Dear Dr. van Hoksbergen,
this is the revison of both versions of the EM contribution on Arguesian lattices. The longer version gives full definition of the (to me) most important concepts not yet covered by the present edition. It might make sense to have these included as separate items.

Best regards
Christian Herrmann

Arguesian lattice, *Desarguesian lattice* - a lattice such that for all a_i, b_i $(a_0+b_0)(a_1+b_1)(a_2+b_2) \leq a_0(a_1+c)+b_0(b_1+c)$ where $c := c_0(c_1+c_2)$ and $c_i := (a_j+a_k)(b_j+b_k)$ - B.Jónsson [16]. A lattice is Arguesian iff its partial order dual is so [19] iff it is modular and $(a_0+b_0)(a_1+b_1) \leq a_2+b_2$ (*central perspectivity*) implies $c_2 \leq c_0+c_1$ (*axial perspectivity*). In an Arguesian lattice, for a_i, b_i such that $a_2 = (a_0+a_2)(a_1+a_2)$ and $b_2 = (b_0+b_2)(b_1+b_2)$ the converse implication is valid [18].

Examples of Arguesian lattices. 1) The lattice of subspaces of a projective space is Arguesian iff Desargues' Assumption is satisfied.

2) A complemented modular lattice is Arguesian iff it is embeddable into a direct product of lattices of vector subspaces.

3) Every lattice $L({}_RM)$ of submodules of an R-module M and any lattice of subobjects of an object in an abelian category.

4) Every lattice of normal subgroups (resp. congruence relations) of a group and any lattice of permuting equivalence relations [16] (also called *linear lattice*).

5) Considering all lattices of congruence relations of algebraic systems in a variety, the Arguesian law is equivalent to the modular law.

All of the above satisfy the *higher dimensional Arguesian laws*, too, [6]. Recursive axiomatizations exist for many of the associated quasi-varieties, e.g. $\mathcal{L}(R)$ of lattices embeddable into some $L({}_RM)$ [12] and linear lattices. But no such quasi-variety ($\supseteq$ some $\mathcal{L}(R)$) can be finitely axiomatized [6].

Yet, the modular law suffices for many purposes, e.g. the description of lattices as subspace lattices of *collinearity spaces on ordered point sets* [10], the theory of congruences and subdirect product decompositions [1,15], methods for computing finitely presented lattices [20], and *gluing* - which analyzes and constructs a modular lattice via a lattice ordered system of convex sublattices and adjunctions between them [5]. Here, the Arguesian law poses strong restrictions [11].

An *n-frame* is a system of generators and relations in a modular lattice mimicking a $n-1$-dimensional projective coordinate system. For $n \geq 4$ ($n \geq 3$ if L is Arguesian) it yields a *coordinate ring* R [2,3,4]. A complemented Arguesian lattice posessing a *large partial 3-frame* or being simple of dimension ≥ 3 is isomorphic to the lattice of principal right ideals of a regular R [17]. The $L({}_RM)$, ${}_RM$ of rank ≥ 3 over completely primary uniserial R have been characterized [18].

Each frame can be generated by 4 elements and a 5th suffices to interpret any finitely generated semi-group into the ring - within any $L({}_RM)$ of sufficiently large dimension [9]; this yields unsolvable 5-generator word problems in any modular quasi-variety containing some $\mathcal{L}(R)$ [13]. Identities in $\mathcal{L}(R)$ match to solvability of linear systems of equations with integer coefficients, whence one can solve the word problem for lattices freely generated by partially ordered sets in many of the $\mathcal{L}(R)$ [14] (not so for 4-generated free modular lattices [7]). Frames (with fixed characteristic of the coordinate ring) are, as systems of generators and relations, projective with respect to onto homomorphisms.

Finitely presented Arguesian lattices have been computed e.g. for frames, 4 generators forming two pairs of complements, and posets not containing disjoint unions $1+1+1+1$ nor $1+2+2$ of 1- resp. 2-element chains [11]. The variety generated by $\mathcal{L}(R)$, R a field, is generated by the finite dimensional $L({}_RM)$ [14]. This allows to compute finitely presented lattices and their subdirect factors using representation theory of partially ordered sets [8] combined with Jónsson's Lemma [1] (subdirectly irreducibles are obtained as homomorphic images of sublattices of ultraproducts of generating lattices) e.g. in the case of 4 generators. Part of the structure extends to Arguesian lattices [9].

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Arguesian lattice, *Desarguesian lattice* - a **lattice** in which the *Arguesian law* is valid, i.e. for all a_i, b_i

$$(a_0+b_0)(a_1+b_1)(a_2+b_2) \leq a_0(a_1+c)+b_0(b_1+c)$$

$c := c_0(c_1 + c_2)$, $c_i := (a_j + a_k)(b_j + b_k)$ for any permutation i, j, k - B.Jónsson [21]. Arguesian lattices form a **variety** since within lattices $p \leq q$ is equivalent to $pq = p$. A lattice is Arguesian iff it is **modular** and $(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ (*central perspectivity*) implies $c_2 \leq c_0 + c_1$ (*axial perspectivity*). In an Arguesian lattice and for a_i, b_i such that $a_2 = (a_0 + a_2)(a_1 + a_2)$ and $b_2 = (b_0 + b_2)(b_1 + b_2)$ the converse implication is valid, too [24]. A lattice is Arguesian iff its **partial order** dual is Arguesian.

Examples of Arguesian lattices. 1) The lattice $L(P)$ of subspaces of a **projective space** P is Arguesian iff **Desargues' Assumption** is satisfied in P.

2) Every lattice $L(_RM)$ of **submodules** of an R-**module** M and any lattice of subobjects of an object in an **abelian category**.

3) Every lattice of **normal subgroups** (resp. **congruence** relations) of a **group** and any lattice of permuting **equivalence** relations [21] (also called *linear lattice*).

4) Considering all lattices of congruence relations of **algebraic systems** in a variety, the Arguesian law is equivalent to the modular law.

5) Every *2-distributive* modular lattice: $w(x+y+z) = w(x+y)+w(x+z)+w(y+z)$, i.e. no **projective plane** in the variety.

The Arguesian law can be characterized in terms of forbidden subconfigurations, but not **sublattices** [17]. Weaker versions involve less variables and higher dimensional versions have increasing strength and number of variables, all are valid in linear lattices [10]. The basic structure theory just relies on the modular law cf **Modular lattice** and [3, 27]. For its rôle in congruence and commutator theory of algebraic systems cf [12]. Large parts of **Dimension Theory** for rings and modules can be conveniently done within modular lattices [29].

Projective spaces[16]. Every modular **lattice with complements** can be embedded into $L(P)$ for a projective space on the set P of its maximal **filters**, actually a sublattice of the **ideal** lattice of the filter lattice (with filters ordered by inverse inclusion), whence preserving all identities. This *Frink embedding* generalizes Stone's representation theorem for **Boolean algebras**. With the Coordinatization theorem of **projective geometry** it follows that any Arguesian relatively complemented lattice can be embedded into a **direct product** of lattices of subspaces of **vector spaces** [22].

Call a compact element p of a modular **algebraic lattice** L a *point* if it is completely join irreducible, i.e. has a unique lower cover p_*. If each element of L is a join of points (e.g. if $\dim L < \infty$), then L can be understood as the subspace lattice of an *ordered linear space* on the set P of points: the order is induced by L, points p, q, r are *collinear* if they are distinct and $p + q = p + r = q + r$, and a *subspace* is a subset X such that $q \leq p \in X$ implies $q \in X$, and $p, q \in X$ with p, q, r collinear implies $r \in X$. This can also be viewed as a **presentation** of L as a **semi-lattice**. Instead of all collinearities one may use a *base of lines*: for each element $l = p + q$ a maximal set of points with pairwise join l. For an abstract ordered linear space one has to require that collinearity is a totally symmetric relation, that collinear points are incomparable, that $p, q \leq s$ and p, q, r collinear implies $r \leq s$, that for $r' \leq r$, $r' \not\leq p, q$, and p, q, r collinear there are $p' \leq p$ and $q' \leq q$ such that p', q', r' are collinear, and, finally, a more elaborate version of the triangle axiom. Then, the subspaces form a lattice L as above and each modular lattice can be naturally embedded into such, preserving identities.

Subdirect products and congruences[3, 20]. Every lattice is a **subdirect product** of subdirectly irreducible (shortly s.i.) **homomorphic** images. By *Jónssons' Lemma*, the s.i. in the variety generated by a class $\mathcal{C}$ are homomorphic images of sublattices of ultraproducts from $\mathcal{C}$. A pair of complementary **central** elements u, v provides a direct decomposition $x \mapsto (xu, xv)$, a neutral element u a subdirect decomposition $x \mapsto (xu, x+u)$.

Any **congruence** θ on a modular lattice L is determined by its set $Q(\theta)$ of quotients where a *quotient* is a pair a/b with $a \geq b$, equivalenty, an interval $[b, a]$. A pair of quotients is *projective* if it belongs to the equivalence relation generated by the $a/b, c/d$ such that $a = b + c$ and $d = bc$. A *subquotient* c/d of a/b is such that $b \leq d \leq c \leq a$. If θ is generated by a set Γ of quotients, then $Q(\theta)$ is the transitive closure of the set of all quotients projective to some subquotient of a quotient in Γ. The congruences form a **Brouwer lattice** with the pseudocomplement θ^* of θ given by the quotients not having any subquotient projective to a subquotient of a quotient in Γ. L is subdirectly decomposed into L/θ and L/θ^* and each s.i. factor of L is a homomorphic image of L/θ or L/θ^*. If $\pi : L \to S$ is onto, $\dim(S) < \infty$, and if $\check{\pi}x = \inf\{a \in L \mid \pi a = x\}$ (which then preserves sups) and the dual $\hat{\pi}$ exist, i.e. for a *bounded image*, then for $\theta = \mathrm{Ker}\pi$ one has θ^* the transitive closure of prime quotients a/b with $a = b + \check{\pi}x$, $b = a\hat{\pi}y$ for some prime x/y in S. For any onto $\psi : L \to M$ with π not factoring through ψ, this *splitting method* yields the relations $\psi\check{\pi}y \leq \psi\hat{\pi}x$ for prime quotients x/y in S. If L is generated by finite E, start with $\sigma_0 x = \inf\{e \in E \mid \pi e \leq x\}$ iterate $\sigma_{k+1}x = \inf \sigma_k r(\sigma_k p + \sigma_k q)$ with p, q, r ranging over all subtriples of lines of a given base, to get $\sigma_{n+1} = \sigma_n = \check{\pi}$ for some n [28].

For $\dim L < \infty$, each congruence is determined by its prime quotients, either those in a given **composition sequence** or those of the form p/p_*, p a point. It follows, that the congruences form a finite **Boolean algebra** and are in 1-1-correspondence with unions of connected components of the point set under the binary relation: $\exists r.$ with p, q, r collinear. Moreover, the s.i. factors L_i of L are simple, i.e. correspond to maximal congruences θ_i, and the dimensions add up: $\dim L = \sum \dim L_i$. The connected components associated with the θ_i^* are disjoint and isomorphic images of the spaces of the L_i via $\check{\pi}_i$. Thus, the space of L can be constructed as the disjoint union of the spaces of the L_i with $p_i \leq q_j$ iff $\pi_j\check{\pi}_i p_i \leq q_j$ where $\pi_j\check{\pi}_i$ depends only on the subdirect product of L_i and L_j and can be computed, in the *scaffolding* construction, as the pointwise largest sup-homomorphism α_{ij} of L_i into L_j such that $\alpha_{ij}\pi_i e \leq \pi_j e$ for a given set of generators e.

Gluing[8]. A *tolerance* relation on a lattice L is a binary relation which is reflexive, symmetric, and compatible, i.e. a subalgebra of $L \times L$. A *block* is a maximal subset with every pair of elements in relation, whence a **convex** sublattice. The set S of blocks has a lattice structure. A convenient way to think of this is a pair σ, γ of embeddings of a (not necessarily modular) *skeleton* lattice S into the filter resp. ideal lattice of L preserving finite sups resp. infs such that $L(x) = \sigma(x) \cap \gamma(x)$ is nonempty for each x, namely one of the blocks. A relevant tolerance for modular lattices is given by the relation that $[ab, a+b]$ be complemented. Its blocks are the maximal relatively complemented convex sublattices of L and S is then the *prime skeleton.* One has a *gluing* if the smallest congruence extending the tolerance is total - which occurs for modular L of $\dim L < \infty$ and the prime skeleton tolerance. The neutrality of $u \in L$ can be shown with suitable S via an order preserving $\alpha : S \to L$ turning L into a gluing with blocks $[u\alpha x, u + \alpha x]$, $x \in S$; this happens if $x \mapsto u\alpha x$ is sup- and $x \mapsto u + \alpha x$ inf-preserving and for each e in some generating set there is $x \in S$ with $e = u\alpha x + ue$.

Every lattice with a tolerance gives rise to a system ϕ_{xy}, ψ_{yx} of **adjunctions** between the blocks $L(x), L(y)$, $x \leq y$ in S, satisfying certain axioms. Namely, $\phi_{xy} a \leq b$ iff $a \leq b$ iff $a \leq \psi_{yx} b$. Conversely, each such system defines a **pre-order** on the disjoint union of the $L(x)$ and, factoring by the associated equivalence relation, a lattice with tolerance having blocks $L(x)$. Gluing always produces a modular lattice from modular blocks, but only in special cases the impact of the Arguesian law and various kinds of representability are understood - a necessary condition is that any pair of adjunctions matching coordinate rings of two frames induces an **anti-isomorphism** [17]. For the combinatorial analysis of subgroup lattices of finite **abelian groups** cf [2].

Coordinates[5, 7]. Von Neumann introduced the lattice theoretic analogue of projective **coordinate** systems: a *n-frame* in consists of independent elements a_i, $a_{ij} = a_{ji}$, $i,j \leq n$, $i \neq j$ such that $a_i a_{ij} = 0$, $\sum a_i = 1$, $a_i + a_{ij} = a_i + a_j$, and $a_{ik} = (a_i + a_j)(a_{ij} + a_{jk})$. There are equivalent variants. Any $b_1 \leq a_1$ provides frames $b_i = a_i a_{1i}, b_{ij} = a_{ij}(b_i + b_j)$ and $c_i = a_i + v, c_{ij} = a_{ij} + v$ where $v = \sum b_i$ of sublattices which can be used to derive frames satisfying relations. The elements r_{ij} such that $r_j a_j = 0$ and $r_{ij} + a_j = a_i + a_j$ form the *coordinate domain* R_{ij}. For a free R-module with basis e_i one has the *canonical frame* Re_i, $R(e_i - e_j)$ and $r_{ij} = R(e_i - re_j)$. If $n \geq 4$ or, in presence of the Arguesian law, $n = 3$ [6] then the R_{ij} are turned into **rings** isomorphic via $r_{ij} \mapsto r_{ik} = (a_i + a_k)(r_{ij} + a_{jk})$ resp. $r_{ij} \mapsto r_{hj} = (a_h + a_j)(r_{ij} + a_{ih})$. with unit a_{ij}

$$r_{ij} \oplus s_{ij} = (a_i + a_j)[(r_{ij} + a_k)(a_{ik} + a_j) + s_{kj}]$$

$$r_{ij} \otimes s_{ij} = (a_i + a_j)(r_{ik} + s_{kj}).$$

Every modular lattice generated by a frame can be generated by 4 elements. Every finitely generated **semi-group** S can be embedded into the muliplicative semi-group of the coordinate ring of a suitable frame in some 5-generated sublattice of $L({}_kV)$ over a given **field** k - finite dimensional if S is finite.

A complemented Arguesian lattice posessing a *large partial 3-frame* (i.e. a 3-frame of a section $[0, u]$ with u having a complement $d = \sum_{i=1}^{m} x_i$, x_i perspective to $y_i \leq a_1$) or being simple of dimension ≥ 3 is isomorphic to the lattice of principal right ideals of some regular ring [23]. Under suitable richness assumptions, lattices $L({}_RM)$ have been characterized for various classes of rings via the Arguesian law and geometric conditions on the lattice e.g. for completely primary **uniserial rings** [24] and left Ore domains. There are results on lattice isomorphisms induced by semilinear maps resp. Morita equivalences and on lattice homomorphisms induced by tensoring [1]. *Abelian lattices*, having certain features of abelian categories, can be embedded into subgroups lattices of **abelian groups**. This includes algebraic modular lattices having an infinite frame.

Equational Theory[5, 7, 8, 20]. The class of all linear lattices resp. the class $\mathcal{L}(R)$ of all lattices embeddable into some $L({}_RM)$ forms a quasi-variety since it arises from a projective class in the sense of Mal'cev. Natural axiom systems and proof theories for quasi-identities have been given cf [10]. The latter present identities via graphs. On the other hand, there is no finitely axiomatized quasi-variety containing $L(k^{(\omega)})$, k some field, and satisfying all higher dimensional Arguesian laws. Also, every quasi-variety of modular lattices containing some $L(k^{(\omega)})$ also contains a 5-generated **finitely presented** lattice with unsolvable **decision problem** for words [18].

Identities are preserved when passing to the ideal lattice, thus one may assume algebraicity. Frames are *projective* systems of generators and relations within modular lattices: for each n there are terms a_i, a_{ij} in the variables x_i, z_{ij} such that the a_i, a_{ij} form a frame in a sublattice for any choice of the x_i, z_{ij} in a modular lattice and $a_i = x_i, a_{ij} = z_{ij}$ if these happen to form a frame, already. This allows to translate divisibility of integer multiples of 1 in a ring, more generally solvability of systems of linear equations with integer coefficients, into lattice identities. The converse has been done in [19] for lattices of submodules: solving the decision problem for words in free lattices in $\mathcal{L}(R)$, whenever R has decidable divisibility of integers (e.g. $R = Z$), and providing a complete list of all varieties $H\mathcal{L}(R)$, each generated by finite dimensional members (related ideas occur in model theory of modules [31]). In contrast, no finitetly axiomatized variety of modular lattices containing $L(Q^{(\omega)})$ is generated by its finite dimensional members. For free lattices with $n \geq 4$ generators in the quasi-varieties of all Arguesian, linear resp. normal subgroup lattices the decision problem remains open - in contrast to the negative answer for modular lattices [11]. The corresponding variety containments, with $H\mathcal{L}(Z)$ included, are all proper [25, 26, 30]. There are R with $\mathcal{L}(R)$ not a variety, but the status for $\mathcal{L}(k)$, k a field, $\mathcal{L}(Z)$, normal subgroup and linear lattices is unknown. Yet, for finite dimensional $L \in H\mathcal{L}(k)$ a retraction into $\mathcal{L}(k)$ is possible. The variety generated by modular lattices of $\dim(L) \leq n$ can be finitely axiomatized, for $n = 3$ the lattice of subvarieties and the covering varieties have been determined [20]. Finitely generated varieties are finitely axiomatizable - this does not extend to quasi-varieties.

Generators and relations[28]. Given a pair u_1, u_2 of complements in a modular lattice L and a subset X such that $x = u_1x + u_2x$ for all $x \in X$ one has u, v central in the sublattice they generate together with X. This applies to a direct decomposition $V = U_1 \oplus U_2$ of a **representation of partially ordered set**, $f : E \to L({}_kV)$, with $X = f(E)$. Hence, for a set E of generators with partial order relation, the s.i. factors of the free lattice in $H\mathcal{L}(k)$ can be obtained via Jónsson's Lemma from the s.i. factors of indecomposable finite dimensional representations. In particular, this carries through for representation-finite E. For E not containing $1+1+1+1$ nor $1+2+2$ these are exactly the s.i. modular lattices generated by such E, namely 2- or 5-element. For $|E| = 4$ one obtains all $L({}_PP^n)$, $n \geq 3$, P the prime subfield, lattices $|L| \leq 6$ and a series of 2-distributives (with 6 labelings by generators) [13]. The latter are exactly the s.i. modular lattices generated by two pairs of complements. Also, the structure of the free lattices in $\mathcal{L}(k)$ over these and other tame E of finite growth is understood cf [4]. Moreover, the word problem for 4-generated finitely presented lattices in $\mathcal{L}(k)$ is solvable. The lattice theoretic approach determines the s.i. factors S, first, using neutral elements and the splitting method.

A large number of finitely presented modular lattices with additional unary operations have been determined in [14, 28] as invariants for the **orbits** of subspaces under the group of **isometric mappings** of a vector space endowed with a **sesquilinear form**. The above methods have been modified to this setting.

The Arguesian lattices generated by a frame can be explicitely determined as certain lattices of subgroups of abelian groups. To some extent the analysis for $|E| = 4$ and other generating posets carries over to Arguesian lattices, but essentially new phenomena occur [15].

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