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A review of some of Bjarni Jónsson's results on representation of arguesian lattices

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Abstract. We review (and slightly extend) Bjarni Jónsson's results on representations of arguesian lattices that are complemented, of low height, or of simple gluing structure.

1. Introduction

Based on preliminary work of Schützenberger, Bjarni Jónsson has introduced [21, 22] a lattice theoretic inequality (i.e., in essence an identity) valid in all lattices of commuting equivalence relations and, on the other hand, valid in the subspace lattice of a projective space if and only if the space is desarguean. Lattices satisfying this inequality are called *arguesian* and have played a major rôle both in Modular Lattice Theory and its applications within Universal Algebra. The focus of this review will be on lattices embeddable into subspace lattices of vector spaces. The author is indebted to the referee for a very careful report and a variety of helpful suggestions.

2. Preliminaries: subspace lattices

All lattices L to be considered will be modular with bounds $0, 1$. For the basics about these and the relations between projective spaces P, their lattices $L(P)$ of subspaces, and subspace lattices $L(V_D)$ of vector spaces V_D, D a division ring, we refer to the literature, e.g., [1, 6].

The *height* of L is $\mathrm{d}(L) = n < \infty$ if some, whence every, maximal chain in L has $n+1$ elements. We put $\mathrm{d}(L) = \infty$, otherwise; also, $\mathrm{d}(a) = \mathrm{d}([0, a]_L)$ is the *height* of a. A map $f\colon L \to M$ is *isometric* if $\mathrm{d}(x) = \mathrm{d}(f(x))$ for all $x \in L$. Any L is a subdirect product of subdirectly irreducibles. If $\mathrm{d}(L) < \infty$, then L is subdirectly irreducible if and only if L is simple.

Consider a set P of *points* endowed with a set G of at least 2-element subsets, the *lines*. We say that pairwise distinct points p, q, r are *collinear* if $\{p, q, r\} \subseteq g$ for some $g \in G$. In order that P (more precisely, (P, G)) is a *projective space*, one requires that any two distinct points are contained in a

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unique line, that any two distinct lines have at most one point in common, and that for any collinear triplets p, x, r and q, y, r with p, q, r pairwise distinct and non-collinear, there is unique z such that p, q, z are collinear. A subset Q of P is a *subspace* if $g \subseteq Q$ whenever g is a line with $|g \cap Q| \geq 2$. The subspaces of P form a complemented (modular) lattice $L(P)$ ordered by inclusion. Points p, q are *perspective* if $p = q$ or if there is r such that p, q, r are collinear. This defines an equivalence relation on P, the classes of which are the *components* P_i. The $L(P_i)$ are subdirectly irreducible and $L(P)$ is canonically isomorphic to a direct product of the $L(P_i)$. P is desarguean if and only if $L(P)$ is arguesian.

A lattice L of finite height is isomorphic to some $L(P)$ if and only if it is complemented (and modular); namely, as P one may choose the set atoms of L and as lines the $P \cap [0, u]$ where $\mathrm{d}(u) = 2$.

Recall that an element c of L is *central* if and only if there is a complement d of c such that $x = xc + xd$ for all $x \in L$; equivalently, $x \mapsto (xc, xd)$ is an isomorphism of L onto $[0, c] \times [0, d]$.

Fact 2.1. For $L = L(P)$ of finite height, the central elements are exactly the $\sum Q$ where Q is a union of components. Any congruence of L is generated by a unique pair $c/0$ with central c. The congruences θ_c, θ_d associated with the central c, d are complementary if and only if c, d are complementary. In this case, θ_c, θ_d is the unique pair of complementary congruences such that $c/0 \in \theta_c$ and $d/0 \in \theta_d$. See e.g., [1, Chapter IV].

Fact 2.2. $L(P)$ of height ≥ 3 is isomorphic to some $L(V_D)$ (and $\mathrm{d}(L(V_D)) = \mathrm{d}(L(P))$) if and only if $L(P)$ is subdirectly irreducible and arguesian; moreover, D is then unique up to isomorphism. Namely, P is desarguean in this case ([22]) and the Coordinatization Theorem applies.

Any division ring D has a *characteristic* $\chi(D)$: the minimum $n > 1$ such that $n1_D = 0_D$ (which is a prime it if exists), and $\chi(D) = 0$, otherwise. Thus, any subdirectly irreducible arguesian $L \cong L(P)$ of height ≥ 3 has a unique *characteristic* $\chi(L) = \chi(D)$ where $L \cong L(V_D)$. The case of lattices L of height ≤ 2 is now dealt with by the following: we may attribute to such L any characteristic c.

Fact 2.3. If L is of height $n \leq 2$, then L embeds isometrically into $L(V_D)$ for any n-dimensional vector space V_D with $|D| \geq |L| - 3$. *Any injective map between the sets of atoms extends to such an embedding.*

Embedding subspace lattices of the same characteristic into a single one becomes possible due to the following facts.

Fact 2.4. For any division ring D and infinite cardinal $\kappa > |D|$, there is a division ring D' of cardinality κ extending D. This follows from the Löwenheim–Skolem Theorem. More meaningful constructions are due to Cohn [5].

Fact 2.5. Division rings D_i (for $i \in I$) embed simultaneously into some division ring D if and only if they have the same characteristic. For $|I| = 2$, this follows from Cohn's proof of the amalgamation property (cf. [4]) for general I, then by the Löwenheim–Skolem Theorem.

Fact 2.6. If V_D is an D-vector space and D' a division ring extending D, then $L(V_D)$ embeds into $L(W_{D'})$ whenever $\dim V_D = \dim W_{D'}$. $W_{D'}$ is isomorphic to $V_D \otimes_D D'$.

Fact 2.7. $\prod_{i\in I} L(V_{iD_i})$ embeds into $\prod_{i\in I} L(V_{iD})$ for any D-vector spaces V_{iD}; the embedding is isometric in case of finite dimension. Namely,

$$(U_i \mid i \in I) \mapsto \sum_{i\in I} U_i.$$

3. Preliminaries: gluing

The *gluing construction* was introduced by Dilworth and Hall; the full power of the construction and the associated structural analysis was demonstrated in [22, 23]. A lattice L is a *gluing* of L_0 and L_1 with *overlap* $[u, v]$ if

$$0 < u \le v < 1 \text{ in } L,\ L_0 = [0, v],\ L_1 = [u, 1],\ \text{and } L = L_0 \cup L_1.$$

Observe that $\mathrm{d}(L) = \mathrm{d}(L_1) + \mathrm{d}(L_2) - d[u, v]$ in case of finite height. Moreover, the lattice structure of L is completely determined by that of the L_i, the elements $u \in L_0$, $v \in L_1$, and the identity map $f\colon [u, 1_{L_0}]_{L_0} \to [0_{L_1}, v]_{L_1}$ (which describes the overlap). Thus, it follows (cf. [22, Lemma 3.4])

Fact 3.1. For L a gluing of L_0 and L_1 with overlap $[u, v]$, if $f_i\colon L_i \to M$ are homomorphisms that coincide on $[u, v]$, then $f_0 \cup f_1\colon L \to M$ is a homomorphism extending both f_i; it is an (isometric) embedding if both f_0, f_1 are. If the θ_i are congruences on L_i (for $i = 0, 1$), that coincide on $[u, v]$, then there is a unique congruence θ on L that restricts to θ_i on L_i for $i = 0, 1$. In particular, L is a subdirect product of L_0 and L_1 if $u = v$, and L is simple if so are the L_i and $u < v$.

Considering L_0 and L_1, separately, $u \in L_0$, $v \in L_1$, and an isomorphism $f\colon [u, 1_{L_0}]_{L_0} \to [0_{L_1}, v]_{L_1}$, a gluing L of L_0 with an isomorphic copy of L_1 arises by identifying $f(x)$ with x for all $x \in [u, 1_{L_0}]$. The resulting lattice will always be modular.

Jónsson considered this construction of a lattice L for $L_i = L(V_{iD_i})$ and $\dim V_{iD_i} = 3$. In [22, Theorem 3.6], he chose $\dim v = 1$ and D_1, D_2 of distinct prime characteristic to obtain an example of a simple lattice of height 5 isomorphic to a lattice of permuting equivalences but not embeddable into the normal subgroup lattice of any group. Allowing also characteristic 0, we will call such L of *type J*. On the other hand, in Case 5 of [23, Theorem 3.1], Jónsson showed that $D_1 \cong D_2$ if L is arguesian and $\dim v = 2$, and that then L embeds isometrically into some $L(V_{D_1})$.

To have a more general view of these and related constructions, recall the following facts about finite height modular lattices: define a^* and a_* as the join of all upper covers and the meet of all lower covers of a, respectively, with the convention that $1^* = 1$ and $0_* = 0$. Then the *(prime) skeleton* $S(L) = \{x \in L \mid (x^*)_* = x\}$ and its associated $S^*(L) = \{y \in L \mid (y_*)^* = y\}$ are complete join and meet sub-semilattices of L, resp., and $x \mapsto x^*$, $y \mapsto y_*$ are mutually inverse isomorphisms between the lattices $S(L)$ and $S^*(L)$. Moreover, L is the union of its intervals $L_x = [x, x^*]$ for $x \in S(L)$, and these *blocks* L_x are the maximal complemented intervals of L. For the case $n \leq 4$, this kind of analysis is already in [23].

Iterating the gluing construction in finitely many steps with L_i that are complemented and of finite height, one obtains an L where the skeleton $S(L)$ is a chain $\{0, \dots, k\}$: this means there are chains

$$0 = u_0 < \cdots < u_k \text{ and } v_0 < \cdots < v_k = 1, \text{ with } u_i < v_i,$$

such that the $L_i = [u_i, v_i]$ are exactly the maximal complemented intervals of L.

Fact 3.2. If L is a modular lattice with $S(L)$ a chain, then L is a subdirect product of simple lattices having chain skeletons. This follows from the Corollary to [27, Lemma 1].

Lemma 3.3. *If L is of finite height with $S(L)$ a chain, then L is simple if and only if all blocks L_i are simple,* $\mathrm{d}(L_i) \geq 2$*, and $|L_i \cap L_{i+1}| > 1$ for all i.*

Proof. This has been stated as Lemma 2 in [27] and proved for the case $|S(L)| = 2$. We shall use a simpler approach relying on the concept of central elements. We define

- $a_1 \perp a_2$ in L if and only if the interval $[a_1a_2, a_1 + a_2]$ of L is complemented and a_1, a_2 are central elements of this interval.
- The pair θ_1, θ_2 of complementary congruences of L *matches* $a_1 \perp a_2$ if the restriction to $[a_1a_2, a_1 + a_2]$ is the pair of congruences determined via Fact 2.1 by the pair a_1, a_2 of central elements of $[a_1a_2, a_1 + a_2]$.

Observe: $(*)$ For any pair of complementary congruences on L, restriction to an interval $[u, v]$ yields a pair of complementary congruences on $[u, v]$.

We prove the following claim by induction on $|S(L)|$ for any L having a chain skeleton.

Claim. *For any $a_1 \perp a_2$ in L, there is a pair θ_1, θ_2 of congruences of L matching $a_1 \perp a_2$.*

Proof of the Claim. In case $|S(L)| = 1$, we have a single complemented $L = L_0$ and may assume $L = L(P)$ for some projective space P with irreducible components P_j for $j = 1, \dots, m$. Choose a complement b of a_1a_2 in $[0, a_1 + a_2]$ and let $b_i = ba_i$. Due to the isomorphism between the intervals, also $b_1 \perp b_2$. Thus, by Fact 2.1, there cannot be points $p_i \leq b_i$ such that $p_1, p_2 \in P_j$ for some j. Now, for $i = 1, 2$, let Q_i be the union of all P_j containing some point $p \leq b_i$, and Q the union of the remaining components. Then $c_1 = \sum Q_1 + \sum Q$ and $c_2 = \sum Q_2$ are complementary central elements, and the associated pair θ_1, θ_2 of congruences matches $a_1 \perp a_2$.

If $|S(L)| > 1$, let $L'_1 = \bigcup_{i=1}^{k} L_i$. Then L is a gluing of L_0 and L'_1 with overlap $[u, v]$ where $u = u_1, v = v_1$. Given $a_1 \perp a_2$ in L, we have $a_1, a_2 \in L_i$ for some i. Thus, we have $a_1, a_2 \in L_0$ or $a_1, a_2 \in L'_1$.

In the first case, choose c_i and θ_i as in the preceding paragraph. Then $c_1 + u, c_2 + u$ are complementary central elements of $[u, 1_{L_o}]$ and it follows $c_1 + u \perp c_2 + u$ in L'_1. By the inductive hypothesis, we have a pair τ_1, τ_2 of congruences on L'_1 matching $c_1 + u \perp c_2 + u$. By $(*)$, the restrictions of θ_1, θ_2 as well as those of τ_1, τ_2 to $[u, v]$ are complementary pairs, both containing $(c_1 + u)/u$, resp., $(c_2 + u)/u$; thus, by Fact 2.1, both pairs restrict to the same pair of congruences of $[u, v]$. For each $i = 1, 2$, let μ_i be the common extension of θ_i and τ_i according to Fact 3.1. Then μ_1, μ_2 is a pair of complementary congruences of L. Indeed, any prime quotient of L belongs to at least one of L_0 or L'_1 and whence to some of the τ_i and θ_i. But $\mu_1 \cap \mu_2 = \mathsf{id}$ due to the way the μ_i restrict to L_0, L'_1, and the intersection of both. Finally, μ_1, μ_2 matches $a_1 \perp a_2$ since so does the restriction θ_1, θ_2 to L_0.

In the second case, by inductive hypothesis, we have congruences θ_1, θ_2 on L'_1 matching $a_1 \perp a_2$. By $(*)$, the restriction to $[u, v]$ gives a complementary pair of congruences, induced by a complementary pair of c_1, c_2 of central elements of $[u, v]$ due to Fact 2.1. In particular, $c_1 \perp c_2$ in L_0 and, by the case of $|S(L)| = 1$, we get complementary congruence τ_1, τ_2 of L_0 matching $c_1 \perp c_2$. As in the preceding case, we conclude that for $i = 1, 2$, there are extensions μ_i of θ_i and τ_i according to Fact 3.1, and thus matching $a_1 \perp a_2$. This completes the proof of the claim. $\square$

Now if some L_i is not simple, by Fact 2.1, it contains distinct atoms $a_1 \perp a_2$; it follows that L has a non-trivial pair of complementary congruences. Conversely, if all blocks are simple and all overlaps non-trivial, then an easy inductive argument on $|S(L)|$ shows that all prime quotients are projective; thus, L is simple. $\square$

Fact 3.4. If L is arguesian of finite height with $S(L)$ a chain, all blocks L_i simple, and $\mathrm{d}(L_i \cap L_{i+1}) \geq 2$ for all i, then L embeds isometrically into some $L(V_D)$ for V_D a D-vector space; moreover, L has unique characteristic $\chi(D)$. This is [27, Cor. 7], based on Case 5 of the proof of [23, Theorem 3.1].

4. Main results

The von Neumann approach to coordinatization via frames is the origin of the following result of [17].

Fact 4.1. There are sets Σ_c of lattice identities, for c prime or $c = 0$, such that Σ_c is valid in $L(V_D)$ with $\dim V_D \geq 3$ if and only if $\chi(D) = c$.

Recall that a *congruence variety* is a lattice variety generated by the congruence lattices of a variety of algebraic structures. The proof of the following result of Freese and Jónsson [12] was crucial for the development of commutator theory in congruence modular varieties as well as for Proposition 4.3 below. Its origins can be traced back to Jónsson's proof that any modular lattice variety having the amalgamation property must be arguesian.

Fact 4.2. Every modular congruence variety is arguesian.

The construction given in Jónsson [22, §3] motivated the lattice identities $\gamma_{n,m}(w_k)$ established in [11]. Let Γ consist of all these where $n = 2$ and k a prime.

Proposition 4.3. *The identities in* Γ *are valid in all modular congruence varieties. Moreover, for any a simple arguesian lattice* L *of finite height, if* L *satisfies* Γ*, then all its simple complemented intervals of height* ≥ 3 *have the same characteristic.*

The converse of the second claim can be shown, easily, using the projectivity of the configurations motivating the identities, but it is not needed here.

Proof. The first claim is the main result of [11]. We just outline the proof of the second. Assume that L is simple arguesian of finite height and contains simple complemented intervals $[u_i, v_i]$, $i = 1, 2$ of height $n_i \geq 3$ and characteristic $c_1 \neq c_2$. Passing to suitable subintervals, we have $n_i = 3$. Also, at least one of the c_i is a prime, say $c_1 = p$. There are spanning frames of order 3, $\vec{a}$ in $[u_1, v_1]$ and $\vec{b}$ in $[u_2, v_2]$. Since L is simple, the quotients $v_1/(a_1 + a_2)$ and b_0/u_2 are projective. With a suitable chain of transpositions via quotients c_i/d_i, one obtains a failure of an identity $\gamma_{2,m}(w_p)$. $\square$

Definition 4.4. Consider the following properties of a modular lattice L.

(i) L is isomorphic to some lattice of permuting equivalences.
(ii) L is arguesian.
(iii) L embeds into $L(P)$ for some desarguean projective space P.
(iv) L embeds into the subgroup lattice of some abelian group.
(v) L embeds into the normal subgroup lattice of some group.
(vi) L is a member of some modular congruence variety.
(vii) L is arguesian and satisfies Γ (cf. Proposition 4.3).
(viii) L is arguesian and not of type J.
(ix) L is arguesian and satisfies Σ_c for some c (cf. Fact 4.1).
(x) L embeds into $L(V_D)$ of some vector space V_D.

The following is due to Jónsson [22, 24, 23, 8] for most parts. The equivalence of (i) and (iv) for L of height ≤ 4 is addressed in [2, 3].

Theorem 4.5. *Consider a modular lattice L and refer to* (i)–(x) *from Definition 4.4.*

(a) *For complemented L,*
  (a1) (i)–(viii) *are pairwise equivalent;*
  (a2) (ix) *and* (x) *are equivalent;*
  (a3) (i)–(x) *are pairwise equivalent if L is subdirectly irreducible.*

(b) *For L of finite height with $S(L)$ a chain,*
  (b1) (i) *and* (ii) *are equivalent;*
  (b2) (iii)–(vii) *are pairwise equivalent;*
  (b3) (iii)–(vii), (ix), (x) *are pairwise equivalent if L is subdirectly irreducible.*

(c) *For L of height $n \leq 5$,*
  (c1) (i) *and* (ii) *are equivalent;*
  (c2) (iii)–(x) *are pairwise equivalent;*
  (c3) (i)–(x) *are pairwise equivalent if $n \leq 4$.*

In (b3), (c2), *and* (c3), *the vector space V_D of* (x) *can be chosen such that* $\dim V_D = \mathrm{d}(L)$ *and* $\chi(D) = \chi(L)$.

First observe that all conditions imply that L is arguesian: (i) $\Rightarrow$ (ii) is in [21]; (iii) $\Rightarrow$ (ii) is in [22, Theorem 1.7]; the implications (x) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) are obvious. (vi) $\Rightarrow$ (ii) is Fact.4.2. Thus, in the sequel we may assume all lattices L to be arguesian. Also (iii) $\Rightarrow$ (iv): any irreducible direct factor $L(P_i)$ embeds into some $L(V_{iD_i})$ (using Fact 2.2 if $\mathrm{d}(L(P_I)) \geq 3$, Facts 2.3 and 2.4 if $\mathrm{d}(L(P_i)) \leq 2$) and $L(P)$ embeds into the subgroup lattice of the direct product of the abelian groups V_i.

Proof of Theorem 4.5(a). In [24], Jónsson has shown that any complemented modular lattice embeds into an $L(P)$ satisfying the same identities as does L. Thus, (a1) follows from the implications derived in the preceding observation. Moreover, if in (iii) L is subdirectly irreducible, it has to embed into one of the $L(P_i)$, and this gives (x). This proves (a3). If (ix) holds, then by Fact 4.1, all $L(P_i)$ have the same characteristic c and by Facts 2.2–2.6, we may choose D such that each $L(P_i)$ is isomorphic to some $L(V_{iD})$. Then Fact 2.7 applies to prove (a2). □

Lemma 4.6. *If L is arguesian of finite height and a gluing of L_i for $i = 0, 1$, having an isometric embedding into $L(V_{iD_i})$ such that $\chi(D_i) = c$ and $\mathrm{d}(L_0 \cap L_1) = 1$, then L admits an isometric embedding into some $L(V_D)$ with $\chi(D) = c$.*

Proof. By Fact 2.5, there is D extending the D_i with $\chi(D) = c$ and such that L_i embeds isometrically into any $L(W_{iD})$ where $\dim W_{iD} = \dim V_{iD_i}$. Choose W_D with $\dim W_D = \dim V_{0D_0} + \dim V_{1D_1} - 1$. Choose a subspace W_0 of W_D of $\dim W_0 = \dim V_{0D_0}$ and U_1 of codimension 1 in W_0. Then one has an isometric embedding $f_0 \colon L_0 \to L(W_{0D})$ and $f_1 \colon L_1 \to L(W_D/U_1)$. Since general linear groups act transitively on their vector spaces, we may assume that $f_1(v) = W_0/U_1)$. Identifying $L(W_D/U_1)$ with the interval $[U_1, W]$ of $L(W_D)$, $f_0 \cup f_1$ defines an isometric embedding of L into $L(W_D)$. $\square$

Proof of Theorem 4.5(b). (b1) is the main result of [27], building on Fact 3.4 and [22, Theorem 3.6]. Observe that (x) implies both (vii) and (ix) in view of Proposition 4.3 and Fact 4.1. We show that either of (vii) and (ix) implies (x) for simple arguesian L with $S(L)$ a chain $\{0, \dots, k\}$. By Lemma 3.3, all L_i are simple. Let θ be the smallest equivalence relation on $S(L)$ containing all pairs $(i, i+1)$ where $\mathrm{d}(L_i \cap L_{i+1}) \geq 2$. For any class X of θ, we have $L_X = \bigcup_{i \in X} L_i$ is either of height 2 and $L_X = L_i$ where $X = \{i\}$, or Fact 3.4 applies to L_X. If the latter occurs, by Proposition 4.3, resp. Fact 4.1, there is a unique characteristic c such that each of those L_X is isomorphic to some $L(V_{XD_X})$ with $\chi(D_X) = c$. Otherwise, we may choose c arbitrary. In any case, in view of Facts 2.3 and 2.4, we may assume that all L_X embed isometrically into some $L(V_{XD_X})$ with all D_X of characteristic c. Now, iterated application of Lemma 4.6 applies to give an isometric embedding of L into $L(V_D)$ with D of characteristic c. This proves (b3). Finally, (b2) follows in view of Fact 3.2 and the implication (iii) $\Rightarrow$ (iv). $\square$

Proof of Theorem 4.5(c). Let L be arguesian of height $n \leq 5$. (c1) is [8, Theorem 4.2]. For $n \leq 4$, inspection of the proof in [23] gives (x). This proves (c3). So let $n = 5$. If L is subdirectly reducible, then L embeds into $L_1 \times L_2$ with L_i of height $n_i > 0$ such that $n_1 + n_2 = 5$, w.l.o.g. $n_1 \leq 4$ and $n_2 \leq 2$. Thus, by (c3), the L_i embed into isometrically into $L(V_{iD_i})$. In view of Facts 2.3–2.6, we may assume $D_1 = D_2 = D$ and obtain an isometric embedding into some $L(V_D)$ by Fact 2.7.

Thus, we may assume that L is simple. For $S(L)$ not a chain, inspection of the proof of [8, Theorem 4.2] shows that L has an isometric embedding into some $L(V_D)$. So suppose that $S(L)$ is a chain but that L does not isometrically embed into any $L(V_D)$; in particular, the proof of (b3) does not apply. Then there must be blocks L_{i_1}, L_{i_2} for $i_1 < i_2$, that are simple of height > 2 and of different characteristic. Choose $i_2 - i_1$ minimal. If $i_2 = i_1 + 1$, then by Facts 3.4 and 4.1, we must have $\mathrm{d}(L_{i_1} \cap L_{i_2}) \leq 1$, and from $\mathrm{d}(L) = 5$ and a dimension count, it follows that $|S(L)| = 2$, so that L is of type J. Otherwise, the $\mathrm{d}(L_j) = 2$ for $i_1 < j < i_2$ and $\mathrm{d}(L_j \cap L_i) = 1$ for $|j - i| = 1$ and a dimension count yields $\mathrm{d}(L) \geq 6$. $\square$

5. Discussion

Haiman [16] has constructed, for any characteristic c, a height 14 simple arguesian lattice having distributive skeleton, satisfying Σ_c and Γ, but not isomorphic to any lattice of commuting equivalences. The subgroup lattices of groups $\mathbb{C}_{p^2}^3$ do not belong to the variety generated by all complemented modular lattices [18]. Thus, for height 6 we may ask the following.

Problem 5.1. Which of the properties (i) and (iv)–(vi) are shared by all arguesian lattices of height 6 and satisfying Γ?

Problem 5.2. Which of the properties (i), (iv)–(vi), and (x) are shared by all arguesian lattices of height 6, having 2-distributive skeleton, and satisfying Γ?

The classes of all lattices having one of the properties (i), (iii)–(v), and (x) (also the subclass with D of fixed characteristic c), respectively, of Theorem 4.5 are well known to be recursively axiomatizable quasivarieties; let us refer to these as the J-quasivarieties. Within the higher arguesian identities, first considered by Bill Lampe, which are valid in all lattices of permuting equivalences, Haiman [16] identified a series D_n such that no class $\mathcal{C}$ satisfying all D_n can be finitely axiomatized provided that it contains all $L(V_D)$, $\dim V_D$ finite and D a fixed prime field (cf. MR1083826 (91m:06016)). The analogous result has been obtained by Freese [10] for $\mathcal{C}$ contained in a modular congruence variety. Thus, what one should ask for is an "effective" axiomatization of each of the J-quasivarieties. Such should provide a positive answer, for the given quasivariety, of the following problem (observe that the answer is positive under the restrictions on L in Theorem 4.5)

Problem 5.3. For which of the J-quasivarieties is there an algorithm deciding membership of finite lattices?

For the J-quasivarieties and the varieties they generate, none but the obvious inclusions are known. Rather obvious non-inclusions can be established using the "characteristic identities" from [17]; for congruence varieties of modules, a complete picture has been given by Hutchinson and Czédli [20]; the related quasivariety inclusions have been analyzed by George Hutchinson in several deep papers. A remarkable non-inclusion result is due to Pálfy and Szabó [28]: an identity valid in all subgroup lattices of abelian groups but not in the normal subgroup lattice of some finite member of the variety generated by the quaternion group.

For none of the J-quasivarieties is it known whether it is a variety. However, there is a counterexample for the quasivariety of submodule lattices over a certain ring [7]. As Theorem 4.5(c) shows, the answer is yes in the following if we restrict to $\dim V_D \leq 5$.

Problem 5.4. Which of the properties (i), (iii)–(vi), and (x), if any, are inherited by homomorphic images of sublattices of lattices $L(V_D)$ with $\dim V_D$ finite (and D a finite prime field)?

Finally, we should mention that the embedding results and problems we have discussed are quite different from questions about isomorphic representations of arguesian lattices as congruence lattices. The most remarkable result of the latter kind is due to Jónsson and Monk [25]: a primary arguesian lattice L is *coordinatizable*, i.e., isomorphic to the submodule lattice of some finitely generated faithful module over a completely primary uniserial ring, provided it has geometric dimension $\mathrm{gd}(L) \geq 3$, Recall that a lattice is *primary* if it is modular of finite height, if any join irreducible p is a *cycle* (i.e., $[0,p]$ a chain), any meet irreducible h a *cocycle* (i.e., $[h,1]$ a chain), and if any interval that is not a chain has at least 3 atoms. The *breadth* of L is the maximal number of independent join irreducibles while the *geometric dimension* $\mathrm{gd}(L)$ refers only to those of maximal height. That the condition $\mathrm{gd}(L) \geq 3$ cannot be omitted in the coordinatization result has been shown by Monk [26]. We use his construction to prove the following.

Theorem 5.5. *There are finite primary lattices of height* 5*, geometric dimension* 2*, and breadth* 3 *that admit an isometric embedding into some* $L(V_D)$ *with* D *a field, but that are not isomorphic to the congruence lattice of any algebra in a congruence modular variety.*

Proof. The primary lattices L constructed by Monk are of height 5, $\mathrm{gd}(L) = 2$, and breadth 3. We say that such are of *Monk type*. The following properties are crucial; they are easy to derive (cf. e.g., [19, Section 2-4]) or to be read from Monk's construction: L is simple, contains $u \leq w$ with $\mathrm{d}(u) = 2$ and $\mathrm{d}(w) = 3$ such that $[0,u] = S(L)$ and $[w,1] = S^*(L)$ are the skeleton and the dual skeleton, and such that all blocks L_x are simple of height 3. Also, there are independent u_1, u_2, u_3 in L with $\mathrm{d}(u_i) = 2$, $[0,u_i] = \{0, p_i, u_i\}$ for $i = 1,2$, and $d(u_3) = 1$ such that $u = p_1 + p_2$, $w = u + u_3$, and $1 = u_1 + u_2 + u_3$. Moreover, u_1 and u_2 have a common complement in $[0, u_1 + u_2]$ and so do p_1 and u_3 in $[0, p_1 + u_3]$. Finally, all intervals of height and breadth 2 have the same cardinality $q + 3$, and we say that q is the *order* L.

Monk has shown that for any field D of cardinality $q = |D| \geq 3$ there are sublattices of $L(V_D)$ with $\dim V_D = 5$, that are of Monk type and order q but not coordinatizable. We show, for finite q, that the latter are also not isomorphic to the congruence lattice $L(A)$ of any algebra A in a congruence modular variety. Namely, we prove that any congruence lattice $L = L(A)$ of an algebra A in a congruence modular variety is coordinatizable, provided that it is of Monk type of finite order q.

First, observe that the u_1+u_2 and p_1+u_3 are abelian congruences, and then that A is abelian (cf. [13]). Thus, we may assume that A is a faithful module M_R, and we write $U_i = u_i$, $P_i = p_i$, $U = u$, and $W = w$. Since L is simple, all its irreducible subquotients are isomorphic as R-modules; moreover, $U_1 \cong U_2$ by perspectivity. Since M_R is faithful, we may assume

$$M = R^2 \times I,\ U_1 = R \times 0^2,\ U_2 = 0 \times R \times 0,\ \text{and } U_3 = 0^2 \times I,$$

where I is the unique proper right ideal of R; in particular, $U = I^2 \times 0$ and $W = I^3$. Thus, M_R is a submodule of $\bar{M}_R = R^3_R$ and L the interval $[0, M]$ of the height 6 lattice $\bar{L} = L(\bar{M}_R)$. Observe that $L(I^3_R)$ is isomorphic to the subspace lattice of an irreducible projective plane with the finite line $L(I^2_R \times 0)$, whence finite. Also, it follows that $\bar{L}$ is simple, too, whence any complemented height 3 interval of $\bar{L}$ is isomorphic to $L(I^3_R)$.

Let N be the radical of $\bar{M}$, i.e., the intersection of all maximal submodules. Then $N \subseteq I^3$, since the $R^2 \times I$, $R \times I \times R$, and $I \times R^2$ are maximal. On the other hand, $\bar{M}_R/N$ is generated by 3 elements (the images of the canonical generators of $\bar{M}_R$) and each element of $\bar{M}_R/N$ generates an irreducible submodule (since $L(\bar{M}_R/N)$ is complemented and since R has unique nontrivial right ideal). It follows that $L(\bar{M}_R/N)$ has height ≤ 3 and so $N = I^3$.

From $N = (\bar{M})_*$ in $\bar{L}$, it follows that $S(\bar{L}) = [0, N]$. On the other hand, $0^* \supseteq N = I^3$ since I is irreducible. Thus, $S^*(\bar{L}) \subseteq [N, \bar{M}]$ and $X \mapsto X^*$ (taken in $\bar{L}$) is an order embedding of $[0, N]$ into $[N, \bar{M}]$. Since both intervals are of height 3 and the same finite cardinality, this map is a lattice isomorphism $\psi\colon [0, N] \to [N, \bar{M}]$, and it follows that $S^*(\bar{L}) = [N, \bar{M}]$. Moreover, all $[X, X^*]$ for $X \in S(L)$ are isomorphic to the subspace lattice of the same irreducible finite projective plane. In particular, all height and breadth 2 intervals of $\bar{L}$ have $q + 1 \geq 3$ atoms. Thus, by Guidici [14] and Tesler [29] (cf. [19, §3]), $\bar{L}$ is a primary lattice. Being the submodule lattice of R^3_R, $\bar{L}$ is also arguesian and of geometric dimension 3. Thus, by [25], $\bar{L}$ is coordinatizable over some completely primary uniserial ring T. But now, $\bar{L}$ contains the 3-frame Re_i, $R(e_i - e_j)$, where e_1, e_2, e_3 is the canonical basis of R^3_R, and R as well as T have to be isomorphic to the von Neumann coordinate ring associated with this frame (cf. [9]). Thus, R is completely primary uniserial, proving coordinatizability of L. $\square$

We conclude with height 4 arguesian lattices L that are not isomorphic to any congruence lattice $L(A)$ for A in a congruence modular variety. Let L be a gluing with overlap $[u, v]$ of height 1 where $L_1 \cong L(D^3_D)$ with a field D of prime order p and L_0 of height 2 with $p + 2$ atoms. Assume $L = L(A)$ with $\gamma = 0_{L_0}$, $\beta = u$, and α a complement of β in L_0. Then the subalgebra α of A^2 also has a modular congruence lattice. As in [15], let θ_i denote the preimage of $\theta \in L(A)$ under the canonical projection π_i of α onto A for $i = 0, 1$. Then $\alpha_0 = \alpha_1$ and $\gamma_i + \beta_0 \cap \beta_1 = \beta_i$, whence $M = [\gamma_0 \cap \gamma_1, \alpha_0]$ is complemented of height 3 with $[\gamma_i, \alpha_0]$ isomorphic to L_0. It follows that M is a projective plane of order $p + 1$. If such exists at all, it is non-desarguean, contradicting [12].

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