RINGS OF QUOTIENTS OF FINITE AW*-ALGEBRAS: REPRESENTATION AND ALGEBRAIC APPROXIMATION

CHRISTIAN HERRMANN AND MARINA SEMENOVA

ABSTRACT. We show that Berberian's $*$-regular extension of a finite AW^*-algebra admits a faithful representation, matching the involution with adjunction, in the $\mathbb{C}$-algebra of endomorphisms of a closed subspace of some ultrapower of a Hilbert space. We derive then that this extension is a homomorphic image of a regular subalgebra of an ultraproduct of matrix $*$-algebras $\mathbb{C}^{n\times n}$.

1. Introduction

Goodearl and Menal [1, Theorem 1.6] have shown that any C^*-algebra C is a homomorphic image of a residually finite dimensional C^*-algebra B. Moreover, if C above is separable, then B is a subdirect product of matrix algebras $\mathbb{C}^{n\times n}$. The first objective of the present paper is to show that one can choose B always as a subalgebra of an ultraproduct of algebras $\mathbb{C}^{n\times n}$ – and to generalize the result to algebras represented in any inner product space. In this note, ultraproducts are those defined in Model Theory.

Another main objective is then to extend this kind of algebraic approximation, with $*$-regular B, to $*$-regular algebras of quotients. Such algebras have been constructed by Berberian [2] (analyzed by Hafner [3], Pyle [4], and Berberian [5]), generalizing the Murray and von Neumann [6] $*$-regular algebra of unbounded operators affiliated with a finite von Neumann algebra factor; and in a more general setting by Handelman [7] and Ara and Menal [8]. Their results, relevant here, are summarized in the following theorem (details are given in Section 7, below). See also [9, Theorem 2.3], [10, Proposition 21.2].

Theorem 1. *Let A be a finite Rickart C^*-algebra. Then A admits a classical ring $Q(A)$ of right quotients. The involution and $\mathbb{C}$-algebra structure of A extend uniquely to $Q(A)$, turning the latter into a $*$-regular $\mathbb{C}$-algebra. Moreover, A and $Q(A)$ have the same projections. If A is, in addition, an AW^*-algebra, then $Q(A)$ is the maximal ring of right quotients of A.*

The main result of the present note is the following.

Theorem 2. *Let A and $Q(A)$ be as in* Theorem 1.

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(i) *There are an inner product space $\hat{V}_{\hat{\mathbb{C}}}$ which is an ultrapower of a Hilbert space $V_{\mathbb{C}}$, a closed $\hat{\mathbb{C}}$-linear subspace U of $\hat{V}$, and a $\mathbb{C}$-algebra embedding ι of $Q(A)$ into the endomorphism algebra of $U_{\hat{\mathbb{C}}}$ such that $\iota(r^*)$ is the adjoint of $\iota(r)$ for any $r \in Q(A)$.*
(ii) *Considering $\mathbb{C}$-algebras with involution and pseudo-inversion, $Q(A)$ is a homomorphic image of a subalgebra of an ultraproduct of algebras $\mathbb{C}^{n\times n}$.*
(iii) *The ortholattice $\mathbb{L}(A)$ of projections of A is a homomorphic image of a sub-ortholattice of an ultraproduct of projection ortholattices of algebras $\mathbb{C}^{n\times n}$.*

Relevant concepts are explained below, the proof is given in Section 7. In (i), in particular, $\hat{\mathbb{C}}$ is an ultrapower of $\mathbb{C}$ and the scalar product on $\hat{V}_{\hat{\mathbb{C}}}$ is obtained from that of $V_{\mathbb{C}}$ by the ultrapower construction. Moreover, if A is separable then $V_{\mathbb{C}}$ can be chosen separable.

The method of proof, of some interest in itself, is a representation of $Q(A)$ within a suitable inner product space, actually a closed subspace of an ultrapower of the Hilbert space V in which A is represented due to the GNS-construction. This is obtained considering $Q(A)$ as a homomorphic image of some abstract algebraic structure, mimicking an algebra of unbounded operators, and this in turn is used to reveal $Q(A)$ as homomorphic image of a subalgebra of a sufficiently saturated elementary extension $\hat{T}$ of the algebra (with unit) T of endomorphisms of V generated by those having finite dimensional image. The algebra $\hat{T}$ can be obtained as an ultrapower of T and admits a representation in an ultrapower of V.

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2. Inner product spaces, $*$-regular rings, and projections

A $*$-*ring* is a ring R (associative with unit) endowed with an *involution*; that is, an anti-automorphism $x \mapsto x^*$ of order 2. We shall consider representations of Λ-algebras R with involution, where Λ is a commutative $*$-ring, within inner product spaces V_F; to define such, we have to assume that F is also a Λ-algebra and that the involutions are related, properly. The adequate concept is that of a $*$-Λ-algebra: a Λ-algebra R which is also a $*$-ring such that $1_\Lambda r = r$ and $(\lambda r)^* = \lambda^* r^*$ for all $\lambda \in \Lambda$ and $r \in R$. For $*$-Λ-algebras, the concepts of homomorphism and subalgebra will refer to both the Λ-algebra structure and the involution. Observe that C^*-algebras are (rather special) $*$-$\mathbb{C}$-algebras.

Our main interest here is the case where Λ is the $*$-ring $\mathbb{C}$ of complex numbers with conjugation and F an elementary extension of $\mathbb{C}$, but there is no extra effort needed if one allows F to be any $*$-Λ-algebra which is a division ring. Then we say that a [right] F-vector space V_F is an *inner product space*, if it is endowed with a *scalar product* $(x, y) \mapsto \langle x \mid y\rangle$ which is an anisotropic (or totally regular) sesquilinear form, hermitean with respect to the involution, cf. [11]. Basic concepts and results for unitary spaces extend canonically. In particular, any F-linear subspace U is an inner product space U_F under the induced scalar product. Let π_U denote the orthogonal projection onto U if it exists; for example, if $\dim U < \infty$.

The endomorphisms φ of V_F form a Λ-algebra, where $(\lambda\varphi)(v) = \varphi(v)\lambda$ for $\lambda \in \Lambda$, $v \in V$ – observe that Λ acts on V by $v\lambda = v(\lambda 1_F)$. Endomorphisms of V_F admitting an adjoint φ^*, with respect to the scalar product, form a subalgebra $\mathsf{End}^*_\Lambda(V_F)$ in the Λ-algebra of all endomorphisms, which is also a $*$-Λ-algebra; indeed, $\lambda^*\varphi^*$ is the adjoint of $\lambda\varphi$. If φ is an endomorphism with $\dim \mathsf{im}\,\varphi < \infty$ then $\varphi \in \mathsf{End}^*_\Lambda(V_F)$ and $\dim \mathsf{im}\,\varphi^* = \dim(\ker\varphi)^\perp = \dim \mathsf{im}\,\varphi$.

A *representation* of a $*$-Λ-algebra R within an inner product space V_F is a homomorphism $\varepsilon\colon R \to \mathsf{End}^*_\Lambda(V_F)$ or, more conveniently, a unitary R-F-bimodule ${}_RV_F$ such that $(\lambda r)v = r(v\lambda)$ and $\langle rv \mid w\rangle = \langle v \mid r^*w\rangle$ for all $r \in R$, $\lambda \in \Lambda$, and v, $w \in V$; namely, $rv = \varepsilon(r)(v)$. In accordance with this, we write application of an endomorphism φ as φv and composition as $\psi\varphi$.

Our main concern will be *faithful* representations; i.e. representations ${}_RV_F$ such that $rv = 0$ for all $v \in V$ if only if $r = 0$. If such a representation exists, then we say that R is *representable* within V_F. The Gelfand-Naimark-Segal construction (cf. [12, §62]) yields the following

Fact 3. *Any (separable) C^*-algebra is representable within a (separable) Hilbert space (as an algebra of bounded operators).*

There are two approaches to $*$-regular rings. For the first, call an ideal I of a ring [*von Neumann*] *regular* if for any $a \in I$, there is $x \in I$ such that $axa = a$; such an element x is called a *quasi-inverse* of a (cf. [13]). Recall the following useful result, cf. [14, Lemma 1.3].

Fact 4. *A ring R is regular if it admits a regular ideal I such that R/I is regular. Any ideal of a regular ring is regular.*

A $*$-ring R is *proper* if $r^*r = 0$ implies $r = 0$ for all $r \in R$. Within any $*$-ring R, a^+ is a [Moore-Penrose] *pseudo-inverse* (or a *Rickart relative inverse*) of a if

$$a = aa^+a, \quad a^+ = a^+aa^+, \quad (aa^+)^* = aa^+, \quad (a^+a)^* = a^+a.$$

Fact 5. *A $*$-ring R is proper and regular if and only if any $a \in R$ admits a pseudo-inverse $a^+ \in R$. In this case, a^+ is uniquely determined by a.*

Fact 5 is well known (combine e.g. [15, XII Satz 2.4], [16, Proposition 88], and [17, Lemma 4]). We choose to define a *$*$-regular ring* R as a $*$-ring with an additional operation $a \mapsto a^+$ such that a^+ is a pseudo-inverse of a. If R is also an $*$-Λ-algebra, then we speak of a *$*$-regular Λ-algebra*. The concepts of subalgebras and homomorphisms of $*$-regular algebras concern pseudo-inversion too. Though, when speaking about representations, only the $*$-Λ-algebra structure matters. With $\Lambda = \mathbb{Z}$ we may subsume all $*$-regular rings. The equivalence between the two concepts of $*$-regularity extends beyond consideration of single algebras.

Lemma 6. *Let R and T be $*$-regular Λ-algebras, S a $*$-Λ-subalgebra of T, and $f\colon S \to R$ a surjective homomorphism [of $*$-Λ-algebras] such that the ideal $\mathsf{ker}\,f$ of S is regular. Then S is closed under pseudo-inversion in T and $f\colon S \to R$ preserves pseudo-inversion; that is, in the context of $*$-regular Λ-algebras, S is a subalgebra of T and f a homomorphism.*

Proof. S is proper, being a $*$-subring of T, and regular by Fact 4. Thus, S is $*$-regular by Fact 5. The uniqueness of pseudo-inverse in T implies that S is closed under pseudo-inversion. Uniqueness of pseudo-inverse in R implies that f preserves pseudo-inversion. $\square$

Fact 7. *Let V_F be an inner product space. The set*

$$\mathsf{End}^*_{\Lambda\mathrm{f}}(V_F) = \{\varphi + \lambda\,\mathsf{id} \mid \lambda \in F,\ \varphi \in \mathsf{End}^*_\Lambda(V_F),\ \dim \mathsf{im}\,\varphi < \infty\}$$

forms a subalgebra of $\mathsf{End}^*_\Lambda(V_F)$ which is a $*$-regular Λ-algebra. In particular, if V_F is finite dimensional, then $\mathsf{End}^*_\Lambda(V_F)$ is a $*$-regular Λ-algebra.

Proof. If $\dim \mathsf{im}\,\varphi < \infty$, then the subspace $U = (\ker\varphi)^\perp$ is finite dimensional, whence $V = U \oplus U^\perp$ and $\varphi|_U$ is an isomorphism of U onto $W = \mathsf{im}\,\varphi$. Since U and W are finite dimensional, the inverse $\psi\colon W \to U$ of $\varphi|_U$ has an adjoint ψ^*. Thus $\pi_U\psi\pi_W$ has $\pi_W\psi^*\pi_U$ as adjoint and belongs to $\mathsf{End}^*_{\Lambda\mathrm{f}}(V_F)$. Moreover, $\varphi = \varphi\pi_U\psi\pi_W\varphi$. Therefore, $I = \{\varphi \in \mathsf{End}^*_\Lambda(V_F) \mid \dim \mathsf{im}\,\varphi < \infty\}$ is a regular ideal of $\mathsf{End}^*_{\Lambda\mathrm{f}}(V_F)$. If $\dim V_F < \infty$ then $I = V_F$. Otherwise, since $\mathsf{End}^*_{\Lambda\mathrm{f}}(V_F)/I$ is isomorphic to F, Fact 6 applies to prove $*$-regularity of $\mathsf{End}^*_{\Lambda\mathrm{f}}(V_F)$. $\square$

The following is granted by the Gram-Schmidt orthonormalization process.

Fact 8. *Let V_F be an inner product space such that $\dim V_F = n < \infty$ and for any $\lambda,\ \mu \in F$, there is $\nu = \nu^* \in F$ such that $\lambda^*\lambda + \mu^*\mu = \nu^2$. Then $\mathsf{End}^*_\Lambda(V_F)$ is isomorphic to the matrix algebra $F^{n\times n}$ with involution $A = (a_{ij}) \mapsto A^*$, where A^* is the transpose of (a^*_{ij}).*

An element e of a $*$-ring is a *projection*, if $e = e^2 = e^*$. Observe that any projection is its own pseudo-inverse and that $e = aa^+$ and $f = a^+a$ are projections if a^+ is a pseudo-inverse of a.

Fact 9. *$ap = 0$ implies $(a^+)^*p = 0$ and $a^*p = 0$ implies $a^+p = 0$ for any $*$-regular ring R, $a \in R$, and projection $p \in R$.*

Proof. For $e = aa^+$ and $f = a^+a$, $ap = 0$ implies $fp = a^+ap = 0$, whence $pf = (fp)^* = 0$. Therefore, $pa^+ = pa^+aa^+ = pfa^+ = 0$ and thus $(a^+)^*p = 0$. From $a^*p = 0$, we get $pa = (a^*p)^* = 0$ whence $pe = paa^+ = 0$. Therefore, $ep = 0$ and $a^+p = a^+aa^+p = a^+ep = 0$. $\square$

For the following fact, see [16, Chapter 2] and [10, §1].

Fact 10. *If R is a regular ring, not necessarily with unit, then the principal left ideals Ra form a (complemented) sublattice $\bar{L}(R)$ of the lattice of all left ideals and for any $a \in R$, $Ra = Re$ for an idempotent e.*

Furthermore, the projections of a $*$-regular Λ-algebra R form an ortholattice $\mathbb{L}(R)$ *where the partial order is given by*

$$e \leq f \quad \textit{if and only if} \quad fe = e \quad \textit{if and only if} \quad ef = e,$$

least and greatest elements are given by the constants 0 and 1 of R. The orthocomplement *is $e' = 1 - e$; namely, e' is a complement of e, $e'' = e$, and $e \leq f$ if and* *only if $f' \leq e'$. Join (supremum) and meet (infimum) are given by equalities*

$$e \cup f = f + (e(1-f))^+ e(1-f), \quad e \cap f = (e' \cup f')'.$$

The map $e \mapsto Re$ is an isomorphism from $\mathbb{L}(R)$ onto $\bar{L}(R)$.

As shown by von Neumann, any modular ortholattice with a certain kind of 'coordinate system' is isomorphic to $\mathbb{L}(R)$ for some $*$-regular algebra R with a system of matrix units. In this sense, one has equivalent structures. Though, $*$-regular rings appear much better suited for the present discussion.

Fact 11. *If S is a subalgebra of a $*$-regular Λ-algebra R then $\mathbb{L}(S)$ is a sub-ortholattice of $\mathbb{L}(R)$. If $\varphi\colon R \to S$ is a homomorphism then its restriction ψ to $\mathbb{L}(R)$ is a homomorphism into $\mathbb{L}(S)$; ψ is surjective if φ is.*

Proof. In view of Fact 10, only surjectivity of φ needs a word: given a projection e in S choose any preimage $a \in R$. Then aa^+ is a projection and $\varphi(aa^+) = ee^+ = e^2 = e$. □

3. Concepts from Model Theory

For a fixed commutative $*$-ring Λ with unit, we are going to consider the classes of all $*$-Λ-algebras and of all $*$-regular Λ-algebras. Their members are viewed as 1-sorted algebraic structures where each $\lambda \in \Lambda$ determines a unary operation $x \mapsto \lambda x$. Moreover, besides this and the ring structure (observe that additive inversion is not required since it can be captured via $-x = (-1_\Lambda)x$), for both types of algebras, we also have a unary operation of involution, while for $*$-regular Λ-algebras, we consider in addition a unary operation of pseudo-inversion. Considering ortholattices we disregard the partial order. Since the three mentioned classes can be defined by identities, they are closed under formation of direct products, subalgebras, and homomorphic images; these concepts refer to the operations of $*$-Λ-algebras and of $*$-regular Λ-algebras respectively.

Let $\mathcal{U}$ be an ultrafilter over a set I. In view of the explicit definition of $\mathbb{L}(R)$ in terms of R (Fact 10), the following statement holds.

Fact 12. $\mathbb{L}\big(\prod_{i\in I} R_i/\mathcal{U}\big) = \prod_{i\in I} \mathbb{L}(R_i)/\mathcal{U}$ *for any $*$-regular Λ-algebras R_i, $i \in I$.*

We will also have to use ultraproducts of inner product spaces V_F and representations ${}_RV_F$, but the presence of scalar products excludes viewing them as 1-sorted algebraic structures. The most convenient way is to view an inner product space V_F as a 2-sorted algebraic structure with sorts V and F, carrying the group operations and the $*$-Λ-algebra operations, respectively. In addition, one has two binary operations $(v,\alpha) \mapsto v\alpha \in V$ and $(u,v) \mapsto \langle u \mid v\rangle \in F$ where u, $v \in V$ and $\alpha \in F$. Considering a representation ${}_RV_F$, we have the $*$-Λ-algebra R as a third sort and, in addition, a binary operation $(r,v) \mapsto rv \in V$ for $r \in R$ and $v \in V$.

The concepts of homomorphism, subalgebra, direct product, and ultraproduct generalize to many-sorted algebraic structures in an obvious way. All constructions are built sort-wise; i.e. the sorts of a direct product (an ultraproduct, etc.) of A_i, $i \in I$, are direct products (ultraproducts, etc.) of the sorts of A_i, $i \in I$. Of course, at the price of undue technical complications, one could also consider many-sorted structures as 1-sorted relational structures.

Given a formula φ in the language, a structure A, and elements $a_1, \ldots, a_n$ of A matching the sorts of the free variables $x_1, \ldots, x_n$ occurring in φ, validity of φ in A under the substitution $x_i \mapsto a_i$ is defined by the same inductive approach as in the 1-sorted case and denoted by

$$A \models \varphi(a_1, \ldots, a_n).$$

Fact 13. *Let $\mathcal{U}$ be an ultrafilter over a set I. The following statements hold:*

(i) *Let Λ be a commutative $*$-ring, let R_i and F_i be $*$-Λ-algebras, let $(V_i)_{F_i}$ be an inner product space, and let ${}_{R_i}(V_i)_{F_i}$ be a faithful representation for all $i \in I$. Then $V_F = \prod_{\mathcal{U}} (V_i)_{F_i}$ is an inner product space and ${}_R V_F = \prod_{\mathcal{U}} {}_{R_i}(V_i)_{F_i}$ is a faithful representation, where $F = \prod_{\mathcal{U}} F_i$ and $R = \prod_{\mathcal{U}} R_i$.*

(ii) *For a $*$-Λ-algebra F and a natural number n, the ultrapower $(F^{n\times n})^I/\mathcal{U}$ of matrix $*$-Λ-algebras is isomorphic to the matrix $*$-Λ-algebra $(F^I/\mathcal{U})^{n\times n}$.*

Proof. (i) is an obvious application of the Łoś Theorem. The isomorphism in (ii) is given by

$$\left[(a_i^{jk})_{j,k=1,\ldots,n} \mid i \in I\right] \mapsto \left([a_i^{jk} \mid i \in I]\right)_{k,j=1,\ldots,n}.$$

$\square$

Fact 14. *Any elementary extension of a representation ${}_R V_F$ is again a representation ${}_{\hat{R}}\hat{V}_{\hat{F}}$, where $\hat{F}$ is an elementary extension of F, $\hat{V}_{\hat{F}}$ – of V_F, and $\hat{R}$ – of R.*

In the proof of our central result, we have to apply a concept of saturated structures. We shall present here a very weak form just sufficient for our purposes. Considering a fixed structure A, add a new constant symbol $\underline{a}$, called a *parameter*, for each $a \in A$. In what follows, $\Sigma(x)$ is a set of formulas with one free variable x in this extended language. Given an embedding $h\colon A \to B$, we call B *modestly saturated over A via h*, if any set of formulas $\Sigma(x)$ with parameters from A which is finitely realized in A (where $\underline{a}$ is interpreted as $\underline{a}^A = a$) is realized in B (where $\underline{a}$ is is interpreted as $\underline{a}^B = h(a)$). The following is a particular case of [18, Corollary 4.3.14].

Fact 15. *Every structure A admits an elementary embedding h into some structure B which is modestly saturated over A via h. One can choose B to be an ultrapower of A and h to be the canonical embedding. Identifying a with $h(a)$, one may assume B to be an elementary extension of A.*

4. Representations of algebras with involution

Theorem 16. *Let a $*$-Λ-algebra R have a faithful representation in an inner product space V_F. Then R is a homomorphic image of a subalgebra S of an ultraproduct of $\mathsf{End}^*_\Lambda(U_F)$ with U ranging over finite dimensional subspaces of V_F. Moreover, if R is $[*$-$]$regular then so is S.*

Recall that in the $*$-regular case, all algebraic constructions also refer to pseudo-inversion; in particular, only $*$-regular $*$-Λ-subalgebras are admitted.

Proof. This proof is a variation of the approach of Tyukavkin [19] and Micol [20].

Choose any set I of finite dimensional subspaces of V_F such that any finite dimensional subspace W of V_F is contained in some member of I. Given a basis B of the vector space V_F, one may choose I to be the set of subspaces U of V_F spanned by finite subsets of B. If the basis is countable and enumerated, one may choose I consisting of the subspaces spanned by initial segments. For $U \in I$, we set $U^+ = \{W \in I \mid U \subseteq W\}$. Notice that $U_1^+ \cap U_2^+ = (U_1 + U_2)^+$. Thus there is an ultrafilter $\mathcal{U}$ on I such that $U^+ \in \mathcal{U}$ for all $U \in I$. To simplify notation, let R_U denote the $*$-regular Λ-algebra $\mathsf{End}^*_\Lambda(U_F)$. Form the direct product $T = \prod_{U\in I} R_U$ and the ultraproduct $\hat{T} = \prod_{U\in I} R_U/\mathcal{U}$. The elements of T and $\hat{T}$ will be denoted as $\sigma = (\sigma_U \mid U \in I)$ and $[\sigma]$. We first relate T with ${}_RV_F$.

Given $\sigma \in T$, $r \in R$, and $U_0 \in I$, we say that $J \in \mathcal{U}$ *witnesses* $\sigma \sim r$ for U_0 if and only if $J \subseteq U_0^+$ and

$$\sigma_U v = rv,\ \sigma_U^* v = r^* v \quad \text{for all } U \in J \text{ and all } v \in U_0.$$

Observe that if J witnesses $\sigma \sim r$ for $U_0 \in I$, then it does so for any $U_1 \subseteq U_0$. We put $\sigma \sim r$ if for any $U_0 \in I$ there is $J \in \mathcal{U}$ witnessing $\sigma \sim r$ for U_0. We also put

$$S = \{\sigma \in T \mid \sigma \sim r \text{ for some } r \in R\}.$$

We prove several auxiliary claims. Let σ, $\tau \in T$ and let r, r_0, r_1, $s \in R$.

Claim 1. *There is a well defined map $g\colon S \to R$, $g(\sigma) = r$, where $\sigma \sim r$.*

Proof of Claim. We have to show that $\sigma \sim r_0$ and $\sigma \sim r_1$ together imply $r_0 = r_1$. Consider an arbitrary $v \in V$ and $U_0 \in I$ containing v. Let J_i witness $\sigma \sim r_i$ for U_0, $i < 2$. Then $J = J_1 \cap J_2$ witnesses $\sigma \sim r_i$ for U_0, $i < 2$, too, and it follows that $r_0 v = \sigma_{U_0} v = r_1 v$. This shows that $r_0 v = r_1 v$ for all $v \in V$. Since ${}_RV_F$ is a faithful representation, we conclude that $r_0 = r_1$. □

Claim 2. *Considering $*$-Λ-algebras, S is a subalgebra of T and $g\colon S \to R$ is a homomorphism.*

Proof of Claim. Let $\sigma \sim r$, $\tau \sim s$, and $\lambda \in \Lambda$. We have to show that

$$(1)\ \sigma^* \sim r^*, \quad (2)\ \lambda\sigma \sim \lambda r, \quad (3)\ \tau + \sigma \sim s + r, \quad (4)\ \tau\sigma \sim sr.$$

Note that (1) is obvious by definition. To prove (2)-(4), we consider $U_0 \in I$ and choose $J \in \mathcal{U}$ witnessing $\sigma \sim r$ and $K \in \mathcal{U}$ witnessing $\tau \sim s$ for U_0. Then $J \cap K \in \mathcal{U}$ witnesses both, $\sigma \sim r$ and $\tau \sim s$, for U_0. Applying linearity, one gets for any $U \in J \cap K$ and any $v \in U_0$:

$$\begin{aligned}
&(\lambda\sigma_U)v = (\sigma_U v)\lambda = (rv)\lambda = (\lambda r)v;\\
&(\lambda\sigma)^*_U v = (\sigma^*_U v)\lambda^* = (r^* v)\lambda^* = (\lambda^* r^*)v = (\lambda r)^* v;\\
&(\sigma+\tau)_U v = \sigma_U v + \tau_U v = rv + sv = (r+s)v;\\
&(\sigma+\tau)^*_U v = \sigma^*_U v + \tau^*_U v = r^* v + s^* v = (r^* + s^*)v = (r+s)^* v.
\end{aligned}$$

Therefore, $J \cap K$ witnesses $\lambda\sigma \sim \lambda r$ and $\sigma + \tau \sim r + s$ for U_0, and (2)-(3) are proved. To prove (4), we choose $U_1 \in I$ such that $U_1 \supseteq rU_0$ and $U_2 \in I$ such that $U_2 \supseteq s^*U_0$. Let $J_0 \in \mathcal{U}$ witness $\tau \sim s$ for U_1 and let $J_1 \in \mathcal{U}$ witness $\sigma \sim r$ for U_2. Then $J' = J \cap K \cap J_0 \cap J_1 \in \mathcal{U}$, and one gets for any $U \in J'$ and any $v \in U_0$:

$$(\tau\sigma)_U v = \tau_U(\sigma_U v) = \tau_U(rv) = s(rv) = (sr)v;$$
$$(\tau\sigma)^*_U v = \sigma^*_U(\tau^*_U v) = \sigma^*_U(s^*v) = r^*(s^*v) = (r^*s^*)v = (sr)^*v,$$

whence J' witnesses $\tau\sigma \sim sr$ for U_0, and (4) is also proved. □

Claim 3. *The map $g\colon S \to R$ is surjective.*

Proof of Claim. Given $r \in R$, let $\varphi = \varepsilon(r)$ denote the associated endomorphism of V_F. For $U \in I$, we set

$$\sigma_U = \pi_U\varphi|_U = \pi_U\varphi\pi_U|_U \in R_U$$

and $\sigma = (\sigma_U \mid U \in I)$, where π_U is the orthogonal projection of V onto U. Observe that $\sigma^*_U = \pi_U\varphi^*\pi_U|_U = \pi_U\varphi^*|_U$. Now given $U_0 \in I$, let $U_1 \in I$ be such that $U_1 \supseteq U_0 + rU_0 + r^*U_0$ and let $J = U_1^+ \in \mathcal{U}$. Then for any $U \in J$ and any $v \in U_0$, one has rv, $r^*v \in U$ and

$$\sigma_U v = \pi_U\big(\varphi(\pi_U v)\big) = \pi_U(\varphi v) = \pi_U(rv) = rv;$$
$$\sigma^*_U v = \pi_U\big(\varphi^*(\pi_U v)\big) = \pi_U(\varphi^* v) = \pi_U(r^*v) = r^*v.$$

Thus J witnesses $\sigma \sim r$ for U_0 and it follows that $g(\sigma) = r$. □

We put

$$\hat{S} = \{[\sigma] \mid \sigma \in S\}.$$

Claim 4. *Considering $*$-Λ-algebras, $\hat{S}$ is a subalgebra of $\hat{T}$ and $f\colon \hat{S} \to R$, $f([\sigma]) = r$, where $\sigma \sim r$, is a well defined onto homomorphism.*

Proof of Claim. In view of Claim 2 it suffices to show that $[\tau] = [\sigma]$ and $\sigma \sim r$ imply $\tau \sim r$. Let $K = \{U \in I \mid \sigma_U = \tau_U\}$; in particular, $K \in \mathcal{U}$. Given $U_0 \in I$, let J witness $\sigma \sim r$ for U_0. Then $J \cap K$ witnesses $\tau \sim r$ for U_0. □

For any $U_0 \in I$, we put $\chi^{U_0} = (\chi^{U_0}_U \mid U \in I) \in T$, where $\chi^{U_0}_U$ is the orthogonal projection of U onto U_0 if $U_0 \subseteq U$ and $\chi^{U_0}_U = 0$ otherwise.

Claim 5. $[\sigma] \in \operatorname{ker} f$ *if and only if*

$$(*) \qquad [\sigma] \cdot [\chi^{U_0}] = [\sigma^*] \cdot [\chi^{U_0}] = 0 \quad \text{for all } U_0 \in I.$$

Proof of Claim. By definition, $[\sigma] \in \operatorname{ker} f$ if and only if $\sigma \sim 0$. So, let $\sigma \sim 0$ and let $U_0 \in I$. Choose $J \in \mathcal{U}$ witnessing $\sigma \sim 0$ for U_0. Then for all $U \in J$ and all $v \in U_0$ one has $\sigma_U v = 0 = \sigma^*_U v$ which implies $\sigma_U\chi^{U_0}_U = 0 = \sigma^*_U\chi^{U_0}_U$. As $J \in \mathcal{U}$, $(*)$ holds.

Conversely, assume that $(*)$ holds and let $U_0 \in I$. This means that $K = \{U \in I \mid \sigma_U\chi^{U_0}_U = 0 = \sigma^*_U\chi^{U_0}_U\} \in \mathcal{U}$. Then $J = K \cap U_0^+ \in \mathcal{U}$ witnesses $\sigma \sim 0$ for U_0. □

Claim 6. $\operatorname{ker} f$ *is regular.*

Proof of Claim. By Fact 7, R_U is $*$-regular for any $U \in I$, whence so is the ultraproduct $\hat{T}$. Observe that for any $U_0 \in I$, $[\chi^{U_0}] \in \hat{T}$ is a projection as $\chi_U^{U_0}$ is a projection in R_U for any $U \in I$. Now let $[\sigma] \in \mathsf{ker}\, f$ and let $[\tau]$ be its pseudo-inverse in $\hat{T}$. Then Claim 5 and Fact 9 imply that

$$[\tau] \cdot [\chi^{U_0}] = [\tau^*] \cdot [\chi^{U_0}] = 0 \text{ for all } U_0 \in I,$$

which in turn implies by Claim 5 that $[\tau] \in \mathsf{ker}\, f$. □

Now we complete the proof of the theorem. The first claim for $*$-Λ-algebras follows from Claim 4 immediately. In the $*$-regular setting, one may apply Lemma 6 due to Claim 6. □

Remark 17. Assume that V_F admits a countable orthonormal basis $v_0, v_1, v_2, \ldots$ (e.g. if $\dim V_F = \omega$ and F satisfies the hypothesis of Fact 8). Let I consist of the subspaces U_n spanned by $\{v_0, \ldots, v_n\}$, $n < \omega$, and let $\mathcal{U}$ extend the cofinite filter. Then one can view $\mathsf{End}^*_\Lambda\big((U_n)_F\big)$, $n < \omega$, as matrix algebras $F^{n\times n}$ uniformly. In this case, $\mathsf{ker}\, f$ consists of elements of the form $[A_n \mid n < \omega]$ such that for any $m < \omega$, there is $J \in \mathcal{U}$ such that the first m rows and m columns of A_n are 0 provided that $U_n \in J$, cf. Tyukavkin [19].

Corollary 18. *Any C^*-algebra is a homomorphic image of a subalgebra of an ultraproduct of algebras $\mathbb{C}^{n\times n}$, $n < \omega$.*

Proof. We apply Theorem 16 to the representation given by the GNS-construction and observe that $\mathsf{End}^*_\Lambda(U_{\mathbb{C}}) \cong \mathbb{C}^{n\times n}$ if U is a finite dimensional subspace of a unitary space (cf. Fact 8). □

The following provides a method to obtain representations of homomorphic images, elaborating on Micol [20, Theorem 3.8].

Proposition 19. *For any regular $*$-Λ-algebra R having a faithful representation within an inner product space V_F, there is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that, for any regular ideal $I = I^*$, R/I admits a faithful representation within some closed subspace of $\hat{V}_{\hat{F}}$.*

Proof. According to Fact 15, there is an ultrapower ${}_{\hat{R}}\hat{V}_{\hat{F}}$ of the faithful representation ${}_RV_F$ which is modestly saturated over ${}_RV_F$ via the canonical embedding. Then $\hat{V}$ is an R-module via the canonical embedding of R into $\hat{R}$ and

$$U = \{v \in \hat{V} \mid av = 0 \text{ for all } a \in I\} = \bigcap_{a\in I} (a\hat{V})^\perp$$

is a closed subspace of $\hat{V}_{\hat{F}}$ and a left (R/I)-module. Moreover, from $I = I^*$ one has

$$\langle (r + I)v \mid w\rangle = \langle v \mid (r^* + I)w\rangle \text{ for all } v, w \in U,$$

which proves that ${}_{R/I}U_{\hat{F}}$ is a representation of R/I.

We show that this representation is faithful; that is, for any $a \notin I$, there has to be $v \in U$ such that $av \neq 0$. Since $v \in U$ means $bv = 0$ for all $b \in I$, we have to show that the set

$$\Sigma(x) = \{\underline{a}x \neq 0\} \cup \{\underline{b}x = 0 \mid b \in I\}$$

of formulas with parameters from $\{a\} \cup I$ and a variable x of type V is satisfiable in ${}_{\hat{R}}\hat{V}_{\hat{F}}$. Due to modest saturation, it suffices to show that for any $b_1, \ldots, b_n \in I$, there is $v \in V$ such that $av \neq 0$ and $b_i v = 0$ for all $i \in \{1, \ldots, n\}$. In view of Fact 10, by regularity of I there is an idempotent $e \in I$ such that $Ie = \sum_{i=1}^{n} Ib_i$; in particular, $b_i e = b_i$ and $b_i v = 0$ for all $i \in \{1, \ldots, n\}$ whenever $ev = 0$. Thus it suffices to show that there is $v \in V$ such that $ev = 0$ but $av \neq 0$.

Assume the contrary, namely that $ev = 0$ implies $av = 0$ for any $v \in V$. For any $w \in V$, let $v = (1 - e)w$. As $ev = 0$, we get $av = 0$ by our assumption. Therefore, $0 = av = a(1 - e)w$ for all $w \in V$ whence $a(1 - e) = 0$, as ${}_R V_F$ is a faithful representation. But then $a = ae \in I$, a contradiction, and we are done. □

5. Algebras of generalized operators

Given a commutative $*$-ring Λ, a *pre-$*$-Λ-algebra* is a set R endowed with binary operations $+$ and $\cdot$, constants 0_R and 1_R, for each $\lambda \in \Lambda$ a unary operation written as $r \mapsto \lambda r$, and a symmetric binary relation $\bowtie$ such that for any $r \in R$ there is $r^* \in R$ with $r \bowtie r^*$ and such that for all r, r^*, s, $s^* \in R$ and all $\lambda \in \Lambda$:

(a) $r \bowtie r^*$ and $s \bowtie s^*$ jointly imply $r + s \bowtie r^* + s^*$;
(b) $r \bowtie r^*$ and $s \bowtie s^*$ jointly imply $r \cdot s \bowtie s^* \cdot r^*$;
(c) $r \bowtie r^*$ implies $\lambda r \bowtie \lambda^* r^*$;
(d) $0_R \bowtie 0_R$ and $1_R \bowtie 1_R$.

An *action* of R on an inner product space V_F, where F is a $*$-Λ-algebra, associates with each $r \in R$ a linear subspace $\mathsf{dom}\, r$ of V_F and an F-linear map $\mathsf{dom}\, r \to V$ written as $v \mapsto rv$. Thus, in particular, for all v, $w \in V$, all $r \in R$, and all $\alpha \in F$:

(e) if u, $v \in \mathsf{dom}\, r$, then $v + w \in \mathsf{dom}\, r$ and $r(v + w) = rv + rw$;
(f) if $u \in \mathsf{dom}\, r$, then $u\alpha \in \mathsf{dom}\, r$ and $r(v\alpha) = (rv)\alpha$.

We write ${}_R V_F$ for the inner product space V with right action of F and left action of R and we consider it as a 3-sorted structure with sorts V, F, and R: the action of R on V is conceived as the ternary relation

$$\{(r, v, w) \in R \times V \times V \mid r \in R,\ v \in \mathsf{dom}\, r,\ w = rv\}.$$

We also write $rX = \{rv \mid v \in X\}$ and $r^{-1}(X) = \{v \in \mathsf{dom}\, r \mid rv \in X\}$ for any $X \subseteq V$.

As additional structure, we assume there is a downward directed set $\mathcal{D}$ of linear subspaces of V_F. We say that the action of R on V_F is *$\mathcal{D}$-supported* if the following hold for all r, $s \in R$ and all $\lambda \in \Lambda$:

(i) $D \subseteq \mathsf{dom}\, r$ for some $D \in \mathcal{D}$;
(ii) for any $D \in \mathcal{D}$, there is $D' \in \mathcal{D}$ such that $D' \subseteq r^{-1}(D)$;
(iii) there is $D \in \mathcal{D}$ such that $D \subseteq \mathsf{dom}\, r \cap \mathsf{dom}\, s \cap \mathsf{dom}(r+s)$ and $(r+s)v = rv+sv$ for all $v \in D$;
(iv) $0_R v = 0$ and $1_R v = v$ for all $v \in V$;
(v) there is $D \in \mathcal{D}$ such that $D \subseteq \mathsf{dom}(r \cdot s) \cap s^{-1}(\mathsf{dom}\, r) \cap \mathsf{dom}\, s$ and $(r \cdot s)v = r(sv)$ for all $v \in D$;
(vi) there is $D \in \mathcal{D}$ such that $D \subseteq \mathsf{dom}\, r \cap \mathsf{dom}(\lambda r)$ and $(\lambda r)v = (rv)\lambda$ for all $v \in D$;

(vii) if $r \bowtie r^*$, then there is $D \in \mathcal{D}$ such that $D \subseteq \mathsf{dom}\, r \cap \mathsf{dom}\, r^*$ and $\langle rv \mid w \rangle = \langle v \mid r^* w \rangle$ for all v, $w \in D$.

Given a $\mathcal{D}$-supported action of R on V_F, we define a binary relation $\approx_{\mathcal{D}}$ on R by

$r \approx_{\mathcal{D}} s$ if and only if for any $r^* \bowtie r$ and any $s^* \bowtie s$, there is $D \in \mathcal{D}$ such that $r \approx_{\mathcal{D}} s$ is *witnessed by D under this proviso*, namely: $D \subseteq \mathsf{dom}\, r \cap \mathsf{dom}\, r^* \cap \mathsf{dom}\, s \cap \mathsf{dom}\, s^*$ and $rv = sv$, $r^* v = s^* v$ for all $v \in D$.

We speak of a Λ-*algebra of generalized operators* on V_F and denote it by $({}_R V_F; \mathcal{D})$ if, in addition, the following holds:

(viii) $r \bowtie t$ and $s \bowtie t$ imply $r \approx_{\mathcal{D}} s$ for all r, s, $t \in R$.

Lemma 20. *If $({}_R V_F; \mathcal{D})$ is a Λ-algebra of generalized operators, then $\approx_{\mathcal{D}}$ is a congruence with respect to the operations defined on R. Moreover, for any r, r^*, s, $s^* \in R$ such that $r \bowtie r^*$ and $s \bowtie s^*$, $r \approx_{\mathcal{D}} s$ implies $r^* \approx_{\mathcal{D}} s^*$.*

Proof. It is straightforward that $\approx_{\mathcal{D}}$ is reflexive and symmetric. We prove that $\approx_{\mathcal{D}}$ is transitive. Let $r \approx_{\mathcal{D}} s$ and $s \approx_{\mathcal{D}} t$. Let also $r \bowtie r^*$ and $t \bowtie t^*$. Then there is $s^* \in R$ such that $s \bowtie s^*$. Let $D_0 \in \mathcal{D}$ witness $r \approx_{\mathcal{D}} s$ under the proviso $r \bowtie r^*$ and $s \bowtie s^*$. Let $D_1 \in \mathcal{D}$ witness $s \approx_{\mathcal{D}} t$ under the proviso $s \bowtie s^*$ and $t \bowtie t^*$. As $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that $D \subseteq D_0 \cap D_1$. Then D witnesses $r \approx_{\mathcal{D}} t$ under the proviso $r \bowtie r^*$ and $t \bowtie t^*$.

For any $\lambda \in \Lambda$, we prove that $\approx_{\mathcal{D}}$ respects the unary operation $\lambda\cdot$. Suppose that $r \approx_{\mathcal{D}} s$ for some r, $s \in R$. To prove that $\lambda r \approx_{\mathcal{D}} \lambda s$, let $t \bowtie \lambda r$ and $u \bowtie \lambda s$. Let also $D' \subseteq \mathsf{dom}\, r \cap \mathsf{dom}\, r^* \cap \mathsf{dom}\, s \cap \mathsf{dom}\, s^*$ witness $r \approx_{\mathcal{D}} s$ under the proviso $r \bowtie r^*$ and $s \bowtie s^*$. Then according to (c) and (viii), $t \approx_{\mathcal{D}} \lambda^* r^*$ and $u \approx_{\mathcal{D}} \lambda^* s^*$; in particular, there are D_{0r}, $D_{0s} \in \mathcal{D}$ such that $tv = (\lambda^* r^*)v$ for all $v \in D_{0r}$ and $uv = (\lambda^* s^*)v$ for all $v \in D_{0s}$. Moreover, according to (vi), there are D_{1r}, D_{2r}, D_{1s}, $D_{2s} \in \mathcal{D}$ such that:

$$
\begin{aligned}
&D_{1r} \subseteq \mathsf{dom}\, r \cap \mathsf{dom}(\lambda r) && \text{and } (\lambda r)v = (rv)\lambda \text{ for all } v \in D_{1r};\\
&D_{2r} \subseteq \mathsf{dom}\, r^* \cap \mathsf{dom}(\lambda^* r^*) && \text{and } (\lambda^* r^*)v = (r^* v)\lambda^* \text{ for all } v \in D_{2r};\\
&D_{1s} \subseteq \mathsf{dom}\, s \cap \mathsf{dom}(\lambda s) && \text{and } (\lambda s)v = (sv)\lambda \text{ for all } v \in D_{1s};\\
&D_{2s} \subseteq \mathsf{dom}\, s^* \cap \mathsf{dom}(\lambda^* s^*) && \text{and } (\lambda^* s^*)v = (s^* v)\lambda^* \text{ for all } v \in D_{2s}.
\end{aligned}
$$

As $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap \bigcap_{i<3} D_{ir} \cap \bigcap_{i<3} D_{is}$. Then one gets for any $v \in D$:

$$
\begin{aligned}
&(\lambda r)v = (rv)\lambda = (sv)\lambda = (\lambda s)v;\\
&tv = (\lambda^* r^*)v = (r^* v)\lambda^* = (s^* v)\lambda^* = (\lambda^* s^*)v = uv.
\end{aligned}
$$

Therefore, D witnesses $\lambda r \approx_{\mathcal{D}} \lambda s$ under the proviso $\lambda r \bowtie t$ and $\lambda s \bowtie u$.

The fact that $\approx_{\mathcal{D}}$ respects $+$ can be established in a similar but easier way using (a) and (iii). We prove now that $\approx_{\mathcal{D}}$ respects $\cdot$. Suppose that $r_0 \approx_{\mathcal{D}} s_0$ and $r_1 \approx_{\mathcal{D}} s_1$. To prove that $r_0 \cdot r_1 \approx_{\mathcal{D}} s_0 \cdot s_1$, let $t \bowtie r_0 \cdot r_1$ and $u \bowtie s_0 \cdot s_1$. Let also $D_0 \subseteq \mathsf{dom}\, r_0 \cap \mathsf{dom}\, r_0^* \cap \mathsf{dom}\, s_0 \cap \mathsf{dom}\, s_0^*$ witness $r_0 \approx_{\mathcal{D}} s_0$ under the proviso $r_0 \bowtie r_0^*$, $s_0 \bowtie s_0^*$, while $D_1 \subseteq \mathsf{dom}\, r_1 \cap \mathsf{dom}\, r_1^* \cap \mathsf{dom}\, s_1 \cap \mathsf{dom}\, s_1^*$ witnesses $r_1 \approx_{\mathcal{D}} s_1$ under the proviso $r_1 \bowtie r_1^*$, $s_1 \bowtie s_1^*$. Then according to (b) and (viii), $t \approx_{\mathcal{D}} r_1^* \cdot r_0^*$ and

$u \approx_{\mathcal{D}} s_1^* \cdot s_0^*$; in particular, there are D_{0r}, $D_{0s} \in \mathcal{D}$ such that $tv = (r_1^* \cdot r_0^*)v$ for all $v \in D_{0r}$ and $uv = (s_1^* \cdot s_0^*)v$ for all $v \in D_{0s}$. Moreover, according to (v), there are D_{1r}, D_{2r}, D_{1s}, $D_{2s} \in \mathcal{D}$ such that:

$$\begin{aligned}
D_{1r} &\subseteq \mathsf{dom}(r_0 \cdot r_1) \cap r_1^{-1}(\mathsf{dom}\, r_0) \cap \mathsf{dom}\, r_1 \\
&\qquad \text{and } (r_0 \cdot r_1)v = r_0(r_1 v) \text{ for all } v \in D_{1r}; \\
D_{2r} &\subseteq \mathsf{dom}(r_1^* \cdot r_0^*) \cap (r_0^*)^{-1}(\mathsf{dom}\, r_1^*) \cap \mathsf{dom}\, r_0^* \\
&\qquad \text{and } (r_1^* \cdot r_0^*)v = r_1^*(r_0^* v) \text{ for all } v \in D_{2r}; \\
D_{1s} &\subseteq \mathsf{dom}(s_0 \cdot s_1) \cap s_1^{-1}(\mathsf{dom}\, s_0) \cap \mathsf{dom}\, s_1 \\
&\qquad \text{and } (s_0 \cdot s_1)v = s_0(s_1 v) \text{ for all } v \in D_{1s}; \\
D_{2s} &\subseteq \mathsf{dom}(s_1^* \cdot s_0^*) \cap (s_0^*)^{-1}(\mathsf{dom}\, s_1^*) \cap \mathsf{dom}\, s_0^* \\
&\qquad \text{and } (s_1^* \cdot s_0^*)v = s_1^*(s_0^* v) \text{ for all } v \in D_{2s}.
\end{aligned}$$

According to (ii), there are D_0', $D_1' \in \mathcal{D}$ such that $D_0' \subseteq r_1^{-1}(D_0)$ and $D_1' \subseteq (r_0^*)^{-1}(D_1)$. In particular, $r_1 D_0' \subseteq D_0$ and $r_0^* D_1' \subseteq D_1$. Since $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that

$$D \subseteq D_0 \cap D_1 \cap D_0' \cap D_1' \cap \bigcap_{i<3} D_{ir} \cap \bigcap_{i<3} D_{is}.$$

Then one gets for any $v \in D$:

$$\begin{aligned}
&(r_0 \cdot r_1)v = r_0(r_1 v) = s_0(r_1 v) = s_0(s_1 v) = (s_0 \cdot s_1)v; \\
&tv = (r_1^* \cdot r_0^*)v = r_1^*(r_0^* v) = s_1^*(r_0^* v) = s_1^*(s_0^* v) = (s_1^* \cdot s_0^*)v = uv.
\end{aligned}$$

Thus D witnesses $r_0 \cdot r_1 \approx_{\mathcal{D}} s_0 \cdot s_1$ under the proviso $r_0 \cdot r_1 \bowtie t$ and $s_0 \cdot s_1 \bowtie u$.

Finally, suppose that $r \bowtie r^*$, $s \bowtie s^*$, and $r \approx_{\mathcal{D}} s$. To prove compatibility with $\bowtie$, let $t \bowtie r^*$ and $u \bowtie s^*$. Then $r \approx_{\mathcal{D}} t$ and $s \approx_{\mathcal{D}} u$ by (viii). As $\approx_{\mathcal{D}}$ is transitive, we conclude that $t \approx_{\mathcal{D}} u$. Thus there is $D \in \mathcal{D}$ which witnesses $t \approx_{\mathcal{D}} u$ under the proviso $t \bowtie r^*$ and $u \bowtie s^*$. Then D witnesses $r^* \approx_{\mathcal{D}} s^*$ under the same proviso. $\square$

It will be shown in Theorem 22 that the factor structure $R/{\approx_{\mathcal{D}}}$ is always a $*$-Λ-algebra.

Fact 21. *If ${}_R V_F$ is a representation of the $*$-Λ-algebra R, then $({}_R V_F; \{V\})$ is a Λ-algebra of generalized operators where $r \bowtie s$ iff $s = r^*$. Here $\approx_{\mathcal{D}}$ is the equality relation.*

Theorem 22. *Let $({}_R V_F; \mathcal{D})$ be a Λ-algebra of generalized operators on an inner product space V_F. Then $R/{\approx_{\mathcal{D}}}$ is a $*$-Λ-algebra. Moreover, if $R/{\approx_{\mathcal{D}}}$ is $*$-regular, then it admits a faithful representation within a closed subspace of some ultrapower of V_F.*

Remark 23. The proof of Theorem 22 is quite similar to that of Theorem 16, and one might conjecture that $R/{\approx_{\mathcal{D}}}$ is a homomorphic image of a subalgebra of an ultraproduct of endomorphism algebras of finite dimensional subspaces of V_F. A difficulty in establishing this comes from the fact that no ultrafilter 'compatible' with $\mathcal{D}$ is at hand. The conjecture can be verified in the case of $F = \mathbb{C}$ (cf. proof of

Theorem 2) but is doubtable in general. Nonetheless, the proof of Theorem 16 (and Tyukavkin's ideas behind) give some intuition for the following proof.

Proof of Theorem 22. Let $T = \mathsf{End}^*_{\Lambda f}(V_F)$; then besides the action of R on V we have also the left action of T on V. The resulting 4-sorted structure is denoted by by ${}_{T,R}V_F$. In particular, T is a $*$-regular algebra by Fact 7 and ${}_TV_F$ is a faithful representation. According to Fact 15, ${}_{T,R}V_F$ admits a modestly saturated elementary extension ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$. By Fact 14, $\hat{T}$ is a $*$-regular algebra and ${}_{\hat{T}}\hat{V}_{\hat{F}}$ is a faithful representation. We put for $\sigma \in \hat{T}$ and $r \in R$

$\sigma \sim r$ iff for any $r^* \bowtie r$ in R, there is $D \in \mathcal{D}$, $D \subseteq \mathsf{dom}\, r \cap \mathsf{dom}\, r^*$ such that $\sigma v = rv$ and $\sigma^* v = r^* v$ for all $v \in D$.

In this case, we say that D *witnesses* $\sigma \sim r$ *under the proviso* $r \bowtie r^*$. We put also

$$S = \{\sigma \in \hat{T} \mid \sigma \sim r \text{ for some } r \in R\}.$$

Recall that relation $\approx_{\mathcal{D}}$ on R, defined for an algebra of generalized operators, is a congruence relation by Lemma 20. Put $[r] = \{s \in R \mid s \approx_{\mathcal{D}} r\}$.

Claim 1. *The map $g\colon S \to R/{\approx_{\mathcal{D}}}$, $g(\sigma) = [r]$, where $\sigma \sim r$, is well defined.*

Proof of Claim. Let D_r and D_s witness $\sigma \sim r$ and $\sigma \sim s$ under the proviso $r \bowtie r^*$ and $s \bowtie s^*$, respectively. Since $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that $D \subseteq D_r \cap D_s$. We get for all $v \in D$ that $rv = \sigma v = sv$ and $r^* v = \sigma^* v = s^* v$, whence $r \approx_{\mathcal{D}} s$. □

Claim 2. *In the language of $*$-Λ-algebras, S is a subalgebra of $\hat{T}$ and $g\colon S \to R/{\approx_{\mathcal{D}}}$ is a homomorphism.*

Proof of Claim. Assume $\sigma \sim r$, $\tau \sim s$. In view of Lemma 20, it suffices to show that

$$(1)\ \sigma^* \sim r^*, \quad (2)\ \lambda\sigma \sim \lambda r, \quad (3)\ \tau + \sigma \sim s + r, \quad (4)\ \tau\sigma \sim s \cdot r.$$

Let $D_r \in \mathcal{D}$ witness $\sigma \sim r$ under the proviso $r \bowtie r^*$. To prove relation (1), we consider any $t \in R$ such that $r^* \bowtie t$. By axiom (viii) we have $t \approx_{\mathcal{D}} r$; the latter is witnessed by some $D' \in \mathcal{D}$ under the proviso $t \bowtie r^*$ and $r \bowtie r^*$. As $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap D_r$. Then D witnesses $\sigma^* \sim r^*$ under the proviso $r^* \bowtie t$, as $\sigma^* v = r^* v$ and $(\sigma^*)^* v = \sigma v = rv = tv$ for all $v \in D$.

Concerning (2), we consider any t with $\lambda r \bowtie t$. By (c) we have $\lambda r \bowtie \lambda^* r^*$, whence $t \approx_{\mathcal{D}} \lambda^* r^*$ by (viii); the latter is witnessed by some $D_0 \in \mathcal{D}$. According to (vi), there is $D_1 \in \mathcal{D}$ such that $D_1 \subseteq \mathsf{dom}\, r \cap \mathsf{dom}(\lambda r)$ and $(\lambda r)v = (rv)\lambda$ for all $v \in D_1$. Since $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that $D \subseteq D_0 \cap D_1 \cap D_r$. One has for all $v \in D$:

$$(\lambda\sigma)v = (\sigma v)\lambda = (rv)\lambda = (\lambda r)v;$$
$$(\lambda\sigma)^* v = (\lambda^*\sigma^*)v = (\sigma^* v)\lambda^* = (r^* v)\lambda^* = (\lambda^* r^*)v = tv,$$

whence D witnesses relation (2) under the proviso $\lambda r \bowtie t$.

To prove (3) and (4), we assume that $D_s \in \mathcal{D}$ witnesses $\tau \sim s$ under the proviso $s \bowtie s^*$. Let $s + r \bowtie t$ for some $t \in R$. According to (a), we have also $s + r \bowtie s^* + r^*$, whence $s^* + r^* \approx_{\mathcal{D}} t$ by (viii). Let D' witness the latter under the proviso $t \bowtie s + r$ and $s^* + r^* \bowtie s + r$. According to (iii), there are D_0, $D_1 \in \mathcal{D}$ such that $D_0 \subseteq \mathsf{dom}\, s \cap \mathsf{dom}\, r \cap \mathsf{dom}(s + r)$, $D_1 \subseteq \mathsf{dom}\, s^* \cap \mathsf{dom}\, r^* \cap \mathsf{dom}(s^* + r^*)$ and

$sv + rv = (s + r)v$ for all $v \in D_0$, $s^*v + r^*v = (s^* + r^*)v$ for all $v \in D_1$. Since $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap D_0 \cap D_1 \cap D_r \cap D_s$. Then one has for all $v \in D$:

$$(\tau + \sigma)v = \tau v + \sigma v = sv + rv = (s + r)v;$$
$$(\tau + \sigma)^*v = (\tau^* + \sigma^*)v = \tau^*v + \sigma^*v = s^*v + r^*v = (s^* + r^*)v = tv,$$

whence D witnesses relation (3) under the proviso $s + r \bowtie t$.

Let $s \cdot r \bowtie t$ for some $t \in R$. According to (b), we have also $s \cdot r \bowtie r^* \cdot s^*$, whence $r^* \cdot s^* \approx_{\mathcal{D}} t$ by (viii). Let D' witness the latter under the proviso $t \bowtie s \cdot r$ and $r^* \cdot s^* \bowtie s \cdot r$. According to (v), there are D_0, $D_1 \in \mathcal{D}$ such that

$$D_0 \subseteq \mathsf{dom}(s \cdot r) \cap r^{-1}(\mathsf{dom}\, s) \cap \mathsf{dom}\, r;$$
$$D_1 \subseteq \mathsf{dom}(r^* \cdot s^*) \cap (s^*)^{-1}(\mathsf{dom}\, r^*) \cap \mathsf{dom}\, s^*$$

and

$$(s \cdot r)v = s(rv) \quad \text{for all } v \in D_0;$$
$$(r^* \cdot s^*)v = r^*(s^*v) \quad \text{for all } v \in D_1.$$

Since $\mathcal{D}$ is directed, there is $D' \in \mathcal{D}$ such that $D' \subseteq D_r \cap D_s \cap D_0 \cap D_1$. Moreover, according to (ii), there is $D \in \mathcal{D}$ such that $D \subseteq D' \cap r^{-1}(D') \cap (s^*)^{-1}(D')$. Thus $D \subseteq D_r$, while $rD \subseteq D_s$. Similarly, $D \subseteq D_s$, while $s^*D \subseteq D_r$.

Therefore, one has for all $v \in D$:

$$(\tau\sigma)v = \tau(\sigma v) = \tau(rv) = s(rv) = (s \cdot r)v;$$
$$(\tau\sigma)^*v = (\sigma^*\tau^*)v = \sigma^*(\tau^*v) = \sigma^*(s^*v) = r^*(s^*v) = (r^* \cdot s^*)v = tv,$$

whence D witnesses relation (4) under the proviso $s \cdot r \bowtie t$.

Of course, $0 \sim 0_R$ and $1 \sim 1_R$ by (iv). □

Claim 3. *The map g is surjective.*

Proof of Claim. Given $r \bowtie r^*$ in R, by (vii) there is $D \in \mathcal{D}$ such that $D \subseteq \mathsf{dom}\, r \cap \mathsf{dom}\, r^*$ and $\langle x \mid r^*y\rangle = \langle rx \mid y\rangle$ for all x, $y \in D$. We show that there is $\sigma \in \hat{T}$ such that $\sigma v = rv$ and $\sigma^*v = r^*v$ for all $v \in D$. Let $v_1, \dots, v_n \in D$ and let U be the subspace of V_F spanned by $v_1, \dots, v_n$. Consider the finite dimensional subspace $W = U + rU + r^*U$ of V and F-linear maps

$$\varphi_0 \colon U \to W, \quad \varphi_0 v = rv, \quad \text{and} \quad \psi_0 \colon U \to W, \quad \psi_0 v = r^*v.$$

In particular, $\langle x \mid \psi_0 y\rangle = \langle \varphi_0 x \mid y\rangle$ for all x, $y \in U$. Choose a basis $u_1, \dots, u_k$ of U and extend it to a basis $u_1, \dots, u_m$ of W. There are unique φ, $\psi \in \mathsf{End}^*_\Lambda(W_F)$ such that

$$\langle \varphi u_i \mid u_j\rangle = \langle u_i \mid \psi u_j\rangle = \begin{cases} \langle u_i \mid \psi_0 u_j\rangle & \text{for } j \leqslant k; \\ \langle \varphi_0 u_i \mid u_j\rangle & \text{for } i \leqslant k; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\psi = \varphi^*$, $\varphi|_U = \varphi_0$, and $\psi^*|_U = \psi_0$. Considering the orthogonal projection $\rho = \pi_W$, we get that $\rho \in T$. Moreover, one has

$$\sigma = \rho\varphi\rho \in T, \quad \sigma^* = \rho\psi\rho \in T; \quad \sigma v = \varphi_0 v = rv, \quad \sigma^* v = \psi_0 v = r^*v$$

for all $v \in U$. Now, consider the set of formulas

$$\Sigma(\xi) = \{(\xi \underline{v} = \underline{r}\, \underline{v}) \ \&\ (\xi^* \underline{v} = \underline{r}^*\, \underline{v}) \mid v \in D\},$$

where ξ is a variable of sort T. If $\Psi(\xi) \subseteq \Sigma(\xi)$ is finite, then only finitely many parameters $\underline{v}_i$, $v_i \in D$, occur in $\Psi(\xi)$ and, as shown above, there is $\sigma \in T$ such that $\Psi(\sigma)$ holds in ${}_R V_F$. Since ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$ is modestly saturated over ${}_{T,R}V_F$, we conclude that there is $\sigma \in \hat{T}$ such that $\Sigma(\sigma)$ holds in ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$, i.e. $\sigma v = rv$ and $\sigma^* v = r^* v$ for all $v \in D$. This proves that $\sigma \sim r$. □

Claim 4. *For any $v \in V$, there is a unique projection $\hat{\pi}_v \in \hat{T}$ such that $\hat{\pi}_v v = v$ and such that for any $w \in \hat{V}$, there is $\lambda \in \hat{F}$ such that $\hat{\pi}_v w = \lambda v$.*

Proof of Claim. There is a unique projection $\pi_v \in T$ such that $\pi_v v = v$ and $\mathsf{im}\, \pi_v$ is the subspace spanned by v; namely the orthogonal projection onto the subspace spanned by v. The claim follows from the fact that ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$ is an elementary extension of ${}_{T,R}V_F$. □

Claim 5. *$\sigma \in \ker g$ if and only if for any $t \bowtie 0_R$, there is $D \in \mathcal{D}$ such that $\sigma \hat{\pi}_v = 0 = \sigma^* \hat{\pi}_v$ for all $v \in D$, where $\hat{\pi}_v$ is as in Claim 4.*

Proof of Claim. Assume that $\sigma \in \ker g$ and $t \bowtie 0_R$. Then $\sigma \sim 0_R$ is witnessed by some $D_0 \in \mathcal{D}$ under the proviso $0_R \bowtie t$. On the other hand, we have by (d) and (viii) that $t \approx_{\mathcal{D}} 0_R$. Let the latter be witnessed by $D_1 \in \mathcal{D}$ under the proviso $0_R \bowtie 0_R$ and $t \bowtie 0_R$. Then there is $D \in \mathcal{D}$ such that $D \subseteq D_0 \cap D_1$ and

$$(*) \qquad \sigma v = 0_R v = tv = \sigma^* v \text{ for all } v \in D.$$

Since $0_R v = 0$, $(*)$ is equivalent to

$$(**) \qquad \sigma \hat{\pi}_v = 0 = \sigma^* \hat{\pi}_v \text{ for all } v \in D.$$

Conversely, consider any $t \bowtie 0_R$ and assume that $(**)$ holds for some $D \in \mathcal{D}$. By (d) and (viii), $t \approx_{\mathcal{D}} 0_R$ is witnessed by some $D_0 \in \mathcal{D}$. Then $\sigma \sim 0_R$ is witnessed by any $D' \in \mathcal{D}$ such that $D' \subseteq D \cap D_0$ under the proviso $t \bowtie 0_R$. □

Claim 6. *$\mathsf{ker}\, g$ is regular.*

Proof of Claim. Since T is $*$-regular so is $\hat{T}$. Thus any $\sigma \in \mathsf{ker}\, g$ has a pseudo-inverse σ^+ in $\hat{T}$. Now, in view of Claim 5 and Fact 9, $\sigma \in \mathsf{ker}\, g$ implies $\sigma^+ \in \mathsf{ker}\, g$. □

We now prove the statements of the theorem. The first one follows from Claims 2 and 3. If $\mathsf{im}\, g = R/{\approx_{\mathcal{D}}}$ is $*$-regular, then $*$-regularity of S follows from Lemma 6. Moreover, since T is faithfully represented in V_F, $\hat{T}$ is faithfully represented in $\hat{V}_{\hat{F}}$ by Fact 14. Therefore, its substructure S is also faithfully represented in $\hat{V}_{\hat{F}}$, and faithful representability of its homomorphic image $R/{\approx_{\mathcal{D}}}$ within a closed subspace U of an ultrapower $\tilde{V}_{\tilde{F}}$ of $\hat{V}_{\hat{F}}$ follows with Proposition 19. Finally, observe that the modestly saturated extension ${}_{\hat{T},\hat{R}}\hat{V}_{\hat{F}}$ can be chosen isomorphic to an ultrapower of ${}_{T,R}V_F$ by Fact 15. In particular, $\tilde{V}_{\tilde{F}}$ is isomorphic to an ultrapower of V_F by Fact 13(i). Composing the representation of $R/{\approx_{\mathcal{D}}}$ in $U_{\tilde{F}}$ with this isomorphism, we get a faithful representation of $R/{\approx_{\mathcal{D}}}$ in a closed subspace of an ultrapower of V_F. □

6. Rings of quotients

We refer to Rowen [21, Chapter 3] for rings of quotients – except that we consider right quotients. Let A be a Λ-algebra, let $\mathcal{I}_r(A)$ denote the set of all right ideals I of A, and let $\mathsf{Hom}(I, A)$ denote the set of all linear maps $f\colon I_A \to A_A$. A right ideal $I \in \mathcal{I}_r(A)$ is *dense* (in A) if for any $J \supseteq I$ in $\mathcal{I}_r(A)$ and any $f \in \mathsf{Hom}(J, A)$, one has $f = 0$ provided that $f|I = 0$.

A subset $\mathcal{E}$ of $\mathcal{I}_r(A)$ is a *set of supports* (for A) if the following hold:

(i) Every member of $\mathcal{E}$ is dense (in A);
(ii) $A \in \mathcal{E}$ and $I \cap J \in \mathcal{E}$ for any $I,\ J \in \mathcal{E}$;
(iii) $f^{-1}(J) \in \mathcal{E}$ for any $I,\ J \in \mathcal{E}$ and any $f \in \mathsf{Hom}(I, A)$.

Observe that (iii) applies in particular to the left multiplication $l_a\colon A \to A$, $l_a(x) = ax$, for any $a \in A$. This includes the special case $a = \lambda 1_A$, where $l_{\lambda 1_A}\colon x \mapsto \lambda x$. Observe that any $I \in \mathcal{I}_r$ is invariant under λ and put $\lambda f = l_{\lambda 1_A} \circ f = f \circ (l_{\lambda 1_A}|I)$ for $f \in \mathsf{Hom}(I, A)$. The following is well known.

Lemma 24. *The set $\mathcal{E}_0$ of all dense right ideals of A is a set of supports for A.*

Given a set $\mathcal{E}$ of supports, define the *algebra* $R(A, \mathcal{E})$ of *abstract quotients over* $\mathcal{E}$ as follows:
$$R(A, \mathcal{E}) = \big\{(f, I) \mid I \in \mathcal{E},\ f \in \mathsf{Hom}(I, A)\big\}.$$
Endow $R(A, \mathcal{E})$ with the operations of a pre-Λ-algebra:
$$\begin{aligned}
&(f, I) + (g, J) = (f|K + g|K,\ K), && \text{where } K = I \cap J;\\
&\lambda(f, I) = \big((\lambda f)|K,\ K\big), && \text{where } K = \lambda^{-1}(I);\\
&(f, I) \cdot (g, J) = \big((f \circ g)|K,\ K\big), && \text{where } K = g^{-1}(I)\\
&0_R = (0, A), \quad 1_R = (\mathsf{id}_A, A)
\end{aligned}$$
Define the binary relation $\equiv_{\mathcal{E}}$ on $R(A, \mathcal{E})$ by
$$(f, I) \equiv_{\mathcal{E}} (g, J) \quad \text{if and only if} \quad f|K = g|K \text{ for some } K \in \mathcal{E} \text{ with } K \subseteq I \cap J.$$
The following facts are either well known or easy to prove.

Proposition 25. *Let A be a Λ-algebra and let $\mathcal{E}$ be a set of supports for A.*

(i) *$(f, I) \equiv_{\mathcal{E}} (g, J)$ if and only if $f|(I \cap J) = g|(I \cap J)$.*
(ii) *$\equiv_{\mathcal{E}}$ is a congruence relation on $R(A, \mathcal{E})$; we denote the factor structure by $Q(A, \mathcal{E})$ and the canonical homomorphism by $\pi_{\mathcal{E}}$.*
(iii) *The map $\omega\colon A \to R(A, \mathcal{E})$, $\omega(a) = (l_a, A)$, is a Λ-algebra embedding; $\equiv_{\mathcal{E}}$ restricts to identity on $\omega(A)$. In particular, $\pi_{\mathcal{E}} \circ \omega$ is a Λ-algebra embedding.*
(iv) *$(f, I) \cdot (l_a, A) \in \omega(A)$ if and only if $a \in I$; in this case, $(f, I) \cdot (l_a, A) = (l_{f(a)}, A)$.*
(v) *$R(A, \mathcal{E})$ is a subalgebra of $R(A, \mathcal{E}_0)$ and $\equiv_{\mathcal{E}}$ is the restriction of $\equiv_{\mathcal{E}_0}$.*
(vi) *$Q(A, \mathcal{E}_0) = Q_{\max}(A)$, the maximal ring of right quotients of A.*
(vii) *$\pi_{\mathcal{E}} \circ \omega$ embeds A into $Q_{\max}(A)$.*

We recall that the set of projections of a $*$-ring is ordered by
$$e \leq e' \quad \text{if and only if} \quad e'e = e \quad \text{if and only if} \quad ee' = e.$$

Theorem 26. *Let A be a $*$-Λ-algebra, let $\mathcal{E}$ be a set of supports for A, and let $R = R(A, \mathcal{E})$ be such that*

(a) *For any $I \in \mathcal{E}$, there is a directed set P_I of projections in I such that $P_I A = \bigcup_{e \in P_I} eA$ is a directed union and $P_I A \in \mathcal{E}$.*
(b) *The involution of A extends to $Q(A, \mathcal{E})$.*
(c) *There is a faithful representation ε of A in the inner product space V_F.*

Then there is an algebra $({}_R V_F; \mathcal{D})$ of generalized operators such that $R/{\approx_{\mathcal{D}}}$ and $Q(A, \mathcal{E})$ are isomorphic as $*$-Λ-algebras.

Proof. To any $I \in \mathcal{E}$, we assign a certain directed set P_I of projections which witnesses (a). We define the action of R on V_F. We put

$$\mathsf{dom}(f, I) = D(I) = \bigcup_{e \in P_I} \mathsf{im}\, \varepsilon(e);$$

$$(f, I)v = \varepsilon(f(e))(v) \quad \text{if } v \in \mathsf{im}\, \varepsilon(e),\ e \in P_I.$$

Observe that $\varepsilon(e)(v) = v$ for any $v \in \mathsf{im}\, \varepsilon(e)$ since e is a projection.

Claim 1. *For any $(f, I) \in R$, the action of (f, I) is well defined.*

Proof of Claim. Assume that $v \in \mathsf{im}\, \varepsilon(e) \cap \mathsf{im}\, \varepsilon(e')$ for some e, $e' \in P_I$. Since P_I is directed, there is $e'' \in P_I$ such that $e''e = e$ and $e''e' = e'$. Then $\mathsf{im}\, \varepsilon(e)$, $\mathsf{im}\, \varepsilon(e') \subseteq \mathsf{im}\, \varepsilon(e'')$ and

$$\varepsilon\big(f(e)\big)(v) = \varepsilon\big(f(e''e)\big)(v) = \varepsilon\big(f(e'')e\big)(v) = \Big(\varepsilon\big(f(e')\big) \circ \varepsilon(e)\Big)(v) =$$
$$= \varepsilon\big(f(e'')\big)\big(\varepsilon(e)(v)\big) = \varepsilon\big(f(e'')\big)(v).$$

Similarly, $\varepsilon\big(f(e')\big)(v) = \varepsilon\big(f(e'')\big)(v)$, whence $\varepsilon\big(f(e)\big)(v) = \varepsilon\big(f(e')\big)(v)$. □

Claim 2. *For any $I \in \mathcal{E}$, $D(I)$ is an F-linear subspace of V_F and*

$$(f, I)\colon\ \mathsf{dom}(f, I) \to V$$

is an F-linear map for any $(f, I) \in R$.

Proof of Claim. Let u, $v \in D(I)$, and let $\lambda \in F$. As P_I is directed, there is $e \in P_I$ such that u, $v \in \mathsf{im}\, \varepsilon(e)$. As $\mathsf{im}\, \varepsilon(e)$ is a subspace of V, $u + v$, $u\lambda \in \mathsf{im}\varepsilon(e)$. Using the fact that ε is a representation, we get:

$$(f, I)(u + v) = \varepsilon\big(f(e)\big)(u + v) = \varepsilon\big(f(e)\big)(u) + \varepsilon\big(f(e)\big)(v) = (f, I)u + (f, I)v;$$
$$(f, I)(u\lambda) = \Big(\varepsilon\big(f(e)\big)(u)\Big)\lambda = \big((f, I)u\big)\lambda.$$

□

For (f, I), $(g, J) \in R$, we put

$$(f, I) \bowtie (g, J) \quad \text{if and only if} \quad \pi_{\mathcal{E}}(g, J) = \big(\pi_{\mathcal{E}}(f, I)\big)^*.$$

Since $\pi_{\mathcal{E}}$ is a pre-Λ-algebra homomorphism and since * is an involution on $Q(A, \mathcal{E})$, (a)-(c) of the definition of a pre-$*$-Λ-algebra hold. Item (d) holds obviously. Let

$$\mathcal{D} = \{D(I) \mid I \in \mathcal{E}\}.$$

Claim 3. *For any (f,I), $(g,J)\in R$, $(f,I)\approx_{\mathcal{D}}(g,J)$ if and only if $(f,I)\equiv_{\mathcal{E}}(g,J)$.*

Proof of Claim. Assume that $(f,I)\approx_{\mathcal{D}}(g,J)$ is witnessed by $D(K)$ for some $K\in\mathcal{E}$ under some proviso. As $K\cap I\cap J\in\mathcal{E}$, we may assume such that $K\subseteq I\cap J$. Then for each $e\in P_K$ and any $v\in\operatorname{im}\varepsilon(e)$, one has

$$\varepsilon\big(f(e)\big)(v)=(f,I)v=(g,J)v=\varepsilon\big(g(e)\big)(v).$$

This implies that for any $u\in V$,

$$\begin{aligned}\varepsilon\big(f(e)\big)(u)=&\varepsilon\big(f(e^2)\big)(u)=\varepsilon\big(f(e)e\big)(u)=\varepsilon\big(f(e)\big)\big(\varepsilon(e)(u)\big)=\\ =&\varepsilon\big(g(e)\big)\big(\varepsilon(e)(u)\big)=\varepsilon\big(g(e)e\big)(u)=\varepsilon\big(g(e^2)\big)(u)=\varepsilon\big(g(e)\big)(u),\end{aligned}$$

whence $f(e)=g(e)$, as ε is a faithful representation. Then for any $a\in eA$, we have

$$f(a)=f(ea)=f(e)a=g(e)a=g(ea)=g(a).$$

According to (a), $P_KA=\bigcup_{e\in p(K)}eA$ is a dense right ideal. As $f|P_KA=g|P_KA$, we conclude that $f|K=g|K$ which proves that $(f,I)\equiv_{\mathcal{E}}(g,J)$.

Conversely, assume $\pi_{\mathcal{E}}(f,I)=\pi_{\mathcal{E}}(g,J)$ and consider any (h_0,K_0), $(h_1,K_1)\in R$ such that $(h_0,K_0)\bowtie(f,I)$ and $(h_1,K_1)\bowtie(g,J)$ in R. By definition, the latter means that

$$\pi_{\mathcal{E}}(h_0,K_0)=\big(\pi_{\mathcal{E}}(f,I)\big)^*=\big(\pi_{\mathcal{E}}(g,J)\big)^*=\pi_{\mathcal{E}}(h_1,K_1).$$

Therefore, for $K\in\mathcal{E}$ such that $K\subseteq I\cap J\cap K_0\cap K_1$, we get $f|K=g|K$ and $h_0|K=h_1|K$. Then $D(K)$ witnesses $(f,I)\approx_{\mathcal{D}}(g,J)$ under the given proviso. $\square$

Claim 4. $({}_RV_F;\approx_{\mathcal{D}})$ *is an algebra of generalized operators.*

Proof of Claim. For any $(f,I)\in R$, $D(I)=\mathsf{dom}(f,I)$, whence (i) follows. If $(f,I)\in R$ and $D(J)\in\mathcal{D}$ for some $J\in\mathcal{E}$, then $K=f^{-1}(P_JA)\in\mathcal{E}$ by (a) and by (iii) of the definition of a set of supports for A. Let $v\in\operatorname{im}\varepsilon(e)$ for some $e\in P_K$. As $K\subseteq I$, we get $f(e)\in P_JA$. Thus $f(e)=e'f(e)$ for some $e'\in P_J$ by (a). Therefore,

$$(f,I)v=\varepsilon\big(f(e)\big)(v)=\varepsilon\big(e'f(e)\big)(v)=\varepsilon(e')\Big(\varepsilon\big(f(e)\big)(v)\Big)\in\operatorname{im}\varepsilon(e')\subseteq D(J).$$

Hence $(f,I)D(K)\subseteq D(J)$ and (ii) holds.

Suppose that (f,I), $(g,J)\in R$ and $K=I\cap J$. Then $K\in\mathcal{E}$ and $D(K)\subseteq\mathsf{dom}(f,I)\cap\mathsf{dom}(g,J)\cap\mathsf{dom}\big((f,I)+(g,J)\big)$. Moreover, for any $e\in P_K$ and any $v\in\operatorname{im}\varepsilon(e)$, we have:

$$\begin{aligned}&\big((f,I)+(g,J)\big)v=\varepsilon\big((f+g)(e)\big)(v)=\varepsilon\big(f(e)+g(e)\big)(v)=\\ &=\Big(\varepsilon\big(f(e)\big)+\varepsilon\big(g(e)\big)\Big)(v)=\varepsilon\big(f(e)\big)(v)+\varepsilon\big(g(e)\big)(v)=(f,I)v+(g,J)v.\end{aligned}$$

Therefore, (iii) holds.

Consider (f,I), $(g,J)\in R$. According to the proof of (ii), $K=g^{-1}(P_IA)\in\mathcal{E}$ and $(g,J)D(K)\subseteq D(I)$. Thus $D(K)\subseteq\mathsf{dom}(g,J)\cap(g,J)^{-1}\big(\mathsf{dom}(f,I)\big)=\mathsf{dom}\big((f,I)\cdot$

$(g, J)\big)$. Moreover by (a), for any $e \in P_K$ and any $v \in \operatorname{im} \varepsilon(e)$, there is $e' \in P_I$ such that $g(e) = e'g(e)$. Therefore, we have for any $v \in \operatorname{im} \varepsilon(e)$:

$$\begin{aligned}\big((f, I) \cdot (g, J)\big)v &= \varepsilon\big((f \circ g)(e)\big)(v) = \varepsilon\big(f\big(g(e)\big)\big)(v) = \varepsilon\big(f\big(e'g(e)\big)\big)(v) = \\ &= \varepsilon\big(f(e')g(e)\big)(v) = \varepsilon\big(f(e')\big)\varepsilon\big(g(e)\big)(v) = (f, I)\big((g, J)v\big),\end{aligned}$$

whence (v) also holds. It is clear that (iv) also holds.

To prove (vi), let $\lambda \in \Lambda$ and let $K = \lambda^{-1}(I) \cap I$. Then $K \in \mathcal{E}$ by (ii)-(iii) of the definition of a set of supports for A. For any $e \in P_K$, we have $\lambda e = \lambda e^2 = e(\lambda e)$. Therefore, we have for any $v \in \operatorname{im} \varepsilon(e)$:

$$\begin{aligned}\big(\lambda(f, I)\big)v &= \varepsilon\big((\lambda f)(e)\big)(v) = \varepsilon\big(f(\lambda e)\big)(v) = \varepsilon\big(f(e \cdot \lambda e)\big)(v) = \\ &= \varepsilon\big(f(e) \cdot \lambda e\big)(v) = \varepsilon\big(f(e)\big)\varepsilon(\lambda e)(v) = \varepsilon\big(f(e)\big)\big(\varepsilon(e)(v)\lambda\big) = \\ &= \varepsilon\big(f(e)\big)(v\lambda) = \varepsilon\big(f(e)\big)(v)\lambda = \big((f, I)v\big)\lambda,\end{aligned}$$

whence (vi) follows.

To prove (vii), we assume that $(f, I) \bowtie (g, J)$. According to Proposition 25(iv), $\omega(f(e)) = (f, I) \cdot \omega(e)$ for any $e \in P_I$. By hypothesis (b), $\pi_{\mathcal{E}}\omega$ is a $*$-homomorphism, whence

$$\begin{aligned}\pi_{\mathcal{E}}\omega\big(f(e)^*\big) &= \pi_{\mathcal{E}}\big(\omega(f(e))\big)^* = \big(\pi_{\mathcal{E}}(f, I) \cdot \pi_{\mathcal{E}}\omega(e)\big)^* = \pi_{\mathcal{E}}\omega(e)^* \cdot \big(\pi_{\mathcal{E}}(f, I)\big)^* = \\ &= \pi_{\mathcal{E}}\omega(e) \cdot \pi_{\mathcal{E}}(g, J) = \pi_{\mathcal{E}}\big(\omega(e) \cdot (g, J)\big) = \pi_{\mathcal{E}}(l_e \circ g, J).\end{aligned}$$

Thus there is $K_0 \in \mathcal{E}$ such that $K_0 \subseteq J$ and for all $a \in K_0$, one has

$$e \cdot g(a) = (l_e \circ g)(a) = l_{f(e)^*}(a) = f(e)^* \cdot a.$$

Let $K = K_0 \cap I \cap J$. Then $K \in \mathcal{E}$ and $D(K) \subseteq D(I) \cap D(J) = \operatorname{dom}(f, I) \cap \operatorname{dom}(g, J)$. Let u, $v \in D(K)$. Then according to (a) there is $e \in P_K$ such that u, $v \in \operatorname{im} \varepsilon(e)$. Since ε is a representation, we have using the above:

$$\begin{aligned}\langle (f, I)u \mid v\rangle &= \langle \varepsilon\big(f(e)\big)(u) \mid v\rangle = \langle u \mid \varepsilon\big(f(e)\big)^*(v)\rangle = \langle u \mid \varepsilon\big(f(e)^*\big)(v)\rangle = \\ &= \langle u \mid \varepsilon\big(f(e)^*\big)\varepsilon(e)(v)\rangle = \langle u \mid \varepsilon\big(f(e)^* \cdot e\big)(v)\rangle = \langle u \mid \varepsilon\big(e \cdot g(e)\big)(v)\rangle = \\ &= \langle u \mid \varepsilon(e)\varepsilon\big(g(e)\big)(v)\rangle = \langle u \mid \varepsilon(e^*)\varepsilon\big(g(e)\big)(v)\rangle = \\ &= \langle \varepsilon(e)(u) \mid \varepsilon\big(g(e)\big)(v)\rangle = \langle u \mid \varepsilon\big(g(e)\big)(v)\rangle = \langle u \mid (g, J)v\rangle.\end{aligned}$$

Therefore, (vii) holds.

If $(g_0, J_0) \bowtie (f, I) \bowtie (g_1, J_1)$, then $\pi_{\mathcal{E}}(g_0, J_0) = \big(\pi_{\mathcal{E}}(f, I)\big)^* = \pi_{\mathcal{E}}(g_1, J_1)$, whence $(g_0, J_0) \equiv_{\mathcal{E}} (g_1, J_1)$ and (viii) follows from Claim 3. □

The above claims prove that $({}_RV_F; \mathcal{D})$ is an algebra of generalized operators. According to Claim 3, $R/{\approx_{\mathcal{D}}} \cong Q(A, \mathcal{E})$. □

7. Proof of main results

For the following concepts, cf. [22] and [10]. Let A be a ring. We put for any $X \subseteq A$:

$$\mathsf{Ann}_r(X) = \{a \in A \mid Xa = 0\};$$
$$\mathsf{Ann}_l(X) = \{a \in A \mid aX = 0\}$$

and call those sets the *right* and the *left annihilator of* X, respectively. A $*$-ring A is *Baer* [*Rickart*] if for any [singleton] subset X of A, $\mathsf{Ann}_r(X) = eA$ for some projection $e \in A$. In this case, the left annihilator of X is also generated by a projection. If, in addition, A is a C^*-algebra then one speaks of an AW^*-*algebra* [*Rickart C^*-algebra*, respectively]. According to [10, 14.22, 14.24], this definition of an AW^*-algebra is equivalent to the one given in [24]. Every von Neumann algebra is an AW^*-algebra. A $*$-ring A is *finite* (it is called $*$-finite in [10]) if $xx^* = 1$ implies $x^*x = 1$ for all $x \in A$. A has *sufficiently many projections*, if any proper right ideal contains a non-zero projection. A satisfies $LP \sim RP$ (notation $LP \sim^* PR$ is used in [10]) if for any $x \in A$ and projections $e, f \in A$ such that $\mathsf{Ann}_r(x) = (1-e)A$ and $\mathsf{Ann}_l(x) = A(1-f)$, there is $y \in A$ such that $e = yy^*$ and $f = y^*y$.

A right ideal I of a ring A is *essential* or *large* in $J \supseteq I$ if $I \cap K \neq 0$ for any $K \in \mathcal{I}_r(A)$ such that $0 \neq K \subseteq J$. Of course, if I is essential in A then any ideal $J \supseteq I$ is. The following statement is well known and straightforward to prove.

Lemma 27. *Let I, $J \in \mathcal{I}_r(A)$ be such that $I \subseteq J$ and J is essential in A. Then I is essential in J if and only if I is essential in A.*

Recall that A is *non-singular* if $\mathsf{Ann}_r(x)$ is essential if and only if $x = 0$. Moreover, for any non-singular ring the notions of 'dense in A' and 'essential in A' right ideals coincide. Any Rickart $*$-ring is obviously non-singular.

Proposition 28. *Assume that A and $Q(A)$ are as in one of the following:*

(i) *A is a finite Rickart C^*-algebra and $Q(A)$ is its classical ring of right quotients;*
(ii) *A is a $*$-Λ-algebra which is a finite Baer $*$-ring satisfying $LP \sim RP$ and having sufficiently many projections and $Q(A)$ is its maximal ring of right quotients.*

Then there is a set $\mathcal{E}$ of supports such that $Q(A, \mathcal{E})$ is isomorphic to $Q(A)$ and such that (a) *and* (b) *of* Theorem 26 *are satisfied, turning $Q(A, \mathcal{E})$ into a $*$-regular Λ-algebra. Moreover, in both cases the involution on A extends uniquely to an involution on $Q(A)$; and, with this involution, $Q(A)$ is $*$-regular.*

Proof. Case (i). Following Handelman [7, p. 177], let $\mathcal{E}$ consist of all right ideals $I \in \mathcal{E}_0$ such that there is a countable set $X \subseteq I$ with $\sum_{x \in X} xA \in \mathcal{E}_0$. According to [7, Proposition 2.1] and Lemma 27, there is a countable orthogonal set $P \subseteq I$ of projections such that the ideal $J = \sum_{e \in P} eA$ is essential in A. Let

$$P_I = \{e_0 + \ldots + e_n \mid n < \omega,\ e_0, \ldots, e_n \in P\}.$$

Then P_I is a directed set of projections and $J = \bigcup_{e \in P_I} eA \in \mathcal{E}_0$. Thus hypothesis (a) of Theorem 26 holds. Moreover, (i)-(iii) of the definition of a set of supports hold by [7, Lemmas 2.3-2.4], and $Q(A,\mathcal{E})$ is regular according to [7, Lemma 2.7]. Furthermore, in [7, Theorem 2.1] a $*$-regular subring R of $Q_H(A) = Q(A,\mathcal{E})$ is constructed such that the involution on R extends the involution on A. Later, it was shown by Ara and Menal [8, p. 129] that $R = Q_H(A)$ is the classical ring of right quotients of A. The uniqueness of extension is obvious in this case.

Case (ii). Let $\mathcal{E} = \mathcal{E}_0$. According to Lemma 24, $\mathcal{E}_0$ is a set of supports. The proof of [4, Corollary 4.10] (cf. [3, Lemma 5]) together with [4, Proposition 4.11] yield that hypothesis (a) of Theorem 26 holds. According to Pyle [23] and Hafner [3, Theorem 2], $Q(A,\mathcal{E}_0)$ is $*$-regular with involution extending the involution of A. Uniqueness of extension of involution follows from [10, Corollaries 21.22 and 21.27], see also [9]. □

Corollary 29. *Let A and $Q(A)$ be as in* Proposition 28 *and let $\Lambda = F = \mathbb{C}$ in case* (i). *For any faithful representation ε of A within a Hilbert space $V_{\mathbb{C}}$, $Q(A)$ is isomorphic to the $*$-$\mathbb{C}$-algebra $R/{\approx_{\mathcal{D}}}$ for some $\mathbb{C}$-algebra $({}_RV_{\mathbb{C}};\mathcal{D})$ of generalized operators on $V_{\mathbb{C}}$.*

Proof of Theorem 2. By the GNS-construction (Fact 3), A has a faithful representation in some Hilbert space $V_{\mathbb{C}}$. By Corollary 29, $Q(A) \cong R/{\approx_{\mathcal{D}}}$ for some algebra $({}_RV_{\mathbb{C}};\mathcal{D})$ of generalized operators. Theorem 22 provides a faithful representation of the latter in some closed subspace U of an ultrapower $\hat{V}_{\hat{\mathbb{C}}} = V_{\mathbb{C}}^I/\mathcal{U}$. This proves (i).

Then by Theorem 16 we get that $Q(A)$ is a homomorphic image of a subalgebra of an ultraproduct $\prod_{k\in K} \mathsf{End}^*_{\Lambda}\big((U_k)_{\hat{\mathbb{C}}}\big)/\mathcal{W}$, where $\dim U_k = n_k < \omega$ for all $k \in K$. By Facts 8 and 13(ii), $\mathsf{End}^*_{\Lambda}\big((U_k)_{\hat{\mathbb{C}}}\big)$ is isomorphic to $(\mathbb{C}^{n_k\times n_k})^I/\mathcal{U}$ for any $k \in K$. All these algebraic constructions respect pseudo-inversion, whence (ii) follows. Furthermore, (ii) in turn implies (iii) due to Facts 11 and 12. □

We finally recall some facts concerning finite AW^*-algebras A. A $*$-regular extension $Q_B(A)$ was constructed by Berberian [2]. Hafner [3] and Pyle [4] showed that, as a ring, $Q_B(A)$ is the maximal ring of right quotients of A. Therefore, this subsumes under case (ii) of Proposition 28. Indeed, according to Berberian [10, 14.31], A satisfies $LP \sim RP$, see also [24, Theorem 5.2] and A has sufficiently many projections by [24, Lemma 2.2].

On the other hand, in [5, proof of Theorem 10] Berberian observed that, for a finite AW^*-algebra, his construction $Q_B(A)$ yields the the classical ring of right quotients of A. We outline a proof in the present framework. As remarked by Ara and Menal [8, p. 129], $Q_H(A)$ consists of all $x \in Q_M(A)$, the maximal ring of right quotients, such that there is an orthogonal sequence of projections e_k with $xe_k \in A$ for all k and $J = \sum_k e_kA$ essential in A. The elements x of $Q_B(A)$ (which is $Q_M(A)$ as a ring) are implemented by OWCs, which are sequences of the form (x_n, f_n) with $x_n \in A$ and $x_nf_m = x_mf_m$ and $x_n^*f_m = x_m^*f_m$ for all $m < n$ where the f_n form an SDD, an ascending chain of projections in A with join 1. Observe that there is an orthogonal sequence e_n of projections with joins $f_n = \sum_{k\leqslant n} e_k$, whence $x_ne_k = x_nf_ke_k = x_ke_k$ and $x^*e_k = x^*e_k$ if $n \geqslant k$. From (the proof of) [2, Theorem 2.1] it is immediate that $xe_k \in Q_B(A)$ is implemented by some OWC (x_ne_k, g_n) which is equivalent to the OWC (x_ke_k, h_n), where $h_n = 0$ for $n < k$ and $h_n = 1$ for $n \geqslant k$. Thus, $xe_k \in A$

for any k and, to derive $Q_B(A) \subseteq Q_H(A)$, it suffices to show that $J = \sum_k e_k A$ is essential in A. So consider a right ideal $K \neq 0$ of A. There is a non-zero projection $e \in K$ and continuity of $\mathbb{L}(A)$ yields $e = e \cap \bigvee_k e_k = e \cap \bigvee_k f_k = \bigvee_k (f_k \cap e)$, whence $e = f_k \cap e = ef_k \in J \cap K$ for some k.

8. Discussion

Evidently, our approach has some similarities with the method established by Elok and Szabó [25] for proving direct finiteness of the group ring $D(G)$ of a (sofic) group, where D is any division ring. The idea is to construct, by means of ultralimits, a pseudo-rank function N on the direct product E of endomorphisms rings of D-vector spaces generated by finite subsets of G and to embed $D(G)$ into the continuous regular ring $E/\mathsf{ker}N$.

More specifically, one may ask to which extent one could replace, in the special case of von Neumann algebras, the model theoretic ultraproducts by von Neumann algebra ultraproducts, cf. e.g. [26], and to gain some insight into the more serious problems concerning these. Though, the saturation property of model theoretic ultraproducts appears to be crucial for our approach.

There is a great variety of results on Baer $*$-rings satisfying certain conditions which imply $*$-regularity of maximal rings of quotients, see e.g. [5, 10, 3, 4, 9]; (i) in Proposition 28 is one of them. In contrast, results on representations of $*$-rings within inner product spaces appear to be located at two extremes: the GNS-construction on the 'continuous side' and the results on rings with minimal right ideals (cf. [27]) on the 'discrete side'. It would be desirable to have results based on a weaker (lattice theoretic) form of continuity.

In a subsequent work we will use the results of the present note for a detailed discussion of classes of $*$-algebras and modular ortholattices representable within inner product spaces over $*$-fields elementarily equivalent to $\mathbb{R}$ and $\mathbb{C}$, respectively, including solvability and complexity of certain decision problems for these classes. In particular, it will be shown that any $Q(A)$ as in Theorem 1 (as well as its projection ortholattice) has decidable equational theory. This is one more indication that the algebras $Q(A)$ are very special members of the class of all $*$-regular rings – recall that $Q(A)$ is directly finite since it has a unit regular extension (Handelman [9, 7]). But it remains open to which extent direct finiteness is inherited by homomorphic images of subalgebras of (model theoretic) ultraproducts of matrix $*$-algebras $\mathbb{C}^{n\times n}$.

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(Christian Herrmann) Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstr. 7, 64289 Darmstadt, Germany

E-mail address: herrmann@mathematik.tu-darmstadt.de

(Marina Semenova) Sobolev Institute of Mathematics, Siberian Branch RAS, Acad. Koptyug prosp. 4, 630090 Novosibirsk, Russia and Novosibirsk State University, Pirogova str. 2, 630090 Novosibirsk, Russia

E-mail address: udav17@gmail.com; semenova@math.nsc.ru