Arguesian lattice

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Desarguesian lattice

A lattice in which the Arguesian law is valid, i.e. for all a_i, b_i,

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0(a_1 + c) + b_0(b_1 + c),$$

$c = c_2(c_0 + c_1)$, $c_i = (a_j + a_k)(b_j + b_k)$ for any permutation i, j, k [a21]. Arguesian lattices form a variety (cf. also Algebraic systems, variety of), since within lattices $p \leq q$ is equivalent to $pq = p$. A lattice is Arguesian if and only if it is a modular lattice and $(a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ (central perspectivity) implies $c_2 \leq c_0 + c_1$ (axial perspectivity). In an Arguesian lattice and for a_i, b_i such that $a_2 = (a_0 + a_2)(a_1 + a_2)$ and $b_2 = (b_0 + b_2)(b_1 + b_2)$, the converse implication is valid too [a24]. A lattice is Arguesian if and only if its partial order dual is Arguesian.

Contents

Examples of Arguesian lattices.

1) The lattice $L(P)$ of subspaces of a projective space P is Arguesian if and only if the Desargues assumption is satisfied in P.

2) Every lattice $L(M)$ of submodules of an R-module M (cf. also Module) and any lattice of subobjects of an object in an Abelian category.

3) Every lattice of normal subgroups (respectively, congruence relations; cf. Normal subgroup; Congruence (in algebra)) of a group and any lattice of

permuting equivalence relations [a21] (also called a linear lattice).

4) Considering all lattices of congruence relations of algebraic systems (cf. Algebraic system) in a variety, the Arguesian law is equivalent to the modular law.

5) Every 2-distributive modular lattice (cf. also Distributive lattice): $w(x+y+z)=w(x+y)+w(x+z)+w(y+z)$, i.e. without a projective plane in the variety.

The Arguesian law can be characterized in terms of forbidden subconfigurations, but not in terms of sublattices [a17]. Weaker versions involve less variables and higher-dimensional versions have increasing strength and number of variables; all are valid in linear lattices [a10]. The basic structure theory relies on the modular law, cf. Modular lattice and [a3], [a27]. For its role in the congruence and commutator theory of algebraic systems, cf. [a12]. Large parts of dimension theory for rings and modules can be conveniently done within modular lattices [a29].

Projective spaces.

See [a16]. Every modular lattice with complements (cf. Lattice with complements) can be embedded into $L(P)$ for a projective space on the set P of its maximal filters (cf. Filter), actually a sublattice of the ideal lattice of the filter lattice (with filters ordered by inverse inclusion), whence preserving all identities. This Frink embedding generalizes the Stone representation theorem for Boolean algebras (cf. Boolean algebra). The coordinatization theorem of projective geometry implies that any Arguesian relatively complemented lattice can be embedded into a direct product of lattices of subspaces of vector spaces (cf. Vector space) [a22].

A compact element p of a modular algebraic lattice L is called a point if it is completely join irreducible, i.e. has a unique lower cover p_*. If each element of L is a join of points (e.g., if $\dim L < \infty$), then L can be understood as the subspace lattice of an ordered linear space on the set P of points: the order is induced by L. Points p, q, r are collinear if they are distinct and $p+q=p+r=q+r$, and a subspace is a subset X such that $q \leq p \in X$ implies $q \in X$, and $p, q \in X$ with p, q, r collinear implies $r \in X$. This can also be viewed as a presentation of L as a semi-lattice. Instead of all collinearities one may use a base of lines: for each element $l=p+q$ a maximal set of points with pairwise join l. For an abstract ordered linear space one has to require that collinearity is a totally symmetric relation, that collinear points are incomparable, that $p, q \leq s$ and p, q, r collinear implies $r \leq s$, that for $r' \leq r$ and p, q, r collinear there are $p' \leq p$ and $q' \leq q$ such that p', q', r' are collinear or $r' < p$ or $r' < q$, and, finally, a more elaborate version of the triangle axiom. Then the subspaces form a lattice L as above and each modular lattice can be naturally embedded into such, preserving identities.

Subdirect products and congruences.

See [a3], [a20]. Every lattice is a subdirect product of subdirectly irreducible homomorphic images (cf. Homomorphism). By Jónsson's lemma, the subdirect irreducibles in the variety generated by a class $\mathcal{C}$ are homomorphic images of sublattices of ultraproducts from $\mathcal{C}$. A pair of complementary central elements u, v provides a direct decomposition $x \mapsto (xu, xv)$, a neutral element u implies a subdirect decomposition $x \mapsto (xu, x+u)$.

Any congruence θ on a modular lattice L is determined by its set $Q(\theta)$ of quotients, where a quotient is a pair a/b with $a \geq b$, equivalently, an interval $[b, a]$. A pair of quotients is projective if it belongs to the equivalence relation generated by a/b, c/d such that $a = b + c$ and $d = bc$. A subquotient c/d of a/b is such that $b \leq d \leq c \leq a$. If θ is generated by a set Γ of quotients, then $Q(\theta)$ is the transitive closure of the set of all quotients projective to some subquotient of a quotient in Γ. The congruences form a Brouwer lattice, with the pseudo-complement $\theta^{\times}$ of θ given by the quotients not having any subquotient projective to a subquotient of a quotient in Γ. L is subdirectly decomposed into L/θ and L/θ^{*} and each subdirectly indecomposable factor of L is a homomorphic image of L/θ or $L/\theta^{\times}$. If $\pi : L \to S$ is onto, $\dim(S) < \infty$, and if $\pi^{\vee} x = \inf\{a \in L : \pi a = x\}$ (which then preserves sups) and the dual $\pi^{\wedge}$ exist, i.e. for a bounded image, then for $\theta = \operatorname{Ker} \pi$ one finds that θ^{*} is the transitive closure of prime quotients a/b with $a = b + \pi^{\vee} x$, $b = a \pi^{\wedge} y$ for some prime quotient x/y in S. For any onto mapping $\psi : L \to M$ with π not factoring through ψ, this splitting method yields the relations $\psi \pi^{\vee} y \leq \psi \pi^{\vee} x$ for prime quotients x/y in S. If L is generated by a finite set E, starting with $\sigma_0 x = \inf\{e \in E : \pi e \leq x\}$ and iterating, $\sigma_{k+1} x = \inf \sigma_k r(\sigma_k p + \sigma_k q)$ with p, q, r ranging over all subtriples of lines of a given base, leads to $\sigma_{n+1} = \sigma_n = \pi^{\vee}$ for some n [a28].

For $\dim L < \infty$, each congruence is determined by its prime quotients, either those in a given composition sequence or those of the form p/p_*, p a point. It follows that the congruences form a finite Boolean algebra and are in one-to-one correspondence with unions of connected components of the point set under the binary relation: $\exists r$ with p, q, r collinear. Moreover, the subdirectly indecomposable factors L_i of L are simple, i.e. correspond to maximal congruences θ_i, and the dimensions add up: $\dim L = \sum \dim L_i$. The connected components associated with the $\theta_i^{\times}$ are disjoint and are isomorphic images of the spaces of the L_i via $\pi_i^{\vee}$. Thus, the space of L can be constructed as the disjoint union of the spaces of the L_i with $p_i \leq q_j$ if and only if $\pi_j \pi_i^{\vee} p_i \leq q_j$ where $\pi_j \pi_i^{\vee}$ depends only on the subdirect product of L_i and L_j and can be computed, in the scaffolding construction, as the pointwise largest sup-homomorphism α_{ij} of L_i into L_j such that $\alpha_{ij} \pi_i e \leq \pi_j e$ for a given set of generators e.

Glueing.

See [a8]. A tolerance relation on a lattice L is a binary relation that is reflexive, symmetric, and compatible, i.e. a subalgebra of $L \times L$. A block is a maximal subset with every pair of elements in relation, whence a convex sublattice. The set S of blocks has a lattice structure. A convenient way to think of this is as a pair σ, γ of embeddings of a (not necessarily modular) skeleton lattice S into the filter, respectively ideal, lattice of L preserving finite sups, respectively infs, such that $L(x) = \sigma(x) \cap \gamma(x)$ is non-empty for each x, namely one of the blocks. A relevant tolerance for modular lattices is given by the relation that $[ab, a+b]$ be complemented. Its blocks are the maximal relatively complemented convex sublattices of L, and S is then the prime skeleton. One has a glueing if the smallest congruence extending the tolerance is total; this occurs for modular L of $\dim L < \infty$ and the prime skeleton tolerance. The neutrality of $u \in L$ can be shown with suitable S via an order-preserving mapping $\alpha : S \to L$ turning L into a glueing with blocks $[u \alpha x, u + \alpha x]$, $x \in S$; this happens if: $x \mapsto u \alpha x$ is sup-preserving, $x \mapsto u + \alpha x$ is inf-preserving, and for each e in some generating set there is an $x \in S$ with $e = u \alpha x + ue$.

Every lattice with a tolerance gives rise to a system ϕ_{xy}, ψ_{yx} of adjunctions between the blocks $L(x)$, $L(y)$, $x \leq y$ in S, satisfying certain axioms. Namely, $\phi_{xy} a \leq b$ if and only if $a \leq b$ if and only if $a \leq \psi_{yx} b$. Conversely, each such system defines a pre-order on the disjoint union of the $L(x)$ and, factoring by the associated equivalence relation, a lattice with tolerance having blocks $L(x)$. Glueing always produces a modular lattice from modular blocks, but only in special cases the impact of the Arguesian law and various kinds of representability are understood (a necessary condition is that any pair of adjunctions matching coordinate rings of two frames induces an anti-isomorphism of partially ordered sets [a17]). For the combinatorial analysis of subgroup lattices of finite Abelian groups, cf. [a2].

Coordinates.

See [a5], [a7]. J. von Neumann introduced the lattice-theoretic analogue of projective coordinate systems: an n-frame consists of independent elements a_i, $a_{ij} = a_{ji}$, $i, j \leq n$, $i \neq j$, such that $a_i a_{ij} = 0$, $\sum a_i = 1$, $a_i + a_{ij} = a_i + a_j$, and $a_{ik} = (a_i + a_k)(a_{ij} + a_{jk})$. There are equivalent variants. Any $b_1 \leq a_1$ provides frames $b_i = a_i a_{1i}$, $b_{ij} = a_{ij}(b_i + b_j)$ and $c_i = a_i + v$, $c_{ij} = a_{ij} + v$, where $v = \sum b_i$, of sublattices which can be used to derive frames satisfying relations. The elements r_{ij} such that $r_{ij} a_j = 0$ and $r_{ij} + a_j = a_i + a_j$ form the coordinate domain R_{ij}. For a free R-module with basis e_i one has the canonical frame Re_i, $R(e_i - e_j)$ and $r_{ij} = R(e_i - re_j)$. If $n \geq 4$ or, in the presence of the Arguesian law, $n = 3$ [a6], then the R_{ij} are turned into rings (cf. Ring) isomorphic via $r_{ij} \mapsto r_{ik} = (a_i + a_k)(r_{ij} + a_{jk})$, respectively $r_{ij} \mapsto r_{kj} = (a_k + a_j)(r_{ij} + a_{ik})$, with unit

a_{ij} and

$$r_{ij} \bigoplus s_{ij} = (a_i + a_j)[(r_{ij} + a_k)(a_{ik} + a_j) + s_{kj}],$$
$$r_{ij} \bigotimes s_{ij} = (a_i - a_j)(r_{ik} + s_{kj}).$$

Every modular lattice generated by a frame can be generated by 4 elements. Every finitely-generated semi-group S can be embedded into the multiplicative semi-group of the coordinate ring of a suitable frame in some 5-generated sublattice of $L(V)$ over a given field k (finite dimensional if S is finite).

A complemented Arguesian lattice possessing a large partial 3-frame (i.e., a 3-frame of a section $[0,u]$ with u having a complement $u = \sum_{i=1}^{\infty} x_i$, x_i perspective to $y_1 \leq a_1$) or being simple of dimension ≥ 3 is isomorphic to the lattice of principal right ideals of some regular ring [a23]. Under suitable richness assumptions, lattices $L(M)$ have been characterized for various classes of rings via the Arguesian law and geometric conditions on the lattice, e.g. for completely primary uniserial rings [a24] and left Ore domains. There are results on lattice isomorphisms induced by semi-linear mappings, respectively Morita equivalences (cf. also Morita equivalence), and on lattice homomorphisms induced by tensoring [a1]. Abelian lattices, having certain features of Abelian categories, can be embedded into subgroup lattices of Abelian groups. This includes algebraic modular lattices having an infinite frame [a32].

Equational theory.

See [a5], [a7], [a8], [a20]. The class of all linear lattices, respectively the class $\mathcal{L}(R)$ of all lattices embeddable into some $L(M)$, forms a quasi-variety, since it arises from a projective class in the sense of Mal'tsev. Natural axiom systems and proof theories for quasi-identities have been given, cf. [a10], [a33]. The latter present identities via graphs. On the other hand, there is no finitely-axiomatized quasi-variety containing $L(k^{(\omega)})$, k some field, and satisfying all higher-dimensional Arguesian laws. Also, every quasi-variety of modular lattices containing some $L(k^{(\omega)})$ also contains a 5-generated finitely-presented lattice with unsolvable decision problem for words [a18].

Identities are preserved when passing to the ideal lattice; thus, one may assume algebraicity. Frames are projective systems of generators and relations within modular lattices: for each n there are terms a_i, a_{ij} in the variables x_i, z_{ij} such that the a_i, a_{ij} form a frame in a sublattice for any choice of the x_i, z_{ij} in a modular lattice and $a_i = x_i$, $a_{ij} = z_{ij}$ if these happen to form a frame already. This allows one to translate divisibility of integer multiples of 1 in a ring (more generally, solvability of systems of linear equations with integer coefficients) into lattice identities. The converse has been done in [a19] for lattices of submodules: solving the decision problem for words in free lattices in $\mathcal{L}(R)$, whenever R has decidable divisibility of integers (e.g. $R=\mathbf{Z}$), and providing a complete list of all varieties $H\mathcal{L}(R)$, each generated by finite-dimensional members (related ideas occur in the model theory of modules [a31]). In contrast, no finitely-axiomatized variety of modular lattices containing $L(Q^{(\omega)})$ is generated by its finite-dimensional members. For free lattices with $n\geq 4$ generators in the quasi-varieties of all Arguesian linear, respectively normal, subgroup lattices the decision problem remains open (in contrast to the negative answer for modular lattices [a11]). The corresponding variety containments, with $H\mathcal{L}(\mathbf{Z})$ included, are all proper [a25], [a26], [a30]. There are rings R with $\mathcal{L}(R)$ not a variety, but the status for $\mathcal{L}(k)$, k a field, $\mathcal{L}(\mathbf{Z})$, normal subgroup and linear lattices is unknown. Yet, for finite-dimensional $L\in H\mathcal{L}(k)$ a retraction into $\mathcal{L}(k)$ is possible. The variety generated by modular lattices of $\dim(L)<n$ can be finitely axiomatized; for $n=3$ the lattice of subvarieties and the covering varieties have been determined [a20]. Finitely-generated varieties are finitely axiomatizable (this does not extend to quasi-varieties).

Generators and relations.

See [a28]. Given a pair u_1, u_2 of complements in a modular lattice L and a subset X such that $x=u_1x=u_2x$ for all $x\in X$, one has that u, v are central in the sublattice they generate together with X. This applies to a direct decomposition $V=U_1\oplus U_2$ of a representation of a partially ordered set, $f:E\to L(V)$, with $X=f(E)$. Hence, for a set E of generators with partial order relation, the subdirectly indecomposable factors of the free lattice in $H\mathcal{L}(k)$ can be obtained via Jónsson's lemma from the subdirectly indecomposable factors of indecomposable finite-dimensional representations. In particular, this carries through for representation-finite E. For E not containing $1-1+1-1$ nor $1-2+2$, these are exactly the subdirectly indecomposable modular lattices generated by such E, namely 2- or 5-element. For $|E|=4$ one obtains all $L(P^n)$, $n\geq 3$, P the prime subfield, lattices with $|L|\leq 6$, and a series of 2-distributives (with 6 labelings by generators) [a13]. The latter are exactly the subdirectly indecomposable modular lattices generated by two pairs of complements. Also, the structure of the free lattices in $\mathcal{L}(k)$ over these and other tame E of finite growth is understood [a4]. Moreover, the word problem for 4-generated finitely-presented lattices in $\mathcal{L}(k)$ is solvable. The lattice-theoretic approach determines the subdirectly indecomposable factors S, first, using neutral elements and the splitting method.

A large number of finitely-presented modular lattices with additional unary operations have been determined in [a14], [a28] as invariants for the orbits (cf. Orbit) of subspaces under the group of isometric mappings (cf. Isometric mapping) of a vector space endowed with a sesquilinear form. The above methods have been modified to this setting.

The Arguesian lattices generated by a frame can be explicitly determined as certain lattices of subgroups of Abelian groups. To some extent the analysis for $E|=4$ and other generating posets carries over to Arguesian lattices, but essentially new phenomena occur [a15].

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