol.**37:4** (1936).

[Birk67] G. Birkhoff: "*Lattice Theory*", vol.**XXV** in *Amer. Math. Soc. Colloq. Publ.*, Providence (1967).

[BVSWZ99] A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White, G.M. Ziegler: "*Oriented matroids*" (2nd Edition), Encyclopedia of Mathematics and Applications, Cambridge University Press (1999).

[BCSS98] L. Blum, F. Cucker, M. Shub, S. Smale: "*Complexity and Real Computation*", Springer (1998).
[BGG01] E. Börger, E. Grädel, Y. Gurevich: "*The Classical Decision Problem*", Springer (2001).
[BCNM06] O. Bournez, F. Cucker, P.J. de Naurois, J.-Y. Marion: "Implicit complexity over an arbitrary structure: Quantifier alternations", pp.210–230 in *Information and Computation* vol.**204:2** (2006).
[Bürg96] P. Bürgisser, M. Clausen, A.M. Shokrollahi: "*Algebraic Complexity Theory*", Springer (1996).
[BSS89] L. Blum, M. Shub, S. Smale: "On a Theory of Computation and Complexity over the Real Numbers: $\mathcal{NP}$–completeness, Recursive Functions, and Universal Machines", pp.1–46 in *Bulletin of the American Mathematical Society* (AMS Bulletin) vol.**21** (1989).
[Boro77] A. Borodin: "On Relating Time and Space to Size and Depth", pp.733–744 in *SIAM J. on Computing* vol.**6** (1977).
[Bub07] J. Bub: "Quantum Computation from a Quantum Logical Perspective", pp.281–296 in *Quantum Information and Computation* vol.**7:2** (2007).
[BüCu06] P. Bürgisser, F. Cucker: "Counting Complexity Classes for Numeric Computations II: Algebraic and Semialgebraic Sets", pp.147–191 in *Journal of Complexity* vol.**22:2** (2006).
[Bürg00] P. Bürgisser: "Cook's versus Valiant's Hypothesis", pp.71–88 in *Theoretical Computer Science* vol.**235** (2000).
[BüSc09] P. Bürgisser, P. Scheiblechner: "On the Complexity of Counting Components of Algebraic Varieties", pp.1114–1136 in *Journal of Symbolic Computation* vol.**44:9** (2009).
[BuSa81] S.N. Burris, H.P. Sankappanavar: "*A Course in Universal Algebra*", Springer (1981) and `http://www.math.uwaterloo.ca/~snburris/htdocs/UALG/univ-algebra.pdf`.
[Cann88] J. Canny: "Some Algebraic and Geometric Computations in PSPACE", pp.460–467 in *Proc. 20th annual ACM Symposium on Theory of Computing* (SToC 1988).
[ChHo99] T. Chadzelek, G. Hotz: "Analytic Machines", pp.151–165 in *Theoretical Computer Science* vol.**219**, Elsevier (1999).
[CKS81] A.K. Chandra, D.C. Kozen, L.J. Stockmeyer: "Alternation", pp.114–133 in *Journal of the ACM* vol.**28:1** (1981).
[Chen09] H. Chen: "A Rendezvous of Logic, Complexity, and Algebra", article no.2 in *ACM Computing Surveys* vol.**42:1** (2009).
[CMW00] B. Coecke, D. Moore, A. Wilce (Editors): "Current research in operational quantum logic", vol.**111** in Series *Fundamental Theories of Physics*, Springer (2000).
[Coec10] B. Coecke: "Quantum Picturalism", pp.59–83 in *Contemporay Physics* vol.**51:1** (2010).
[Cook71] S. A. Cook: "The complexity of Theorem Proving Procedures", pp.151–158 in *Proc. of the 3rd IEEE Symp. on the Foundations of Computer Science* (1971).
[CLOS07] D.A. Cox, J.B. Little, D. O'Shea: *Ideals, Varieties, and Algorithms* (3rd Edition), Springer (2007).
[Cuck93] F. Cucker: "On the Complexity of Quantifier Elimination: the Structural Approach", pp.400–408 in *The Computer Journal* vol.**36:5** (1993).
[CuBr07] F. Cucker, I. Briquel: "A Note on Parallel and Alternating Time", pp.594–602 in *Journal of Complexity* vol.**23** (2007).
[CuRo92] F. Cucker, F. Rosselló: "On the Complexity of Some Problems for the Blum, Shub & Smale model", pp.117–129 in *Proc. LATIN'92*, Springer LNCS vol.**583**.
[CzSk10] G. Czédli, B. Skublics: "The Ring of an Outer von Neumann Frame in Modular Lattices", pp.187–202 in *Algebra Universalis* vol.**64:1** (2010).
[DHMW05] J.M. Dunn, T.J. Hagge, L.S. Moss, Z. Wang: "Quantum Logic as Motivated by Quantum Computing", pp.353–359 in *Journal of Symbolic Logic* vol.**70:2** (2005).
[Duck69] Ch. J. Duckenfield: "A Continuous Geometry as a Mathematical Model for Quantum Mechanics", pp.217–263 in *Comment. Math. Univ. Carolinae* vol.**10** (1969).
[EGL07] K. Engesser, D.M. Gabbay, D. Lehmann (Editors): "*Handbook of Quantum Logic and Quantum Structures – Quantum Structures*", Elsevier (2007).
[Fare01] D.R. Farenick: "*Algebras of Linear Transformations*", Springer, New York (2001).
[FoKo98] H. Fournier, P. Koiran: "Are Lower Bounds Easier over the Reals?", pp.507–513 in *Proc. 13th Annual ACM Symposium on Theory of Computing* (SToC'98).
[FoKo00] H. Fournier, P. Koiran: "Lower Bounds Are Not Easier over the Reals", pp.832–843 in *Proc. 27th International Colloquium on Automata, Languages and Programming* (ICALP 2000), Springer LNCS vol.**1853**.

[GaJo79] M.R. Garey, D.S. Johnson: "*Computers and Intractability: A Guide to the Theory of $\mathcal{NP}$–completeness*", Freeman (1979).

[Gelf61] I.M. Gel'fand: "*Lectures on Linear Algebra*", Interscience, New York (1961).

[GiHe91] M. Giusti, J. Heintz: "Algorithmes – disons rapides – pour la décomposition d'une variété algébrique en composantes irréductibles et équidimensionnelles", pp.169–193 in *Effective Methods in Algebraic Geometry (Proceedings of MEGA'90)*, T. Mora and C. Traverso editors, Birkhäuser (1991).

[Good94] J.B. Goode: "Accessible Telephone Directories", pp.92–105 in *The Journal of Symbolic Logic* vol.**59:1** (1994).

[GoOR04] J.E. Goodman, J. O'Rourke (Editors): "*Handbook of Discrete and Computational Geometry*", Chapman & Hall (2004).

[GrMe96] E. Grädel, K. Meer: "Descriptive Complexity Theory over the Real Numbers", pp.381–403 in *Mathematics of Numerical Analysis: Real Number Algorithms* vol.**32** (1996).

[Grig88] D.Y. Grigoriev: "Complexity of Deciding Tarski Algebra", pp.65–108 in *Journal of Symbolic Computation* vol.**5** (1988).

[Hagg07] T.J. Hagge: "$QL(\mathbb{C}^n)$ Determines n", pp.1194–1196 in *Journal of Symbolic Logic* vol.**72:4** (2007).

[Halm58] P.R.. Halmos: "*Finite Dimensional Vector Spaces*", Van Nostrand, Princeton (1958).

[Her10a] C. Herrmann: "On the equational theory of projection lattices of finite von-Neumann factors", pp.1102–1110 in *J. Symbolic Logic* vol.**75:3** (2010).

[Her10b] C. Herrmann: "Generators for Complemented Modular Lattices and the von-Neumann-Jónsson Coordinatization Theorems", pp.45–64 in *Algebra Universalis* vol.**63:1** (2010).

[Herr05] C. Herrmann, F. Micol, M.S. Roddy: "On n-distributive Modular Ortholattices", pp.143–147 in *Algebra Universalis* vol.**53** (2005).

[HRSZ11] C. Herrmann, M.S. Roddy, M. Semenova, M. Ziegler: "Rings of quotients and projection lattices of finite $\mathbf{AW}^*$-algebras: Faithful representation and equational theory", submitted.

[HeZi11] C. Herrmann, M. Ziegler: "Computational Complexity of Quantum Satisfiability", pp.175–184 in *Proc. 26th Annual IEEE Symposium on Logic in Computer Science* (2011).

[HRS90] J. Heintz, M.-F. Roy, P. Solernó: "Sur la complexité du principe de Tarski–Seidenberg", pp.101–126 in *Bull. Soc. Math. France* vol.**118** (1990).

[Hilb03] D. Hilbert: "Grundlagen der Geometrie". 2nd. ed. Teubner, Leipzig (1903).

[Hodg93] W. Hodges: "*Model Theory*", Encyclopedia of Mathematics and its Applications vol. **42**, Cambridge University Press (1993).

[Huhn72] A.P. Huhn: "Schwach distributive Verbände I", pp.297–305 in *Acta Scientiarum Mathematicarum* vol.**33** (1972).

[Jaco53] N. Jacobson: "*Lectures in Abstract Algebra, Volume I–III*", Van Nostrand (1953).

[JKSS04] G. Jeronimo, T. Krick, J. Sabia, M. Sombra: "The Computational Complexity of the Chow Form", pp.41–117 in *Found. Comput. Math.* vol.**4:1** (2004).

[Jóns60] B. Jónsson: "Representations of complemented modular lattices", pp.64–94 in *Trans. Amer. Math. Soc.* vol. **97** (1960),

[Kalm83] G. Kalmbach: "*Orthomodular Lattices*", Academic Pr. (1983)

[Karp72] R.M. Karp: "Reducibility Among Combinatorial Problems", pp.85–103 in (R.E. Miller and J.W. Thatcher, editors) *Compexity of Computer Computations*, New York (1972).

[Koir96] P. Koiran: "Hilbert's Nullstellensatz is in the Polynomial Hierarchy", pp.273–286 in *Journal of Complexity* vol.**12:4** (1996).

[Koir99] P. Koiran: "The Real Dimension Problem is $\mathcal{NP}_{\mathbb{R}}$–complete", pp.227–238 in *J. Complexity* vol.**15:2** (1999).

[Lang99] S. Lang: "*Algebra*", Addison Wesley (1999).

[Lece00] G. Lecerf: "Computing an Equidimensional Decomposition of an Algebraic Variety by means of Geometric Resolutions", pp.209–216 in *Proceedings of the 2000 International Symposium on Symbolic and Algebraic Computation* (ISAAC), ACM.

[Mack63] G.W. Mackey: "Mathematical Foundations of Quantum Mechanics", *Mathematical Physics Monograph Series* (1963).

[Marx07] M. Marx: "Complexity of Modal Logic", pp.139–179 in (P. Blackburn, J.v. Benthem, F. Wolte, Editors) *Handbook of Modal Logic* (2007).

[MaTo97] G. Matera, J.M.T. Torres: "The Space Complexity of Elimination Theory: Upper Bounds", pp.267–276 in *Selected Papers of a Conference on Foundations of Computational Mathematics*, Springer (1997).

[Mati70] Y.V. Matiyasevich: "Enumerable Sets are Diophantine", pp.354–357 in *Soviet Math.Dokl.* vol.**11** (1970).

[Maye07] R. Mayet: "Equations Holding in Hilbert Spaces", pp.1257–1287 in *International Journal of Theoretical Physics* vol.**45:7** (2007).

[MaRo87] R. Mayet, M.S. Roddy: "n-Distributivity in Ortholattices", pp.285–294 in (Czermak et al. eds.) *Contributions to General Algebra* vol.**5** (1987).

[MeMi97] K. Meer, C. Michaux: "A Survey on Real Structural Complexity Theory", pp.113–148 in *Bulletin of the Belgian Mathematical Society* vol.**4** (1997).

[Megi09] N.D. Megill: "Quantum Logic Explorer Home Page" at *Metamath*; `http://us.metamath.org/qlegif/mmql.html` (2009).

[Mich89] C. Michaux: "Une remarque á propos des machines sur $\mathbb{R}$ introduites par Blum, Shub et Smale", pp.435–437 in *C. R. Acad. Sci. Paris Sér. I Math.* vol.**309** (1989).

[Mnëv88] N.E. Mnëv: "The Universality Theorems on the Classification Problem of Configuration Varieties and Convex Polytopes Varieties", pp.527–544 in *Topology and Geometry — Rohlin Seminar*, Springer Lecture Notes in Mathematics vol.**1346** (1988).

[MnZi93] N.E. Mnëv, G.M. Ziegler: "Combinatorial Models for the Finite-dimensional Grassmannians", pp.241–250 in *Discrete & Computational Geometry* vol.**3:10**, Springer (1993).

[MuNe36] F. J. Murray, J. von Neumann: "On Rings of Operators", pp.116–229 in *Ann. Math.*, vol.**37** (1936),

[Murray and von Neumann 1943] F. J. Murray, J. von Neumann "On Rings of Operators IV", pp.761–808 in *Ann. Math.*, vol. **44** (1943).

[Neum32] J.von Neumann: "*Mathematical Foundations of Quantum Mechanics*", Princeton (1932); reprinted 1955.

[Neum58] J. von Neumann "The Non-Isomorphism of Certain Continuous Rings ", pp.485–496 in *Ann. Math.*, vol. **67** (1958).

[Neum60] J.von Neumann: "*Continuous Geometry*", Princeton Math.Series no.**25** (1960).

[Neum81] J.von Neumann: "*Continuous Geometries with a Transition Probability*", Memoirs Amer. Math. Soc. no.**252** (1981).

[Papa94] C. Papadimitriou: "*Computational Complexity*", Addison-Wesley (1994).

[PaMe07] M. Pavačić, N.D. Megill: "Quantum Logic and Quantum Computation", pp.755–792 in [EGL07].

[Piro64] C. Piron: "Axiomatique Quantique", pp.439–468 in *Helv. Phys. Acta* vol.**37** (1964).

[Poiz95] B. Poizat: "*Les petits cailloux: Une approche modèle-theéoretique de l—algorithmie*", Aléas (1995).

[Poon09] B. Poonen: "Characterizing Integers among Rational Numbers with a Universal-Existential Formula", in *American Journal of Mathematics* vol.**131:3** (2009).

[Pres07] A. Prestel: "On Sóler's Charcaterization of Hilbert Spaces pp.225–238 in *Manuscripta Math.* vol **86:2** (1995).

[Putn69] H. Putnam: "Is Logic Empirical?", pp.216–241 in *Boston Studies in the Philosophy of Science* vol.**5** (1969); reprinted as "The Logic of Quantum Mechanics", pp.174–197 in *Mathematics, Matter and Method* (1975).

[Pyka00] J. Pykacz: "Quantum Logic as a Basis for Computations", pp.839–850 in *International Journal of Theoretical Physics* vol.**39:3** (2000).

[Rede07] M. Redei: "The Birkhoff-von Neumann Concept of Quantum Logic", pp.1–22 in [EGL07].

[Rich99] J. Richter-Gebert: "The Universality Theorems for Oriented Matroids and Polytopes", pp.269–292 in *Contemporary Mathematics* vol.**223** (1999).

[Rodd89] M.S. Roddy: "On the word problem for orthocomplemented modular lattices", pp.961–1004 in *Canadian Journal of Mathematics* vol.**37** (1989).

[Roma06] L. Román: "A Characterization of Quantic Quantifiers in Orthomodular Lattices", pp.206–217 in *Theory and Applications of Categories* vol.**16:10** (2006).

[Rota97] G.-C. Rota: "The Many Lives of Lattice Theory", pp.1440–1445 in *Notices of the AMS* (1997).

[Rowe88] L.H. Rowen: "*Ring Theory, Vol. I*", Academi Press (1988).

[Scha78] T.J. Schaefer: "The Complexity of Satisfiability Problems", pp.216–226 in *Proc. 10th Annual ACM Symposium on Theory of Computing* (SToC'1978).

[Scha10] M. Schaefer: "Complexity of Some Geometric and Topological Problems", pp.334–344 in *Proc. 17th Int. Symp. on Graph Drawing*, Springer LNCS vol.**5849** (2010).

[Sche12] P. Scheiblechner: "Castelnuovo-Mumford Regularity and Computing the de Rham Cohomology of Smooth Projective Varieties", to appear in *Foundations of Computational Mathematics*.

[Shoe67] J.R. SHOENFIELD: "*Mathematical Logic*", Addison-Wesley (1967).

[Shor91] P. SHOR: "Stretchability of Pseudolines is $\mathcal{NP}$–hard", pp.531–554 in *Applied Geometry and Discrete Mathematics — The Victor Klee Festschrift* (P. Gritzmann and B. Sturmfels Edts), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol.**4** (1991).

[Smal98] S. SMALE: "Mathematical Problems for the Next Century", pp.7–15 in *Math. Intelligencer* vol.**20:2** (1998).

[Svoz98] K. SVOZIL: "*Quantum Logic*", Springer (1998).

[Take79] M. TAKESAKI: "*Theory of Operator Algebras I*", Springer (1979).

[TMR53] A. TARSKI, A. MOSTOSWKI, R.M. ROBINSON: "*Undecidable Theories*", North-Holland Publishing (1953).

[TuZu00] J.V. TUCKER, J.I. ZUCKER: "Computable functions and semicomputable sets on many-sorted algebras", pp.317–523 in *Handbook of Logic in Computer Science* vol.**5** (S. Abramsky, D.M. Gabbay, T.S.E. Maybaum Edts), Oxford Science Publications (2000).

[Vali79] L.G. VALIANT: "The Complexity of Computing the Permanent", pp.189–201 in *Theoretical Computer Science* (Elsevier) vol.**8:2** (1979).

[Weiz85] C.-F. VON WEIZSÄCKER: "*Aufbau der Physik*", Munich (1985); translated as "*The Structure of Physics*", Heidelberg (2006).

[Ying05] M. YING: "A Theory of Computation based on Quantum Logic I", pp.134–207 in *Theoretical Computer Science* vol.**344** (2005).

[Zass48] H. ZASSENHAUS: "Über einen Algortihmus zur Bestimmung der Raumgruppen", pp.117–141 in *Comment. Math. Helv.* vol.**21** (1948).

[Zhan93] X.-D. ZHANG: "Complexity of Neural Network Learning in the Real Number Model", pp.146–149 in *Proc. Workshop on Physics and Computation* (1992).