Equational and universal Horn theory of modular lattices

Equationaly defined and universal Horn classes of modular lattices (i.e. quasivarieties and varieties) arise in many natural contexts including ring theory, database dependencies, and construction of experimental desingns. The question to be considered are

- axiomatizability of classes
- decidability of theories
- closure under homomorphism
- combinatorial subclasses
- related subjects

Axiomatizability. The state of affairs is as follows

- Each of the following classes (an many others) can be axiomatized by a recursive set of universal Horn sentences. The class of all lattices embeddable into: complemented modular resp. Arguesian lattices, submodule lattice of R-modules ($L(R)$), lattices of normal subgroups, lattices of permuting equivalences (type 1). (Malcev; Jónsson; Schein; Hutchinson 73; Haiman 84; Finberg, Mainetti and Rota 96)
- None of these classes can be finitley axiomatized. Even more: No quasivariety of type 1 lattices and containing some $L(F)$, F a field, can be axiomatized by sentences in a finite number of variables (Herrmann and Poguntke 74; Haiman 91)
- In general, these classes are distinct (Jönsson 54; Freese, Herrmann and Huhn 81; Pálfy and Szabó 95)
- There are finite type-1 lattices which are not in the variety generated by lattices representable on finite sets (Herrmann 87)

Still, the question whether there exist "effective" axiomatizations remains open. Therefore we ask

- For which of the above mentioned quasivarieties (resp, the varieties they generate) there exists an algorithm deciding membership of finite lattices?
- Is the equational theory of type-1 lattices representable on finite sets recursively axiomatizable (whence decidable)?

Decidability. The state of affairs is as follows

- The free modular lattice on 4 or more generators (Freese 80; Herrmann 83) has an unsolvabe word problem
- The lattice variety generated by subgroup lattices of abelian groups has a decidable equational theory and so do many natural subvarieties - including the varieties generated by $L(R)$, R recursive. (Herrmann and Huhn 74; Czedli and Hutchinson 78)
- Every modular lattice variety containing $L(F)$ for some field F has a 5-generated finitely presented member with unsolvable word problem (Hutchinson 73, 77; Lipshitz 74)
- The 4-generator word problem is solvable for the lattice variety generated by complemented modular lattices. (Gel'fand and Ponomarev 70; Herrmann 73)

The negative results on free modular lattices definitely depend on constructions incompatible with Desargues' Law. The positive results on submodule lattices depend on the theory of elementary divisors resp. representation theory of quivers. Filling the gap (or rather abyss) includes to

- consider decidability of equational theory resp. 4-generator word problem for
  - Argeuesian lattices
  - type 1 lattices
  - subgroup lattices of abelian groups

Homomorphic closure. The state of affairs is as follows

- The classes of type-1 and of sublattices of complemented modular lattices are closed under forming ideal lattices (Nation 87; Herrmann and Semenova 03).
- The classes $L(R)$ are not closed under homomorphic images, in general (Czedli and Hutchinson 87).

Thus, homomorphic closure may be expected only for quite restricted or quite general classes. Therefore we ask

- which off the following classes is closed under homomprphic images
  - type 1 lattices?
  - sublattices of complemented modular lattices?

Combinatorial subclasses. The most important classes of modular lattices with a purely combinatorial flavour are given by breadth 2 and 2-distributive lattices. The state of affairs is as follows

- Finitely generated finite height 2-distributive modular lattice are finite (Herrmann 90)
- 2-distributive modular lattices are in $L(R)$ for every R (Jónsson and Nation 87, Herrmann, Pickering, and Roddy 94)
- there is a simple finitely generated modular breadth-2 lattice with no prime quotients and not contained in the quasivariety generated by finite height modular lattices (Schmidt 75)
- finite height 2-distributive modular lattices are of finite representation type if and only if they are "acyclic". All such are bounded homomorphic images within modular lattices (Herrmann and Wild 91)

This gives rise to the following questions for the variety of 2-distributive modular lattices resp. variety generated by breadth-2 modular lattices

- Is the variety generated by its finite members?
- Is the equational resp. universal Horn theory decidable?
- Does the finite height conjecture hold (varieties of finite height in the lattice of varieties are generated by finite lattices)?
- Are subdirectly irreducible acyclic lattices splitting in the variety (resp. in the variety of all modular lattices)?

Related subjects. Type 1 lattices are closely related to the theory of relation algebra, databases, experimental designs, stochastic independence, and invariant theory. The most significant results in this context are

- The equational theory of the reduct of representable relation algebras involving product, meet, conversion, and identity, only, is decidable but does not axiomatize the class (Andreka and Bredikhin 95)
- Any variety of relation algebras containing the complex algebra of an infinite group has undecidable equational theory (Andreka, Givant, and Nemeti 97)
- Representability is not decidable for finite relation algebras (Hirsch and Hodkinson 99)
- The implication problem for embedded multivalued dependencies is unsolvable (Herrmann 95)
- Lattices of countable and "continuous" lattices of measurable commuting σ-algebras are type 1 (Yan 99)

Of interest are fragments of the universal Horn logic of subreducts of relation algebras endowed at least with the relational product.

- Is the universal Horn theory of the relational product decidable?
- Is there a decision procedure to test whether a given finite set of generators with product and meet relations generates distributive and permuting lattices of equivalences, only?
- For which quasivarieties of reducts is representability of finite algebras decidable?

References

[1]