Corrigendum: Gluings of modular lattices

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[1]

Abstract

In [2] it has been stated that for every S-cover of a lattice L there is an extension L' of L in the variety of L and a bounded S-cover of L' which restricts to the given S-cover and has each block $L'(x)$ in the variety of $L(x)$. A correct proof of this statement is given, here.

The notion of gluing of lattices resp. the decomposition of a lattice under this point of view is due to R.P.Dilworth. The latter can be captured as follows: Given lattices L and S, a *bounded S-cover* of L consists of a join homomorphism $\sigma : S \to L$ and a meet homomorphism $\pi : S \to L$ such that $L(x) := [\sigma x, \pi x] \neq \emptyset$ for all $x \in S$ and $L = \bigcup_{x \in S} L(x)$.

In order to generalize this as to include the block decomposition given by an arbitray tolerance relation, in [2] a pair σ, π is defined to be an *S-cover* of L, if $\sigma : S \to \mathsf{F}(L)$ is a join homomorphism, $\pi : S \to \mathsf{I}(L)$ a meet homomorphism, $L(x) := \sigma x \cap \pi x \neq \emptyset$ for all $x \in S$ and $L = \bigcup_{x \in S} L(x)$. Here, $\mathsf{I}(L)$ denotes the ideal lattice of L and $\mathsf{F}(L)$ the lattice of all filters ordered by dual inclusion. If σx and πx are principal for all $x \in S$ then σ and π can be considered maps from S into L and one obtains a bounded S-cover of L. As observed by E.T.Schmidt in his review [6] the following is the essence of Lemma 3.3. in [2]. Recall that $\mathsf{HSP}_u\mathcal{C}$, the class of all homomorphic images of sublattices of ultraproducts of lattices in $\mathcal{C}$, is the model class of the positive-universal theory of $\mathcal{C}$.

Proposition 1 *For every S-cover of a lattice L there is an extension L' of L in $\mathsf{HSP}_u\{L\}$ and a bounded S-cover of L' which block $L'(x)$ in $\mathsf{HSP}_u\{L(x)\}$. Moreover, this is an extension of the given S-cover, i.e. $L(x) = L'(x) \cap L$ for $x \in S$.*

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This requires an extension of L and embeddings of $\mathsf{I}(L)$ and $\mathsf{F}(L)$ into this extension. Let $\downarrow X$ and $\uparrow X$ resp. denote the ideal resp. filter generated by X and $\iota_L : L \to \mathsf{I}(L)$ and $\varepsilon_L : L \to \mathsf{F}(L)$ the canonical embeddings $\iota_l a =\downarrow a$ and $\varepsilon_L a =\uparrow a$. Recall that $\mathsf{I}(L)$ and $\mathsf{F}(L)$ belong to $\mathsf{HSP}_u\{L\}$ cf Sachs [5] and Baker and Hales [1]. We also have an embedding (lifting the embedding of L into $\mathsf{I}(L)$ to the filter lattices cf Freese [4] Lemma 5.1)

$$\eta_L : \mathsf{F}(L) \to \mathsf{FI}(L) \quad \eta F =\uparrow \{\downarrow a \mid a \in F\}$$

Observe that

$$\eta F \leq \varepsilon_{\mathsf{I}(L)} I \;\Leftrightarrow\; I \cap F \neq \emptyset \;\text{ for } I \in \mathsf{I}(L),\; F \in \mathsf{F}(L)$$

Indeed, $\eta F \leq \varepsilon_{\mathsf{I}(L)} I$ means $I \in \eta F$ i.e. $I \geq\downarrow a$ for some $a \in F$ and so $I \cap F \neq \emptyset$.

In [2] the extension has been choosen as $\mathsf{D}(L)$, the direct limit over the lattices L_n ($n \in \mathbb{N}$) with the canonical embeddings where $L_0 = L$ and $L_{n+1} = \mathsf{I}(L_n)$ if n is even, $L_{n+1} = \mathsf{F}(L_n)$ if n os odd cf Dilworth and Fresse [3]. Denoting by λ_n the associated embedding of L_n into $\mathsf{D}(L)$, we have the embeddings

$$\lambda = \lambda_1 : \mathsf{I}(L) \to \mathsf{D}(L), \quad \mu = \lambda_2 \circ \eta : \mathsf{F}(L) \to \mathsf{D}(L)$$

Given an S-cover σ, π of L define as in [2]

$$\mathsf{D}(\sigma) = \lambda \circ \sigma, \;\; \mathsf{D}(\pi) = \mu \circ \pi, \;\; \mathsf{D}(L, \sigma, \pi) = \bigcup_{x \in S} [\mathsf{D}(\sigma)(x),\; \mathsf{D}(\pi)(x)] \subseteq \mathsf{D}(L)$$

to obtain an extension of L satisfying all the requirements but $L'(x) \in \mathsf{HSP}_u\{L(x)\}$. The statement that $[\mathsf{D}(\sigma)(x),\; \mathsf{D}(\pi)(x)]$ were isomorphic to $\mathsf{D}(\sigma x \cap \pi x)$ was derived from the assumption that $\beta_2 \circ \alpha_1$ is the canonical embedding of L_1 into L_3 where $\alpha_1 : L_1 \to \mathsf{IF}(L)$ is the cononical lifting of ε_L and β_2 the canonical lifting of η. As observed by B.Jónsson, this would force that all ideals of L are principal. As a consequence, we have to pass to sublattices both of $\mathsf{D}(L)$ and $\mathsf{D}(L, \sigma, \pi)$. For this purpose we have the following lemma.

Lemma 2 *Let σ, π be a bounded S-cover of L. For each $x \in S$ let $U(x)$ be a sublattice of $L(x)$ with bottom σx and top πx. Assume that for all $x, y, z \in S$*

$$(*) \;\; a \in U(x),\; y \leq x \leq z \;\textit{ implies }\; a + \sigma z \in U(z) \textit{ and } a \cdot \pi y \in U(y)$$

Then $M = \bigcup_{x \in S} U(x)$ is a sublattice of L with S-cover σ, π and $M(x) = U(x)$.

Proof. Given $a \in U(x)$ and $b \in U(y)$ let $z = x + y$ and observe that $a + b = a + \sigma x + b + \sigma y = a + \sigma z + b + \sigma z \in U(z)$. $\square$.

Given a sublattice U of L we define

$$\mathsf{I}|U(L) = \{J \in \mathsf{I}(L) \mid \emptyset \neq J = \downarrow(I \cap U)\}$$

and $\mathsf{F}|U(L)$, dually.

Lemma 3 *$\mathsf{I}|U(L)$ is a sublattice of $\mathsf{I}(L)$ isomorphic to $\mathsf{I}(U)$ via $X \mapsto \downarrow X$ with inverse $J \mapsto J \cap U$.*

Proof. If $J_i \in \mathsf{I}|U(L)$ and $a \in J_1 \cap J_2$ then there are $a_i \in J_i \cap U$ with $a \leq a_i$ whence $a \leq a_1 a_2 \in J_1 \cap J_2 \cap U$. $\square$

Given an S-cover σ, π of L define for $x \in S$

$$\sigma^{\#}x = \uparrow\{\downarrow a \mid a \in L(x)\} = \eta\sigma x, \quad \pi^{\#}x = \uparrow \pi x = \varepsilon_{\mathsf{I}(L)}\pi x \text{ in } \mathsf{FI}(L)$$

$$\mathsf{D}^{\#}(x) = \mathsf{F}|(\mathsf{I}|(L(x))(L))$$

Clearly, $\sigma^{\#}x \in \mathsf{D}^{\#}(x)$. Now, let $a \in \pi(x)$, Since $L(x) \neq \emptyset$, choosing $b \in L(x)$ we get $a \leq a+b \in L(x)$. Thus $\pi x \in \mathsf{I}|L(x)(L)$ and so $\pi^{\#}x \in \mathsf{D}^{\#}(x)$. Observe that for all $I \in \mathsf{I}|L(x)(L)$ one has

$$\downarrow a \leq I \leq \pi x \quad \text{for some } a \in L(x)$$

whence for any filter $\mathcal{F}$ generated by a set of such ideals

$$\sigma^{\#}x \leq \mathcal{F} \leq \pi^{\#}x$$

From Lemma 3 and its dual it follows that $\mathsf{D}^{\#}(x)$ is a sublattice of $\mathsf{FI}(L)$ and isomorphic to $\mathsf{IF}(L(x))$. Thus, $\mathsf{D}^{\#}(x)$ in $\mathsf{HSP}_u\{L(x)\}$. Now, define

$$D^{\#}(L, \sigma, \pi) = \bigcup_{x \in S} D^{\#}(x).$$

Clearly, $\sigma^{\#}$ is a join homomorphism and $\pi^{\#}$ a meet homomorphism of S into $\mathsf{FI}(L)$. We claim that for $\mathcal{F}$ as above and $z \leq x \leq y$ in S one has

$$\mathcal{F} + \sigma^{\#}y \in \mathsf{D}^{\#}(y) \text{ and } \mathcal{F} \cdot \pi^{\#}z \in \mathsf{D}^{\#}(z)$$

Indeed, for any $J \in \mathcal{F} \cap \sigma^{\#}y$ there is $K \in \mathcal{F} \cap \mathsf{D}^{\#}(x)$ with $J \geq K$ and $a \in L(y)$ with $J \geq \downarrow a$ and follows $J \geq K + \downarrow a$. Now, if $b \in K + \downarrow a$ then $b \leq c + a$ for some $c \in K$ whence $c \leq d$ for some $d \in K \cap L(x)$ and so

$b \leq d + a \in L(y) \cap (K + \downarrow a)$, in other words $K + \downarrow a \in \mathsf{I}|L(y)(L)$. This proves $\mathcal{F} + \sigma^{\#} y \in \mathsf{D}^{\#}(y)$.

For the second claim, consider $J \in \mathcal{F} \cdot \pi^{\#} z$. Then $J \geq K \cap \pi z$ for some $K \in \mathcal{F}$. For any $a \in K \cap \pi z$ there is $b \in K \cap L(x)$ and $c \in L(z)$ such that $a \leq bc$. Now, $bc \in \sigma z$ since σz is a filter and $\sigma z \leq \sigma x$, Iherefore $bc \in L(z)$ proving $K \cap \pi z \in \mathsf{I}|L(z)(L)$. It follows $\mathcal{F} \cdot \pi^{\#} z \in \mathsf{D}^{\#}(z)$.

By Lemma 2 we may conclude that $\mathsf{D}^{\#}(L, \sigma, \pi)$ is a sublattice of $\mathsf{FI}(L)$ whence in $\mathsf{HSP}_u\{L\}$. Moreover, the image $\gamma(L)$ of L under the embedding $\gamma = \varepsilon_{\mathsf{I}(F)} \circ \iota_L$ is contained in $\mathsf{D}^{\#}(L, \sigma, \pi)$ and $\gamma(L(x)) = \mathsf{D}^{\#}(x) \cap \gamma(L)$ for $x \in S$. This proves the proposition. $\square$

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