A Geometric Description of Modular Lattices

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1 Introduction

Baer [1] observed that modular lattices of finite length (for example subgroup lattices of abelian groups) can be conceived as subspace lattices of a projective geometry structure on an ordered point set; the set of join irreducibles which in this case are the cyclic subgroups of prime power order. That modular lattices of finite length can be recaptured from the order on the points and, in addition, the incidence of points with 'lines', the joins of two points, or the blocks of collinear points has been elaborated by Kurinnoi [18] , Faigle and Herrmann [7], Benson and Conway [2] , and , in the general framework of the 'core' of a lattice, by Duquenne [5]. In [7] an axiomatization in terms of point-line incidence has been given.

Here, we consider, more generally, modular lattices in which every element is the join of completely join irreducible 'points'. We prove the isomorphy of an algebraic lattice of this kind and the associated subspace lattice and give a first order characterization of the associated 'ordered spaces' in terms of collinearity and order which appears more natural and powerful. The crucial axioms are a 'triangle axiom', which includes the degenerate cases, and a strengthened 'line regularity axiom', both derived from [7]. As a consequence, using Skolemization, we get that any variety of modular lattices is generated by subspace latices of countable spaces.

The central concept, connecting the geometric structure and the lattice structure, is that of a line interval $(p+q)/(\underline{p}+\underline{q})$ where p and q are points and $\underline{p}, \underline{q}$ their unique lower covers. Given a line interval, any choice of one incident point per atom of the line interval produces a line of the space.

Under the descending chain condition, or countability of the point set, it is possible to choose one line per line interval in such a way that the lattice can be recovered from the order on the points and this 'base of lines'. The finite dimensional case has been dealt with in Herrmann and Wild [12].

As an application we consider modular lattices which are 2-distributive, i.e. satisfy the identity

$$x(y+z+w) \leq x(y+z) + x(y+w) + x(z+w)$$

or, equivalently, do not contain a projective plane in their variety, Huhn [13]. So these lattices constitute the properly lattice theoretic, 'coordinate free' part of modular lattice theory. Extending a theorem of Jónsson and Nation [17] from the finite length case we have the following.

Theorem 1.1 *Every 2-distributive modular lattice can be embedded in a vector space lattice over a field of arbitrary characteristic.*

Proof. The proof follows from Corollary 6.1 and theorem 8.1 below and the fact that that the lattices embeddable into subspace lattices of vector spaces of characteristic p form a universal class.

Let $\mathcal{D}_2$ be the class of 2-distributive modular lattices and $\mathcal{V}_p$ the variety generated by subspace lattices of vector spaces of a given characteristic p. Now, let p and q be two distinct characteristics. By the theorem, $\mathcal{D}_2$ is contained in $\mathcal{V}_p \cap \mathcal{V}_q$. Since characteristic of a projective plane can be expressed equationally, this intersection cannot contain any plane, so it equals $\mathcal{D}_2$.

Corollary 1.1 *For distinct characteristics p and q, $\mathcal{V}_p \cap \mathcal{V}_q = \mathcal{D}_2$.*

2 Subspace lattices

An *ordered space* consists of a set of *points*, P, endowed with a partial order, $\leq$, and a ternary, totally symmetric *collinearity relation* with the following two properties. Collinear points are pairwise incomparable, and, if p, q, r are collinear and $p, q \leq s$ then $r \leq s$. We will sometimes write $C(p, q, r)$ for 'p, q, r are collinear'. A *subspace* is an order ideal containing along with any two points of a collinear triple also the third. The set of all subspaces of P ordered by containment forms an algebraic spatial lattice, $L(P)$. Here, a *spatial lattice* is a lattice M in which every element is a join of completely join irreducible elements, or *points*, cf. [15]. So, if M is algebraic, then the points are compact. Conversely, with any spatial lattice M we associate the set P_M of all its points, with the induced order, and with the collinearity relation: $C(p, q, r)$ iff p, q, r are pairwise incomparable, and $p+q = p+r = q+r$. This yields the ordered space $S(M)$ of M. M is canonically meet embedded into $L(S(M))$. On the other hand, any ordered space P is canonically isomorphic to $S(L(P))$.

The concepts of ordered space and subspace can also be expressed in terms of *lines*, maximal at least 2-element sets of incomparable points, any three of which are collinear. It suffices, as in Benson and Conway [2], to consider *proper*, i.e. at least 3-element lines. Then $L(P)$ consists of the 2-subspaces in the sense of Buekenhout [3].

A complemented length 2 interval $l/\underline{l}$ in a lattice M is called a *line interval* if $x + \underline{l} = l$ implies $x = l$ for all x in M. For a point p let $\underline{p}$ denote its unique lower cover.

Lemma 2.1 *In a modular spatial lattice $l/\underline{l}$ is a line interval if and only if there are incomparable points p, q such that $l = p + q$ and $\underline{l} = \underline{p} + \underline{q}$. If so, then $l = x + y$, $\underline{l} = \underline{x} + \underline{y}$ for all pairs $x, y \leq l$, $x, y \not\leq \underline{l}$ of points such that $x + \underline{l} \neq y + \underline{l}$. In particular, $\underline{l} = \prod\{x \in M : x \prec l\}$.*

Proof. Let $l/\underline{l}$ be a line interval and let s and t be nonequal covers of $\underline{l}$ in $l/\underline{l}$. There exist points p, q with $\underline{l} + q = s$ and $\underline{l} + p = t$. Setting $x = p + q$ we get $\underline{l} + x = l$ whence $x = l$. Now, by modularity, $ps = p\underline{l} = \underline{p}$, $qt = q\underline{l} = \underline{q}$, and therefore, $\underline{p} + \underline{q} = st(p + q) = \underline{l}$. It remains to show that if p, q are incomparable points then $l/\underline{l}$ is a line interval, where $l = p + q$ and $\underline{l} = \underline{p} + \underline{q}$. Obviously $l/\underline{l}$ is a complemented length 2 interval, it remains to show that if $x + \underline{l} = l$ then $x = l$. First we will derive contradictions from $x \neq l$ in two special cases.

Case 1: $x \geq p$.
In this case $x + \underline{q} = x + \underline{p} + \underline{q} = x + \underline{l} = l = x + q$. If $x \not\geq q$ then $xq = x\underline{q}$, contradicting modularity.

Case 2: $x \preceq l, x \not\geq p, q$.
Then $px \geq \underline{p}$ and $qx \geq \underline{q}$ whence $x \geq \underline{p} + \underline{q} = \underline{l}$ and $x = x + \underline{l} = l$.

By Case 1 and symmetry, it suffices, for the general case, to consider $x \not\geq p, q$. Applying Case 1 to $x+p$ we get $x+p = l$. From $px \leq \underline{p}$ it follows that $\underline{p}+x \prec l$. If $\underline{p}+x \not\geq q$ then Case 2 applies to $\underline{p}+x$ so $\underline{p}+x = l$, a contradiction. Therefore $\underline{p} + x \geq q$ whence $\underline{p} + x = \underline{l} + q$ and $\underline{l} + x = \underline{q} + \underline{p} + x = \underline{q} + q + \underline{l} = \underline{l} + q < l$, another contradiction.

Lemma 2.2 *In a modular spatial lattice if r is a point and if $r \leq a + b$, $r \not\leq a$ and $r \not\leq b$ then there are points $p \leq a$ and $q \leq b$ such that p, q, r are collinear. If $a = a(b + r)$ is a point then we can assume $p = a$.*

Proof. The proof of theorem 4.2 of Faigle and Herrmann [7] is valid for spatial lattices too. It allows us to assume that $a = p$ is a point and $p + b = r + b$. In the next paragraph we will argue that the sublattice generated by $b, p, r, \underline{p}$, and $\underline{r}$ is a homomorphic image of the lattice pictured in Fig. 1. Having shown this one can easily see from Fig. 1 that $p \leq r$ would imply $b \leq r$, and hence $r = p + b$. But this contradicts the fact that r is a point. Hence by lemma 2.1, $(p + r)/(\underline{p} + \underline{r})$ is a line interval, so any point q with $q \leq b(p + r)$, $q \not\leq b(\underline{p} + \underline{r})$ may be used.

First observe that $br = b\underline{r}$ whence by modularity and hypothesis $b + r \succ b+\underline{r} \not\geq r, p$. Therefore, $\underline{p} = p(b+\underline{r})$ and we deal with the sublattice generated by $b, p, r, \underline{r}$ which satisfy the relations $p+b = r+b$ and $rp+rb \leq \underline{r} \leq r$. Now, Fig. 1 shows the free modular lattice F with these generators and relations. To see this we may refer to Wille [19] that only 2- and 5-element subdirectly irreducible factors are possible and check that the 6 factors of F are the only ones which satisfy the relations.

Lemma 2.3 *Let M be a modular spatial lattice embedded into a modular algebraic lattice L such that the compact elements of L are contained in M. Then ψ given by*

$$\psi(a) = \{p \in P_M : p \leq a\}$$

is a homomorphism of L onto $L(S(M))$ and $\psi|_M$ is faithful.

Proof. Obviously $\psi|_M$ is faithful. Let $a, b \in L$ and let $p \in \psi(a) \cap \psi(b)$. Then $p \leq a$ and $p \leq b$ and so $p \leq ab$ and meets are preserved. We claim that every point p of M is a compact element of L, and therefore ψ is compatible with directed joins. Let $p \in P_M$. Since L is algebraic, p is a join of a set X of

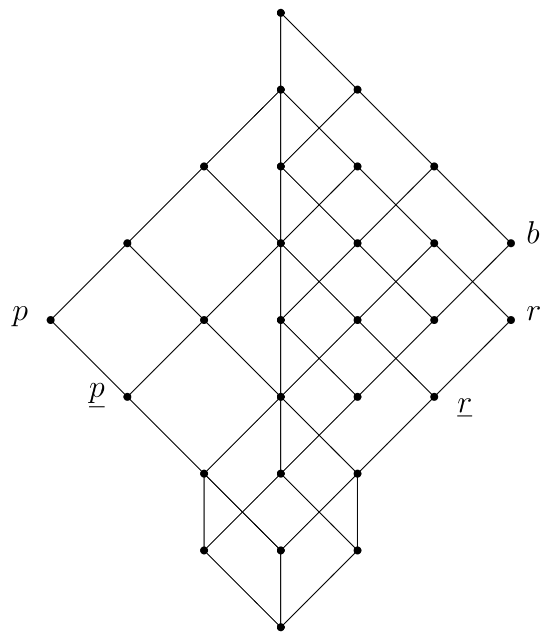

Figure 1:

compact elements of L. By hypothesis $X \subseteq L$ and either $p \in X$ or $x < p$ for all $x \in X$. The latter would yield $\sum_L X \leq \underline{p}$, a contradiction.

To show ψ preserves joins of compact elements, let $a, b \in L$ be compact and let $r \in \psi(a+b)$. Since L is algebraic $a = \sum_{i \in I} a_i$ with the a_i compact and $b = \sum_{j \in J} b_j$ with the b_j compact. Then $a, b \in M$ and by 2.2, $r \leq a'$, or $r \leq b'$, or there exist points p and q with $p \leq a'$, $q \leq b'$ and $C(p, q, r)$. In any of these cases $r \in \psi(a) + \psi(b)$.

The ontoness of ψ now follows immediately from the fact that P_M is join dense in $L(S(M))$.

Theorem 2.1 *A modular spatial lattice M embeds canonically into $L(S(M))$ which is a homomorphic image of the ideal lattice of M. If M is algebraic then this embedding is an isomorphism.*

Proof. We apply 2.3 and in the first instance take L to be the ideal lattice of M. If M is algebraic we set $L = M$.

Corollary 2.1 *(Faigle [6]). Every modular lattice embeds into a modular algebraic spatial lattice generating the same variety.*

Proof. A lattice generates the same variety as its ideal and filter lattices and the filter lattice is a spatial lattice.

3 Axiomatization

An ordered space is *projective* if it satisfies the regularity and triangle axioms given below.

Regularity axiom: For any collinear p, q, r and $r' \leq r$, $r' \not\leq p$ and $r' \not\leq q$ there are $p' \leq p$ and $q' \leq q$ such that p', q', r' are collinear.

A six-tuple (a, b, c, p, q, x) of pairwise incomparable points is called a *triangle configuration* iff it satisfies the following list of collinearities, and no others, see the figure.

$$C(a, c, p), C(b, c, q), C(a, b, x), C(p, q, x)$$

Triangle axiom: If a, c, p and b, c, q are collinear then at least one of the following holds;

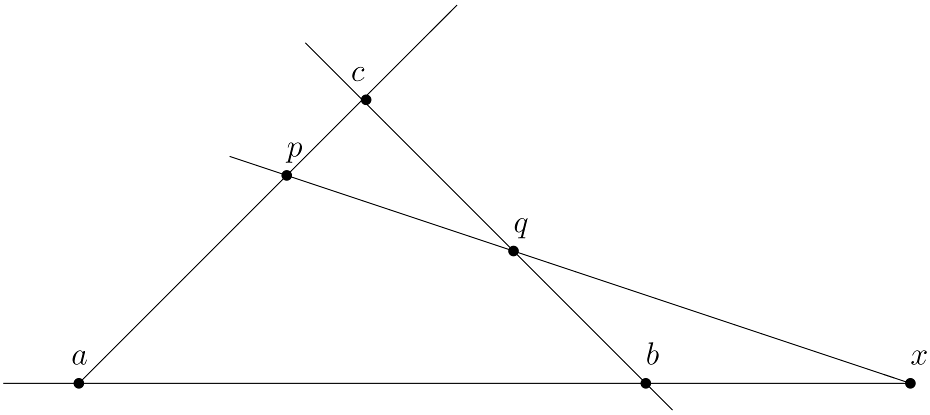

Figure 2: Triangle Configuration

1. there is an x so that (a, b, c, p, q, x) is a triangle configuration,
2. there is an $a' \leq a$ such that b, q, a' are collinear,
3. b, q, p are collinear,
4. there are $a' \leq a$ and $p' \leq p$ such that q, a', p' are collinear,
5. $q \leq a$ or $q \leq p$.

Lemma 3.1 *The join of subspaces of a projective ordered space is given by*

$$S + T = S \cup T \cup \{r : \text{ There exists } x \in S, y \in T \text{ with } C(r, x, y)\}.$$

Proof. We have to show that

$$S \vee T = S \cup T \cup \{r : \text{ There exists } x \in S, y \in T \text{ with } C(r, x, y)\}.$$

is a subspace. Regularity implies that $S \vee T$ is an order ideal and so it remains to show that it is linearly closed: if $r_1, r_2 \in S \vee T$ and if $C(r_1, r_2, r)$ then $r \in S \vee T$. We begin with the special case where at least one of r_1, r_2 is in one of S or T. The proof will be given for $r_2 = s \in S$, the other cases have similar proofs. If $r_1 \in S \cup T$ we get $r \in S$, since S is a subspace, or $r \in S \vee T$ by definition.

Suppose we have $C(s_1, r_1, t_1), C(s, r_1, r)$ with $s_1, s \in S$ and $t_1 \in T$. We will use the triangle axiom to show $r \in S \vee T$. Let us apply the triangle axiom to the pair of triples listed just above. Then one of the following must occur:

(1) There is an x so that (s_1, s, r_1, t_1, r, x) is a triangle configuration. In this case $C(s_1, s, x)$ and $C(x, t_1, r)$, hence $x \in S$ and $r \in S \vee T$.

(2) There exists $s_1' \leq s_1$ with $C(s, r, s_1')$ and hence $r \in S$.

(3) $C(s, r, t_1)$, this implies $r \in S \vee T$.

(4) There are $s_1' \leq s_1, t_1' \leq t_1$ with $C(s_1', t_1', r)$ and $r \in S \vee T$.

(5) $r \leq s_1$ whence $r \in S$ or $r \leq t_1$ whence $r \in T$.

For the general case we consider r_1, r_2 with $C(r_1, s_1, t_1), C(r_2, s_2, t_2)$, for some $s_1, s_2 \in S, t_1, t_2 \in T$. We will apply the triangle axiom three times. The first two times possibility (1) will lead to another application of the triangle axiom. The other possibilities will all either lead directly to $r \in S \vee T$ or will reduce to the case already handled above.

We apply the triangle axiom to the collinearities, $C(s_2, r_2, t_2), C(r_1, r_2, r)$ to get one of:

(1) There exists y so that $(s_2, r_1, r_2, t_2, r, y)$ is a triangle configuration. We will apply the triangle axiom again here, but let us deal with the other cases first.

(2) There is an $s_2' \leq s_2$ with (r_1, r, s_2'). This case has already been dealt with.

(3) $C(r_1, r, t_2)$ is another instance of the special case already dealt with.

(4) There are $s_2' \leq s_2$, $t_2' \leq t_2$ with $C(r, s_2', t_2')$, which gives $r \in S \vee T$.

(5) $r \leq s_2$ and then $r \in S$, or $r \leq t_2$ and then $r \in T$.

Assuming that (1) above occurs we apply the triangle axiom to the collinearities, $C(s_1, r_1, t_1), C(s_2, r_1, y)$. We obtain one of the following:

(1) There exists s so that $(s_1, s_2, r_1, , t_1, y, s)$ is a triangle configuration. In particular, $s \in S$. We will use the triangle axiom one more time but again let us delay its use until we've handled the remaining possibilities. In all of these we will show $y \in S \vee T$. Since $C(y, t_2, r)$, if we replace r_1 with y and r_2

with t_2, then we have an instance of the special case handled above whence $r \in S \vee T$.

(2) There is an $s_1' \le s_1$ with $C(s_1', s_2, y)$. This implies $y \in S$.

(3) $C(s_2, t_1, y)$. In this case $y \in S \vee T$.

(4) There are $s_1' \le s_1, t_1' \le t_1$ with $C(s_1', t_1', y)$, so $y \in S \vee T$.

(5) $y \le s_1$ and we have $y \in S$, or $y \le t_1$ and $y \in T$.

Let us assume that (1) occurs here as well. We apply the triangle axiom once more to the collinearities $C(t_1, y, s), C(t_2, y, r)$ to get one of the following:

(1) There is a t so that (t_1, t_2, y, s, r, t) is a triangle configuration. In particular, $t \in T$, $s \in S$, and $C(s, t, r)$ imply $r \in S \vee T$.

(2) There is a $t_1' \le t_1$ with $C(t_1', t_2, r)$. This gives $r \in T$.

(3) $C(t_2, s, r)$ gives $r \in S \vee T$.

(4) There are $t_1' \le t_1, s' \le s$ with $C(t_1', s', r)$ and $r \in S \vee T$.

(5) $r \le t_1$ and $r \in T$, or $r \le s$ and $r \in S$.

Theorem 3.1 *The subspace lattice of a projective ordered space is modular.*

Proof. The proof of proposition (3.4) in [7] can be followed virtually word for word.

Theorem 3.2 *The ordered space associated with a modular spatial lattice is projective.*

Proof. Regularity is immediate by lemma 2.2. Let us write $a/b \nearrow c/d$ and $c/d \searrow a/b$ if $c = a + d$ and $b = cd$. Assume $C(a, c, p), C(b, c, q)$.

Let

$$l = a + c = a + p = c + p, \quad \underline{l} = \underline{a} + \underline{c} = \underline{a} + \underline{p} = \underline{c} + \underline{p}$$

$$k = b + c = b + q = c + q, \quad \underline{k} = \underline{b} + \underline{c} = \underline{b} + \underline{q} = \underline{c} + \underline{q}$$

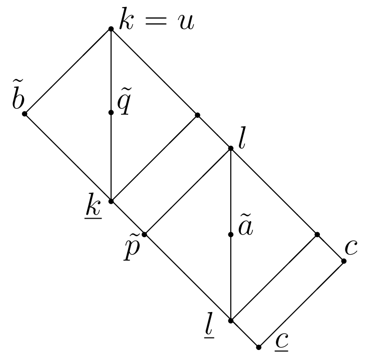

Figure 3: Case 2

$$\tilde{a} = a + \underline{l}, \quad \tilde{p} = p + \underline{l}, \quad \tilde{b} = b + \underline{k}, \quad \tilde{q} = q + \underline{k}$$

$$u = l + k, \quad v = \underline{l} + \underline{k}, \quad w = \tilde{a}\tilde{b}.$$

If $l \geq k$ then, by lemma 2.2 applied to $q \leq a + p$, possibility (5) or (4) of the triangle axiom takes place.

Otherwise, we have the following transpositions

$$c/\underline{c} \nearrow k/\tilde{b} \text{ and } c/\underline{c} \nearrow l/\tilde{a}.$$

The sublattice generated by these quotients is actually generated by $c, \tilde{a}, \tilde{b}$. An elementary analysis of the free modular lattice on these generators together with the conditions $\tilde{a} \prec l$, $\tilde{b} \prec k$, $k \not\leq l$ and $\tilde{a}c = \tilde{b}c = \underline{c}$ shows that one of the four sets of relations, case 2 through case 5 below (we have counted $l \geq k$ as case 1) must hold, cf. Jónsson [16], Grätzer [10]. In view of lemma 2.1, in any of these cases one easily determines the possible extensions to the sublattice generated by the additional elements $\tilde{p}$ and $\tilde{q}$.

Case 2: $l \leq k$, $\tilde{a} \leq \tilde{b}$, cf. figure 3 . In this case we have $C(p, b, q)$ by lemma 2.1 and hence (3) of the triangle axiom.

Case 3: $l \leq k$, $l \leq \tilde{a} + \tilde{b}$, cf. figure 4. Here we have two possible extensions. By lemma 2.1, in both cases we have $C(b, q, a)$ and hence (2) of the triangle axiom.

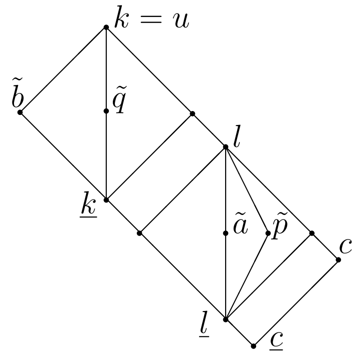

Figure 4: Case 3

Case 4: $l \not\leq k$, $l \leq \tilde{a} + \tilde{b}$, cf. figure 5. Here by lemma 2.1 we have $a + \underline{c} = \tilde{a}$. Hence $ak \leq w$ would give $\tilde{a}k = ak + \underline{c} \leq w$. We also have, in case 4, $c/\underline{c} \nearrow kl/\underline{k}l \searrow \tilde{a}k/w$. Hence, if $\tilde{a}k = w$ then $c = \underline{c}$, a contradiction, and we have shown $ak \not\leq w$. So choose any point x, with $x \leq ak$, $x \not\leq w$. It follows that $x \not\leq b$ since $x \leq b$ would give $x \leq \tilde{a}\tilde{b} = w$. Similarly, $x \leq q$ gives $x \leq \tilde{a}\tilde{q} = w$. From lemma 2.1 we have C(b,q,x). This is possibility (2) of the triangle axiom.

Case 5: $l \not\leq k$, $l \not\leq \tilde{a} + \tilde{b}$, cf. figure 6, where u/v is a projective plane which is depicted only partially.

Since $a + b + v = \tilde{a} + \tilde{b}$ the quotient $(\tilde{a} + \tilde{b})/v$ transposes down to $(a + b)/v(a + b)$ which is therefore of length 2, too, and thus turns out to be the line interval $(a+b)/(\underline{a}+\underline{b})$, cf. 2.1. Assume $(a+b)(p+q) \leq v$. It follows, from just above, that $(a+b)(p+q) \leq \underline{a}+\underline{b}$. This gives, $u/(\tilde{p}+\tilde{q}) \searrow (a+b)/(\underline{a}+\underline{b})$, since now $(a+b)(\tilde{p}+\tilde{q}) = (a+b)(p+\underline{a}+q+\underline{b}) = \underline{a}+\underline{b}+(a+b)(p+q) = \underline{a}+\underline{b}$. But $u/(\tilde{p} + \tilde{q})$ is a prime quotient and $(a + b)/(\underline{a} + \underline{b})$ is not, a contradiction. Hence, $(a + b)(p + q) \not\leq v$.

Choose a point $x \leq (a+b)(p+q)$, and $x \not\leq v$. We claim that (a, b, c, p, q, x) is a triangle configuration.

Clearly $x \not\leq \underline{l}, \underline{k}$ and $x \leq l, k$. Also $x \not\leq \tilde{a}, \tilde{b}, \tilde{p}, \tilde{q}$ since, for example, $\tilde{a}(\tilde{p} + \tilde{q}) = \tilde{a}(\tilde{p} + l\tilde{q}) = \tilde{a}(\tilde{p} + w) = \tilde{a}\tilde{p} \leq v$. We can therefore apply lemma 2.1 to get the collinearities C(a,b,x) and C(p,q,x) besides the given C(a,c,p) and C(b,c,q). One can easily show that no other collinearities are possible.

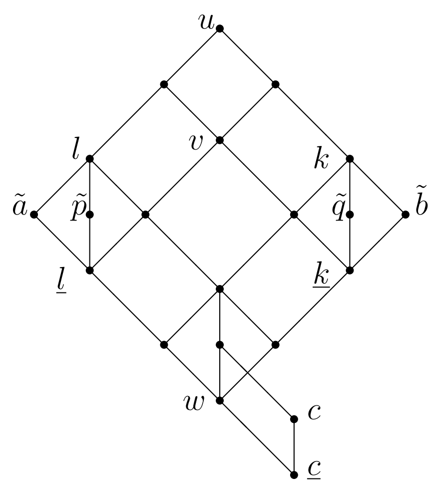

Figure 5: Case 4

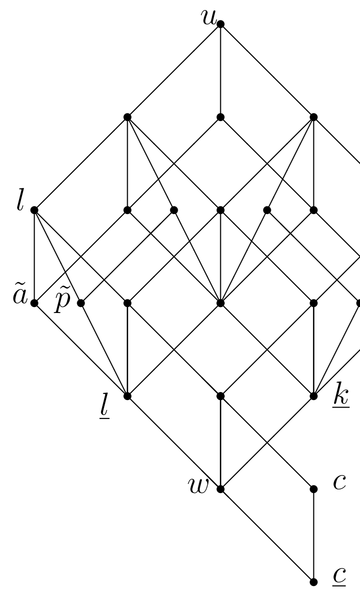

Figure 6: Case 5

4 Lattice identities

In this section we will prove,

Theorem 4.1 *Every modular lattice M can be embedded, within its variety, into the subspace lattice of a projective ordered space whose point set Q is at most countable or the cardinality of M.*

To prove the theorem we need,

Lemma 4.1 *For every lattice identity α there is a sentence α' in the first order language of ordered spaces so that α' holds in a projective ordered space if and only if α holds in its lattice of subspaces.*

We will associate such a sentence with each lattice inequality $p \leq q$. The conjunction of the sentence associated with $p \leq q$ and the sentence associated with $q \leq p$ will provide a sentence for the identity $p = q$. The procedure is similar to that of the proof that identities are preserved in the ideal lattice, cf. [4]. Let $u_1, ..., u_n$ be the variables occurring in $p \leq q$ and introduce a 'point' variable x_{ij} corresponding to the j'th occurrence of u_i in p, (x_{ij}) will denote the array of all the x_{ij}'s. (By using the absorption law we can ensure that the same variables occur in p as in q.)

Lemma 4.2 *For each subterm r of p there exists a formula $\tilde{r} = \tilde{r}(y, (x_{ij}))$, whose free variables come from $(y) \cup (x_{ij})$, such that for every projective ordered space P, every point $a \in P$ and all subspaces $U_1, ..., U_n$ in P,*

$$a \in r(U_1, ..., U_n)$$

iff

there exists an array (a_{ij}) with, for each i, j, $a_{ij} \in U_i$, and so that

$$\tilde{r}(a, (a_{ij})) \text{ holds in } P.$$

Proof. The proof is by an easy induction on the length of r. If r is the j'th occurrence of u_i then let $\tilde{r}$ be the formula

$$y \leq x_{ij}.$$

If $r = st$ then let $\tilde{r}$ be the formula

$$\tilde{s} \wedge \tilde{t}.$$

If $r = s + t$ then let $\tilde{r}$ be the formula,

$$(\exists y_s, y_t)[C(y, y_s, y_t) \wedge \tilde{s}(y_s, (x_{ij})) \wedge \tilde{t}(y_t, (x_{ij}))] \vee \tilde{s}(y, (x_{ij})) \vee \tilde{t}(y, (x_{ij})).$$

Verifying that these formulae work is straightforward (the last formula comes directly from lemma 2.2).

Lemma 4.3 *For every subterm r of q there exists a formula $\hat{r} = \hat{r}(y, (x_{ij}))$, whose free variables come from $(y) \cup (x_{ij})$, so that for every projective ordered space P and array $(a) \cup (a_{ij})$ in P,*

$$a \ \in \ r(\Sigma_j a_{1j}, ..., \Sigma_j a_{nj})$$

iff

$$\hat{r}(a, (a_{ij})) \textit{ holds in } P.$$

Proof. To start the induction we need for each m, a formula $\beta_m(z, z_1, ..., z_m)$, so that for all points $a, a_1, ..., a_m$ in P,

$$a \in \Sigma_{i=1}^{m} a_i$$

iff

$$\beta_m(a, a_1, ..., a_m) \text{ holds in P.}$$

For each k let v_k, v'_k be two new variables. Let β_1 be the formula, $z \leq z_1$, and for $m > 1$ let β_m be the formula,

$$(\exists v_m, v'_m)[C(z, v_m, v'_m) \wedge (v_m \leq z_m) \wedge \beta_{m-1}(v'_m, z_1, ..., z_{m-1})] \\ \vee \beta_1(z, z_m) \vee \beta_{m-1}(z, z_1, ..., z_{m-1}).$$

If r is an occurrence of u_i in q and if u_i occurs in p exactly m-times let $\hat{r}$ be

$$\beta_m(y, x_{i1}, ..., x_{im}).$$

As in lemma 4.2, if $r = st$ then $\hat{r}$ is $\hat{s} \wedge \hat{t}$, and if $r = s + t$ then $\hat{r}$ is,

$$(\exists y_s, y_t)[C(y, y_s, y_t) \wedge \hat{s}(y_s, (x_{ij})) \wedge \hat{t}(y_t, (x_{ij}))] \vee \hat{s}(y, (x_{ij})) \vee \hat{t}(y, (x_{ij})).$$

We are now in a position to prove lemma 4.1 .

Proof (of lemma 4.1). The sentence corresponding to the inequality $p \leq q$ is

$$(\forall y, (x_{ij}))[\tilde{p}(y, (x_{ij})) \to \hat{q}(y, (x_{ij}))]. \tag{1}$$

Suppose $p \leq q$ in $L(P)$, and suppose for some interpretation of y and the x_{ij}, $\tilde{p}(y, (x_{ij}))$ holds in P. Then by lemma 4.2,

$$y \in p(\Sigma_j x_{1j}, ..., \Sigma_j x_{nj}).$$

Since $p(\Sigma_j x_{1j}, ..., \Sigma_j x_{nj}) \subseteq q(\Sigma_j x_{1j}, ..., \Sigma_j x_{nj})$, we have, by lemma 4.3, $\hat{q}(y, (x_{ij}))$. Hence the sentence (1) is valid in P.

Conversely, suppose the sentence (1) is valid in P and let $a \in p(U_1, ..., U_n)$. Then by lemma 4.2 there exists an array (a_{ij}), with $a_{ij} \in U_i$ so that, $\tilde{p}(a, (a_{ij}))$ holds in P. It follows that $\hat{q}(a, (a_{ij}))$ holds in P as well, and therefore by lemma 4.3

$$a \ \in \ q(\Sigma_j a_{1j}, ...\Sigma_j a_{nj}) \ \subseteq \ q(U_1, ..., U_n).$$

Proof (of theorem 4.1). In view of Corollary 2.1 we may assume that M is a sublattice of a modular algebraic spatial lattice L in the variety generated by M . We will construct an elementary substructure Q of $S(L)$ having the claimed cardinality such that $\phi(a) = \{p \in Q \ : \ p \leq a \ \text{ in } \ L\}$ defines a lattice embedding of M into $L(Q)$. By theorem 3.2, $S(L)$ and so Q are projective ordered spaces. Now L is isomorphic to $L(S(L))$ by theorem 2.1, so by lemma 4.1 $L(Q)$ belongs to the variety generated by M. We construct Q as follows:

Choose Q_0 so that for each $a < b$ in M there is $p \in Q_0$ with $p \leq b$, $p \not\leq a$.

Suppose we are given Q_n and n is even. By the downward Löwenheim-Skolem-Tarski theorem there is an elementary substructure, Q_{n+1}, of $S(L)$

whose cardinality is countable or of the cardinality of Q_n, whichever is greater.

Suppose we are given Q_n and n is odd. For a, b in M and r in Q_n with $r \leq a+b$ there are $p \leq a$ and $q \leq b$ in P with p, q, r collinear. Q_{n+1} is formed from Q_n by adjoining a suitable p and q for every such triple a, b, r. The cardinality of Q_{n+1} is at most countable or the cardinality of Q_n, whichever is greater.

Let $Q = \cup_{i=1}^{\infty} Q_n$.

It remains to show that ϕ is a lattice embedding. It is clear that ϕ is an order embedding and that it preserves meets. That it preserves joins as well is a consequence of the construction. Explicitly, let $a, b \in M$ and let $r \in \phi(a+b)$. Then, by construction, there are $p, q \in Q$ with $p \leq a$, $q \leq b$ and p, q, r collinear. From 3.1 we see that $r \in \phi(a) + \phi(b)$.

5 Decomposition.

Let P be a projective ordered space. Relating all points on a proper line, we call such points *perspective*, and passing to the transitive closure gives a decomposition of P into connected *components*, Q_i, $i \in I$, some of which may correspond to isolated points, i.e. points which are not on any proper lines. With the induced order and space,

Proposition 5.1 *Each component Q_i is a projective ordered space and the complete lattice homomorphisms*

$$pr_i : L(P) \to L(Q_i) \quad S \mapsto (S \cap Q_i : i \in I)$$

define a subdirect decomposition of the $L(P)$ into the subdirectly irreducible factors $L(Q_i)$.

Proof. Inspection of the axioms shows that each component is a projective ordered space. Let $S \not\subseteq T$ be subspaces of P. There is a point $p \in S - T$ and $p \in Q_i$ for some i whence $pr_i(S) \neq pr_i(T)$.

The map pr_i is clearly a map of $L(P)$ into $L(Q_i)$ preserving arbitrary intersections and directed joins. Let S, T be subspaces of P and let $r \in (S+T) \cap Q_i$. If $r \notin S \cup T$ then by lemma 3.1 there exists $s \in S, t \in T$ with

$C(r, s, t)$. Since $r \in Q_i$ it follows $s, t \in Q_i$ and $r \in (S \cap Q_i) + (T \cap Q_i)$. Hence, in view of algebraicity, arbitrary joins are preserved.

For a point p in Q_i the subspace $\{q \in Q_i : q \leq p\}$ of Q_i is the image of the subspace $\{q \in P : q \leq p\}$ of P ; thus since the points of Q_i are join dense, the map pr_i is also onto.

By 3.1, each $L(Q_i)$ is modular. If $p \neq q$ are on a proper line $p/\underline{p}$ and $q/\underline{q}$ are projective via $p + q/\underline{p} + \underline{q} + r$ where r is any third point on the line. Since every proper quotient of $L(Q_i)$ contains a quotient transposing to a $p/\underline{p}$, subdirect irreducibility follows, cf [4], chapter 10. This completes the proof of the proposition.

Conversely, we can compose projective ordered spaces. Let $(Q_i, \leq_i, C_i)$, $i \in I$, be ordered projective spaces, Q the disjoint union of the Q_i , the relation C the disjoint union of the C_i, and $\leq$ an order on Q having restriction $\leq_i$ to Q_i for all i.

Proposition 5.2 $(Q, \leq, C)$ *is an ordered projective space if and only if* $C_i(p, q, r)$ *and* $p, q \leq s \notin Q_i$ *implies* $r \leq s$ *and, secondly,* $C_i(p, q, r)$ *and* $r \geq r' \in Q_j, j \neq i$*, implies* $r' \leq p$*,*$r' \leq q$*, or* $C_j(p', q', r')$ *for some* $p' \leq_i p$*,* $q' \leq_i q$ *.*

For the proof just observe that this characterizes the $(Q, \leq, C)$ which are regular ordered spaces and that the triangle axiom is satisfied automatically since its hypothesis concerns points in a common component, only.

Corollary 5.1 *If* L *is a spatial modular lattice then* L *is connected under the transitive closure of perspectivity.*

Corollary 5.2 *Every variety of modular lattices is generated by its subdirectly irreducible spatial algebraic members, cf. [8].*

One easily derives a characterization of the scaffoldings of finite length modular lattices (see Ganter, Poguntke, and Wille [9] for the definition and a special case). The space associated with a finite length lattice can be considered as a relative substructure of the scaffolding. Hence, one has to rephrase the axioms of an ordered space and regularity into the language of scaffoldings. This is easily done in view of lemma 2.2.

6 Bases of lines

Let M be a modular algebraic spatial lattice with point set P_M. The set T_M of *line tops* of M consists of all joins $p+q+r$ where p, q, r is a collinear triple. In view of lemma 2.1 these are just the upper bounds l of at least 5-element line intervals. The associated lower bound was denoted by $\underline{l}$. We say that p *copes* with the *task* (a, l), $a \prec l$, if $p \leq a$ and $p \not\leq \underline{l}$.

Consider a system Λ of proper lines of M which is *irredundant* in the sense that $\lambda \mapsto \sum \lambda$ yields an injective map of Λ into the set T_M of line tops. We want to single out those Λ which capture the join structure of M, in a fairly strong sense. For that purpose call a map ϕ of P_M into a complete join semilattice L *compatible* with Λ, if

$$\phi(r) \leq \phi(p) + \phi(q) \quad \text{for all } \lambda \in \Lambda \text{ and distinct } p, q, r \in \lambda.$$

Call Λ a *base of lines* if for all maps ϕ which are compatible with Λ, the map $\phi : P_M \to L$ defined by

$$\overline{\phi}(x) = \sum\{\phi(p) : p \leq x\}$$

defines a Λ-compatible map from P_M into L. This then means, that $\overline{\phi}$ extends to a complete join homomorphism of M into L.

Λ induces a Λ-collinearity relation on P_M, and this in turn gives P_M a Λ-subspace structure, cf. section 2. Let us call the lattice of Λ-subspaces, $L(P_M, \Lambda)$. Trivially, the identity map $id : P_M \to L(P_M, \Lambda)$ is compatible with Λ. Thus, if Λ is a base of lines then $\overline{id} : P_M \to L(P_M, \Lambda)$ is compatible with Λ. Using the fact that the elements of P_M are compact in M, cf. lemma 2.3, one can easily show that the extension of $\overline{id}$ to M is an isomorphism.

Theorem 6.1 *A modular algebraic spatial lattice admits a base of lines provided it has countable point set P_M or one of P_M and T_M satisfy the descending chain condition.*

For M of finite length this is the combination of (2.4) and (2.5) used in [12]. More generally, under the descending chain condition any irredundant Λ will work provided that $\lambda \mapsto \sum \lambda$ is a bijection of Λ onto T_M. If we have the descending chain condition for T_M then the join compatibility of the order preserving map $\overline{\phi}$ is proved with the inductive approach taken in the proof of

(2.5) of [12]. Now observe that in view of lemma 2.2 every infinite descending chain in T_M produces an infinite descending chain in P_M.

We believe that a modular algebraic spatial lattice having no base of lines can easily be constructed for which T_M has the structure of a binary tree with all branches infinite. The above theorem and the theory developed in sections 2,3,4 is summarized with the following:

Corollary 6.1 *Every modular lattice belongs to a universal subclass of its variety which is generated by subspace lattices of projective ordered spaces admitting a base of lines.*

Proof. A modular lattice is a member of a universal class of modular lattices iff each of its finitely generated sublattices is. By theorem 4.1, every countable modular lattice M can be embedded, within its variety, into the subspace lattice of a projective ordered space whose point set is countable.

The main effort of this section will consist in the proof of the theorem in case of countable P_M. For this we need a mechanism for selecting Λ.

A *preference* E is a linear order on a subset $dom(E)$ of P_M. We say that p is E-*incident* with l and write pEl if p is the first point in $dom(E)$ to cope with $(p + \underline{l}, l)$. We say that E is *finite* if $dom(E)$ is. An *extension* E^+ of E is a preference for which E is an initial segment. If E^+ is an extension of E then pEl implies pE^+l.

If the distinct points p, q, r are all E-incident with some $l \in T_M$ then we say p, q, r are E-*collinear*. The E-collinearity relation induces a subspace structure on P_M, the E-subspace structure, cf. section 2. For points p, q, $E(p, q)$ is the E-subspace generated by p, q. Maximal, at least two element sets of points, any three of which are E-collinear are E-*lines*. The definition of E-incidence guarantees that the set of E-lines, Λ_E, is irredundant. Furthermore, if

$$\{p \in P_M : \quad p \text{ is on some proper line}\} \subseteq dom(E)$$

then E-lines are lines of M in the original sense. We will give a procedure for constructing a preference E for which Λ_E is a base of lines. First let us give a version of compatability for preferences.

A map ϕ of P_M into a complete join semilattice L is *compatible* with E if $\phi(r) \leq \phi(p) + \phi(q)$ for all E-collinear triples p, q, r.

Let $p_1, p_2, ...$ be an enumeration of the points of P_M. We will call this the *reference enumeration* and it will remain fixed for the remainder of the section.

Let E be a finite preference. The *E-predecessor tree* of a perspective pair (p, q) of points, and the *E-predecessor relation*, will be defined inductively. The induction will be on the set

$$P(E; p, q) = \{r \in E : \ r \leq p + \underline{q} \ \text{ or } \ r \leq q + \underline{p}\}.$$

(p, q) is always a member of its E-predecessor tree. If $P(E; p, q) = \emptyset$ then (p, q) is the only element of its E-predecessor tree.

For the inductive step:

If there is a (necessarily unique) $p' \in P_M$ satisfying

$$p' \neq p, \ p'El, \ p' + \underline{l} = p + \underline{l}$$

then $p' \not\leq q$ and $p' \not\leq p$. Hence, we have $p' \leq p + q$, $p' \not\leq p$, $p' \not\leq q$, $p = p(q+p')$, and so by lemma 2.2 there exists a point $q^{\surd} \leq q$ so that $p, p', q^{\surd}$ are collinear. We choose the first such $q^{\surd}$ in the reference enumeration. Observe that $p' \notin P(E; p, q^{\surd})$ and that $p, q^{\surd}$ are perspective and so, by inductive hypothesis, the E-predecessor tree of $(p, q^{\surd})$ is defined. The elements of the E-predecessor tree of $(p, q^{\surd})$ are elements of the E-predecessor tree of (p, q) and are all E-predecessors of (p, q). We call $(p, q^{\surd})$ a *left lower cover* of (p, q).

Symmetrically, if there is a (necessarily unique) $q' \in P_M$ satisfying

$$q' \neq q, \ q'El, \ q' + \underline{l} = q + \underline{l}$$

then $q' \not\leq p$ and $q' \not\leq q$. Hence, there exists a first element of the reference enumeration $p^{\surd} \leq p$ so that $q, q', p^{\surd}$ are collinear. The elements of the E-predecessor tree of $(p^{\surd}, q)$ are elements of the E-predecessor tree of (p, q) and are all E-predecessors of (p, q). We call $(p^{\surd}, q)$ a *right lower cover* of (p, q).

If neither of these occur then (p, q) is the only member of its E-predecessor tree.

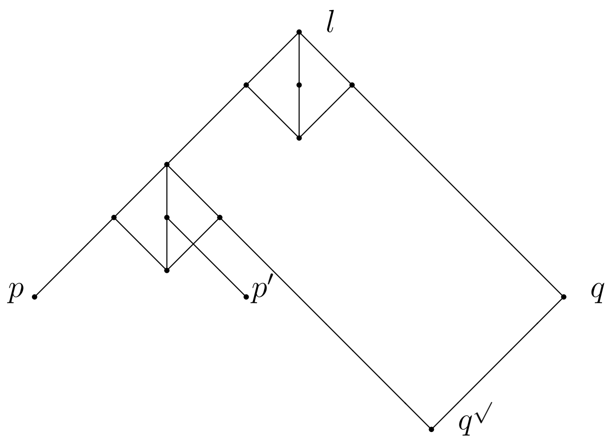

Figure 7: A partial Hasse diagram of M depicting the left lower cover $(p, q^{\surd})$ of (p, q).

Lemma 6.1 *The E-predecessor tree of a pair (p, q) is a finite ordered set under the reflexive closure of the predecessor relation. If (x, y) is an E-predecessor of (v, w) then $x \leq v$ and $y \leq w$. In particular, $x \leq p$ and $y \leq q$. Furthermore, the predecessor relation is the transitive closure of the union of the left and right lower cover relations.*

A *left terminal* of the E-predecessor tree is an element of the predecessor tree with no left lower cover. A *right terminal* of the E-predecessor tree is an element with no right lower cover. We set $U(E; p, q) = U_p \cup U_q$ where,

$$U_p = \{x \in P_M : \quad (x, y) \text{ is a left terminal of the E-predecessor tree}\}$$

$$U_q = \{y \in P_M : \quad (x, y) \text{ is a right terminal of the } E\text{-predecessor tree}\}.$$

Lemma 6.2 *Let E be a finite preference. If p, q, r are collinear and if $rE(p+q)$ then there is a finite extension $E^+ = (E; p, q, r)^+$ of E with the property that for any ϕ compatible with E^+, $\phi(r) \leq \overline{\phi}(p) + \overline{\phi}(q)$.*

Proof. We extend the preference E to the preference E^+ whose domain is $dom(E) \cup U(E; p, q) \cup \{p, q, r\}$ in such a manner that the elements of

$U(E;p,q) \cup \{p,q,r\}$ which are not already in $dom(E)$ are adjoined to E in the order in which they appear in the reference enumeration.

We claim that the following more general assertion is true. Let ϕ be compatible with E^+. If (x,y) is in the E-predecessor tree of (p,q) and $zE(x+y)$ then $\phi(z) \leq \overline{\phi}(p) + \overline{\phi}(q)$. This is done by induction on the number of predecessors of (x,y). If (x,y) has no predecessors then consider $(x+\underline{y}, x+y)$. Either $xE(x+y)$ or there is no $x' \in E$ coping with $(x+\underline{y}, x+y)$. If the second case occurs then $x \in U_p$ and hence $x \in dom(E^+)$. It is possible that some other point is added to E before x which also copes with this task. But since all points added are below p or below q this causes no problem. In both cases there is a point w with $w \leq p$ or $w \leq q$, coping with $(x+\underline{y}, x+y)$ and with wE^+x+y. In particular, $\phi(w) \leq \overline{\phi}(p) + \overline{\phi}(q)$. Symmetrically, there is some point s coping with $(\underline{x}+y, x+y)$, with $\phi(s) \leq \overline{\phi}(p) + \overline{\phi}(q)$. Now, w, s, z are E^+-collinear and hence $\phi(z) \leq \overline{\phi}(w) + \overline{\phi}(s) \leq \overline{\phi}(p) + \overline{\phi}(q)$.

The inductive step is similar: If (x,y) has a left lower cover of the form $(x, y^{\checkmark})$ then we have, by inductive hypothesis, $\phi(x') \leq \overline{\phi}(p) + \overline{\phi}(q)$. If (x,y) has no left lower cover then we can find a w as above coping with $(x+\underline{y}, x+y)$, with $\phi(w) \leq \overline{\phi}(p) + \overline{\phi}(q)$. Symmetrically, if (x,y) has a right lower cover $(x^{\checkmark}, y)$ then $\phi(y') \leq \overline{\phi}(p) + \overline{\phi}(q)$. If (x,y) has no right lower cover then we can obtain an s as above coping with $(\underline{x}+y, x+y)$, with $\phi(s) \leq \overline{\phi}(p) + \overline{\phi}(q)$. In any of the three possible resulting combinations of the cases one can easily show that $\phi(z) \leq \overline{\phi}(p) + \overline{\phi}(q)$.

Let E be a finite preference and let p, r be a pair of perspective points. The *E-sequence of r relative to p* is defined as follows.

If there is no element of E which copes with $(r+\underline{p}, p+r)$ then the E-sequence of r relative to p is empty.

If there is such an element, then let r_0 be the first such.

Inductively, assume r_{k-1} is defined. If $r_{k-1} = r$ then we have completed the sequence. If $r_{k-1} \neq r$ then we look for r_k so that $r_k E(r + r_{k-1})$. If there is no such point then we set $r_k = r$ and say that r is not a *proper member* of its own E-sequence relative to p.

If $0 < i < j$ then, by lemma 2.1, $r_j + \underline{r}_{i-1} = r_i + \underline{r_{i-1}} = r + \underline{r}_{i-1}$ and $\underline{p} + r_i = \underline{p} + r$. This implies that r_0 precedes r_i and r_i precedes r_j in E. Since

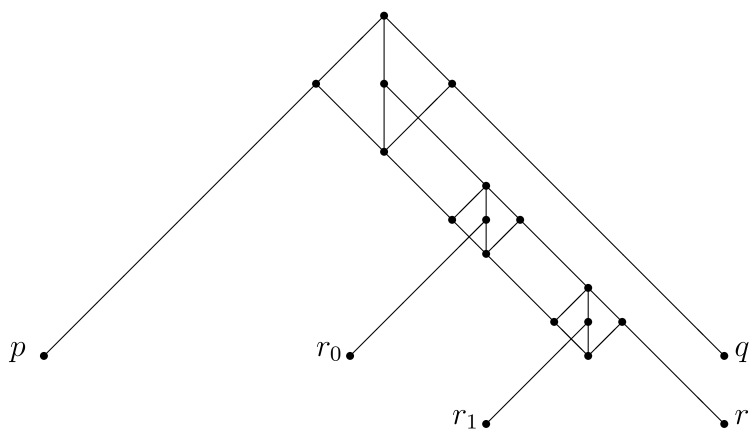

Figure 8: A partial Hasse diagram of M depicting the first two elements of the E-sequence of r relative to p.

E is finite, we obtain a finite sequence r_i, $0 \leq i \leq n$, with $r = r_n$. Note that r is a proper member of its E-sequence relative to p iff $r \in dom(E)$.

Let $r \in P_M$ be collinear with p and q. The $(E; p, q)$-*grid* of r, a collection of ordered pairs of points, and the *immediate ancestor* relation, are defined as follows:

For each r_i, $0 < i \leq n$, we consider whether or not there is a point x with $xE(r_{i-1} + r)$ and coping with $((\underline{p} + r)(r_{i-1} + r), r_{i-1} + r)$. If such exists and, if $x \not\leq p$ and $x \not\leq q$, then we let p_x and q_x be the first points of the reference enumeration with $p_x \leq p$, $q_x \leq q$ and with p_x, q_x, x collinear. Since $x \leq p + q$, $x \not\leq p$ and $x \not\leq q$ such points exist by lemma 2.2. Let x_j, $0 \leq j \leq k$, be the E-sequence of x relative to p_x and adjoin to the $(E; p, q)$-grid of r all the pairs (p_x, x_j), these are all immediate ancestors of (p, r_i). Note that x is a proper member of its own E-sequence relative to p_x.

Inductively suppose (p_x, x_i), $0 < i$, is in the $(E; p, q)$-grid of r. We consider whether or not there is a point y with $yE(x_{i-1} + x)$ and coping with $((\underline{p_x} + \underline{x})(x_{i-1} + x), x_{i-1} + x)$. If $y \not\leq p$ and $y \not\leq q$ then we let p_y and q_y be the first points of the reference enumeration with $p_y \leq p_x$, $q_y \leq q$ and with p_y, q_y, y collinear. Since $y \leq p_x + q$, $y \not\leq p_x$ and $y \not\leq q$ such points exist by lemma 2.2. Let y_j, $0 \leq j \leq m$, be the E-sequence of y relative to p_y. We adjoin all the pairs (p_y, y_j) to the I-grid of r and call each (p_y, y_j) an *immediate ancestor* of (p_x, x_i). Note that y is a proper member of its E-sequence relative to p_y, and note also that p_y is a function of p_x, q and y, not just one of y.

The *ancestor relation* is the transitive closure of the immediate ancestor relation.

Lemma 6.3 *The $(E; p, q)$-grid of r with the reflexive closure of the ancestor relation is a finite partially ordered set.*

Proof. If (p_y, y_j) is an ancestor of (p_x, x) then $p_x + y_j = p_x + p_y + y_j = p_x + p_y + y \leq p_x + y \leq p_x + \underline{x} < p_x + x$. This computation shows that the reflexive closure of the ancestor relation is antisymmetric and hence a partial order. With this order the grid has the elements (p, r_i) as maximal elements and any element (p_x, x_i) has as lower covers all the elements (p_y, y_j), its immediate ancestors. Each member of an I-sequence is in E so that $|E|$ is a bound on the number of maximal elements and on the number of lower covers that an element may have. Similarly, a descending chain, determines different elements of E, as the calculation at the beginning of this proof shows. Thus the length of descending chains is bounded by $|E|$ as well.

Lemma 6.4 *Let E be a finite preference and p, q, r a collinear triple. Then there is a finite extension $E^{\checkmark} = (E; p, q, r)^{\checkmark}$ of E with the property that for any ϕ compatible with $E^{\checkmark}$ we have $\phi(r) \leq \overline{\phi}(p) + \overline{\phi}(q)$.*

Proof. Let (p_x, x_i) be a minimal element of the $(E; p, q)$-grid or r. There are three ways in which this can occur.

(i) $i = 0$.

(ii) There is a point $y = y_{(p_x, x_i)}$ with $yE(x_{i-1} + x)$ and coping with $((\underline{p_x} + \underline{x})(x_{i-1} + x), x_{i-1} + x)$, but $y \leq p$ or $y \leq q$.

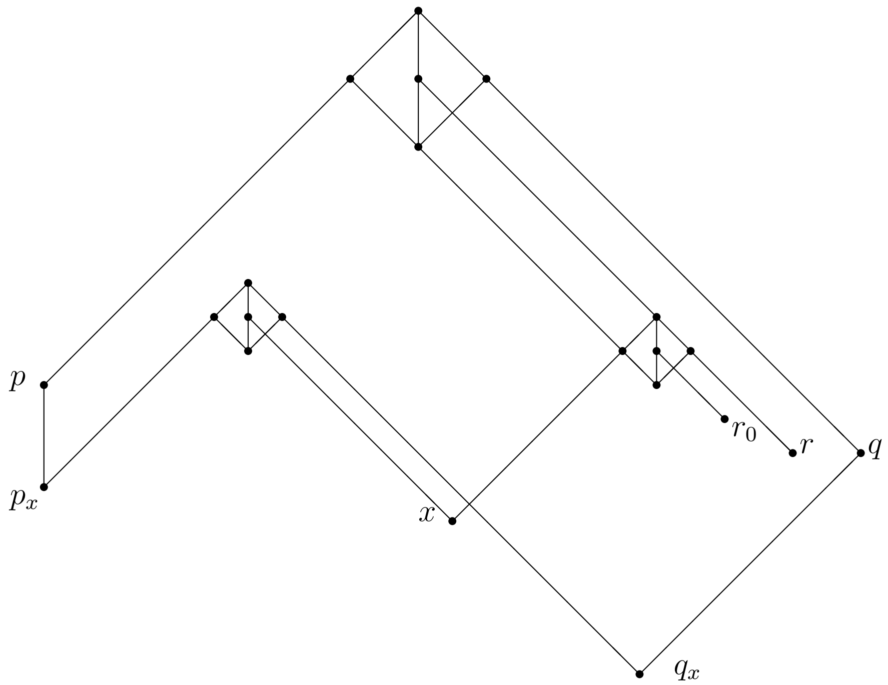

Figure 9: A partial Hasse diagram of M depicting the immediate ancestor (p_x, x) of (p, r_1), r_1 is not shown. The other immediate ancestors of (p, r_1) are the elements (p_x, x_i), where x_i is in the E-sequence of x relative to p_x.

(iii) There is no such point y.

For the moment we restrict our attention to the third case. In this case let $y = y_{(p_x,x_i)}$ be the first point of the reference enumeration below p which copes with $((\underline{p_x} + \underline{x})(x_{i-1} + x), x_{i-1} + x)$. Such a point exists by lemma 2.2. Let

$$Y = \{y_{(p_x,x_i)} : \ (p_x, x_i) \text{ is minimal in the } (E; p, q)\text{-grid of } r \text{ and case (iii) occurs.}\}$$

We order the elements of $Y \cup \{r\}$ according to their order of appearance in the reference enumeration and adjoin these elements, except r if $r \in dom(E)$ already, to E to get the extension E^0.

We list the pairs (p_x, q_x), where (p_x, x_i) is a member of the $(E; p, q)$ grid of r, in some manner (say, according to their occurence in the lexicographical ordering with the order on the components being the restriction of the ordering of the reference enumeration).

If (p_x, q_x) is the i'th element of this list then $E^i = (E^{i-1}; p_x, q_x)^+$.

$E^{\surd}$ is the final element of this finite sequence of extensions of E.

We claim that the lemma is valid for the second component of every element of the grid. Specifically, if ϕ is compatible with $E^{\surd}$ and (p_x, x_i) is an element of the $(E; p, q)$-grid of r then $\phi(x_i) \leq \overline{\phi}(p) + \overline{\phi}(q)$. We prove this by induction on the pairs (h, i) ordered lexicographically, where h is the maximum length of a chain below (p_x, x_i) in the $(E; p, q)$ grid of r.

Our first case, $h = 0$, includes the first step of the induction, $h = 0$, $i = 0$. Consider an element of the grid of the form (p_x, x_0). The pair (p_x, q_x) occurs somewhere in the list of such pairs, and hence $E^{n+1} = (E^n; p_x, q_x)^+$ for some n. Since $E^{\surd}$ is an extension of E^{n+1}, ϕ is compatible with E^{n+1}. We can now obtain the result by applying lemma 6.2 to E^n and p_x, q_x, x_0. This gives $\phi(x_0) \leq \overline{\phi}(p_x) + \overline{\phi}(q_x) \leq \overline{\phi}(p) + \overline{\phi}(q)$.

Consider now the case $h = 0$ and $i > 0$. By inductive hypothesis, $\phi(x_{i-1}) \leq \overline{\phi}(p) + \overline{\phi}(q)$. By definition, $y = y_{(p_x,x_i)} \leq p$, and hence $\phi(y) \leq \overline{\phi}(p) + \overline{\phi}(q)$. By construction, y, x_{i-1}, x_i are E^0-collinear and hence $E^{\surd}$-collinear. Since ϕ is compatible with $E^{\surd}$, we have $\phi(x_i) \leq \phi(y) + \phi(x_{i-1})$. It follows that $\phi(x_i) \leq \overline{\phi}(p) + \overline{\phi}(q)$.

The case $h > 0$, $i > 0$, now follows in a similar manner. By inductive hypothesis, $\phi(x_{i-1}) \leq \overline{\phi}(p) + \overline{\phi}(q)$. Also by inductive hypothesis, $\phi(y) \leq \overline{\phi}(p) + \overline{\phi}(q)$, where (p_y, y_j) are the immediate ancestors of (p_x, x_i). The compatability of ϕ and the $E^{\checkmark}$-collinearity of x_{i-1}, x_i, y imply that $\phi(x_i) \leq \phi(y) + \phi(x_{i-1})$, and hence that $\phi(x_i) \leq \overline{\phi}(p) + \overline{\phi}(q)$.

Proof of the theorem. Let $T_1, T_2, ...$ be an enumeration of the set of all collinear triples of points. We will define preferences E_n, inductively as follows:

$E_0 = \emptyset$

Assume that E_i is defined and let $T_{i+1} = (p, q, r)$. Then $E_{i+1} = (E_i; p, q, r)^{\checkmark}$, cf. lemma 6.4.

We claim that Λ_E is a base of lines. Let ϕ be compatible with Λ_E. Then ϕ is compatible with E, and hence with each E_n.

Assume that $\hat{p}, \hat{q}, \hat{r}$ are E-collinear, i.e. $\hat{p}, \hat{q}, \hat{r} \in \lambda$ for some $\lambda \in \Lambda_E$, and let $r \leq \hat{r}$. If $r \leq \hat{p}$ or if $r \leq \hat{q}$ then $\phi(r) \leq \overline{\phi}(\hat{p}) + \overline{\phi}(\hat{q})$ trivially. Otherwise, by lemma 2.2, there are points $p \leq \hat{p}$ and $q \leq \hat{q}$ with p, q, r collinear. Let $T_n = (p, q, r)$. Since ϕ is compatible with E_n we have, by lemma 6.4, that $\phi(r) \leq \overline{\phi}(p) + \overline{\phi}(q) \leq \overline{\phi}(\hat{p}) + \overline{\phi}(\hat{q})$. It follows that $\overline{\phi}(\hat{r}) \leq \overline{\phi}(\hat{p}) + \overline{\phi}(\hat{q})$, and the proof is complete.

7 2-Distributivity

For this and the following section let M be a modular spatial lattice with point set P_M . Let Λ_M be the set of all lines of M. For a line λ let $\overline{\lambda}/\underline{\lambda}$ denote the associated line interval , i.e. $\overline{\lambda} = \Sigma\{p : p \in \lambda\}$ and $\underline{\lambda} = \Sigma\{\underline{p} : p \in \lambda\}$, the meet of all lower covers of $\overline{\lambda}$. Let $a \prec b$ in M and $Q = \{p \in P_M : p \leq b,\ \ p \not\leq a\}$

Lemma 7.1 *Every line λ with $\overline{\lambda} \leq b$, and $\overline{\lambda} \not\leq a$ contains a unique point p, with $p \leq a$. Every two distinct points $q, r \in Q$ are on a proper line.*

Proof. $a\overline{\lambda} \prec \overline{\lambda}$, and the first claim follows from lemma 2.1. For the second part, observe that q and r are incomparable since, for example, $q < r$ would imply $q \leq \underline{r} = ra$. From the first part and lemma 2.1, there is some point $p \leq a$ with p, q, r collinear. This completes the proof.

An (a, b)-*cycle* is a sequence $\{\lambda_i \ : 0 \leq i < n\}$ in Λ_M with each $\lambda_i \cap Q$ containing at least 2-elements, all $\overline{\lambda_i}$ pairwise distinct, and $\lambda_i \cap \lambda_j \cap Q \neq \emptyset$ if and only if $i = j$ or $i - j \equiv \pm 1$ modulo n.

Theorem 7.1 *For a modular spatial lattice M the following are equivalent.*

(i) M is 2-distributive

(ii) If q is a point and $q \leq \sum_{i=1}^{n} a_i$ then $q \leq a_i$ for some i, or there are $i \neq j$ and points $p_i \leq a_i$, $p_j \leq a_j$ such that q, p_i, p_j are collinear.

(iii) If a, c, p and b, c, q are collinear, then $q \leq a+b$, $q \leq b+p$, or $q \leq a+p$.

(iv) $S(M)$ contains no triangle configuration.

(v) There is no (a, b)-cycle in M.

Proof.

(i) implies (v). This is basically due to Jónsson and Nation [17] the additional reasoning is given in the proof of '(1) implies (2)' of (5.1) of [12].

(v) implies (iv). Let (a, b, c, p, q, x) be a triangle configuration. We claim that for each of the four collinear triples the points of the triple are the only ones in the configuration which, in M, are below the associated line top.

Assume, for example, that $c \leq a+b+x = l$. The meet of any two of $a+\underline{l}$, $b+\underline{l}$ and $x+\underline{l}$ is $\underline{l}$. If $c \not\leq \underline{l}$ then, for example, $c \not\leq a+\underline{l}$ and $c \not\leq b+\underline{l}$. But, by lemma 2.1, $C(a, b, c)$, a contradiction. Thus $c \leq \underline{l}$, whence $p \leq a + \underline{l}$. Since b, p, x are not collinear it follows $p \leq \underline{l}$ and by $C(a, c, p)$ we get $a \leq \underline{l}$ which is impossible. The other possibilities can be handled in a similar manner.

Now, let $u = l + c = l + p = l + q$ and $v = l + \underline{c} = l + \underline{p} = l + \underline{q}$. By modularity and join irreducibility we get from the above that $c, p, q \not\leq v \prec u$. The three pairs from c, p, q give rise to three distinct line tops, so we would obtain three lines constituting a (u, v)-cycle in M.

(iv) implies (iii). If (iv) holds only cases 2-5 of the triangle axiom can apply, and (iii) is satisfied.

(iii) implies (ii). Assume (iii). If $n \leq 2$ then (ii) is satisfied by lemma 2.2. We will prove the claim for $n = 3$, the general result follows by a straightforward induction.

Let $q \leq a_1 + a_2 + a_3$. If $q \leq a_i + a_j$ then the result follows from lemma 2.2. Otherwise, by lemma 2.2, there are points $b \leq a_1$ and $c \leq a_2 + a_3$ such that b, c, q are collinear and, again by lemma 2.2, there are points $a \leq a_2$, $p \leq a_3$ with a, c, p collinear. By (iii), $q \leq a + b$, $q \leq b + q$ or $q \leq a + p$. Another application of lemma 2.2 proves the result.

(ii) implies (i). The 2-distributivity of $L(S(M))$ follows from (ii) easily, whence that of M via theorem 2.1.

We conclude this section with some definitions and a combinatorial lemma needed in the final section. Let M be a 2-distributive modular spatial lattice and Λ an *irredundant* set of proper lines. In particular distinct lines in Λ cannot have more than one point in common. Let P_Λ be the union of all lines in Λ. Call a point $p \notin P_\Lambda$ *isolated* (with respect to Λ).

A *chain* (respectively *cycle*) in (Q, Λ), Q a subset of P_M, is a sequence $\{\lambda_i : 0 \leq i < n\}$ in Λ with all $\lambda_i \cap Q$ at least 2-element and $\lambda_i \cap \lambda_j \cap Q \neq \emptyset$ if and only if $|i - j| \leq 1$ (respectively $i = j$ or $i - j \equiv \pm 1$ modulo n). We say that the chain *joins* λ_0 with λ_{n-1}. Two lines of Λ belong to the same *component* of (Q, Λ) if and only if they are joined by a chain in (Q, Λ). The set of all points in Q lying on the lines of a component will be also be called a component.

Consider $\Gamma \subseteq \Lambda$, $P_\Gamma \subseteq U \subseteq P_\Lambda$ and let $\Gamma' = \Lambda - \Gamma$. A *depth function* for Γ and U is a map d from Λ into the natural numbers such that:

(i) $d(\lambda) = 0$ iff $\lambda \in \Gamma$.

(ii) If $d(\lambda) = 1$ then λ contains exactly one point from U.

(iii) If $d(\lambda) > 1$ then either there are exactly two points on λ which are in U or there is one point on λ which is in U and one point on a line of depth less than $d(\lambda)$; also there are no other points on λ which are on lines of lesser depth.

(iv) For each $p \in P_\Lambda - U$ there is a unique line of minimum depth containing p.

For a in M let $P_a = \{p \in P_M : p \leq a\}$ and let $\Lambda_a = \{\lambda \in \Lambda : \overline{\lambda} \leq a\}$.

Lemma 7.2 *Let $\overline{\lambda} \leq b$ for all $\lambda \in \Lambda$ and let a and c be lower covers of b in*

M . Then there are depth functions for $\Lambda_a \subseteq \Lambda$ and $U = P_a \cap P_\Lambda$ and for $\Lambda_a \cup \Lambda_c \subseteq \Lambda$ and $U = (P_a \cup P_c) \cap P_\Lambda$. In the first case, for any line of depth greater than zero there is a unique chain joining it to a line of depth 1.

Proof. First observe that by lemma 7.1 each line in Λ'_a contains exactly one point from P_{Λ_a} and that (P_M, Λ'_a) is cycle free since any such cycle would be an (a, b)-cycle contrary to theorem 7.1.

In the first case, let all lines from Λ_a be of depth 0. Distinguish one line in each component of $(Q_{\Lambda_a}, \Lambda'_a)$ giving it depth 1. For each of the remaining lines there is a unique chain joining it with a line of depth 1, because (P_M, Λ'_a) is cycle free. Let the depth of such a line be the number of lines in the chain. Each such line of depth $k > 1$ will contain exactly one point on a line of depth $k - 1$ and one point on a line of depth 0. It will contain no other points on a line of depth less than k. The final statement of the lemma follows directly from this construction.

In the second case let $a \neq c$ and $\Gamma = \Lambda_a \cup \Lambda_c$. Let Ω consist of those lines from Γ' which contain a point from P_{ac}, and then no other point from $P_a \cup P_c$. Let Δ consist of the remaining lines in Γ' having exactly one point from each P_a and P_c but none from P_{ac}.

(1) No two lines λ_1, λ_2 from Δ intersect in a point $r \in P_\Lambda - P_\Gamma$.

Assume that such an r exists and let $p_i \in \lambda_i$ with $p_i \leq a$ and $p_i \not\leq c$. If $p_1 = p_2$ then $\overline{\lambda_1} = \overline{\lambda_2}$ contradicting the irredundancy of Λ. If $p_1 \neq p_2$ then there is a line $\lambda \in \Lambda_M$ containing p_1 and p_2. Since $\overline{\lambda} \not\leq c$, the sequence $\lambda_1, \lambda_2, \lambda$ is a (c, b)-cycle, contrary to theorem 7.1.

Let $R = P_b - (P_a \cup P_c)$ and observe that each line in Ω contains at least two points of R. Of course, (R, Ω) is cycle free since such a cycle would be both an (a, b)-cycle and a (c, b)-cycle. Let $S = P_\Delta - (P_a \cup P_c))$.

(2) Each component of (R, Ω) contains at most one $r \in S$.

Let $\lambda_1, ..., \lambda_n$ be a chain in (R, Ω) and $r_i \in \lambda_i \cap S$ for $i = 1, n$ with $r_1 \neq r_n$. By definition of Δ there exist $p_i \in P_a - P_{ac}$, $q_i \in P_c - P_{ac}$ so that p_i, q_i, r_i on $\gamma_i \in \Delta$, $i = 1, n$. If $p_1 = p_n$ then $\gamma_1, \lambda_1, \lambda_2, ..., \lambda_n, \gamma_n$ is a (c, b)-cycle. Otherwise, by lemma 7.1 there is a line $\mu \in \Lambda_M$ joining p_1 and p_n. Since, $\overline{\mu} = p_1 + p_n \leq a$, the lines $\mu, \gamma_1, \lambda_1, \lambda_2, ..., \lambda_n, \gamma_n$ form a (c, b)-cycle.

Now, define the lines in Γ and Δ to have depth 0 and 2, respectively. If a component of (R, Ω) contains a point of S then choose a single line containing this point and give it depth 3. If a component contains no point from S then choose a line of the component arbitrarily and give it depth 1. For each of the remaining lines there is a unique (since (R, Ω) is cycle free) chain, consisting of m lines in (R, Ω) joining it with a line of depth $d \leq 3$. Let the depth of such a line be defined as $m + 2$. The observations (1) and (2) above ensure that we have defined a depth function.

8 Representation

In this section we will set up the machinery for a vector space representation of the line spaces associated with 2-distributive modular algebraic spatial lattices. Let M be a modular algebraic spatial lattice, Λ a set of lines of M, k a field with $|k| + 1 \geq |\lambda|$ for all $\lambda \in \Lambda$, and E a k-vector space. For $X \subseteq E$ let $< X >$ be the subspace of E generated by X. As in section 6, any map $\phi : P_M \to E$ provides us with an order preserving map $\overline{\phi} : M \to L(E)$ where $\overline{\phi}(a) =< \phi(p) : \ p \leq a >$.

Call $\phi : P_M \to E$ a *representation* of (M, Λ) if for all $p, q, r \in P_M$ and $a \in M$

(1) $\phi(p) \in \overline{\phi}(a)$ implies $p \leq a$,

(2) $p, q, r \in \lambda, \ \ \lambda \in \Lambda, \ \ p \neq q$ implies $\phi(r) \in < \phi(p), \phi(q) >$,

(3) $\overline{\phi}$ is a meet homomorphism of M into $L(E)$.

Our candidate for a representation is built as follows:

Introduce vector space generator symbols e_p, one for each $p \in P_M$. For a proper line λ of M fix a system Σ_λ of linear relations

$$\alpha_t e_p + \beta_t e_q + \gamma_t e_r = 0,$$

$\alpha_t, \beta_t, \gamma_t \in k$, one for each 3-element subset $t = \{p, q, r\}$ of λ such that

$$\underline{\lambda} + p \mapsto e_p$$

is a representation of the lattice $(\overline{\lambda}/\underline{\lambda})$, endowed with a single line consisting of all atoms, in the free k-vector space E_λ with generators $e_p, \ \ p \in \lambda$ and relations Σ_λ. The cardinality restriction, $|k| > |\lambda|$ implies that such a representation exists; it can be derived from any embedding of the length two lattice $\overline{\lambda}/\underline{\lambda}$ into the subspace lattice of a 2-dimensional k-vector space. In fact, the given 2-dimensional vector space is free on Σ_λ.

Let E be the k-vector space with presentation consisting of, as generators,

$$\{e_p \ : \ p \in P_M\},$$

and relations,

$$\Sigma_\Lambda = \cup(\Sigma_\lambda, \ \ \lambda \in \Lambda).$$

The map

$$p \to \phi(p) = e_p \in E$$

is called *canonical* for k, M, and Λ. If k, M, and the Σ_λ are fixed, then for each Λ we have a uniquely determined $E = E_\Lambda$ and $\phi = \phi_\Lambda$.

Observe that property (2) is an immediate consequence of the construction and that for $\Gamma \subseteq \Lambda$ there is a canonical linear map

$$f_{\Gamma\Lambda} : E_\Gamma \to E_\Lambda,$$

onto E_Λ, such that for all $p \in P_M$,

$$\phi_\Lambda(p) = f_{\Gamma\Lambda}\phi_\Gamma(p).$$

Let F_Λ denote the subspace generated by the e_p, $p \in P_\Lambda$, in E_Λ. Isolated points do not appear in any of the relations in Σ_Λ , so they yield direct summands:

$$E_\Lambda = F_\Lambda \oplus \bigoplus(ke_p : p \notin P_\Lambda).$$

Lemma 8.1 *Let d be a depth function for $\Gamma \subseteq \Lambda$ and $P_\Gamma \subseteq U \subseteq P_\Lambda$, and let X be a selection of one point, not in U, from each $\lambda \in \Gamma'$ with $d(\lambda) = 1$. Then there is a linear isomorphism g of E_Λ onto*

$$F_\Gamma \oplus \bigoplus(ke_p : p \in U - P_\Gamma) \oplus \bigoplus(ke_p : p \in X) \oplus \bigoplus(ke_p : p \notin P_\Lambda)$$

such that $g \circ f_{\Gamma\Lambda} : (F_\Gamma \oplus \bigoplus(ke_p : p \in U - P_\Gamma))$ is the canonical embedding into the direct sum and $g(e_p) = e_p$, for all $p \in X$ and all $p \notin P_\Lambda$.

Proof. It suffices to show that there is a unique way to define, in the direct sum, vectors e_p, for $p \in P_\Lambda$, $p \notin P_\Gamma \cup X$, in such a way that we get a realization of Σ_Λ. Proceeding inductively, assume that this definition is done for all points on lines of depth less than d for given $d > 0$. Each line λ of depth d contains exactly two points s, t for which e_s, e_t are already defined and these are the only ones which may lie on any other line of depth d. These facts follow directly from the properties of a depth function and from the fact that, for $d = 1$, one of s and t is in P_Γ, the other in X.

The linear relations Σ_λ determine the remaining assignments of the points on λ to vectors on $< e_s, e_t >$ uniquely. Because only two points have been assigned already, there is such an assignment and it is faithful; this was precisely the reason we defined the relations Σ_λ the way we did. So we can extend our definition to all points on lines of depth d, simultaneously.

Theorem 8.1 *Let k be a field and M a 2-distributive modular algebraic spatial lattice with an irredundant set, Λ, of proper lines each of cardinality at least $|k| + 1$. There exists a canonical map $\phi : P_M \to L(E)$, and every such map is a representation of (M, Λ). If Λ is a base of lines then $\overline{\phi}$ is a cover preserving lattice embedding of M into $L(E)$*

Proof. If Λ is a base of lines then property (2) implies that the map $\overline{\phi}$ is a join homomorphism of M into $L(E)$. So we are left to verify that ϕ enjoys properties (1) and (3).

For $\Lambda = \emptyset$ everything is trivial. Next, let Λ be nonempty and finite, assume the claim is true for all proper subsets of Λ, and let $b = \sum\{\overline{\lambda} : \lambda \in \Lambda\}$. As a join of finitely many points, b is compact. As remarked above 8.1,

$$\overline{\phi}_\Lambda(x) = \overline{\phi}_\Lambda(bx) \oplus \bigoplus(ke_p : p \leq x, \;\; p \not\leq b),$$

for any $x \in M$.

From lemmas 7.2 and 8.1 it follows that for any lower cover a of b we have

(4) $f_{\Lambda_a\Lambda}$ is faithful on $\overline{\phi}_{\Lambda_a}(a)$ and

$$\overline{\phi}_\Lambda(x) = f_{\Lambda_a\Lambda}(\overline{\phi}_{\Lambda_a}(x)) < \overline{\phi}_\Lambda(b), \text{ for all } x \leq a,$$

Let $x \in M$ and $p \in P_M$ with $\overline{\phi}_\Lambda(p) \leq \overline{\phi}_\Lambda(x)$. If $p \not\leq b$ then, from the observation just above and the comment above lemma 8.1, $p \leq x$. If $p \leq b$

then, again from the observation just above, $e_p \in \overline{\phi}_\Lambda(bx)$. If $b \leq x$ then $p \leq x$. Otherwise $bx \leq b$. Since b is compact, there is an $a \in M$ with $bx \leq a \prec b$ and we have $\overline{\phi}_\Lambda(p) \leq \overline{\phi}_\Lambda(a)$.

We wish to apply lemmas 7.2 and 8.1, with $U = P_a \cap P_\Lambda$, to get $\overline{\phi}_{\Lambda_a}(p) \leq \overline{\phi}_{\Lambda_a}(bx)$. But to do this we first need to ensure that $p \leq a$.

Assume $p \not\leq a$. Then there is a line λ of minimum depth containing p. If $e_p \in \overline{\phi}_\Lambda(a)$ then λ contains two points whose images are in $\overline{\phi}_\Lambda(a)$, p and an element of U. It follows that the image of every point of λ is in $\overline{\phi}_\Lambda(a)$. We can proceed inductively along the unique chain joining λ with a line of depth 1 (which exists by lemma 7.2), and obtain an element r of X whose image is in $\overline{\phi}_\Lambda(a)$. This is in contradiction to lemma 8.1.

Now $|\Lambda_a| < |\Lambda|$, because $\sum_{\lambda\in\Lambda_a} \overline{\lambda} \leq a < b = \sum_{\lambda\in\Lambda} \overline{\lambda}$. Hence, by the inductive hypothesis, $p \leq bx$. This proves (1) for the given finite Λ.

Now, consider two distinct lower covers a and c of b and let $\Gamma = \Lambda_a \cup \Lambda_c$. Let B be the amalgamated free coproduct in the category of k-vector spaces of $\overline{\phi}_{\Lambda_a}(a)$ and $\overline{\phi}_{\Lambda_c}(c)$ over $\overline{\phi}_{\Lambda_{ac}}(ac)$ along the embeddings (viz. (4)) $f_{\Lambda_{ac}\Lambda_a}|\overline{\phi}_{\Lambda_{ac}}(ac)$ and $f_{\Lambda_{ac}\Lambda_c}|\overline{\phi}_{\Lambda_{ac}}(ac)$. Then E_Γ is canonically isomorphic to $B \oplus \bigoplus(ke_p : p \notin P_a \cup P_c)$ since this satisfies Σ_Γ and is as free as possible. Using lemmas 7.2 and 8.1 another time we get that E_Λ is, for suitable X, canonically isomorphic to

$$B \oplus \bigoplus(ke_p : p \not\leq b \text{ or } p \in X).$$

Since amalgamated coproducts in the category of k-vector spaces are separating we have,

(5) for any two distinct lower covers a, c of b

$$\overline{\phi}_\Lambda(ac) = \overline{\phi}_\Lambda(a)\overline{\phi}_\Lambda(c).$$

Now consider $c, d \in M$ and $u \in \overline{\phi}_\Lambda(c) \cap \overline{\phi}_\Lambda(d)$; then there exist unique $s, t \in E_\Lambda$ with $s \in \overline{\phi}_\Lambda(c{+}d)$ and $t \in \bigoplus(ke_p : p \leq c{+}d,\ p \not\leq b)$ so that $u = s{+}t$. But there also exist $v, w \in E_\Lambda$ with $v \in \overline{\phi}_\Lambda(bc)$ and $w \in \bigoplus(ke_p : p \leq c,\ p \not\leq b)$ with $u = v + w$. The uniqueness of s and t imply $s = v$ and $t = w$. Similarly, $s \in \overline{\phi}_\Lambda(bd)$ and $t \in \bigoplus(ke_p : p \leq d,\ p \not\leq b)$. We have shown,

$$\overline{\phi}_\Lambda(c)\overline{\phi}_\Lambda(d) = \overline{\phi}_\Lambda(bc)\overline{\phi}_\Lambda(bd) \oplus \bigoplus(ke_p : p \leq cd,\ \ p \not\leq b),$$

and therefore,

(6) $\overline{\phi}_\Lambda(c)\overline{\phi}_\Lambda(d) \leq \overline{\phi}_\Lambda(bc)\overline{\phi}_\Lambda(bd) + \overline{\phi}_\Lambda(cd)$.

To complete the proof of the theorem for the finite case we must show that (3) holds. By (6) it suffices to consider $x, y \leq b$. Let $a \in M$ with $x \leq a \prec b$ and let $c \in M$ with $y \leq c \prec b$; these exist by the compactness of b. By the inductive hypothesis and (4), we can assume that (3) holds for any pair of elements less than or equal to a or c. Hence, by (5), (4), and induction

$$\overline{\phi}_\Lambda(x)\overline{\phi}_\Lambda(y) = \overline{\phi}_\Lambda(xa)\overline{\phi}_\Lambda(yc) = \overline{\phi}_\Lambda(x)\ \overline{\phi}_\Lambda(a)\overline{\phi}_\Lambda(c)\ \overline{\phi}_\Lambda(y) = \overline{\phi}_\Lambda(x)\ \overline{\phi}_\Lambda(ac)\overline{\phi}_\Lambda(y)$$
$$= \overline{\phi}_\Lambda(xac)\overline{\phi}_\Lambda(y) = \overline{\phi}_\Lambda(xc)\overline{\phi}_\Lambda(y) = \overline{\phi}_\Lambda(xcy) = \overline{\phi}_\Lambda(xy).$$

Now, assume that Λ is infinite. By definition, E_Λ is the k-vector space with presentation

$$(\{e_p : p \in P_M\} : \bigcup(\Sigma_\lambda : \lambda \in \Lambda)).$$

It follows that any relation which holds in E_Λ is a consequence of only finitely many of the relations of this presentation and therefore it holds in E_Γ for some finite $\Gamma \subseteq \Lambda$. Again, since (2) is built into the construction we only have to show that (1) and (3) hold.

Now suppose $v \in \overline{\phi}_\Lambda(x) \cap \overline{\phi}_\Lambda(y)$ for some $x, y \in M$. Then there are two representations of v in E_Λ,

$$v = \sum_{i=1}^{n} \alpha_i e_{p_i} = \sum_{i=1}^{m} \beta_i e_{q_i},$$

for some $\alpha_i, \beta_i \in k$, $p_i \in P_M$ and $q_i \in P_M$. Since this is a relation holding in E_Λ it must also hold in E_Γ for some finite $\Gamma \subseteq \Lambda$. This implies that we have $u \in \overline{\phi}_\Gamma(x) \cap \overline{\phi}_\Gamma(y)$ with $f_{\Gamma\Lambda}(u) = v$. From the finite case we know that $u \in \overline{\phi}_\Gamma(xy)$, i.e. in E_Γ we can write $u = \sum_{i=1}^{l} \delta_i e_{r_i}$ with $\delta_i \in k$ and $r_i \in P_{xy}$. It follows from linearity of $f_{\Gamma\Lambda}$ that $v = \sum_{i=1}^{k} \delta_i e r_i$ in E_Λ, and hence (3) holds. In particular, if y is a point and $v = e_y \in \overline{\phi}_\Lambda(x)$ then we have $u = e_y \in \overline{\phi}_\Gamma(x)$ whence $y \leq x$ by the finite instance of (1).

Finally, if $p \in P_M$ then $\overline{\phi}_{\Lambda_p}(\underline{p}) \prec \overline{\phi}_{\Lambda_p}(\underline{p}) + < e_p >= \overline{\phi}_{\Lambda_p}(p)$. Hence, by (4) above, $\overline{\phi}_\Lambda(\underline{p}) \prec \overline{\phi}_\Lambda(p)$. Every cover in M transposes down to a cover of the form $\underline{p} \prec p$, $p \in P_M$, and hence, if $\overline{\phi}_\Lambda$ is a lattice embedding, then $\overline{\phi}_\Lambda$ will preserve covers.

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