On forbidden minors for matroids representable in finite characteristic

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Abstract

For finite matroids, (skew) representability (in given characteristic) cannot be characterized in terms of finitely many forbidden minors. Skew representability cannot be decided, even.

1 Introduction

For a class $\mathcal{F}$ of fields or division rings let $M\mathcal{F}$ denote the class of all matroids representable over some F in $\mathcal{F}$. A *forbidden minor characterization* of a class $\mathcal{M}$ of matroids consists in a list of finite matroids such that a matroid is in $\mathcal{M}$ if and only if it does not contain any matroid from the list as a minor. The characterization is *finitary* if the list is finite. Important such characterizations have been established, mostly by Tutte (cf [6]), including the classes $M\mathcal{F}$ where $\mathcal{F}$ consists just of the 2- resp. 3-element field cf Kahn and Seymour [7]. On the other hand, Vámos [13] has shown that for $\mathcal{F}$ containing all fields of characteristic 0 there is no finitary characterization - even if one allows a subdivision of the list such that the matroid is not in the class if and only if it contains at least one minor from each sublist. Pursuing an approach familiar in the study of modular lattices (cf [3],[2]) we get the following.

Theorem 1 *Let $\mathcal{F}$ contain an infinite division ring or for each integer d a finite field having at least degree d over its prime subfield. Then there is no finitary characterization of $M\mathcal{F}$ in terms of forbidden minors. If $\mathcal{F}$ is closed* *under ultraproducts, then there are infinitely many finite matroids minimal (with respect to the formation of minors) not in $M\mathcal{F}$.*

The statement remains true relatively to the class of all matroids given by any combination of the following properties: finite, finite rank, rank ≤ 3, combinatorial geometry. Though, it remains open whether $M\{F\}$ has a finitary characterization for each finite field F as conjectured by Rota [11].

According to Vámos [12] membership in $M\mathcal{F}$ is decidable for finite matroids if $\mathcal{F}$ is the class of all fields. This extends to any recursively axiomatizable universal class of fields generated by its finite members. In contrast, decidability fails for $\mathcal{F}$ the class of all division rings (of given characteristic), due to Macintyre [8] and the following proposition - here, a division ring is understood as a ring with unit and inversion where $0^{-1} = 0$. This implies that there is no effective characterization of skew representability.

Proposition 2 *Let $\mathcal{F}$ any class of division rings with undecidable universal Horn theory. Then membership in $M\mathcal{F}$ is undecidable for finite matroids.*

2 Linear spaces

A *linear space* P is given by a set of points (also denoted by P) and a system of at least 2-element subsets of P, called blocks or lines, such that any two points are on a unique line - such are also known as "planar configurations" or "geometries" cf Kahn [5]. Of course, a linear space is determined by its points and proper (i.e. at least 3-element) lines, already. Given a set P of points in a projective space of any dimension, one obtains the *induced linear space* where the proper lines are the at least 3-element restrictions to P of lines from the projective space (that's why we prefer this terminology).

For any linear space, the set consisting of all points, all lines, as well as the empty set and the total set P yields a rank ≤ 3 geometric lattice ordered by inclusion. This will be the matroid associated with the linear space.

Observation 3 *For a finite linear space P induced in a (left) projective geometry over a division ring F the associated matroid is representable over every extension of sufficiently large (right) dimension over F.*

Namely, P is an iterated truncation of the subgeometry induced on P and the proof in[1] 7.4.10 can be modified.

Given two linear spaces P and P' with disjoint point sets and a bijection γ between a line l from P and a line l' from P' the *glued space* $P\gamma P'$ consists of the points from P and P', identifying γp with p, and has the proper lines from P as well as P', in particular the line $l = l'$.

Fixing $n \geq 4$ and a field F, let V an n-dimensional F-vector space with basis $e_1, \ldots, e_n$. Let P consist of the projective points on the plane $Q = Fe_1 + Fe_2 + Fe_3$ or on the line $l_i = Fe_i + Fe_{i+1}$ or of the form $p_i = F(e_{i-1} - e_{i+1})$ for some $i < n$. Let P' be derived, analogously, from a second (disjoint) F-vector space of dimension n. Consider a permutation α of F fixing 0 and 1. It induces a bijection γ between the points on the line l_{n-1} and those on l'_{n-1} such that $\gamma Fe_n = Fe'_n$, and $\gamma F(e_{n-1} - re_n) = F(e'_{n-1} - \alpha re'_n)$ for r in F. Let $M_{n\alpha}$ denote the matroid associated with the glued linear space $P\gamma P'$.

Lemma 4 *$M_{n\alpha}$ is a rank 3 combinatorial geometry. If it is representable over the division ring R then F embeds into R and α is an automorphism of F.*

Proof. Let ρ be a representation of $M_{n\alpha}$ over R which we conceive as join and rank preserving embedding of the geometric lattice into the lattice of subspaces of a 3-dimensional R-vector space W. In particular, ρ provides an representation of P and so an embedding of the projective plane Q over F into the subspace lattice of W. Therefore, there is an embedding β of F into R and a β-semilinear map f of Q into W inducing $\rho|Q$. On the other hand, for $i < n-1$, the perspectivity $p \mapsto l_{i-1} \cap (p + p_i)$ between the subspaces l_i and l_{i-1} of V is induced by an F-linear map f_i. In the same way, the perspectivity $p \mapsto \rho l_{i-1} \cap (p + \rho p_i)$ from ρl_i onto ρl_{i-1} is induced by an R-linear map g_i and $g_i \rho = \rho f_i$.

Let ρ' the map of P' into the subspace lattice of W such that $\rho' p = \rho\gamma^{-1} p$ for p on l' and $\rho' p = \rho p$, otherwise. Then ρ' is a respresentation of P' in W giving rise to f', f'_i, g'_i, analoguously. With these one can compose an F-semilinear map h of the subspace l of V onto the subspace l' of V'

$$h = {f'}_{n-1}^{-1} \cdots {f'}_3^{-1} {f'}^{-1} g'_3 \cdots g'_{n-1} g_{n-1}^{-1} \cdots g_3^{-1} f f_3 \cdots f_{n-1}.$$

Then, h induces the map $\rho'^{-1}\rho|l$ which is nothing else than γ. Thus, α is an automorphism. The converse is obvious.

Lemma 5 *Any at most $n/27$-element minor of an $M_{n\alpha}$ is representable over a transcendental extension of F.*

Proof. Let N be such a minor. For N of rank at most 2 the claim is trivial. N of rank 3 can be considered as a linear space and by Observation 3 it suffices to show that it is isomorphic to one induced in the projective space associated with the vector space W which is the amalgamation of V and V' over $l \cong l'$. More precisely, let W such that there are embeddings $\epsilon V \to W$ and $\epsilon' : V' \to W$ where the ϵe_i and $\epsilon' e'_i$ $(i < n-1)$ together with $\epsilon e_i = \epsilon' e'_i$ $(i = n-1, n)$ form a basis. An easy counting argument yields that there is a j such that N does not meet l'_j nor contain any of p'_j and p'_{j+1}. Now, define δ such that $\delta p = \epsilon p$ for p in P and $\delta p = \epsilon' p$ for p in Q' or on l'_i for $i < j$ or $p = p'_i$ or $p = Fe'_i$. Then, extend δ to the l'_i, $i > j$, by downward induction:

$$\delta p = (\delta F e'_i + \delta F e'_{i+1}) \cap (\delta q + \delta p_i)$$

if p on l'_i, q on l'_{i+1}, and p, q, p_i are collinear . Clearly, the domain of δ includes N, all collinearities are preserved, and no new ones are introduced (due to independency).

3 Axiomatic correspondences

When considering axiomatizability questions for classes of matroids it is convenient to consider matroids as first order structures as in Vámos [13]: for each n there is an n-ary predicate singling out the independent n-tuples. The concept of representation then can be seen as an axiomatic correspondence (in the sense of Mal'cev [9] cf [3]) between matroids and vector spaces i.e. a class of three sorted structures with sorts for matroid points, vectors, and scalars - subject to certain first order axioms. In particular, the representation is encoded into a binary relation between matroid points and vectors (for a representation in the projective space one relates to a point all vectors in a one dimensional vector subspace). Ultraproducts (cf [10], [4]) can be formed by taking ultraproducts of the sorts, separately. The following are more or less well known.

Lemma 6 *Let M be a finite matroid representable over a transcendental extension field of a field F. If F is the rationals, then M is representable* *over F. If F is finite, then M is representable over a finite extension of F as well as a simply transcendental extension of F.*

Lemma 7 *For any finite matroid M there is an integer d such that for any prime p, if M is representable over a field of characteristic p, then M is representable over $GF(p^d)$. If a finite matroid is representable over some field, then it is representable over some field of finite characteristic.*

The proof relies on the fact that a finite matroid embedded into an ultraproduct of matroids can be embedded into some (even almost all) factors - embeddability can be expressed by a first order sentence and Łos' Theorem (cf [4]) can be applied. Also ,observe that any (nontrivial) ultrapower of F contains an element a transcendental over F; namely, if F is the rationals or is a simply transcendental extension, by t, of a subfield take a represented by the sequence $1, 2, \ldots$ resp. $t, t^2, \ldots$. Moreover, every ultraproduct of the $GF(p^n)$, p fixed, contains an element transcendental over $GF(p)$ - e.g. represented by a sequence of primitives. Now, the first lemma is immediate by an iterated application of these observations and the fact that a representation of a finite matroid uses a finitely generated subfield, only.

For the second lemma we refer to 7.3.4 in [1] reading off the bound from a constructive proof of the Nullstellensatz or using the fact that the algebraic numbers are embedded into an ultraproduct of the algebraic closures of the $GF(p)$, p ranging over any infinite set of primes.

4 Forbidden minors

Clearly, we may assume that $\mathcal{F}$ is closed under taking division subrings and that it contains an infinite field or infinitely many finite fields (since a simply generated division ring is commutative). Assume that there were a finitary characterization of $M\mathcal{F}$ with m a bound on the size of matroids in this characterization.

The first case splits into 3 subcases: $\mathcal{F}$ contains a) all finite fields of a fixed characteristic, or b) a simply transcendental extension of a finite prime field, or c) the rationals. In that case let F one of these having at least 4 elements. Then there is a permutation α of F which is not an automorphism. For $n \geq 27m$ all minors of $M_{n\alpha}$ having size at most m are representable over a transcendental extension of F (Lemma 5) whence they are in $M\mathcal{F}$ by Lemma

6. In view of the assumed characterization, this would mean that $M_{n\alpha}$ is in $M\mathcal{F}$ contradicting Lemma 4.

In the second case, $\mathcal{F}$ contains only finite fields, for each p there is a maximum degree d_p taken by any F in $\mathcal{F}$ over $GF(p)$, and there is no bound on the d_p. Let M_p be the projective plane over $GF(p^{d_p+1})$. Then none of the M_p is in $M\mathcal{F}$. Let d a common bound from Lemma 7 for all matroids of size at most m and $d_p \geq d$. Then all minors of M_p having at most m-elements are representable over $GF(p^{d_p})$ which is in $\mathcal{F}$, a contradiction.

If $\mathcal{F}$ is closed under ultraproducts, a matroid is in $M\mathcal{F}$ if and only each of its finite submatroids is - this is an immediate consequence of Mal'cev's embedding of a structure into an ultraproduct of its finite relative substructures. So, there is a characterization by a list of finite minimal forbidden minors - and the list cannot be finite.

5 Undecidability

Using Mal'cev correspondences, again, one easily sees that if $\mathcal{F}$ is recursively axiomatizable then so is $M\mathcal{F}$. In particular, the finite forbidden minors form an enumerable set. On the other hand, so do the finite members of $M\mathcal{F}$ in view of Lemma 7. This yields decidability. Of course, better algorithms can be derived from constructive versions of the lemma.

Consider a universal Horn formula σ in the language of division rings given as $R \to v = w$ where R is a finite conjunction of relations $s = t$. Let T consist of $0, 1$ and all subterms occurring in σ. Let M be a finite geometry associated with a linear space having basis a_1, a_2, a_3 and for $1 \leq i < j \leq 3$ a (additional) "unit" point c_{ij} on the line $a_i \vee a_j$ such that c_{12}, c_{13}, c_{23} are collinear - shortly a "frame" Φ. Let μ a map from T into $(a_1 \vee a_2) - a_2$ with $\mu 0 = a_1$, $\mu 1 = c_{12}$, $\mu t^{-1} = a_1$ if $\mu t = a_1$, and $\mu s = \mu t$ if $s = t$ is a relation from R, but $\mu v \neq \mu w$. Assume that with every composed term u in T there are points p, q in M such that the following collinearities hold (just written as juxtaposition): $pa_3\mu s$, pa_2c_{13}, $qc_{13}\mu t$, qa_2a_3, $pq\mu u$ if $u = s + t$; $pc_{23}\mu t$, pa_1a_3, qa_1c_{23}, qpa_2, $qa_3\mu u$ if $u = -t$; $pc_{23}\mu s$, pa_1a_3, $qc_{13}\mu t$, qa_2a_3, $pq\mu u$ if $u = st$; $pc_{23}\mu t$, pa_1a_3, qpc_{12}, qa_2a_3, $qc_{13}\mu u$ if $u = t^{-1}$ and $\mu t \neq a_1$.

The matroid M or rather (M, Φ, μ) is called a *test matroid* for σ is it contains only the points required by the above i.e. at most $3|\mu(T)| + 6$ points

. For any finite geometry it can be checked whether, for some Φ and μ, it is a test matroid for σ or not. Thus, for any σ the isomorphism types of its test matroids can be effectively determined. The proof of the Proposition is now immediate by the following.

Lemma 8 *σ is valid in F if and only if no test matroid for σ can be represented over F.*

Proof. If ν is an interpretation falsifying σ in F let e_1, e_2, e_3 a basis of an F-vector space, $a_i = Fe_i$, $c_{ij} = F(e_i - e_j)$, and $\mu t = F(e_1 - \nu(t)e_2)$ for $t \in T$. Add the auxiliary points $F(e_1 - \nu s e_2 - e_3)$, $F(-\nu t e_2 + e_3)$ if $s + t$ in T; $F(e_1 - \nu t e_3)$, $F(e_1 + \nu t e_2 - \nu t e_3)$ if $-t$ in T; $F(e_1 - \nu s e_3)$,$F(-\nu t e_2 + e_3)$ if st in T; $F(e_1 - \nu t e_3)$, $F(e_2 - \nu t e_3)$ if t^{-1} in T and $\nu t \neq 0$. Then one gets a subgeometry of the projective plane which is a test matroid for σ.

Conversely, assume that some test matroid (M, Φ, μ) for σ is represented in the F-vector space V. Then we may assume that M is a subgeometry of the projective plane and since frames are uniquely coordinatizable over F we may choose a basis such that we get the above coordinate description for the frame. Then the description also applies to the μt with a map ν from T to F defined, accordingly. Now, in view of the collinearities this coordinate description extends to the auxiliary points and yields $\nu s + \nu t = \nu(s + t)$ for $s + t$ in T and so on, i.e. ν is an interpretation of T in the division ring F. By hypothesis, the relations R are satsified under ν, but $\nu v \neq \nu w$.

References

[1] T.Brylwaski, Constructions, in: N.White, ed., Theory of Matroids (Encycl. of Math. 26, Cambridge University Press 1986) 127-222

[2] A.Day, C.Herrmann, B.Jónsson, J.B.Nation, and D.Pickering, Small non-arguesian lattices, Algebra Universalis, to appear

[3] C.Herrmann and W.Poguntke, The class of sublattices of normal subgroup lattices is not elementary, Algebra Universalis 4 (1974) 280-286

[4] Ch.U.Jensen and H.Lenzing, *Model Theoretic Algebra* (Gordon and Breach, Amsterdam 1989)

[5] J.Kahn, Characteristic sets of matroids, J.London Math.Soc. 26 (1982) 207-217

[6] J.Kahn, A geometric approach to forbidden minors for GF(3), J. Comb. Th. A 37 (1984) 1-12

[7] J.Kahn and P.Seymour, On forbidden minors for GF(3), Proc. Amer. Math. Soc. 102 (1988) 437-440

[8] A.Macintyre, The word problem for division rings, J.Symb.Logic 38 (1973) 428-436

[9] A.I.Mal'cev, The Metamathematics of Algebraic Systems (North Holland, Amsterdam 1971)

[10] A.I.Mal'cev, Algebraic Systems (Springer, Berlin 1973)

[11] G.-C.Rota, Combinatorial theory old and new, in: Proc. Int. Congr. of Math. (Nice 1970) 229-233

[12] P.Vámos, A necessary and sufficient condition for a matroid to be linear, in: Mbius Algebras (Proc. Conf. Waterloo, 1971) 170-174

[13] P.Vámos, The missing axiom of matroid theory is lost forever, J. London Math. Soc. 18 (1978) 403-408

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