Frankl's Conjecture for lower semimodular lattices

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Running title: On Frankl's Conjecture

Abstract

A stronger lattice theoretic version of Frankl's Conjecture on union closed families is verified for each of lower semimodular, sectionally complemented, and self-dual lattices.

Frankl's Conjecture on union closed families of sets can be equivalently stated as a conjecture on finite lattices, cf Poonen [4]. Namely, for a finite lattice L, let $\chi(L) = m \cdot |L|^{-1}$ where m is the minimum size of an upper section $[p, 1]$ with a join irreducible p. Then the conjecture and a stronger version due to Poonen [4] read as follows: For any finite lattice L with $|L| > 1$

(a) $\chi(L) \leq \frac{1}{2}$.

(b) If $\chi(L) \geq \frac{1}{2}$ then L is boolean.

(b) implies (a) since $\chi(L) = \frac{1}{2}$ for boolean L. (a) has been proved by Poonen [4] for distributive and for sectionally complemented lattices, (b) by Abe and Nakano [1] for modular lattices. Using their approach we show

Theorem 1 *Let L be a finite lattice, $|L| > 1$. Then L is boolean provided that $\chi(L) \geq \frac{1}{2}$ and that, in addition, one of the following holds*

- *L is lower semimodular*
- *all sections $[0, x]$ of L are complemented lattices*
- *$\chi(L^*) \geq \frac{1}{2}$ where L^* is the dual of L.*

In particular, it follows that for every L at least one of L and L^* satisfy Frankl's Conjecture (a) - a closely related result has been obtained by Johnson and Vaughan [3]. The key is the following definition, due to [1], of the subset $D(L)$ of L

> $x \in D(L)$ if and only if for every $z \geq x$ there is $w \not\geq x$ such that $z = x + w$.

The proof is based on the following observations for $|L| > 1$

(1) $x > 0$ if $x \in D(L)$.

(2) $x \in D(L)$ iff for every $z \geq x$ there is a lower cover $w \not\geq x$ of z.

(3) If $x \in D(L)$ then $|[x, 1]| \leq \frac{1}{2}|L|$.

(4) If $x \in D(L)$ and $|[x, 1]| = \frac{1}{2}|L|$ then $L \cong [x, 1] \times \{0, x\}$.

(5) $1 \in D(L)$.

(6) If L is sectionally complemented then $D(L) = L \setminus \{0\}$.

(7) If L is lower semimodular and $x \in D(L)$ minimal, then x is join irreducible.

(8) $\chi(L_1 \times L_2) = \min\{\chi(L_1), \chi(L_2)\}$ if $|L_i| > 1$.

Proof. Joins and meets are written as $x + y$ and $x \cdot y$, respectively. (1),(3), (5) and (8) are obvious, (6) was observed in [4], proof of Prop.3.

Ad (2): Consider $z \geq x$. If $x \in D(L)$ choose $w \not\geq x$ maximal such that $x + w = z$. Then for any $w < y \leq z$ one has $x \leq y$ whence $y = z$, i.e. w is a lower cover of z. Conversely, if $w \not\leq x$ is a lower cover of z, then $z = x + w$.

Ad (4): For any $z \geq x$ there is unique $w = \psi(z) \not\geq x$ such that $z = x + \psi(z)$. Thus, ψ is a map of $[x, 1]$ into $[0, u]$ where $u = \psi(1)$. By the definition, the join homomorphism $\phi(w) = x + w$ is a left inverse of ψ. Since $|[0, u]| \leq |[x, 1]$, it follows that ψ is an isomorphism and L is the disjoint union of $[0, u]$ and $[x, 1]$. Moreover, x is an atom whence $L \cong [0, u] \times \{0, x\}$.

Ad (7): Suppose that x has distinct lower covers y_1, y_2. Since $y_i \notin D(L)$ there are $z_i \geq y_i$ with no lower cover $w_i \not\geq y_i$ - for $i = 1, 2$. Put $z = z_1 + z_2$ and choose a lower cover $w \not\geq x$ according to (2). Since $x = y_1 + y_2$ we have $w \not\geq y_i$ for some i. Then $w \not\geq z_i$ and, by lower semimodularity, $w_i = z_i \cdot w \not\geq y_i$ is a lower cover of z_i. Contradiction.

Proof of the Theorem. Let L be lower semimodular resp. sectionally complemented, $|L| > 1$. Choose $x \in D(L)$ minimal. x is join irreducible by (7) resp. (6). Apply (3) to get $[x, 1] \leq \frac{1}{2}|L|$ and $\chi(L) \leq \frac{1}{2}$.

Now assume, in addition, that $\chi(L) = \frac{1}{2}$. $|L| = 2$ if $x = 1$. Otherwise, $|[x, 1]| = \frac{1}{2}$ and by (4) we get a direct decomposition of L into a two element lattice 2 and a lower section L' - which is lower semimodular resp. sectionally complemented, too, and $|L'| > 1$. By (8) we have $\chi(L') = \frac{1}{2}$. Therefore, we may apply induction to conclude that L' is boolean. Then so is L.

Finally, consider L such that $\chi(L) \geq \frac{1}{2}$ and $\chi(L^*) \geq \frac{1}{2}$. Choose a maximal join irreducible p and a meet irreducible $h \not\geq p$. By hypothesis, $|[p, 1]| \geq \frac{1}{2}|L|$ and $|[0, h]| \geq \frac{1}{2}|L|$. Since these two intervals are disjoint, they both have size $\frac{1}{2}|L|$ and their union is L. By maximality of p, all join irreducibles but p have to be in $[0, h]$. Consider $x \geq p$. Since x is a join of join irreducibles, it follows $x = p + y$ for some $y \leq h$. In other words, $p \in D(L)$ and by (4) we get a direct decomposition of $L \cong L' \times 2$. By (8) we have $\chi(L') \geq \frac{1}{2}$ and $\chi(L'^*) \geq \frac{1}{2}$ and may apply induction.

Abe and Nakano [1] have provided an example of an atomistic, dually atomistic, and consistent lattice L such that $D(L)$ contains no join irreducible element. Also, the smallest non-modular upper semimodular lattice contains a minimal element of $D(L)$ which is not join irreducible.

References

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