On the Coordinatization of Primary Arguesian Lattices of Low Geometric Dimension

Christian Herrmann

Dedicated to Bjarni Jónsson and George S. Monk.

Technische Universität Darmstadt
Fachbereich Mathematik
email: herrmann@mathematik.tu-darmstadt.de

Abstract

Correcting claims made in Herrmann and Takach (2005), we give lattice theoretic characterizations of lattices, L, isomorphic to submodule lattices of finitely generated modules over commutative completely primary uniserial rings and of those isomorphic to subgroup lattices of finite abelian p-groups. Dealing with coordinatization over arbitrary completely primary uniserial rings, we have to exclude the case that L has breadth ≥ 3 and all but 2 basis elements are atoms. Primary Arguesian lattices L of the latter type are shown to admit a cover preserving embedding into the subspace lattice of some vector space. The approach is that of Herrmann and Takach (2005) but takes into account Monk's construction of non-coordinatizable primary Arguesian lattices of the exceptional types.

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1 Introduction

We refer to Herrmann and Takach (2005) for definitions and basic results. There it has been claimed that any Arguesian primary lattice L of geometric dimension 2 and breadth ≥ 3 is coordinatizable as the submodule lattice of some finitely generated faithful module over a completely primary uniserial (shortly CPU) ring. Though, counterexamples of type $[2,2,1]$ had been given by Monk (1969), already, cf. Herrmann (2006) and Nation (2006). Here, we say that a primary lattice L has *type* $[h_1,\ldots,h_m]$, where $h_i \geq h_{i+1}$, if $h_i = \mathsf{h}(b_i)$ is the height of b_i for some/any (ordered) basis $b_1,\ldots,b_m$ such that $\mathsf{h}(b_i) \geq \mathsf{h}(b_{i+1})$. We say that a primary lattice of type $[h_1,h_2,1,\ldots,1]$ with $h_2 > 1$ is of *Monk type*. As it turns out, the results of Herrmann and Takach (2005) remain valid for lattices L not of Monk type; for such of Monk type, to obtain complete isomorphism invariants, it suffices to add the condition of 'Monk primality', to be defined below, and to require that the socle of L is coordinatized over a field.

First, recall that, in a modular lattice L of finite height, by a^* and a_* we denote the join of all upper covers resp. the meet of all lower covers of a, with the convention that $1^* = 1$ and $0_* = 0$. Also recall that we write $a+b$ for joins and $ab = a \cap b$ for meets. Now consider a primary lattice L of type $[2,2,1]$ and an (ordered) basis b_1,b_2,b_3 of L. Let $\bot = 0$, $u = b_{1*}$, $\top = u + b_{2*}$, $S = \{\bot, u, \top\}$, and L_x the interval sublattice $[x, x^*]$, for $x \in S$. Consider

a 2-step projectivity $(b_1+b_2)/\top \cong (b_1+b_3+\top)/\top$ in $L_\top$,
a 3-step projectivity $(b_1+b_3+\top)/\top \cong (b_3+\top)/u$ in L_u,
and a 5-step projectivity $(b_3+\top)/u \cong \top/0$ in $L_\bot$.

Denote by ψ the lattice isomorphism of $[\top, b_1+b_2]$ onto $[0,\top]$ which is induced by the composition of these. We say that L is *Monk primary* if for any basis there are projectivities as above such that for the induced isomorphism ψ one has

$$\psi x = x_* \quad \text{for all } x \in [\top, b_1+b_2].$$

Call a lattice *Monk-primary* if all its primary type $[2,2,1]$ intervals are Monk primary. Observe that this property can be expressed by a first oder axiom in lattice language. As shown in Monk (1969), for any lattice $L({}_RM)$ of type $[2,2,1]$ with socle U and radical W there is a semilinear map $f : M/U \to W$ between R/P-vector spaces (P the maximal ideal of R) such that $f(X)$ is the radical of X for any $X \in [U,M]$ (so this map reveals $L({}_RM)$ as an S-glued sum over its prime skeleton $S = [0,W]$. Though, Monk primality requires f to be linear - which we could show only for lattices coordinatizable over CPU rings R with an element p such that $P = Rp$ and $p + P^2$ central in R/P^2.

The 'extended type' a of primary Arguesian lattice L of breadth ≥ 3 is $[h_1,\ldots,h_m;\hat{R}]$ where $[h_1,,\ldots,h_m]$ is its type and $\hat{R}$ is the isomorphism type of any *coordinate ring* R of L, a CPU ring such that the type $[h_3, h_3, h_3, \ldots, h_m]$ section of L is coordinatized by a faithful R-module. Observe that R is unique up to isomorphism and that R is a division ring in the case of Monk type.

The claims of §9 in Herrmann and Takach (2005) become correct if restricted to the class of primary lattices which are not of Monk type or are of Monk type and Monk primary with coordinate field. That is, extended type is a complete isomorphism invariant for this class of lattices. For lattices of type $[h_1, 1, \ldots, 1]$ this is Thm.4 in Antonov and Nazyrova (1998) and implicitly in the proof of the main result in Nation and Pickering (1987).

It follows that Monk primary Arguesian lattices of breadth ≥ 3 with commutative coordinate ring are those coordinatizable over some commutative CPU ring. From that, we derive an internal characterization of lattices isomorphic to subgroup lattices of finite abelian p-groups.

In Section 12 it is shown that a primary lattice of Monk type is Arguesian if and only if it admits a cover preserving embedding into the subspace lattice $L({}_DV)$ of some vector space.

This erratum presupposes the concepts from Herrmann and Takach (2005) and those results which are explicitely mentioned (in the form Lemma A.1.1). We will do so at the appropriate places in order to facilitate reading.

The basic approach remains unchanged; namely, to consider a (gluing) decomposition of a primary lattice L into 'blocks': maximal intervals of geometric dimension ≥ 3. Coordinatization of these yields a 'local coordinatization' of L.

The basic shortcoming of Herrmann and Takach (2005) was not to give heed to matching the bounds of blocks when reducing an isomorphism between local coordinatizations of L and L' to a local coordinatization of L: in Proposition A.1.2 and its application in Corollary A.8.2. Thus, both results need this stronger hypothesis and the latter has to be verified in the proof of the main result, Corollary A.9.2 - which follows the old lines, otherwise. The task is to define $\Phi_\top$ to obtain the proper matching of bounds of blocks. In the case not of Monk-type, $\Phi_\top$ from the old proof will do due to a simple combinatorial result presented in Section 4 (the overlaps of blocks are 'sufficiently large' to provide this proper matching for free). In the case of Monk type, the linear maps given by the definition of Monk-primality have to be used in the definition of $\Phi_\top$, commutativity of the coordinate ring R is needed to make this work.

Also, the hypotheses in Lemma A.1.1 were insufficient: one has to require, in addition, that $x \mapsto \varphi_x \sigma x$ and $x \mapsto \varphi_x \pi x$ preserve joins and meets, respectively. Though, in all applications, namely in Corollary 2.3 (stated informally in §A.1) and Lemma A.9.3, this additional hypothesis was satisfied.

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2 Gluings of lattices

Compare §A.1. Recall that a lattice L is an S-*glued sum*, S another lattice, if there are join resp. meet embeddings σ, π of S into L such that L is the union of its *blocks* $L_x = [\sigma x, \pi x]$. Lemma A.1.1 has to be corrected as follows.

LEMMA 2.1. *Let L be modular lattice of finite height and an S-glued sum of the L_x. Let L' be any lattice.*

1. *Let α and β be a join resp. meet preserving map of S into L' and, for each $x \in S$, $\varphi_x : L_x \to L'$ a homomorphism such that $\varphi_x \sigma x = \alpha x$, $\varphi_x \pi x = \beta x$ for all $x \in S$ and such that φ_x and φ_y coincide on $L_x \cap L_y$ for any $x \prec y$ in S. Then there is a unique homomorphism $\varphi : L \to L'$ extending all φ_x. φ is an embedding if so are all φ_x.*

2. *Let L' an S'-glued sum of the L'_x, $x \in S'$. Let $\delta : S \to S'$ be an isomorphism and, for each $x \in S$, $\varphi_x : L_x \to L'_{\delta x}$ an isomorphism such that φ_x and φ_y coincide on $L_x \cap L_y$ for any $x \prec y$ in S. Then there is a unique isomorphism $\varphi : L \to L'$ extending all φ_x.*

1. is a consequence of the fact that such L has a presentation given by the lattice structure of the L_x, the join relations $\sigma x + \sigma y = \sigma(x + y)$ and the meet relations $\pi x \cap \pi y = \pi(x \cap y)$ where $x, y \in S$. In a special case, this was already observed in Lemma 3.4. of Jonsson (1954)
Proof. 1. We first show that φ_x and φ_y coincide on $L_x \cap L_y$ for any x, y in S. Observe that $L_x \cap L_y \subseteq L_{xy} \cap L_{x+y}$ since $a \in L_x \cap L_y$ implies $\sigma(x + y) = \sigma x + \sigma y \leq a \leq \pi x \cap \pi y = \pi(x \cap y)$. Thus, it suffices to consider the case $x < y$ and the claim follows by induction on the length of a maximal chain in the interval $[x, y]$ of S. It follows that there is a unique map $\varphi : L \to L'$ extending all φ_x.

We show in several steps that φ is join preserving. First, observe that

$$\varphi\sigma x + \varphi\sigma y = \alpha x + \alpha y = \alpha(x + y) = \varphi\sigma(x + y).$$

Now, we show by induction on the length of a maximal chain in $[x,z]$ that

$$\varphi(a+\sigma z)=\varphi a+\varphi\sigma z \quad \text{ for } a\in L_x \text{ and } x\leq z.$$

If $x=z$ we use that φ_x preserves joins. Let $x\leq y\prec z$. Then $\sigma z\in L_y$ and $\varphi(a+\sigma z)=\varphi_y(a+\sigma y+\sigma z)=\varphi_y(a+\sigma y)+\varphi_y\sigma z=\varphi a+\varphi\sigma y+\varphi\sigma z=\varphi a+\varphi\sigma z$. Finally, consider $a\in L_x$ and $b\in L_y$ and let $z=x+y$. Then $a+b\in L_z=[\sigma z,\pi z]$ since σ preserves joins and π preserves order. It follows $\varphi(a+b)=\varphi(a+\sigma z+b)=\varphi_z(a+\sigma z+b)=\varphi_z(a+\sigma z)+\varphi_z(b+\sigma z)=\varphi a+\varphi\sigma z+\varphi b=\varphi a+\varphi\sigma x+\varphi\sigma y+\varphi b=\varphi(a+\sigma x)+\varphi(a+\sigma y)=\varphi a+\varphi b$. By duality, φ is meet preserving.

Assume that the φ_x are embeddings. To prove φ an embedding, it suffices to show that $a\leq b$ and $\varphi a=\varphi b$ jointly imply $a=b$. We may assume $a\in L_x$ and $b\in L_z$ with $x\leq z$. Again, we use induction. The claim is obvious for $x=z$. Let $x\leq y\prec z$. Then $\varphi a\leq\varphi(a+\sigma z)=\varphi a+\varphi\sigma z\leq\varphi b=\varphi a$ whence $a=a+\sigma z$ by inductive hypothesis since $a+\sigma z\in L_y$. Thus, we have $a\in L_z$ and apply the injectivity of φ_z.

2. Assume that L' is an S'-glued sum via σ' and π'. Observe that the hypotheses of 1. are satisfied with $\alpha x=\sigma'\delta x=\varphi_x\sigma x$ and $\beta x=\pi'\delta x=\varphi_x\pi x$ since the bottom and top elements of L_x and $L'_{\delta x}$ are matched by φ_x. And φ is surjective since so are the φ_x and since L' is required to be the union of the images of the φ_x. □

Recall from Section A.1 that a *local coordinatization* of an S-glued sum L associates with each $x\in S$ a coordinatization R_x, M_x, ω_x of $L_x=[\sigma x,\pi x]$, i.e. an isomorphism $\omega_x: L_x\to L({}_{R_x}M_x)$. The associated *gluing maps*

$$\gamma_{xy}:[\omega_x\sigma y)_{L({}_{R_x}M_x)}\to(\omega_y\pi x]_{L({}_{R_y}M_y)}$$

are isomorphisms given as the restrictions of $\omega_y\omega_x^{-1}$. A *linear* local coordinatization is such that $R_x=R$ for all x and all gluing maps are induced by linear isomorphisms between subquotients. Given an S'-glued sum L' (with maps σ' and π') with linear local coordinatization R, M'_x, ω'_x and gluing maps γ'_{xy}, a *linear isomorphism* (by abuse of language we say: of L onto L') is constituted by an isomorphism $\delta: S\to S'$ and linear isomorphisms

$$\Phi_x:\ {}_RM_x\to\ {}_RM'_{\delta x}\quad(x\in S)$$

such that (with induced lattice isomorphisms $\hat{\Phi}_x$)

$$\hat{\Phi}_y\gamma_{xy}=\gamma'_{\delta x\delta y}\hat{\Phi}_x|_{[\omega_x\sigma y)}\quad \text{for } x\prec y \text{ in } S$$

or, equivalently,

$$\gamma'^{-1}_{\delta x\delta y}\hat{\Phi}_y|_{\mathsf{im}\gamma_{xy}} = \hat{\Phi}_x\gamma^{-1}_{xy}.$$

PROPOSITION 2.2. *Given an isomorphism $\delta : S \to S'$ and linear local coordinatizations R, M_x, ω_x $(x \in S)$ of L and R, M'_x, ω'_x $(x \in S')$ of L', the R-linear isomorphisms $\Phi_x : {}_RM_x \to M'_{\delta x}$ $(x \in S)$ constitute a linear isomorphism if and only if the isomorphisms $\hat{\Phi}_x\omega_x : L_x \to L({}_RM'_{\delta x})$ $(x \in S)$ constitute a linear local coordinatization of L such that*

$$(*)\ \hat{\Phi}_x\omega_x\sigma y = \omega'_{\delta x}\sigma'\delta y \quad \textit{and} \quad (**)\ \hat{\Phi}_y\omega_y\pi x = \omega'_{\delta y}\pi'\delta x \textit{ for all } x \prec y \textit{ in } S.$$

Proof. Assume that the Φ_x are given. Consider $x \prec y$ in S. Since $\hat{\Phi}_x$ is an isomorphism, $(*)$ means that the image under $\hat{\Phi}_x$ of the domain $[\omega_x\sigma y)_{L({}_RM_x)}$ of γ_{xy} coincides with the domain $[\omega'_{\delta x}\sigma'\delta y)_{L({}_RM'_{\delta x})}$ of $\gamma'_{\delta x\delta y}$, i.e. that $\hat{\Phi}_x$ restricts to an isomorphism ψ_x between these intervals. Similarly, since $\hat{\Phi}_y$ is an isomorphism, $(**)$ means that the image under $\hat{\Phi}_y$ of the image $(\omega_y\pi x]_{L({}_RM_y)}$ of γ_{xy} is the image $(\omega'_{\delta y}\pi'\delta x]_{L({}_RM'_{\delta y})}$ of $\gamma'_{\delta x\delta y}$, i.e. that $\hat{\Phi}_y$ restricts to an isomorphism χ_x between these intervals. Also observe that $\gamma_{xy} = \omega_{yx}\omega^{-1}_{xy}$ where ω_{xy} and ω_{yx} are the restrictions of ω_x and ω_y to $L_x \cap L_y = [\sigma y, \pi x]_L$.

Now, assume that $(*)$ and $(**)$ hold. Then

$$\chi_y\gamma_{xy} = \gamma'_{\delta x\delta y}\psi_x \text{ if and only if } \gamma'_{\delta x\delta y} = \chi_y\gamma_{xy}\psi_x^{-1}.$$

The identity required for linear isomorphims follows since

$$\hat{\Phi}_y\gamma_{xy} = \chi_y\gamma_{xy} \text{ and } \gamma'_{\delta x\delta y}\hat{\Phi}_x = \gamma'_{\delta x\delta y}\psi_x$$

and

$$\chi_y\gamma_{xy}\psi_x^{-1} = \chi_y\omega_{yx}\omega^{-1}_{xy}\psi_x^{-1} = \chi_y\omega_{yx}(\psi_x\omega_{xy})^{-1} = \hat{\Phi}_y\omega_y(\hat{\Phi}_x\omega_x)^{-1}.$$

It remains to derive $(*)$ and $(**)$ from the definition of linear isomorphism. First, observe that

$$\gamma'_{\delta x\delta y}\omega'_{\delta x}\sigma'\delta y = 0_{L({}_RM'_{\delta y})}$$

since R, M'_z, ω'_z, $(z \in S')$ is a linear local coordinatization of the S'-glued sum L'. As $\hat{\Phi}_y : L({}_RM_y) \to L({}_RM'_{\delta y})$ is an isomorphism, it follows that

$$\hat{\Phi}_y^{-1}\gamma'_{\delta x\delta y}\omega'_{\delta x}\sigma'\delta y = 0_{L({}_RM_y)} = \gamma_{xy}\omega_x\sigma y.$$

Thus,

$$\gamma'_{\delta x\delta y}\omega'_{\delta x}\sigma'\delta y = \hat{\Phi}_y\gamma_{xy}\omega_x\sigma y = \gamma'_{\delta x\delta y}\hat{\Phi}_x\omega_x\sigma y$$

by the definition of linear isomorphism; now, injectivity of $\gamma'_{\delta x\delta y}$ implies $(*)$. In particular, as observed above, the domain of $\gamma'_{\delta x\delta y}$ is the image under $\hat{\Phi}_x$ of the domain of γ_{xy}. It follows that

$$(\omega_{\delta y}\pi'\delta x] = \mathsf{im}\gamma'_{\delta x\delta y} = \mathsf{im}(\gamma'_{\delta x\delta y}\hat{\Phi}_{x|\mathsf{dom}\gamma_{xy}})$$

$$= \mathsf{im}(\hat{\Phi}_y\gamma_{xy}) = \hat{\Phi}_y(\mathsf{im}\gamma_{xy}) = \hat{\Phi}_y((\omega_y\pi x])$$

whence $(**)$. ☐

COROLLARY 2.3. *Any linear isomorphism between linear local coordinatization of lattices L and L' induces a (unique) lattice isomorphism $\varphi : L \to L'$ such that $\varphi_{|L_x} = {\omega'}^{-1}_{\delta x}\hat{\Phi}_x\omega_x$.*

These lattice isomorphisms φ have been called 'locally linear isomorphisms' in Herrmann and Takach (2005) if σ and σ' are identity maps and and $\delta = \varphi|S$. In the sequel, we prefer to call such *locally linear lattice isomorphisms.*
Proof. $\varphi_x = {\omega'}^{-1}_{\delta x}\hat{\Phi}_x\omega_x$ is an isomorphism of L_x onto $L'_{\delta x}$. We have to verify the compatibility condition in the hypothesis of Lemma 2.1 (2). So consider $x \prec y$ in S and $a \in L_x \cap L_y$. Then

$$\varphi_y\ a = {\omega'}^{-1}_{\delta y}\hat{\Phi}_y\omega_y\ a = {\omega'}^{-1}_{\delta y}\hat{\Phi}_y\gamma_{xy}\omega_x\ a =$$

$$= {\omega'}^{-1}_{\delta y}\gamma'_{\delta x\,\delta y}\hat{\Phi}_x\omega_x\ a = {\omega'}^{-1}_{\delta x}\hat{\Phi}_x\omega_x\ a = \varphi_x\ a. \quad \square$$

3 Semi-primary lattices

§A2 is valid without changes. We give a summary. Given a basis $b_1, \dots, b_m$ of a semi-primary lattice, one has each $C_i = (b_i]$ a chain and the union of these chains generates a distributive cover preserving sublattice D of L, with $\prod_{i=1}^m C_i$ isomorphic to D via $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m x_i$. The elements of D are said to *fit* into the basis $b_1, \dots, b_m$. Given two such, $u \le v$ where $v = \sum_{i=1}^m v_i$ with $v_i \in C_i$, the $u+v_1, \dots, u+v_m$ form a basis of $[u, v]$ (omitting the terms $u + v_i$ where $u \le v_i$), the *induced basis* of $[u, v]$. Unless stated otherwise, bases will be *ordered*, i.e. with heights $h_i = \mathsf{h}(b_i) \ge \mathsf{h}(b_{i+1})$ and $[h_1, \dots, h_m]$ the *type* of L. This convention does not apply to induced bases.

4 Geometric decomposition

§A.3 and §A.4 remain valid and will be heavily used. In the sequel, L will be always semi-primary of type $[h_1, \ldots, h_m]$ and the concepts of gluing decomposition of L into blocks L_x, of local coordinatization, and linear isomorphism will always refer to the geometric decomposition according to the 'geometric skeleton' $S_+(L)$ - with smallest and greatest elements $\bot = 0$ and $\top = 1_+$. The following recall from Theorem A.4.1 and Corollaries A.4.2 and A.4.4 the basic facts about the geometric decomposition. $\mathsf{tp}(u)$ denotes the type of $[0, u]$.

COROLLARY 4.1. *Let $b_1, \ldots, b_m$ be any basis of L, $m \geq 2$. Let $n = h_3$ if $m \geq 3$ and $n = 1$ otherwise.*

1. *The geometric skeleton and its dual are given as $S_+(L) = (1_+] = (b_1^{h_1-n} + b_2^{h_2-n}]$ and $S^+(L) = [0^+) = [b_1^n + b_2^n + \sum_{i>2} b_i)$.*

2. *The maps $x \mapsto x^+$ and $y \mapsto y_+$ are mutually inverse isomorphisms between $S_+(L)$ and $S^+(L)$. L is a $S_+(L)$-glued sum of its intervals $L_x = [x, x^+]$, $x \in S_+(L)$.*

3. *For $y \in S^+(L)$ and $x \in L$ one has $x = y_+$ if and only if $[x, y]$ is of type $[h_3, h_3, h_3, \ldots, h_m]$ in case $m \geq 3$, of type $[1, 1]$ in case $m = 2$.*

4. *Let $m \geq 3$. $S_+(L)$ is a chain if and only if $h_1 > h_2 = h_3$. Otherwise, $S_+(L)$ is primary of breadth 2. $S_+(L)$ is of height and breadth 2 if and only if $h_1 = h_2 = h_3 + 1$.*

5. *L is of Monk type if and only if $m \geq 3$, $S_+(L)$ is of breadth 2, and $a^+ = a^*$ for all $a \in L$. In this case, also $a_+ = a_*$ for all $a \in L$, the geometric skeleton and the prime skeleton coincide, and so do their duals.*

COROLLARY 4.2. *If u is a coatom of $S_+(L)$ then u is of one of the types below and there is an ordered basis $a_1, \ldots, a_m$ of L with h_i such that*

$$\begin{array}{lll} u = a_1^{h_1-n-1} & u^+ = a_1^{h_1-1} + \sum_{i>1} a_i & \text{if } \mathsf{tp}(u) = [h_1 - n - 1], \\ u = a_1^{h_1-n-1} + a_2^{h_2-n} & u^+ = a_1^{h_1-1} + \sum_{i>1} a_i & \text{if } \mathsf{tp}(u) = [h_1 - n - 1,\ h_2 - n], \\ u = a_1^{h_1-n} + a_2^{h_2-n-1} & u^+ = a_2^{h_2-1} + \sum_{i\neq 2} a_i & \text{if } \mathsf{tp}(u) = [h_1 - n,\ h_2 - n - 1]. \end{array}$$

COROLLARY 4.3.

1. *If $u \leq v$ in $S_+(L)$ then $[u, v^+]$ is of breadth m and $[u, v] = S_+([u, v^+])$.*
2. *If L is not of Monk type and $u \leq v$ in $S_+(L)$ then $[u, v^+]$ is also not of Monk type.*
3. *If L is of Monk type and $[u, v]$ a height and breadth 2 interval in $S_+(L)$ then $[u, v^+]$ is of type $[2, 2, 1, \ldots, 1]$.*
4. *If u is a coatom in $S_+(L)$ then $S_+(L) \subseteq (u^+]$.*

Proof. 1. follows from Corollary 4.1 (1) and (2). By Corollary 4.1 (3), in case of $m \geq 3$ one has all $[x, x^+]$, $x \in S_+(L)$ of type $[h_3, h_3, h_3, \ldots, h_m]$ and $h_3 = 1$ if and only if L is of Monk type. 3. By Corollary 4.1 (1) one has $\mathsf{h}(b_i) = 2$ for $i = 1, 2$. 4. If u is a coatom of $S_+(L)$ then u^+ is a coatom of L (since $S^+(L)$ is an upper section). By Lemma A.2.3 there is a (unordered) basis $b_1, b_2, \ldots$ of L such that $u^+ = b_{1*} + \sum_{i>1} b_i$. Thus, $1_+ \leq u^+$ being the meet of the $b_j^{h_j - h_3} + \sum_{i \neq j} b_i$ (cf. Corollary 4.1 (1)); whence $S_+(L) = (1_+] \subseteq (u^+]$. □

We add some new material in the context of §A.4. Within a given L we define, inductively,

$$a_{\circledast 0} = a, \quad a_{\circledast k+1} = (a_{\circledast k})_* = (a_*)_{\circledast k-1}.$$

For any $u \in L$ and $a \leq u$ the definitions of $a_{\circledast k}$ in L and $(u]$ coincide.

LEMMA 4.4. *Let L be of type $[n, n]$ and $k < n$. Then for any $x \geq 1_*$ in L one has $[x_{\circledast k}, x]$ of type $[k, k]$.*

Proof. Given $x \geq 1_*$, there is a basis b_1, b_2 of L such that $x = b_1^{n-i} + b_2^{n-j}$ with $i, j \in \{0, 1\}$. Any basis will do if $x \in \{1, 1_*\}$; otherwise, due to breadth 2, x is a coatom of L and such basis exists by Lemma A.2.3. By induction it follows that $x_{\circledast k} = b_1^{n-i-k} + b_2^{n-j-k} =: x_k$. Indeed, $[x_k, x_{k-1}]$ is of height and breadth 2 whence $x_k = (x_{k-1})_*$ since L has breadth 2. Now, $b_1^{n-i} + x_k$, $b_2^{n-j} + x_k$ is a basis of $[x_k, x]$. □

LEMMA 4.5. *Let $m \geq 3$ and $n = h_3$. Then $h_1 = h_2 = n + 1$ if and only if $S_+(L)$ is of height and breadth 2. Assume $h_1 = h_2 = n+1$. Then for any basis $b_1, \ldots, b_m$ of L and $c = \sum_{i>2} b_i$ one has that $x \mapsto x(b_1 + b_2)$ and $y \mapsto y + c$*

are mutually inverse isomorphisms between $S^+(L)$ *and* $[b_1^n + b_2^n,\, b_1 + b_2]$. *Moreover,*

$$x_+ = (x(b_1+b_2))_{\circledast n} \quad \textit{for all } x \in S^+(L).$$

Also, for $n > 1$ *(i.e. in non-Monk type) one has*

$$(x(b_1+b_2))_* \in L_\top \quad \textit{for all } x \in S^+(L).$$

Proof. First, observe that the sum $(b_1+b_2)+c$ is direct and apply Corollary 4.1 (1). Given $x \in S^+(L) = [b_1^n + b_2^n + c,\, 1]$ we have $x \geq c$ and, by Lemma 4.4, $[y_{\circledast n}, y]$ of type $[n,n]$ where $y = x(b_1+b_2) \geq b_1^n + b_2^n = (b_1+b_2)_*$. Thus, $[y_{\circledast n}, x]$ is of type $[h_3, h_3, h_3, \ldots h_m]$ whence $y_{\circledast n} = x_+$ in view of Corollary 4.1 (3). And, if $n > 1$ then $y_* \geq (b_1^n + b_2^n)_* = b_1^{n-1} + b_2^{n-1} \geq b_1^1 + b_2^1 = \top$. □

LEMMA 4.6. *Let* L, L' *be semi-primary of the same type non-Monk type as in Lemma 4.5. Given bases* $b_1, \ldots, b_m$ *of* L *and* $b'_1, \ldots, b'_m$ *of* L'*, assume that there are isomorphisms* $\varphi_\perp : L_\perp \to L'_{\perp'}$ *and* $\varphi_\top : L_\top \to L'_{\top'}$ *preserving the induced bases and such that*

$$\varphi_{\perp | L_\perp \cap L_\top} = \varphi_{\top | L_\perp \cap L_\top}.$$

Then

$$\varphi_\top x = (\varphi_\perp(x_+))^+ \quad \textit{for all } x \in S^+(L).$$

Proof. From Cor.4.1.1 we have $\top = b_1^1 + b_2^1$, $\perp^+ = b_{1*} + b_{2*} + \sum_{i>2} b_i$, and the analogues for L'. Also, by Cor.4.1.1, for $x \in S^+(L)$ we have $x' := \varphi_\top x \in S^+(L')$. By Lemma 4.5 it follows

$$x_+ = z_{\circledast h_3 - 1} \text{ where } y = ((x(b_1+b_2)),\ z = y_* \text{ in } L$$

$$x'_+ = z'_{\circledast h_3 - 1} \text{ where } y' = ((x'(b'_1+b'_2)),\ z' = y'_* \text{ in } L'.$$

Observe that $y \in L_\top$, $y' \in L'_{\top'}$, and $\varphi_\top y = y'$. Now, $[z, y]$ and $[z', y']$ are breadth and height 2 intervals of L and L' but, by Lemma 4.5, also of $L_\top$ and $L'_{\top'}$, respectively; thus, $y_* = z$ and $y'_* = z'$ hold also within the latter and it follows $\varphi_\top z = z'$. By definition, $z \leq (b_1+b_2)_* = b_{1*} + b_{2*} \in L_\perp$ whence $\varphi_\perp z = z'$ and it follows $\varphi_\perp x_+ = \varphi_\perp z_{\circledast h_3 - 1} = z'_{\circledast h_3 - 1} = x'_+$. The claim is now a consequence of Corollary 4.1 (2). □

LEMMA 4.7. *If* L *is Monk primary of type* $[2, 2, 1, \ldots]$ *and* $b_1, \ldots, b_m$ *a basis of* L *with projective isomorphism* $\psi : [b_{1*} + b_{2*},\ b_1 + b_2] \to [0, b_{1*} + b_{2*}]$ *witnessing Monk primality w.r.t. the basis* b_1, b_2, b_3 *of* $(b_1 + b_2 + b_3]$*, then*

$$x_+ = \psi(x(b_1+b_2)) \quad \textit{for all } x \in S^+(L).$$

Proof. By Lemma 4.5 we have $x_+ = (x(b_1+b_2))_*$. □

5 Completely primary uniserial rings

Recall that a ring is completely primary uniserial (CPU) if it has a maximal ideal P such that $P^n = 0$ for some (smallest) n, the *rank* of R, and such that any left and and right ideal is of the form P^k for some k. Then $R^* = R \setminus P$ is the group of units of R. We presuppose all results of §A.5 and add the following.

COROLLARY 5.1. *If R is a commutative CPU ring then for any $k \geq$ rank R there is a commutative CPU ring S such that R is a homomorphic image of S.*

Proof. The monoid M in the proof of Theorem A.5.2 has a set of pairwise commuting generators. $\square$

6 Submodule lattices

In the sequel, let R be a CPU ring, ${}_RM$ be a faithful finitely generated R-module, and $L({}_RM)$ its lattice of submodules. L is said to be *coordinatized over* R if $L \cong L({}_RM)$. The type of $L({}_RM)$ is also referred to as the *type* of ${}_RM$. Observe that for any type $[h_1, \ldots, h_m]$ there is ${}_RM$ of this type if and only if R is of rank h_1 (e.g. $M = \prod_{i=1}^m P^{h_1 - h_m}$); moreover, ${}_RM$ is unique up to linear isomorphism in this case.

We make use of all results of §A.6. Though, observe that in the definition of 'axis' Y and Z have to be interchanged and that X is an axis for Y and Z if and only if there is a linear map $\varphi : Y \to Z$ such that $X = \{y + \varphi y \mid y \in Y\}$. This observation also can be used to prove Lemma A.6.3. We complete the results of §A.6 by one with focus on Monk-primality.

LEMMA 6.1. *The lattice $L({}_RM)$ is Monk-primary provided that R contains an element p such $P = Rp$ is the maximal ideal of R and $p + P^2$ central in R/P^2.*

Proof. Since intervals of $L({}_RM)$ are isomorphic to some $L({}_SN)$, S a homomorphic image of R and ${}_RN$ a subquotient of ${}_RM$, it suffices to consider the case that $L = L({}_RM)$ is of type $[2, 2, 1]$ with basis b_1, b_2, b_3. Then $P^2 = 0$ and p is central in R. Choose a basis e_1, e_2, e_3 of ${}_RM$ such that $b_i = Re_i$. Then $b_{i*} = Rpe_i$. Put

$$A = b_{1*} + b_{2*} = Rpe_1 + Rpe_2, \quad B = b_1 + b_2 = Re_1 + Re_2$$

$$C = b_{1*} + b_{2*} + b_3 = Rpe_1 + Rpe_2 + Re_3.$$

The projectivities $\hat{\Psi}_x$, $x = \top, u, \perp$, required by Monk primality are induced by the linear isomorphisms Ψ_x composed of the canonical isomorphisms given by the following lists of transpositions

$$B/A \nearrow M/(R(e_2 - e_3) + C) \searrow (Re_1 + Re_3 + A)/A$$

$$(Re_1+Re_3+A)/A \searrow (Re_1+Re_3)/Rpe_1 \nearrow (Re_1+C)/R(e_1-pe_2) \searrow C/Rpe_1$$

$$C/Rpe_1 \searrow (Rpe_2 + Re_3)/0 \nearrow C/R(pe_1 - pe_2) \searrow$$

$$\searrow (Rpe_1 + Rpe_3)/0 \nearrow C/R(pe_2 - e_3) \searrow A/0.$$

Observe that

$$\Psi_\top(e_1 + A) = e_1 + A, \quad \Psi_\top(e_2 + A) = e_3 + A$$

$$\Psi_u(e_1 + A) = pe_2 + Rpe_1, \quad \Psi_u(e_3 + A) = e_3 + A$$

$$\Psi_\perp(pe_2 + Rpe_1) = pe_1, \quad \Psi_\perp(e_3 + A) = pe_2.$$

Thus,

$$\Psi(e_i + A) = pe_i \text{ for } i = 1, 2 \text{ where } \Psi = \Psi_\perp \Psi_u \Psi_\top.$$

On the other hand, since p is central, $\Pi v = pv$ defines an R-linear map of B onto A with kernel A, so that we have the induced isomorphism $\tilde{\Pi} : B/A \to A$. Since Ψ and $\tilde{\Pi}$ coincide on the basis $e_1 + A$, $e_2 + A$ of B/A, we conclude $\Psi = \tilde{\Pi}$. Finally, consider $A < X < B$. Then $0 < \Pi(X) < A$. Since $L(B)$ is of type $[2, 2]$ and X of height 3 there are cycles Y and Z of height 2 and 1 such that $X = Y \oplus Z$. Thus $\hat{\Psi}(X) = \Pi(X) = pY + pZ = pY = Y \cap (pY \oplus Z) = X_*$, the radical of X. □

7 2-gluings.

§A.7 deals with the gluing of two intervals of geometric dimension ≥ 3. These results are curcial and will be completed by the following.

COROLLARY 7.1. *Let L, L' be primary Arguesian of type $[h_1, \ldots, h_m]$ with $m \geq 3$ and $h_2 = h_3 = h_1 - 1$, in particular $S_+(L) = \{\perp, \top\}$ and $S_+(L') = \{\perp', \top'\}$. Let R, M_x, ω_x and $R, M'_{x'}, \omega_{x'}$ be linear local coordinatizations and $b_1, \ldots, b_m$ and $b'_1, \ldots, b'_m$ be bases of L and L' respectively. Let $\Phi_\perp : M_\perp \to M'_{\perp'}$ be a linear isomorphism matching the induced bases of $L_\perp$ and $L'_{\perp'}$. Then there exists $\Phi_\top : M_\top \to M'_{\top'}$ such that $\Phi_\perp, \Phi_\top$ constitute a locally linear lattice isomorphism matching the given bases.*

Proof. With $c = b_1$ and $c' = b'_1$ this follows from Corollary A.7.3, immediately.

COROLLARY 7.2. *The coordinate ring R of a primary Arguesian lattice of Monk type is a division ring.*

Proof. By Cor.4.1 (5), any basis elements of $L_\perp$ are of height 1, whence $P = 0$ for the maximal ideal of R. $\square$

8 2×2-gluings.

Theorem A.8.1 and Corollary A.8.3 remain unchanged but Corollary A.8.2 has to be corrected as follows.

COROLLARY 8.1. *Given primary Arguesian lattices L and L' of breadth ≥ 3 and of the same type with geometric skeletons of height and breadth 2, let S and S' be 4-element sublattices of $S_+(L)$ and $S_+(L')$, respectively. Consider linear local coordinatizations R, M_x, ω_x of L_S and $R, M'_{x'}, \omega'_{x'}$ of $L'_{S'}$ over the same R, linear isomorphisms $\Phi_x : {}_RM_x \to {}_RM'_{\delta x}$, an isomorphism $\delta : S \to S'$, and an atom v of S such that*

$$(\delta v)^+ = {\omega'}^{-1}_{\top'} \hat{\Phi}_\top \omega_\top \ (v^+).$$

Then the Φ_x yield a linear isomorphism between L_S and $L'_{S'}$ provided they yield linear isomorphisms between L_U and $L'_{\delta(U)}$ where $U = S \setminus \{v\}$ and $U = \{\perp, v\}$.

Proof. Let u be the second atom of S, $u' = \delta u$, and $v' = \delta v$. By Proposition 2.2, the conditions $(*)$ and $(**)$ are satisfied for all $x \prec y$ except $v \prec \top$. For the latter, $(**)$ is satisfied by hypothesis since $\pi v = v^+$ and $\pi' v' = {v'}^+$. By Proposition 2.2 the $\hat{\Phi}_x \omega_x$ yield linear local coordinatizations of the L_U, whence by Theorem A.8.1 also for L. Thus, in order to apply Proposition 2.2 in the converse direction, it suffices to verify $(*)$ for $v \prec \top$.

First, observe that $\sigma \top = \top \in \bigcap_{x \in S} L_x$ and $\sigma' \top' = \top' \in \bigcap_{x \in S'} L'_x$. Thus, in particular,

$$\omega'_{v'} \top' = \gamma'_{\perp' v'} {\gamma'}^{-1}_{\perp' u'} \omega'_{u'} \top'.$$

Now, $(*)$ for $u \prec \top$ reads as

$$\omega'_{u'} \top' = \hat{\Phi}_u \omega_u \top$$

and, indeed,

$$\omega'_{v'}\top' = \gamma'_{\perp' v'}{\gamma'}^{-1}_{\perp' u'}\hat{\Phi}_u\omega_u\top = \gamma'_{\perp' v'}\hat{\Phi}_\perp\gamma^{-1}_{\perp u}\omega_u\top = \hat{\Phi}_v\gamma_{\perp v}\gamma^{-1}_{\perp u}\omega_u\top = \hat{\Phi}_v\omega_v\top$$

since the Φ_x constitute linear isomorphisms between the L_U and $L_{\delta(U)}$. $\square$

Also, we complete the section focussing on Monk-primality.

LEMMA 8.2. *Let L and L' be Monk primary of the same type $[2, 2, 1, \ldots]$ with linear local coordinatizations R, M_x, ω_x, $x \in S(L)$ and R, M'_x, ω'_x, $x \in S(L')$ where R is commutative. Let $b_1, \ldots, b_m$ and $b'_1, \ldots, b'_m$ be bases of L and L', $u = b_{1*}$, $u' = b'_{1*}$, and let $\Phi_\perp : {}_RM_\perp \to {}_RM'_{\perp'}$ and $\Phi_u : {}_RM_u \to {}_RM'_{u'}$ induce a linear local lattice isomorphism $\varphi : L_\perp \cup L_u \to L'_{\perp'} \cup L'_{u'}$ such that $\varphi b_i = b'_i$ for $i \neq 2$ and $\varphi b_{2*} = b'_{2*}$. Then $\varphi(S(L)) = S(L')$ and there is $\Phi_\top : {}_RM_\top \to {}_RM'_{\top'}$ such that Φ_x ($x \in S = \{\perp, u, \top\}$) induce a linear local lattice isomorphism $\bar{\varphi}$ of L_S onto $L'_{S'}$, $S' = \varphi(S)$, such that*

$$\bar{\varphi}(b_2 + \top) = b'_2 + \top' \quad \textit{and} \quad \bar{\varphi}x = (\bar{\varphi}(x_+))^+ \quad \textit{for all } x \in S^+(L).$$

Proof. For any $x \in S$, the projectivity within L_x, given by Monk primality, determines a linear isomorphism Ψ_x between the corresponding subquotients of ${}_RM_x$. Then

$$\Psi = \Psi_\perp\Gamma^{-1}_{\perp u}\Psi_u\Gamma^{-1}_{u\top}\Psi_\top : \omega_\top(b_1 + b_2) \to \omega_\perp\top$$

is an R-linear isomorphism and, in view of Lemma 4.7,

$$\hat{\Psi}(x(b_1 + b_2)) = x_+ \quad \text{for all } x \in S^+(L).$$

Similarly, we have Ψ' for L'. Thus,

$$\Omega = {\Psi'}^{-1}\Phi_\perp\Psi : \omega_\top(b_1 + b_2) \to \omega'_\top(b'_1 + b'_2)$$

is an R-linear isomorphism such that $\hat{\Omega}b_i = b'_i$ for $i = 1, 2$ and

$$(*) \quad \hat{\Omega}(x(b_1 + b_2)) = (\varphi(x_+))^+)(b'_1 + b'_2) \ \text{ for all } x \in S^+(L).$$

On the other hand,

$$\Phi = \Gamma'_{u'\top'}\Phi_u\Gamma^{-1}_{u\top} : \omega_\top u^+ \to \omega_{\top'}{u'}^+$$

is a linear isomorphism such that $\hat{\Phi}(b_1) = b'_1 = \hat{\Omega}(b_1)$. Choose a basis $\mathbf{e}^\top$ of $M_\top$ such that $Re^\top_i = \omega_\top b_i$, let ${e'}^{\top'}_i = \Phi e_i$ for $i \neq 2$ and ${e'}^{\top'}_2 = \Omega e^\top_2$. This yields a basis of $M'_\top$ such that $R{e'}_i^{\top'} = b'_i$. Then $\Omega e_1^\top = r{e'}_1^{\top'}$ for some unit r in R. Define the linear isomorphism $\Phi_\top : M_\top \to M'_{\top'}$ by

$$\Phi_\top e_i^\top = {e'}_i^{\top'} \text{ for } i \neq 2 \text{ and } \Phi_\top e_2^\top = r{e'}_2^{\top'}.$$

Then $\Phi_\top$ extends Φ so that $\Phi_\perp, \Phi_u, \Phi_\top$ constitute a locally linear lattice isomorphism $\bar{\varphi}$ as required. On the other hand, due to commutativity of R, the scalar multiple $r\Omega$ is an R-linear map which coincides with $\Phi_\top$ on the domain of Ω. Therefore, $\hat{\Phi}_\top$ extends $\hat{\Omega}$. The claim follows by $(*)$ and Lemma 4.5. □

9 Isomorphism invariants

Recall that any modular lattice L of finite height h and breadth ≤ 2 admits a cover preserving embedding into the subspace $L({}_DV)$ if $h = \dim_D V$ and $|D| + 3 \geq |I|$ for any height 2 interval I. In particular, such lattices are Arguesian. Theorem A.9.1 and Corollary A.9.2 have to be restated taking into account the requirement of Monk-primality.

THEOREM 9.1. *Extended type is a complete isomorphism invariant for each of the following classes of primary lattices*

1. *uniform primary lattices of breadth* ≤ 2*;*
2. *primary Arguesian lattices of breadth* ≥ 3 *and not of Monk type;*
3. *Monk-primary Arguesian lattices of Monk type with coordinate field.*

COROLLARY 9.2. *For any two lattices* L, L'*, as in Thmeorem 9.1, of the same extended type and for any ordered bases of* L *and* L' *there is a basis preserving isomorphism. Moreover, if* L, L' *are of breadth* ≥ 3 *with given linear local coordinatizations then the isomorphism can be chosen locally linear.*

Proof. The reasoning of Section A.9 is valid if L is of breadth ≤ 2: observe that in the proof of Lemma A.9.3 the hypotheses of the corrected Lemma 1.1 are verified; an alternative proof for this case is given in Section 11. The latter can be seen as an easy variant of the proof for the case of breadth ≥ 3.

In the case of L, L' of breadth ≥ 3 and the same extended type, in view of Corollaries A.6.6 and A.8.3 we may choose linear local coordinatizations M_x $(x \in S_+(L))$ and M'_x $(x \in S^+(L'))$ of L and L' over the same CPU ring R. We show the following by induction on $S_+(L)$.

(i) If $u \in S_+(L)$ and $u' \in S_+(L')$ are coatoms of the same type fitting into bases of L and of L' according to Corollary 4.2, then every locally linear lattice isomorphism $\varphi : (u^+] \to (u'^+]$ preserving the induced bases can be extended to a basis preserving locally linear lattice isomorphism of L onto L'.

(ii) If $u \in S_+(L)$ and $u' \in S_+(L')$ are coatoms of the same type then every locally linear lattice isomorphism of $(u^+]$ onto $(u'^+]$ can be extended to a locally linear lattice isomorphism of L onto L'.

(iii) For any ordered bases of L and L' there exists a basis preserving locally linear lattice isomorphism of L onto L'.

We prove these claims by induction on the height of $S_+(L)$. Observe that by Corollary 4.1 (1) $S_+(L)$ and $S_+(L')$ are of the same type, too. If $S_+(L)$ is 1-element, then (i) and (ii) are void and ${}_RM_\perp \cong {}_RM'_\perp$ since they are of the same type; moreover, any ordered bases can be matched via a linear isomorphism.

In the inductive step, consider the premise of (i). By Corollary 4.3 (4) we have $S_+(L) \subseteq (u^+]$ and $S_+(L') \subseteq (u'^+]$. Since φ matches bases, due to Corollary 4.1 (1), it restricts to an isomorphism $\delta : S_+(L) \to S_+(L')$. Let φ be induced by the linear isomorphism $\Phi_x : {}_RM_x \to {}_RM'_{\delta x}\,(x \in (u])$. We have to define Φ_x for the remaining $x \in S_+(L)$ so that we obtain a linear isomorphism inducing a basis preserving locally linear lattice isomorphism.

If $S_+(L)$ is a chain, then u and $u' = \varphi u$ are the unique coatoms of $S_+(L)$ and $S_+(L')$. Apply Corollary 7.1 to the lattices $[u)$ and $[u')$ with the induced bases to obtain $\Phi_\top$ and so the required extension of φ.

Otherwise, both $S_+(L)$ and $S_+(L')$ are primary of breadth 2, and admit a unique second coatom v resp. v' fitting into the given basis; thus $\delta v = \varphi v = v'$. Let $w = u \cap v = \top_*$ and $w' = u' \cap v' = \delta w = \top'_*$. In particular, $S_+(L) \subseteq [0, w^+]$ and $S_+(L') \subseteq [0, w'^+]$, again by Corollary 4.3 (4). Observe that, by Corollary 4.3 (1), $[w)$ resp. $[w')$ have geometric skeletons $[w, \top]$ and $[w', \top']$ of height and breadth 2 and ordered bases $b_1, b_2, , b_3, \ldots$ and $b'_1, b'_2, b'_3, \ldots$ induced by the given ones of L and L', respectively, such that $u \leq b_1$, $v \leq b_2$, $u' \leq b'_1$, and $v' \leq b'_2$ are atoms in $[w)$ and $[w')$, respectively. Moreover, δ restricts to an isomorphism of $S_+([w))$ onto $S_+([w'))$.

In case L not of Monk type, choose $\Phi_\top$ applying Corollary 7.1 to $[u)$ and $[u')$ with the induced bases. In case of Monk type choose $\Phi_\top$ applying Lemma 8.2 to $[w)$ and $[w')$ with the above bases. Thus, $\Phi_x\,(x \in S)$ yield a locally linear lattice isomorphism χ from L_S onto $L'_{S'}$ where $S = \{w, u, \top\}$ and $S' = \{w', u', \top'\}$. Moreover, by Lemma 4.6 resp. 8.2 we conclude that χ restricts to an isomorphism of $S^+([w)$ onto $S^+([w'))$ such that $\chi x = (\chi x_+)^+$. Thus by Corollary 4.1 (2)

$$\omega'^{-1}_{\top'} \hat{\Phi}_\top \omega_T x^+ = \chi x^+ = (\varphi x)^+ = (\delta x)^+ \quad \text{for all } x \in S_+([w)).$$

Now, for any coatom $z \neq u$ of $S_+(L)$ we may apply the inductive hypotheses to the coatoms $w \in (z]$ and $w' \in (\delta z]$ and the restriction of φ to $(w^+]$. Namely, we apply (i) for $z = v$ and (ii), else. Thus, for each $z \in S_+(L)$ we have a well defined $\Phi_z : {}_R M_z \to {}_R M'_{\delta z}$. The compatibility condition $\hat{\Phi}_y \gamma_{xy} = \gamma'_{xy} \hat{\Phi}_{x\,|\mathsf{dom}\gamma_{xy}}$ is satisfied a fortiori if $y \neq \top$ (since $y \leq z$ for some coatom z of $S_+(L)$) or if $x = u$; in case $y = \top$ and $x \neq u$ we apply Corollary 8.1 with $S = \{w, u, x, \top\}$.

The induced isomorphism of L onto L' is basis preserving, since the isomorphisms on $(u^+]$ and $(v^+]$ are basis preserving and since the basis of L is contained in $(u^+] \cup (v^+]$. To prove (ii) just choose bases to fit u and u' according to Corollary 4.2 and apply (i). To prove (iii) choose coatoms u and u' fitting into the induced bases of $S_+(L)$ and $S_+(L')$, analogously. In particular, u and u' are of the same type and so are u^+ and u'^+. By the inductive hypothesis (ii) there is a locally linear lattice isomorphism φ of $(u^+]$ onto $(u'^+]$ preserving the induced bases. Hence, we can apply (i) to get the required isomorphism of L onto L'. □

10 Coordinatization

Theorem A.10.1 and Corollary A.10.2 have to be restated as follows.

THEOREM 10.1.

1. *For any extended type $[k, l; q]$, $q \geq 2$, there exist up to isomorphism exactly one semi-primary lattice L of this type. L is coordinatizable, if and only if q is infinite or of the form $q = p^d + 1$, p a prime. In this case, one has L coordinatizable over $F[x]/(x^k)$, F any field such that $|F| + 1 \geq q$.*

2. *For any extended non-Monk type $[h_1, \ldots, h_m; \hat{R}]$, $m \geq 3$, there is up to isomorphism exactly one primary Arguesian lattice L of this type. In particular, a primary lattice L of non-Monk type and breadth ≥ 3 is*

coordinatizable ($L \cong L({}_S M)$) if and only if it is Arguesian; here, S can be chosen commutative if R is so.

3. *For a primary lattice L of Monk type $[h_1, h_2, 1, \ldots]$ the following are equivalent*

   (i) L is Monk primary Arguesian with commutative coordinate ring.

   (ii) L is coordinatizable over S with $p \in S$ such that $P = Sp$ is the maximal ideal of S and $p + P^2$ central in S/P^2.

   (iii) L is coordinatizable over $F[x]/(x^{h_1})$, F the coordinate field of L.

Proof. 1. This is the first claim of Theorem 10.1 Herrmann and Takach (2005) and the proof, given there, applies. In 2. choose by Thmeorem A.5.2 a CPU ring S having rank of L and R as a homomorphic image (and S commutative if R is so, cf. Corollary 5.1). Choose ${}_S M$ of the type of L. Then L and $L({}_S M)$ have the same extended type and Theorem 9.1 applies to yield $L \cong L({}_S M)$.

In 3. (iii) implies (ii), trivially, and (i) follows from (ii) by Lemma 6.1. If (i) holds, and if ${}_S M$ is chosen of the type of L where $S = F[x]/(x^{h_1})$, then Theorem 9.1 applies to prove $L \cong L({}_S M)$. □

Recall (cf. Freese (1979)) that a (von Neumann) 3-frame in a modular lattice is given by independent elements a_1, a_2, a_3 and common complements $c_{ij} = c_{ji}$ of a_i and a_j in $(a_i + a_j]$ for $i \neq j$ such that $c_{ik} = (a_i + a_j)(c_{ij} + c_{jk})$ for pairwise distinct i, j, k. We say that the frame is of *height* n if the a_i are cycles of height n. Following von Neumann, there are lattice terms $\oplus$ and $\otimes$ in variables x, y, x_i, x_{ij} and, for any 3-frame in a modular lattice one considers the set of complements of a_2 in $(a_1 + a_2]$ endowed with the binary operations $(r, s) \mapsto \oplus(r, s, a_1 \ldots, c_{23})$ and $(r, s) \mapsto \otimes(r, s, a_1 \ldots, c_{23})$: this is the *coordinate domain* of the frame. The frame has *characteristic* k if $c_l(a_1, \ldots, c_{23}) = a_1$ if and only if $l = k$, where the terms $c_k = c_k(x_1, \ldots, x_{23})$ are inductively defined by

$$c_1 = x_{12}, \quad c_{k+1} = c_k \oplus x_{12}.$$

For any free R-module with basis e_1, e_2, e_3 one obtains a 3-frame $a_i = Re_i$ and $c_{ij} = R(e_i - e_j)$ such that $r \mapsto R(e_1 - re_2)$ is an isomorphism of R onto the coordinate domain. In particular, R has characteristic k if and only if the coordinate domain of the frame has characteristic k.

COROLLARY 10.2. *Let L be a primary lattice of type $[h_1, \ldots, h_m]$.*

1. *If L is Arguesian of breadth ≥ 3 then the coordinate ring is isomorphic to the coordinate ring of any 3-frame of height h_3 in L.*

2. *L is coordinatizable over some commutative CPU ring if and only if one of the following holds*

   (1) L is of breadth 2 and q-uniform with q infinite of $q = p^d + 1$ for some d and prime p.

   (2) L is of breadth ≥ 3, Monk primary, Arguesian, and any height 3 frame has commutative coordinate domain.

3. *For a fixed prime p, L is isomorphic to the subgroup lattice of some finite abelian p-group if and only if one of the following holds*

   (1) L is of breadth 2 and $p+1$-uniform for some prime p.

   (2) L is of breadth ≥ 3, Arguesian, not of Monk type, $p+1$-uniform, and (*): if there is a 3-frame of height n but none of height $n+1$ then some/any 3-frame of height n has characteristic p^{h_3}.

   (3) L is of Monk type, Arguesian, Monk primary, and $p+1$-uniform.

4. *Except primality, all lattice properties in 2. and 3. can be expressed by sets of first order formulas. Considering lattices of height bounded by given h, all properties in 3. (including primality) can be expressed by a single formula.*

The characterization of subgroup lattices of finite abelian groups is now immediate via the primary decomposition.
Proof. 1. follows from the preceding remarks. In 2. the breadth 2 case is dealt with by Theorem 9.1 (1). In the breadth ≥ 3 case, necessity follows from Lemma 6.1, while sufficiency follows from Theorem 9.1 (2) and (3).

3. Again, necessity is obvious: in (2) observe that any frame of height n has to belong to $L_\perp$ whence $n \leq h_3$ and that $L_\perp$ contains 3-frames of height h_3. To show sufficiency via Theorem 9.1 we verify that L and $L(\prod_{i=1}^m \mathbb{Z}/(p^{h_i})$ have the same extended type. This is obvious in (1) while in (3) $L_\perp \cong L({}_F F^m) \cong L((\mathbb{Z}/(p))^m$ with p-element field F. It remains to show that in (2) the coordinate ring R is isomorphic to $\mathbb{Z}/(p^n)$, $n = h_3$. In $L({}_R M_\perp) \cong L_\perp$ one has a 3-frame of height n associated with a basis of ${}_R M_\perp$. Let R_0 denote the smallest subring of R. By hypothesis, $|R_0| = p^n$ whence $R_0 \cong \mathbb{Z}/(p^n)$. Having the same extended type $[n, n; p+1]$, the lattices $L({}_R(Re_1 +$

Re_2)), and $L((\mathbb{Z}/(p^n))^2$ are isomorphic with $Re_i \mapsto (\mathbb{Z}/(p^n))e_i$ (Cor.9.2). This isomorphism provides a bijection between the sets of complements of Re_2 and $(\mathbb{Z}/(p^n))e_2$ respectively and it follows $|R| = p^n = |R_0|$ whence $R = R_0$.

4. Obviously, the Arguesian law, Monk primality, and q-uniformity for fixed q can be expressed by a single formula, each. The same holds for commutativity of the coordinate domain of a 3-frame, for 'breadth $\geq m$' (existence of boolean sublattice with m atoms), and for Monk type: there exists a 3-frame of height 1 but none of height 2. To express uniformity, one needs one formula for each finite $q \geq 3$: if there is some interval of height and breadth 2 with q elements then all such intervals must have q elements. Similarly, excluding q not of the form $p^d + 1$ one needs one formula for each p and d. Finally, in (*) the second condition is given by a single formula whereas the first needs infinitely many. If h is a bound on height, in (*) it suffices to consider $n \leq h$; also, in defining primality, it suffices to consider joins and meets of at most h elements. □

The following is Corollary A.10.4.

COROLLARY 10.3. *A primary lattice is Arguesian if and only if it admits a cover preserving embedding into a coordinatizable lattice.*

Proof. If the primary Arguesian lattice L is not coordinatizable, then it is either of of Monk type or of extended type $[h_1, h_2; q]$ with q finite. In the first case, the claim, follows from Thmeorem 12.1, below. In the second, there is a cover preserving embedding of L into the subspace lattice of some vector space (cf. Herrmann (1973)). □

11 Semi-primary lattices of breadth 2

In this and in the following section we consider semi-primary lattices L of breadth 2 or of Monk type. In particular, $S_+(L) = S(L)$ is the prime skeleton, $a^+ = a^*$ for all a, and the blocks L_x of the geometric decomposition are the maximal complemented intervals of L.

LEMMA 11.1. *Let L be semi-primary of breadth 2, L' modular with $\mathsf{h}(L) = \mathsf{h}(L')$, and $\tau : [1_*, 1] \to [t, 1']$ an embedding into a height 2 interval of L'. Assume that for any coatom u of L there is given an embedding $\psi_u : (u] \to (\tau u]$ such that ψ_u and ψ_v coincide on $(1_*]$ for all u, v. Then there is a unique* *embedding $\psi : L \to L'$ extending all ψ_u. Moreover, if L' is also semi-primary of breadth 2 and if τ and the ψ_u are isomorphisms then ψ is an isomorphism.*

Proof. $\psi a = \psi_u a$ for $a \in (u]$ and $\psi 1 = 1'$ is a well defined injective map. The ψ_u are cover preserving, hence so is ψ and it suffices to show that ψ preserves meets. $\tau u > \psi_u 1_* = \psi_v 1_* < \tau v$ for any two coatoms of L whence $\psi 1_* = t$. Thus, given $a \leq u$ and $b \leq v$, we have $\psi a \cap \psi b = t \cap \psi a \cap \psi b = \psi 1_* \cap \psi a \cap \psi b = \psi_u(a \cap 1_*) \cap \psi_v(b \cap 1_*) = \psi_u(a \cap 1_*) \cap \psi_u(b \cap 1_*) = \psi_u(a \cap 1_* \cap b \cap 1_*) \leq \psi(a \cap b)$. The remaining meets are obviously preserved.

If L' is also semi-primary of breadth 2 and if all ψ_u and τ are isomorphisms, then ψ is bijective and restricts to an isomorphism $\delta : S(L) \to S(L')$ (in view of $t = 1'_*$) and we may apply Lemma 1.1 with $\varphi_{1_*} = \tau$ and $\varphi_z = \psi_u|L_z$ if $z \leq u_*$, u a coatom of L - observe that $z^* \leq u$ whence ψ_u maps L_z onto a height 2 subinterval of $L'_{\delta z}$ and equality follows from $L'_{\delta z}$ having height 2. $\square$

LEMMA 11.2. *Extended type is a complete isomorphism invariant for uniform semi-primary lattices of breadth* 2.

Proof. We show the following by induction on height.

(i) If $u \in L$ and $u' \in L'$ are coatoms of the same type fitting into bases of L and of L' according to Corollary 4.2, then every isomorphism $\psi_u : (u] \to (u']$ preserving the induced bases can be extended to a basis preserving isomorphism of L onto L'.

(ii) If $u \in L$ and $u' \in L'$ are coatoms of the same type then every isomorphism ψ_u of $(u]$ onto $(u']$ can be extended to an isomorphism of L onto L'.

(iii) For any ordered bases of L and L' there exists a basis preserving isomorphism of L onto L'.

The cases, where $u = 0$ or L is a chain, are trivial. Otherwise, we may choose bases a_1, a_2 and a'_1, a'_2 such that $u = a_1 + a_{2*}$ and $u' = a'_1 + a'_{2*}$. Consider an isomorphism $\psi_u : (u] \to (u']$ mapping a_1 onto a'_1 and a_{2*} onto a'_{2*} (such exists by inductive hypothesis (ii)). By Corollary4.1 (1), ψ_u restricts to an isomorphism of $S(L)$ onto $S(L')$. Define the isomorphism $\tau : [1_*, 1] \to [1'_*, 1']$ by $\tau v = (\psi_u v_*)^*$ for $v \prec 1$. Then v and τv are of the same type, whence by inductive hypothesis (iii) for each $v \prec 1$ there is an isomorphism $\psi_v : (v] \to (\tau v]$ extending $\psi_u|(1_*]$. Lemma 11.1 yields the extension ψ to an isomorphism of L onto L'. Moreover, for $v = a_{1*} + a_2$, in choosing ψ_v we may apply inductive hypothesis (i) to the basis a_{1*}, a_2 and a'_{1*}, a'_2 to obtain $\psi a_2 = a'_2$. $\square$

12 Embedding results

Recall that any modular lattice of breadth 2 and finite height admits a cover preserving embedding into $L({}_DV)$, D any division ring such that $|D| \geq |[x, x^*]| - 3$ for all x in the prime skeleton $S(L)$ (cf. Herrmann (1973)).

THEOREM 12.1. *Let L be a primary Arguesian lattice of breadth $m \geq 3$ and extended type $[h_1, h_2, 1, \dots, 1; \hat{D}]$. Then D is a division ring and L admits an embedding into $L({}_DV)$, $\dim_D V = \mathsf{h}(L)$.*

Proof. By Corollary4.1 (1), $S_+(L) = S(L)$ is primary of breadth ≤ 2; observe that $m = \mathsf{h}(L_x)$, $x \in S(L)$. By Corollary 7.2, D is a division ring. We show the following by induction on $\mathsf{h}(L) = \dim_D V$ for any given linear local coordinatization D, M_x, ω_x of L.

(i) There are cover preserving embeddings $\delta : S(L) \to L({}_DV)$ and $\delta^* : S^*(L) \to L({}_DV)$, $\delta 0 = 0$, $\delta^* 1 = V$, and a linear isomorphism Φ_x ($x \in S(L)$) between the given local coordinatization of L onto the canonical one of the sublattice $\bigcup_{x \in S_+(L)} [\delta x, \delta^* x^*]$ of $L({}_DV)$ - observe that $\dim_D \delta^* x^* / \delta x = m$.

(ii) Given coatoms u of $S_+(L)$ and U of $L({}_DV)$, and δ_u, δ^*_u, and Φ_x as in (i) w.r.t. $(u^*]$ (with D, M_x, ω_x, $x \in S(u^*]$)) and the vector space U, there is an extension to L as required in (i).

The case $S(L) = \{0\}$ is obvious. In the inductive step, given coatoms u of $S(L)$ and U of $L({}_DV)$ apply the inductive hypothesis (i) to provide the data assumed in (ii). Continue to derive the claim of (i) and (ii). By Corollary 2.3 these data yield an embedding φ of $(u^*]$ into $L({}_DU)$. Observe tat $\delta_u x = \varphi x$ for $x \in S(u^*]$ and $\top \leq u^*$ by Corollary 4.3 (4). Define $\delta x = \varphi x$ for $x \in S(L)$, $\delta^* x = \delta^*_u x^*$ for $x \leq u$, and $\delta^* 1 = V$. Choose $\Phi_\top$ according to Corollary A.7.2 to obtain a linear isomorphism between the linear local coordinatizations of $L_u \cup L_\top$ and $[\delta u, U] \cup [\delta \top, V] \subseteq L({}_DV)$.

The proof is complete if u is the unique coatom of $S(L)$. Otherwise, choose a second coatom v of $S(L)$ and let $w = uv$. Choose a coatom W

in $[\delta\top, U]$. Then there is an isomorphism $f : [w, \top]$ onto $[W, V]$ such that $fu = U$. Define $\delta^* x = fx$ for $x \in [w, \top]$. Now, any coatom x of $S(L)$ is an atom of $[w, \top]$ and for $x \neq u$ inductive hypothesis (ii) applies to $(x^*]$ and the coatom w of $S(x^*] = (x]$. This yields $\delta^* y$ and Φ_y for $y \leq x$, $y \not\leq w$. By Lemma 11.1, δ^* is an embedding of $S^*(L)$ into $[\varphi 0^*, V]$. By Corollary 8.1, the Φ_y ($y \in S_x = \{w, u, x, \top\}$) yield a linear isomorphism between the local coordinatizations of L_{S_x} and of $\bigcup_{y \in S_x} [\delta y, \delta^* y] \subseteq L({}_D V)$. Thus, these combine to a linear isomorphism as required in (i) and (ii). $\square$

13 Discussion

The result that semi-primary lattices are those having prime skeleton a chain has first been published in Appendice D of Giudici (1995) and by Tesler (1995); it has been refined by Regonati and Sarti (2000). Coordinatizability of primary Arguesian lattices of geometric dimension ≥ 3 has first been shown by Monk (1966) in the von Neumann approach. A direct proof that any primary lattice of geometric dimension ≥ 4 is Arguesian has been given Monk (1969). A thorough discussion of coordinatizability questions can be found in Giudici (1995). Primary lattices of breadth 2 which are isomorphic to subgroup lattices of finite abelian groups have been also characterized in Anishchenko (1965). An 'external' characterization of subgroup lattices of abelian groups is given by Contiu (2012), based on the characterization of subgroup lattices of groups, due to Yakovlev (1974), and referring to an embedding into a sufficiently rich lattice.

In Antonov and Nazyrova (2005) it has been claimed that any primary Arguesian lattice of geometric dimension ≥ 3 can be embedded into the subspace lattice of some vector space. A counterexample is given by the subgroup lattice of A^3, A the cyclic group of order 4, cf. Nation (2006b). The same counterexample applies to the claim of Antonov and Nazyrova (2002) that any finite primary Arguesian lattice L of geometric dimension ≥ 3 can be coordinatized over a factor ring of the polynomial ring $F[x]$, F the coordinatizing field for the socle of L.

Finally, we have to discuss primary Arguesian lattices L of Monk type $[h_1, \ldots, h_m; \hat{D}]$. As shown, above, there is up to isomorphism exactly one Monk primary lattice of this type (and one can deduce coordinatizability) if D is commutative.

For the case of type $[2, 2, 1, \ldots; \hat{D}]$ primary Arguesian lattices, L, it has been proposed by the referee that the following two equivalent conditions (motivated by Monk's example) should characterize coordinatizability:

(i) For some/any D, D' of isomorphism type $\hat{D}$ and for some/any isomorphisms $\omega_\perp : L_\perp \to L({}_DV_\perp)$ and $\omega_\top : L_\top \to L({}_{D'}V_\top)$ there is a semilinear bijection $\psi : {}_{D'}V_\top/\omega_\top(0^*) \to \omega_\perp(\top) \subseteq {}_DV_\perp$ such that $\omega_\perp(z_*) = \psi(\omega_\top(z))$ for all $z \in S^*(L)$.

(ii) For some/any 3-frames $a_i^\perp, c_{ij}^\perp$ and $a_i^\top, c_{ij}^\top$ in height 3 subintervals $[0,u]$ of $L_\perp$ and $[v,1]$ of $L_\top$ such that $a_3^\top = 0^*$, $a_i^\perp = (a_i^\top + 0^*)_*$ for $i = 1, 2$, and $c_{12}^\perp = (c_{12}^\top + 0^*)_*$ one has that the map $z \mapsto (z + 0^*)_*$ is an isomorphism of coordinate rings of the frames.

Observe that (ii) can be expressed by a sentence in the first order language of lattices. The characterization implies that for $|D| \leq 4$ any L is coordinatizable (since any isomorphism between $L({}_DV)$ and $L({}_{D'}V')$, $\dim {}_DV = 2$, is induced by a semilinear map) while for $|D| > 5$ Monk's construction applies to yield non-coordinatizables.

The equivalence of (i) and (ii) (and of 'some' and 'any') follows from the Fundamental Theorem of Projective Geometry and basic facts on coordinatization via frames. As observed by the referee, for $L = L({}_RM)$ we have $X_* = \text{ rad } X = pX$ and the map $f : M/S \to \text{rad}M$, $S = \text{soc M}$, between D-vector spaces, $D = R/P$, is α semilinear where α is such that $rp \in p\alpha(r+P)$. Thus, L satisfies (i). And, conversely, any pair (D, α) occurs, up to isomorphism, in this way: define R as the skew polinomial ring $D[p]$ with additional relations $p^2 = 0$ and $rp = p\alpha(r)$ for $r \in D$.

Now, consider Arguesian L of fixed type $[2, 2, 1, \ldots, \hat{D}]$. Let us say that L is of *type* (D, α) if L admits a linear local coordinatization $\omega_x : L_x \to L({}_DV_x)$ for which (i) takes place with an α-semilinear map. In view of Corollary A.8.3, any L we consider satisfies (i) if and only if it is of some type (D, α). Clearly, type (D, α) is preserved under locally linear lattice isomorphism: given such onto L' with $\omega'_x : L'_x \to L({}_DV'_x)$ define $\psi' = \Phi_\perp \psi \tilde{\Phi}_\top^{-1}$ where $\tilde{\Phi}_\top : V_\top/\omega_\top(0^*) \to V'_\top/\omega'_\top(0^*)$ is induced by $\Phi_\top$. We claim that lattices L, L' of the same type (D, α) are isomorphic - and this will finish the proof of the characterization.

In view of Thmeorem 12.1 we may assume that L is given as the union of intervals $[\psi(X), X]$ of $L({}_DV)$, $X \in [U, V]$, where $\dim {}_DV = \mathsf{h}(L) = m + 2$, codim $U = 2 = \dim W$, and $\psi : V/U \to W$ an α-semilinear bijection. Similarly for L'. Choose an isomorphism $\varphi_U : U \to U'$ such that $\varphi(W) = W'$.

Choose v_1, v_2 such that $v_1 + U, v_2 + U$ is a basis of V/U. Now, choose v_1', v_2' such that $\psi'(v_i') = \varphi_0(\psi(v_i + U))$ for $i = 1, 2$. Then φ_0 extends to a linear isomorphism φ such that $\varphi(v_i) = v_i'$. Let $\tilde{\varphi} : V/U \to V'/U'$ the induced linear isomorphism. It follows that the α-semilinear maps $\varphi\psi$ and $\psi'\tilde{\varphi}$ coincide since they do so on the basis v_1, v_2. Thus, there is a linear isomorphism connecting the data of L and that of L', whence $L \cong L'$.

PROBLEM 13.1 *Considering coordinatizable lattices of Monk type in general, is there an internal characterization or a characterization as cover preserving sublattices of lattices $L({}_DV)$?*

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