On the number of join irreducibles and acyclicity in finite modular lattices

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Abstract. Bases of lines provide useful presentations of finite height modular lattices, acyclic ones being related to amenable properties in equational and representation theory. It is shown that some (equivalently: any) base of L is acyclic if and only if L has exactly $2d(L) - s(L)$ join irreducibles; moreover, that this is the minimal possible number for any L. Here $d(L)$ denotes the height and $s(L)$ the number of maximal congruences of L. We elaborate the proof outlined in [5], parts of which also had been reworked by Wild [13].

1. Introduction

That finite height modular lattices can be understood in terms of some geometry, was well known to principal contributors to the theory, just to mention Baer, Dilworth, and Jónsson: The set of *points* is the set $J(L)$ of join irreducibles and lines l correspond to (certain) joins $\bar{l}$, the *line-tops*, of two points. Kurinnoi [11] and Benson and Conway [1] showed that indeed this geometry determines the lattice up to isomorphism - in [1] a *line* l is a maximal at least 3-element set of points any two of which have the same join $\bar{l}$. If Λ is some set of lines, then one may consider the "partially ordered partial linear space" $(J(L), \leq, \Lambda)$ and its *subspaces*, i.e. order ideals X of $J(L)$ such that $l \subseteq X$ for all $l \in \Lambda$ with $|X \cap l| \geq 2$. These form a closure system on $J(L)$, whence a lattice $L(J(L), \leq, \Lambda)$. The above mentioned results now read: L is isomorphic to $L(J(L), \leq, \Lambda)$ (via $a \mapsto \{p \in J(L) \mid p \leq a\}$ if Λ is the set of all lines. An axiomatic characterization of these geometries was given in [4].

In their proof that any 2-distributive L admits a representation in the subspace lattice of some vector space, Jónsson and Nation [10] studied for a given coatom u the geometry induced on the set J'' of points $p \not\leq u$. In particular they showed that for lines l_1, l_2 with a common point in J'' either the line-tops $\bar{l}_1$ and $\bar{l}_2$ are comparable or that a failure of dual 2-distributivity occurs which then yields an irreducible height 3 interval and projectivity within $[0, u]$ for points below u incident with l_1 resp. l_2. This result became crucial for the generalization of the geometric description to spatial lattices [6] and for applications [5, 7].

Acyclic lattices were introduced by Marcel Wild in his thesis [12]: he showed that any index preserving isomorphism between two acyclic lattices of subspaces of some vector space may be induced by a linear isomorphism - a property of good use in the classification of orthogonal geometries in infinite dimensions. Also, much

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of the general theory appears the first time in [12]. The most important change of view is that, as far as the isomorphism onto the lattice of subspaces is concerned, the set of all lines may be replaced by certain much smaller subsets, the *bases* Λ of lines, where for any line-top a there is exactly one $l \in \Lambda$ with $\bar{l} = a$. The join relations given by a base of lines together with the order on points provide a (rather small) presentation of L as a join-semilattice - cf. (3) below. The price is absence of an axiomatic description.

Pairwise distinct $l_1, \ldots, l_m$, $m \geq 3$, form a *cycle* if $l_i \cap l_j \neq \emptyset$ if and only if $j - i \equiv \pm 1 \ \mathsf{mod}\ m$. A set Λ of lines is *acyclic* if it consists of 3-element lines and is cycle free (in [13], cycle free Λ are called 'acyclic'). L is *acyclic* if it admits some acyclic base of lines.

These investigations have been continued in [5], focussing on equational and representation theory. A sample of combinatorial results is the following where $d(L)$ is the height, $s(L)$ the number of maximal congruence relations, $j(L) = |J(L)|$, and $i(L)$ the number of line-tops i.e. $i(L) = |\Lambda|$ where Λ is any base of lines of L.

Theorem 1. *Given a modular lattice L of finite height and a base Λ of lines the following hold.*

(i) *Λ is cycle free and finite iff $i(L) \leq d(L) - s(L)$ iff $i(L) = d(L) - s(L)$.*
(ii) *Λ is acyclic and finite iff $j(L) \leq 2d(L) - s(L)$ iff $j(L) = 2d(L) - s(L)$.*

Corollary 2. *$i(L) \geq d(L) - s(L)$ and $j(L) \geq 2d(L) - s(L)$ in any finite modular lattice.*

Corollary 3. *If some base of lines is finite and cycle free resp. acyclic, then so are all.*

Corollary 4. *L is acyclic if and only if so is its dual.*

Corollary 5. *If L admits a finite cycle free base of lines, then L is 2-distributive.*

Observe that Corollary 2 and Corollary 3 follow from the Theorem, immediately, and Corollary 4 referring to Dilworth's Theorem that $j(L)$ is also the number of meet irreducibles. A direct proof of the second part of Corollary 2 has been given by Wild [13].

The purpose of the present note is to give a unified proof referring to the introductory section 2 of [5], only, together with some elementary combinatorics and well known facts about modular lattices - to be presented in sections 2-5. The proof of Theorem 1, to be given in section 6, follows the basic ideas of [5, Sect.5-6]: To match the minimality of $d(L) - s(L)$ resp. $2d(L) - s(L)$ with a structural characterization, taking advantage of Dilworth's Theorem, the Jónsson-Nation analysis, and Huhn's characterization of 2-distributivity.

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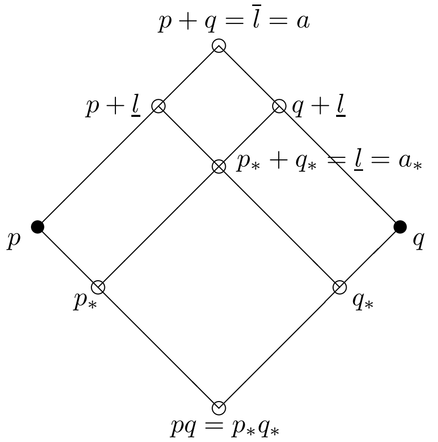
2. Modular lattice geometry

We follow Crawley and Dilworth [2] for general concepts of lattice theory but write $a + b$ for joins and ab for meets. We write $x \prec y$ if x is a lower cover of y. Join

FIGURE 1



irreducibles will be denoted by $p, q, r, \ldots$ (with unique $p_* \prec p$) and are also called *points*. We write $p \approx q$ if and only if p/p_* is projective to q/q_*. We call a map ϕ from $J(L)$ into a complete join semilattice M a Λ-*morphism*, if $\phi r \leq \phi p + \phi q$ for any pairwise distinct $p, q, r \in l \in \Lambda$. The following are basic for the geometric view on modular lattices of finite height.

(1) For any $a, b \in L$ and $r \in J(L)$ with $r \leq a+b$, $r \not\leq a$, and $r \not\leq b$, there are $p \leq a$ and $q \leq b$ in $J(L)$ such that p, q, r are collinear.

(2) For any incomparable $p, q \in J(L)$, one has a sublattice of L satisfying the relations of Fig.1; also $d([p_* + q_*,\, p+q]_L) = 2$. For any line l, there is $\underline{l}$ such that $\underline{l} = p_* + q_*$ for all $p, q \in l$, $p \neq q$. Moreover, $[\underline{l}, \overline{l}]$ has as least 3 atoms, $p \approx q$ for all $p, q \in l$, $p \neq q$, and $x \succ \underline{l}$ for any $x \prec \overline{l}$. Also, choosing for each $x \prec \overline{l}$ a point $p_x \leq x$, $p_x \not\leq \underline{l}$, one obtains a line g such that $\overline{g} = \overline{l}$.

(3) If Λ is a base of lines for L and $\phi : J(L) \to M$ a Λ-morphism, then $\sigma a = \sum\{\phi p \mid p \in J(L), p \leq a\}$ defines a join homomorphism $\sigma : L \to M$.

For a line-top a we write $a_* = \prod\{x \mid x \prec a\}$, i.e. $a_* = \underline{l}$ if l is a line with $a = \overline{l}$. Also, for $p \in J(L)$, we write $p \in a$ if $a_* \prec p + a_*$, i.e. if $p \in l$ for some line l with $\overline{l} = a$.

Proof. (1) is the dual of Corollary 2 in Edmondson [3] - which deals even with the more general setting of ideal lattices of modular lattices. It can also be derived, iterating the procedure given by [4, Thm.4.2] (cf. [5, Lemma 2.1]). The first claim in (2) is immediate by modularity, the rest by [5, Corollary 2.3]. The proof of (3) follows the lines of the proof of [5, Thm.2.5] as remarked in the comment to [6, Thm.6.1]. For convenience, it is carried out, here. Clearly, σ is order preserving and, in view of (1), it suffices to show that $r \leq p+q$ implies $\sigma r \leq \sigma p + \sigma q$ for any $p, q, r \in J(L)$. This is done by order induction on the sum of the ranks of $p+q$, $p+r$, and $q+r$. Again by (1), we may assume that p, q, r are collinear. Since Λ is a base, in view of (2) there are $l \in \Lambda$ and $p', q', r' \in l$ such that $p + \underline{l} = p' + \underline{l}$, $q + \underline{l} = q' + \underline{l}$, and $r + \underline{l} = r' + \underline{l}$. By definition of Λ-morphism, one has $\sigma r' \leq \sigma p' + \sigma q'$. Applying the inductive hypothesis to $p + q \geq p'$, $p + q \geq q'$, and $p + r' \geq r$, one obtains $\sigma p' \leq \sigma p + \sigma q$, $\sigma q' \leq \sigma p + \sigma q$, and $\sigma r \leq \sigma p + \sigma r' \leq \sigma p + \sigma p' + \sigma q' \leq \sigma p + \sigma q$. $\square$

3. Cycle free sets of lines

Pairwise distinct lines $l_1, \ldots, l_m$ form a *path* if, for any $i \neq j$, one has $l_i \cap l_j \neq \emptyset$ if and only if $|i - j| = 1$. A set Λ of lines is *connected* if, for any $l, g \in \Lambda$, there is a path from l to g. Define $P(\Lambda) = \{p \mid p \in l \in \Lambda\}$.

(4) Given sets Λ_i $(0 \leq i \leq n)$ of lines such that $P(\Lambda_i) \cap P(\Lambda_j) = \emptyset$ for all $0 < i < j$ and such that for any $i > 0$, there is exactly one $l_i \in \Lambda_0$ with $l_i \cap P(\Lambda_i) \neq \emptyset$; moreover, if Λ_0 and all $\Lambda_i \cup \{l_i\}$ are cycle free, then so is $\bigcup_{i=0}^{n} \Lambda_i$.

(5) If Λ is a cycle free set of lines with $\overline{l} \leq u$ for all $l \in \Lambda$ and if l is a line with $\overline{l} \not\leq u$, then $\Lambda \cup \{l\}$ is cycle free.

Proof. Assume that, under the hyptheses of (4), there were a cycle $g_1, \ldots, g_m$ of lines. Wihout loss of generality, $g_1, \ldots, g_k \in \Lambda_0$ and $g_{k+1} \in \Lambda_1$. Choose $k \leq h \leq m$ maximal with $g_h \in \Lambda_1$. Then $g_h \cap P(\Lambda_i) \neq \emptyset$ for some $i \neq 1$, since we have a cycle; but only $i = 0$ is allowed. By the uniqueness required, $g_h = g_k = l_i$ and it follows $g_j \in \Lambda_0$ for $j > k$. Thus, there would be a cycle in $\Lambda_0 \cup \{l_i\}$, a contradiction.

Now, assume that $l, g_1, \ldots, g_k$ were a cycle with $g_j \in \Lambda$. If $k > 2$ then $g_1 \cap g_k = \emptyset$ whence $|l \cap [0, u]| \geq 2$ and $\overline{l} \leq u$, a contradiction. If $k = 2$, then there are $p_i \neq q$ $(i = 1, 2)$ such that $\{q\} = g_1 \cap g_2$ and $\{p_i\} = l \cap g_i$. Thus $p_1 = p_2$ (since $\overline{l} \not\leq u$), whence $g_1 = g_2$, again a contradiction. $\square$

Now, consider a finite, cycle free, and connected set Λ of lines.

(6) There is $l \in \Lambda$ such that $|l \cap P(\Lambda \setminus \{l\})| = 1$.

(7) $|P(\Lambda)| = (\sum_{l \in \Lambda} |l|) - |\Lambda| + 1$.

(8) For any division ring F such that $|F| + 1 \geq |l|$ for all $l \in \Lambda$, there is a vector space V of $\dim V = |\Lambda| + 1$ and an injective Λ-morphism ϕ from $P(\Lambda)$ into the subspace lattice $L(V)$ of V, such that $\dim \phi p = 1$ for all $p \in P(\Lambda)$ and $V = \sum \{\phi p \mid p \in P(\Lambda)\}$.

(9) $|\Lambda| \leq d(L) - 1$ if Λ is a base of lines of L.

(8) and (9) originate from Wild [12], there discussed within the setting of matroids, to wit, the linear matroid induced on $P(\Lambda)$. Also, it was shown that, in the situation of (9), there is actually an embedding of L into $L(V)$ cf. [5, Thm.6.3].

Proof. In (6), if there were no such l, then starting with any l, finiteness would yield a cycle. (7) is shown by induction on $|\Lambda|$: Choose l according to (6) and apply the inductive hypothesis to $\Lambda' = \Lambda \setminus \{l\}$. The same kind of induction is applied to (8): given $\phi' : P(\Lambda') \to V'$ let $V = V' \oplus F v_1$, $\varepsilon : l \to F \setminus \{0\}$ injective, and $\phi' p_0 = F v_0$ where $\{p_0\} = l \cap P(\Lambda')$. Extend ϕ' to ϕ with $\phi p = F(v_0 + \varepsilon(p) v_1)$ for $p \in l$. To prove (9), construct V, ϕ, and $\sigma : L \to L(V)$ according to (8) and (3). Then $V = \sum \{\sigma p \mid p \in J(L)\} = \sigma 1$ and, given a maximal chain $0 = a_0 \prec a_1 \prec$

$\ldots \prec a_d = 1$ of L, it follows $\dim \sigma a_{i+1} \leq 1 + \dim \sigma a_i$ for all i (since $a_{i+1} = a_i + p_i$ for some $p_i \in J(L)$), whence $\dim V = \dim \sigma 1 \leq d = d(L)$. $\square$

4. Subdirect decomposition

$n = s(L)$ means that L admits a irredundant subdirect decomposition into subdirectly irreducible (whence simple) $L_1, \ldots, L_n$. Recall that

(10) $d(L) = \sum_{i=1}^{n} d(L_i)$

and that projectivity is an equivalence relation on prime quotients (which implies that the congruence lattice is 2^n-element Boolean). Thus, $\approx$ is an equivalence relation on $J(L)$, the classes (*components*) Q_i of which are in 1-1-correspondence with the minimal congruences θ_i of L resp. the $L_i \cong L/\theta_i'$ (θ_i', the complement of θ_i). A detailed analysis of subdirect products has been given in Wille [14]: The canonical projections $\pi_i : L \to L_i$ (with kernel θ_i') have lower and upper adjoints $\sigma_i, \gamma_i : L_i \to L$ (providing the smallest resp. greatest preimage) which are join resp. meet embeddings. For convenience, we identify L_i with $\sigma_i(L_i)$, whence $x \in L_i$ if and only if $\pi_i x = x$. The following is in essence contained in Zusatz 2.3 and section 6 of [14].

(11) $J(L)$ is the disjoint union of the $J(L_i)$ and $J(L_i) = \{p \in J(L) \mid p/p_* \in \theta_i\}$.
(12) $L_i = \{a \in L \mid a/x \in \theta_i$ for all $x \prec a\}$ and $L_i \cap L_j = \emptyset$ for $i \neq j$.

Proof. For convenience, we give a proof. Consider $p \in J(L)$. There is unique i such that $p/p_* \in \theta_i$. Then $\pi_i p_*$ is the smallest element of the θ_i'-class of p_* and, for any $x \in L_i$ with $x < p$, one has $x \leq p_*$, whence $x = \pi_i x \leq \pi_i p_*$. Thus, $\pi_i p_*$ is the unique lower cover of p in L_i.

Conversely, consider $p \in J(L_i)$ with unique lower cover y in the lattice L_i and $x = \gamma_i y$, so $x \not\geq p$. Now, let $z \prec p$. Then $\pi_i z \leq \pi_i y = y$, whence $z \leq \gamma_i \pi_i z \leq \gamma_i y = x$ and $z \leq px$. Thus, px is the unique lower cover p_* of p in L. Moreover, $p/p_* \notin \theta_i'$, whence $p/p^* \in \theta_i$.

Now, if $a \in L_i \setminus J(L_i)$ and $x \prec a$, then $\pi_i x$ is a lower cover of a in L_i and there is $p \in J(L_i)$ with $p + x = a$. Then $px = p_*$ and $p/px \in \theta_i$, whence $a/x \in \theta_i$. Conversely, assume $a/x \in \theta_i$ for all $x \prec a$. If one had $\pi_i a < a$, then $\pi_i a \leq x$ for some $x \prec a$ and $a/x \in \theta_i \cap \theta_i'$, a contradiction. Thus, $a = \pi_i a$ and $a \in L_i$. $\square$

Relating the congruence structure to the geometry, with $\Lambda_i = \{l \in \Lambda \mid l \cap Q_i \neq \emptyset\}$ one has

(13) With any base Λ of lines, one may associate an equivalence relation $\sim_\Lambda$ on $J(L)$: the transitive closure of the relation given by pairs p, q such that $p, q \in l$ for some $l \in \Lambda$. Then $p \approx q$ if and only if $p \sim_\Lambda q$.
(14) Any base Λ of lines of L is the disjoint union of the Λ_i and $P(\Lambda_i) = Q_i$. Moreover, the Λ_i are the maximal connected subsets of Λ.
(15) a is a line-top of L if and only if it is a line-top of some (unique) L_i.
(16) Λ is a base of lines for L if and only if the Λ_i are bases of lines for the L_i.
(17) Λ is cycle free resp. acyclic if and only if so are all Λ_i.

Proof. (13) is [5, Prop.2.6]. (14) follows, immediately, and (15) with (11-12). Again, (16) and (17) are immediate consequences. □

5. The Jónsson-Nation analysis

Recall that a lattice L is 2-*distributive* if for all $u, x, y, z \in L$

$$u(x + y + z) = u(x + y) + u(x + z) + u(y + z).$$

(18) A modular lattice L of finite height is 2-distributive if and only if it does not contain an interval sublattice which is an irreducible projective plane. In particular, L is 2-distributive if and only is so is its dual.

This was shown by András Huhn [9]. In their proof [10] of the vector space representation for 2-distributive L, Bjarni Jónsson and J.B. Nation considered a designated coatom u and analyzed the structure of join irreducibles relatively to u. Define

$$J = J(L),\ L' = [0, u],\ J' = J(L'),\ J'' = J \setminus J'$$

The first observation is that

(19) $u + x = 1$ and $ux \prec x$ for any $x \not\leq u$.

(20) The points $p \in J''$ are exactly the minimal elements of $L \setminus L'$ and, in particular, pairwise incomparable. Moreover, $p_* = up$ for all $p \in J''$ and $px \leq u$ if $p \not\leq x$.

(21) $x = ux + p$ where $p \in J''$ and $p \leq x \not\leq u$, in particular $p + q = p + u(p + q)$ for all $p, q \in J''$.

Proof. (19) follows from modularity, immediately. If $p \in J''$ then $up \prec p$ whence p minimal in $L \setminus L'$. Conversely, the minimal elements of $L \setminus L'$ have to be join irreducible since L' is closed under joins. Thus, (20) holds. (21) follows from (19). □

Further on, Jónsson and Nation considered $p, q, r \in J''$ such that $p + q$ and $p + r$ are incomparable; they derived, by modularity,

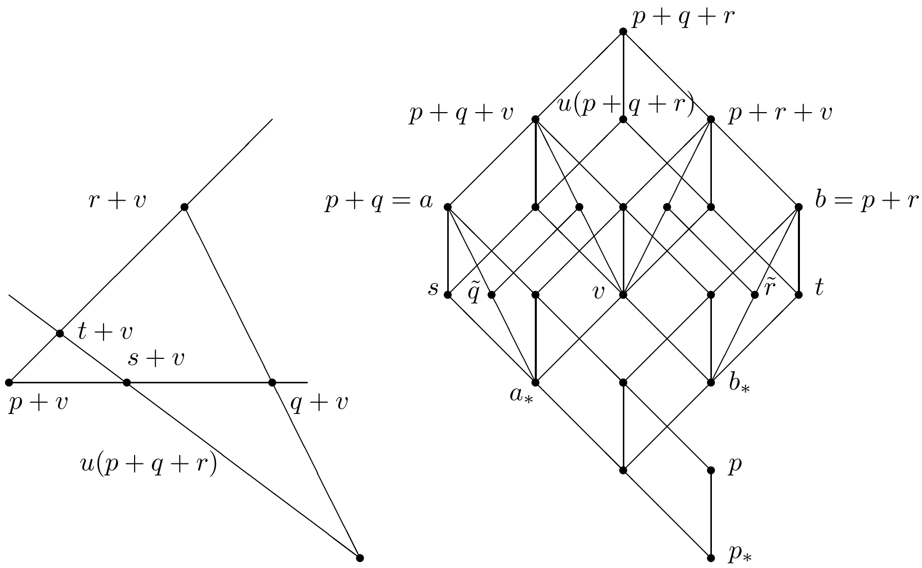
(22) $(p + q)(p + r)(q + r) = q(p + r) + r(p + q) \leq uq + ur$,

and observed a failure of the dual 2-distributive law for the substitution $u,\ p + q,\ p + r,\ q + r$. According to Huhn [8], with $v = up + uq + ur$ this leads to a spanning dual 3-diamond $u(p+q+r),\ p+q+v,\ p+r+v,\ q+r+v$ in the interval $[v, p+q+r]$ and to an irreducible projective plane. This is captured in the following (cf. Fig.2, where $\tilde{x} = x + up$).

(23) For $p, q, r \in J''$ with $p + q$ and $p + r$ incomparable, one has an irreducible projective plane $[v, p + q + r]$ with coatoms $v + p + q$ and $v + p + r$, where $v = up + uq + ur$.

Proof. Assume $p+q$ and $p+r$ incomparable. By (19-21) and modularity, $u(p+q+r)$, $p + q + v$, $p + r + v$, and $q + r + v$ are lower covers of $p + q + r$ whence $[v, p + q + r]$ has height at most 3. We have to show that these coatoms are also in general position, i.e. that any 3 of them behave as coatoms of an 8-element boolean 0-1-sublattice of $[v, p + q + r]$. Indeed, with the relations between u, p, q, r given

FIGURE 2



by (19-22) and with modularity, one has $u(p+q+r)+(p+q+v)(p+r+v) = (p+q+v)[u+p+(q+v))(p+r+v)] = p+q+r$, $p+q+v+(p+r+v)(q+r+v) = p+q+v+r+(p+v)(q+r+v) = p+q+r$, $(p+q+v)(p+r+v)(q+r+v) = (q+v)(p+r+v)+(r+v)(p+q+v) = v+q(p+r+v)+r(p+q+v) = v+q(p+r)+r(p+q) = v$, and $u(p+q+r)(p+q+v)(p+r+v) = u(p+v+q(p+r)) = up+v+uq(p+r) = v$ while the remaining relations follow, similarly. Also, $v < p+q+r$. Thus, $[v, p+q+r]$ is an irreducible projective plane (cf. [6, Thm.3.5] and [7, Lemma 2]). See Fig.2 where $\tilde{q} = q+p_* = q+a_*$ and $\tilde{r} = r+p_* = r+b_*$. $\square$

We continue studying the inductive approach via $d(L)$ with a fixed coatom u. We write $p \in a \cap X$ if $p \in a$ and $p \in X$. Let T denote the set of line-tops $a \not\leq u$.

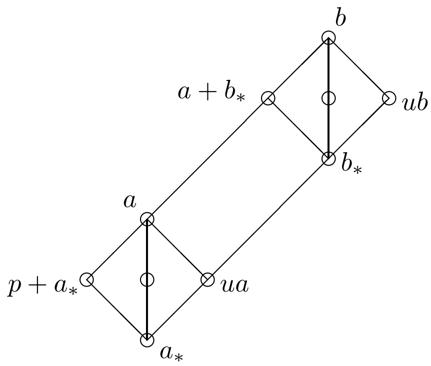
(24) $a \in T$ if and only if there are $p \neq q$ in J'' with $a = p+q$. Moreover, for any $p \in a \cap J''$ there is $q \in a \cap J''$ with $p+q = a$; also $|l \cap J'| = \{r\}$ where $r \in l$ with $r + a_* = ua$ for any line l with $\overline{l} = a$.

(25) If $a < b$ in T, then $[a_*, a] \cup [b_*, b]$ is a sublattice of L and $p \in b$ for any $p \in a \cap J''$ (cf. Fig.3).

(26) If $a, b \in T$ are incomparable and $p \in a$, $p \in b$ for some $p \in J''$, then there are $s \in a \cap J'$ and $t \in b \cap J'$ such that $s \approx t$ in L'.

Proof. If $p, q \in l \cap J'$ and $p \neq q$ then $\overline{l} = p+q \leq u$. If $p \neq q$ in J'', then by (19) and (2) $v := p_* + q_* \prec u(p+q)$, hence there is $r \in J'$ with $r+v = u(p+q)$. By (1), there are $p' \leq p$ and $q' \leq q$ such that p', q', r are collinear; but then $p' = p$ and

FIGURE 3



$q' = q$ since, otherwise, $r \leq v$. This proves (24). In (25), the prime quotient a/ua transposes up to $(a+b_*)/b_*$ and the claim follows. In (26), by (24) we have $a = p+q$ and $b = p + r$ for some $q, r \in J''$. By (23), $[v, a + b]$ is an irreducible projective plane with coatoms $v + a$ and $v + b$, where $v = a_* + b_* \leq u$. Since $a_* \prec ua \prec a$ and $b_* \prec ub \prec b$ by (19) and (2), there are $s, t \in J$ with $s + a_* = ua$ and $t + b_* = ub$. Then $s + v$ and $t + v$ are atoms of $[v, u(p + q + r)]$ and have a common complement x within this interval of an irreducible projective plane. Thus, s/s_* and t/t_* both transpose up to $u(p + q + r)/x$ whence $s \approx t$ in L'. $\square$

A partially ordered set is a *forest* if all its principal filters are chains. We say that $F \subseteq T$ has the *JN-property* if there may be comparable $a, b \in F$, only, such that $p \in a$ and $p \in b$ for some $p \in J''$.

(27) If L is 2-distributive, then T has the JN-property.
(28) If $F \subseteq T$ has the JN-property, then F is a forest.
(29) Assume that the following condition holds for $F \subseteq T$:
$(*)$ If $a, b \in F$ and $s \approx t$ in L' for some $s \in a \cap J'$ and $t \in b \cap J'$, then $a = b$.
Then F has the JN-property and, if F is finite, $|J''| \geq |F| + 1$.
(30) A set Λ of lines is cycle free provided that $l \mapsto \bar{l}$ is an injective map from Λ into some $F \subseteq T$ with F enjoying the JN-property.

Proof. (27) is due to Jónsson and Nation [10] and follows from (18) and (23). Assume that $F \subseteq T$ enjoys the JN-property and $c < a, b$ in F. By (25), there is $p \in J''$ with $p \in a$ and $p \in b$ and, by the JN-property, it follows that a, b are comparable. Thus, F is a forest.

Now, consider F satsifying $(*)$. In view of (26), F has the JN-property. Define $A(F) = \{x \in L \mid ua \neq x \prec a \in F\}$ and $M(F)$ as the set of minimal elements of $A(F)$. We show by induction on $|F|$ that $|M(F)| \geq |F| + 1$. Indeed, choose $a \in F$ minimal and b its unique upper cover in F. By (25) and induction it follows

$$|M(F)| \geq |M(F \setminus \{a\})| - 1 + 2 \geq |F| - 1 + 1 - 1 + 2 = |F| + 1.$$

For any $x \in M(F)$, there is unique $a_x \in F$ such that $x \prec a_x$ and there is $p_x \in J''$ such that $p_x + a_* = x$. Assume $p_x = p_y$. Due to the JN-property, a and b are comparable, e.g. $a \leq b$, whence also $a_* \leq b_*$ and $x = p_x + a_* \leq p_y + b_* = y$. By minimality, $x = y$ showing that $x \mapsto p_x$ is injective. It follows $|J''| \geq |M(F)| \geq |F| + 1$.

In order to prove (30), we may assume that Λ is finite and use induction on $|F|$. Considering a cycle $l_1, \ldots, l_m$ in Λ, we may assume that $\bar{l}_1$ is minimal in F. Let $l_1 \cap l_2 = \{p\}$ and $l_1 \cap l_m = \{q\}$, in particular, $p \neq q$ and $p, q \not\leq u$ in view of (24). By the JN-property, we have $\bar{l}_1 < \bar{l}_2$ and $\bar{l}_1 < \bar{l}_m$. Since F is a forest by (28), we have e.g. $\bar{l}_2 < \bar{l}_m$. Thus, omitting l_1 and replacing l_m by $(l_m \setminus \{q\}) \cup \{p\}$ (which can be done in view of (25)), we get a cycle with line-tops $\bar{l}_i$ ($i > 1$) - which is impossible by inductive hypothesis.

□

6. Proof of Theorem 1 and Corollary 5

Proof. Thm.1 is shown by order induction on $d(L)$. We also include the statement of Corollary 2 and the following

(31) If $j(L) \leq 2d(L) - s(L)$ then $j(I) = 2d(I) - s(I)$ for all interval sublattices I of L.

First, assume that L has an irredundant subdirect decomposition into factors $L_i, (i = 1, \ldots, n)$ with $n = s(L) \geq 2$ and with Λ_i according to (14-16). Then by (9), (10), (16), and (11)

$$|\Lambda| = \sum_{i=1}^{n} |\Lambda_i| \geq \sum_{i=1}^{n} (d(L_i) - 1) = d(L) - n$$

$$j(L) = \sum_{i=1}^{n} j(L_i) \geq \sum_{i=1}^{n} (2d(L_i) - 1) = 2d(L) - n$$

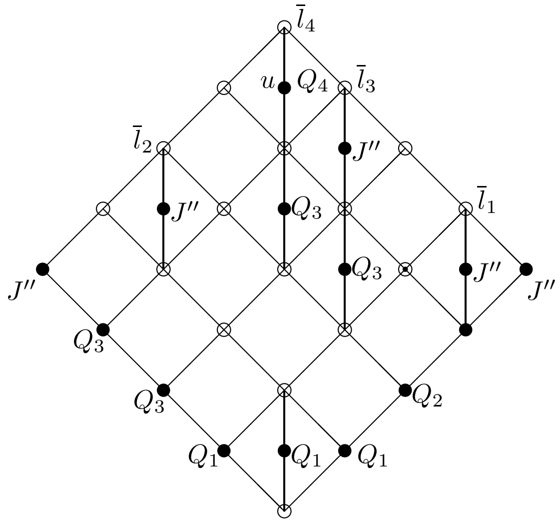
in view of the inequalities for corresponding summands given by inductive hypothesis. Now, $|\Lambda| \leq d(L) - n$ (resp. $j(L) \leq 2d(L) - n$) if and only if the respective middle inequality is an equality if and only $|\Lambda_i| = d(L_i) - 1$ for all i (resp. $j(L_i) = 2d(L_i) - 1$ for all i). The latter is equivalent (by inductive hypothesis) to all Λ_i being cycle free resp. acyclic and this, in turn by (17), to Λ having the same property.

Now, suppose that L is simple, i.e. $s(L) = 1$. The claim is obviously true for $d(L) = 1$. So assume $d(L) > 1$ and Λ a finite base of lines. Consider any coatom u and

$$J = J(L),\ L' = [0, u],\ J' = J(L'),\ J'' = J \setminus J',\ \Lambda' = \{l \in \Lambda \mid \bar{l} \leq u\},\ \Lambda'' = \Lambda \setminus \Lambda'$$

cf. Fig.4. Evidently, Λ' is a base of lines of L'. Let Q_i ($i = 1, \ldots, c$) denote the components of J', with base $\Lambda_i \subseteq \Lambda'$ of lines according to (16). By (14), the partial linear space (J', Λ') is the disjoint sum of the (Q_i, Λ_i). Since L is simple, J has a single connected component due to (13). Thus, for each i there is $l_i \in \Lambda''$ with $l_i \cap Q_i \neq \emptyset$ and, by (24), $l_i \cap Q_j = \emptyset$ for $j \neq i$. Choose such l_i for each i and let

Figure 4



$\Lambda_0 = \{l_1, \ldots, l_c\}$. Then $|\Lambda_0| = c$ and $F = \{\bar{l} \mid l \in \Lambda_0\}$ satisfies condition $(*)$ in (29), whence F enjoys the JN-property and

(32) $|\Lambda''| \geq c$ and $|J''| \geq |F| + 1 = c + 1$.

Also, by (30), Λ_0 is cycle free. On the other hand, by inductive hypothesis one has

(33) $|\Lambda'| \geq d(L') - c = d(L) - 1 - c$ and $j(L') \geq 2d(L') - c = 2d(L) - 2 - c$

Consider L such that $i(L) \leq d(L) - 1$. Then

$$d(L) - 1 \geq |\Lambda| = |\Lambda'| + |\Lambda''| \geq |\Lambda'| + c \geq d(L) - 1 - c + c = d(L) - 1$$

and equality holds, everywhere. In particular, $|\Lambda''| = c$, whence $\Lambda'' = \Lambda_0$ is cycle free. It follows, with (4) and (5), that Λ is cycle free.

Next, suppose that $j(L) \leq 2d(L) - 1$. Firstly, it follows with (32) and (33)

$$j(L') + c + 1 \leq j(L) \leq 2d(L) - 1 \leq j(L') + c + 1$$

whence

(34) $j(L) = 2d(L) - 1$ and $j(L') = 2d(L') - c = 2d(L') - s(L')$.

This applies for all $u \prec 1$ and $L' = [0, u]$. By inductive hypothesis (31), we have $j(I) = 2d(I) - s(I)$ for all interval sublattices $I = [0, b]$ of L. Since, by Dilworth's Theorem, $j(L)$ is also the number of meet irreducibles, by duality and induction we have $j(I) = 2d(I) - s(I)$ for all interval sublattices I of L.

In particular, this excludes any I of height 2 with more than 3 atoms whence, by (2), all lines are 3-element. Also, this excludes any I of height 3 which is an irreducible projective plane (having at least $7 > 2 \cdot 3 - 1$ points). According to (18), this implies that L is 2-distributive. Thus, in view of (27) and (30), Λ'' is cycle free and, again by, (4) and (5), Λ is acyclic.

Conversely, if Λ is a finite and cycle free base of lines, then by (9) $|\Lambda| \leq d(L) - 1$ and equality holds as just shown. If, in addition, Λ is acyclic, then by (7) $j(L) \leq 3(d(L) - 1) - (d(L) - 1) + 1 = 2d(L) - 1$ and, again, equality holds.

To prove Corollary 5, similarly to the use of (31), proceed by induction on the claim: $i(L) \leq d(L) - 1$ implies $i(I) \leq d(I) - 1$ for all interval sublattices I of L. This, of course, excludes I which are projective planes. Corollary 5 follows with (18). □

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