Abstract.

The origings of regular and $*$-regular rings lie in the works of J. von Neumann and F.J. Murray on operator algebras, von Neumann-algebras and projection lattices. They constitute a strong connection between operator theory, ring theory and lattice theory.

This paper aims at the following results:

1. The class of all $*$-regular rings forms a variety.

2. A subdirectly irreducible $*$-regular ring R is faithfully representable (i.e. isomorphic to a subring of an endomorphisms ring of vector spaces, where the involution is given by adjunction with respect to a scalar product on the vector space) if so is its ortholattice of projections.

Contents

1 Introduction

In the present section, we will fix notational conventions and introduce the different concepts of representability.

The term *ring* is used for rings with or without unit. Rings are denoted by R, S, T, C. We consider rings with an involution $^* : R \to R$ (an anti-automorphism of order two). We consider the unary map as part of the signature of the ring R. A (von Neumann-)regular ring is a ring such that every element x has at least one quasi-inverse y, i.e., for $x \in R$ there exists an $y \in R$ such that $xyx = x$. A $*$-regular ring is a regular involutive ring satisfying the implication $xx^* = 0 \Rightarrow x = 0$.

The term *idempotent* is used for a ring element x satisfying $x^2 = x$, the term *projection* is used for a ring element x satisfying $x^2 = x^* = x$. We use the letters p, q for projections and e, f, g for idempotents and projections.

The term *lattice* is used for a partially ordered set with binary operations join and meet. These operations are denoted by $+$ and $\cdot$. All lattices considered have a smallest element 0. By a *bounded lattice*, we mean a lattice with top and bottom. We use the terms *interval* and *section* of a lattice in the usual way. Intervals and sections are bounded lattices in their own right, with the inherited operations. We use the notation $a \oplus b$ or $\bigoplus a_i$ for the join of independent elements a and b or for the join of the independent family $\{a_i : i \in I\}$. We define the *height* $h(L)$ of a lattice as usual to be the supremum of all cardinalities $|C| - 1$, C a chain in L.

In this paper, we deal mainly with *modular* lattices. Of particular interest are (*relatively or sectionally*) *complemented modular lattices* and (*sectional*) *modular ortholattices*. We use the abbreviations CML and MOL, respectively. We denote the *orthocomplementation* on a MOL L by $^\perp : L \to L$.

1.1 Rings and Lattices

In this section, we recall well-known results about regular rings and the connections between regular rings and complemented modular lattices.

Theorem 1.1. *A ring with unit is regular if and only if the set of all its principal right ideals is a complemented modular lattice.*

If R does not contain a unit, the equivalence holds for complemented *replaced by* relatively complemented*.*

For a ring R, we denote the set of all its principal right (left) ideals by $\overline{L}(R_R)$ (by $\overline{L}({}_RR)$).

Lemma 1.2. *A ring with unit (without unit) is $*$-regular if and only if $\overline{L}(R_R)$ is a (sectional) MOL.*

▷ Proof. Folklore. If R is $*$-regular, every principal right ideal is generated by a projection. The orthogonality on $\overline{L}(R_R)$ is given by

$$aR \perp bR \Leftrightarrow b^*a = 0$$

If R contains a unit, then the orthocomplement of eR, e a projection in R, is given by $(1-e)R$. ◁

Proposition 1.3. *If R is regular, then the lattices $\overline{L}(R_R)$ and $\overline{L}({}_RR)$ of principal right ideals and principal left ideals respectively, are anti-isomorphic.*

If R is $*$-regular, $\overline{L}(R_R)$ and $\overline{L}({}_RR)$ are isomorphic.

▷ Proof. See [Mic03], [Skor64] and [Mae58]. ◁

Lemma 1.4. *If R is a simple $*$-regular ring (without unit), then $\overline{L}(R_R)$ is a simple (sectional) MOL.*

▷ Proof. See [HR99, Theorem 2.5]. ◁

Lemma 1.5. *In a simple MOL, each non-trivial interval $[0, a]$ is simple.*

▷ Proof. [Jón60, Lemma 2.2]. ◁

Lemma 1.6. *If R is $*$-regular and e a projection in R, then the set eRe is a $*$-regular subring (with unit e) of R.*

▷ Proof. Since e is a projection, eRe is a subring and the involution on R restricts to an involution on eRe. For regularity, consider $x \in eRe$. Take an quasi-inverse y of x in R and reflect that eye is also an quasi-inverse of x. ◁

Let R be a $*$-regular ring and e a projection in R. We write R_e for the $*$-regular subring eRe. Furthermore, we define the *height* $h(R)$ by the height $h(\overline{L}(R_R))$ of its principal ideal lattice $\overline{L}(R_R)$.

Lemma 1.7. *Let R be a $*$-regular ring and e a projection in R. Then the lattice $\overline{L}((R_e)_{R_e})$ of all principal right ideals in $R_e = eRe$ is isomorphic to the section $[0, eR] \subseteq \overline{L}(R_R)$.*

$\triangleright$ Proof. [Jón60, Lemma 8.2]. $\triangleleft$

Lemma 1.8. *If R is a simple $*$-regular ring and e a projection in R, then R_e is a simple $*$-regular ring with unit e.*

$\triangleright$ Proof. It is left to show simplicity. For a non-vanishing ideal A in eRe, consider the ideal generated by A in R. $\triangleleft$

Lemma 1.9. *Let R be a $*$-regular ring. Then for each $x \in R$, there exists a projection $e_x \in R$ such that $e_x x e_x = x$.*

$\triangleright$ Proof. Let p_x be the projection that generates the left ideal generated by x and q_x be the projection that generates the right ideal generated by x. Take $e_x := p_x \vee q_x$ to be the supremum in the lattice of all projections of R. $\triangleleft$

1.2 Frames

In this section, we recall the notion of perspectivity of elements of a lattice and the concept of a frame. The reader familiar with frames of modular lattices might give the following descriptions only a short glance and then jump to Definition 1.16 of a *stable orthogonal frame* and Corollary 1.22 that a simple MOL of height at least n contains a stable orthogonal (n, k)-frame.

Two elements a, b of a lattice L are called *perspective (to each other)* if they have a common complement c in L. If a and b are perspective, we write $a \sim b$. We say that a is *subperspective to b* or *perspective to a part of b* if there exists an element $d \leq b$ such that $a \sim d$. We write $a \precsim b$. An element c establishing a (sub)perspectivity between elements a and b is called an *axis of (sub)perspectivity* between a and b. If $a \precsim b$, the part $d \leq b$ such that $a \sim d$ is called the *image of a under the perspectivity between a and b*.

There exist different notions of a *frame*. In [vN60], von Neumann defined a *homogeneous basis* for a CML L (p. 93) and a *(normalised) system of axes of perspectivity* for a given homogeneous basis (p. 118). The combined system was called a *(normalised) frame* for L. Equivalently, G. Bergmann and A. Huhn introduced the notion of a *n-frame* (originally, a $(n-1)$-*diamond*) in a modular lattice (see the survey articles [Day82], [Day84] or the article of C. Herrmann in memory of A. Day [Herr95]).

The notion of a frame was subject to further development and generalisation. See [Jón60] for the introduction of a *partial frame*, a *large partial frame* and a *global frame*. In [Jón60], Jónsson defined a *large partial n-frame* in a bounded modular lattice B to be a subset of B consisting of independent elements $a_0, \dots, a_{n-1}, d$ and the entries of a symmetric matrix $c = (c_{ij})_{i,j<n}$ such that the supremum of $a_0, \dots, a_{n-1}$ and d equals the unit element 1_B, d consists of a sum of finitely many elements each of which is subperspective to a_0, and c_{ij} is an axis of perspectivity between a_j and a_i.

We adapt the definition of Jónsson in the following way: Decomposing d into k summands, each of which is subperspective to a_0, we incorporate these summands and their axes of subperspectivity to a_0 into the frame. Furthermore, we demand that the spanning elements of the frame are independent.

Definition 1.10. Large partial (n, k)-frame
A *large partial frame of format* (n, k) in a bounded modular lattice L is a subset

$$\Phi := \{a_i, a_{0i} : 0 \leq i < n + k\} \subseteq L$$

such that the following conditions are satisfied.

1. $\bigoplus\limits_{i<n+k} a_i = 1_L$
2. $a_0 + a_i = a_0 \oplus a_{0i} = a_i \oplus a_{0i}$ for $i = 1, \dots, n-1$
3. $a_0(a_i + a_{0i}) + a_i = a_i \oplus a_{0i} = a_0(a_i + a_{0i}) \oplus a_{0i}$ for $i = n, \dots, n+k-1$

That is, Φ contains $n + k$ independent elements a_i spanning the lattice L (condition (1)). Conditions (2) and (3) state that $a_1, \dots, a_{n-1}$ are perspective to a_0 and $a_n, \dots, a_{n+k-1}$ are subperspective to a_0, where the axes of (sub)perspectivity are just the a_{0i}. In particular, we have $a_0 \cdot a_{0i} = a_i \cdot a_{0i} = 0$ for all i.

The axes of perspectivity between a_i, a_j for indices $i, j < n$ can be constructed via the axes a_{0i}, a_{0j}: We have $a_{ji} = [a_{0j} + a_{0i}] \cdot [a_j + a_i]$ and consequently, we have $a_{ki} = [a_{kj} + a_{ji}] \cdot [a_k + a_i]$ for $i, j, k < n$. Likewise, we can construct the axis of subperspectivity a_{ji} between a_i and a_j for indices i, j such that $j < n$ and $n \leq i < n + k$.

For short, we call Φ a *large partial (n, k)-frame* or an *(n, k)-frame*, dropping the attribute *large partial* for the ease of notation and to avoid confusion with the notion of a large partial n-frame in the sense of Jónsson.

In the following, we state some helpful results and develop the appropriate notion of a frame for a modular ortholattice.

Lemma 1.11. *Let L be a CML and assume that $a_0, a, b \in L$ are elements such that $a_0 \leq a$, $a_0 \cdot b = 0$, and b is subperspective to a_0. Then the relative complement d of ab in $[0, b]$ is subperspective to a_0 and $a \oplus d = a + b$.*

Lemma 1.12. *Let L be a CML. If L contains a large partial n-frame in the sense of Jónsson, then L contains an (n, k)-frame.*

$\triangleright$ Proof. Construct summands $a_i, n \leq i < n + k$ of d with the desired properties (subperspective to a_0 and such that $a_0, \ldots, a_{n+k-1}$ are independent) inductively. $\triangleleft$

Next, we introduce the concept of a stable frame. The main difference is that we incorporate all the axes of (sub)perspectivity (see Definition 1.10) and a set of relative complements.

Definition 1.13. Stable (n, k)-frame
Let L be a CML. A subset

$$\Phi = \{a_i, a_{ij} : 0 \leq i < n, 0 \leq j < n + k\} \cup \{z_{ij} : j < n, n \leq i < n + k\} \subseteq L$$

will be called a *stable (n, k)-frame Φ in L* if

1. $\{a_i, a_{0i} : 0 \leq i < n + k\} \subseteq L$ is a (n, k)-frame in L
2. for $i, j \in I$, $i < n$, a_{ij} is the axis of (sub)perspectivity between a_j and a_i
3. for each pair (i, j) of indices with $j < n$ and $n \leq i < n+k$, the element z_{ij} is a complement of b_{ji} in $[0, a_j]$, where b_{ji} is the image of a_i under the subperspectivity a_{ji} between a_i and a_j.

Lemma 1.14. *Let L be a CML.*
If L contains an (n, k)-frame, L contains a stable (n, k)-frame.

$\triangleright$ Proof. Choose the necessary relative complements. $\triangleleft$

Definition 1.15. Orthogonal (n, k)-frame
Let L be a MOL.
An (n, k)-frame Φ in L is called an *orthogonal (n, k)-frame* if the following additional condition is satisfied:

$$\forall i \in \{0, \ldots, n + k - 1\}. \qquad a_i^{\perp} = \sum_{j \neq i} a_j$$

Definition 1.16. Stable orthogonal (n, k)-frame
Let L be a MOL. A *stable orthogonal (n, k)-frame* is a stable frame such that

1. Φ as a frame satisfies the condition of Definition 1.15, and
2. the relative complements z_{ij} are relative orthocomplements.

1.2.1 Orthogonalisation of a (n,k)-Frame via Jónsson

Now we want to show that the notion of a (stable) orthogonal (n,k)-frame is the appropriate one for a MOL: We will proove that a given frame can be orthogonalised. We will base the proof on arguments and results presented in [Jón60]. In fact, one could choose an alternative approach via ideas of Fred Wehrung, presented in [Weh98], using the notion of a *normal equivalence* in a modular lattice and the concept of a *normal* modular lattice.

Lemma 1.17. *Let L be a MOL and a, b projective elements in L. Then there exist four elements b_0, b_1, b_2, b_3 in L such that b is the direct orthogonal sum of b_0, b_1, b_2, b_3 and each b_i is perspective to a part of a.*

$\triangleright$ Proof. The lines of argument follow Jónsson's proof of Lemma 1.4 in [Jón60]. The only difference is that we choose the relative complements in Jónsson's proof to be relative orthocomplements in the considered intervals. $\triangleleft$

Lemma 1.18. *Let L be a MOL and a_0, a, b elements in L such that $a_0 \leq a$, $a \cdot b = 0$, and $b \lesssim a_0$. Then b decomposes into a direct orthogonal sum of five elements b_0, b_1, b_2, b_3, b_4 such that each b_i is subperspective to a_0.*

$\triangleright$ Proof. We choose $c := b \cdot a^{\perp}$ and d as the relative orthocomplement of c in $[0, (a+b) \cdot a^{\perp}]$. As part of b, c is subperspective to a_0. Furthermore, one can show that d is projective to a part $x \leq a_0$. Consequently, by Lemma 1.17, we can decompose d into the direct orthogonal sum of 4 elements $b_1, \ldots, b_4$, each of which is subperspective to a_0. Together with $b_0 := c$, we have the desired result. $\triangleleft$

Lemma 1.19. *Let L be a simple MOL and $a, b \in L$ non-trivial independent elements. Then there exist non-trivial elements a_0, b_0 with $a_0 \leq a, b_0 \leq b$ such that a_0 and b_0 are perspective to each other. In particular, this holds if $b \leq a^{\perp}$.*

$\triangleright$ Proof. Since L is simple, the neutral ideal generated by a is the whole lattice. Then b is the sum of finitely many elements, each of which is perspective to a part of a. Choose one such non-trivial summand as b_0 and the corresponding perspective part of a as a_0. $\triangleleft$

Lemma 1.20. *Let L be a simple MOL with $h(L) \geq n$. Then there exists a large partial n-frame Φ (in the sense of Jónsson) such that the first n elements $a_0, \ldots, a_{n-1}$ of Φ are orthogonal, that is, we have*

$$a_k \leq \Big(\bigoplus_{i<k} a_i\Big)^{\perp}$$

▷ Proof. By induction. ◁

Lemma 1.21. *Let L be a MOL containing a (n,k)-frame Φ such that $a_0, \dots a_{n-1}$ are orthogonal, that is, if for all $k < n$, we have $a_k \leq (\oplus_{i<k} a_i)^\perp$. Then L contains an orthogonal (n,k')-frame for some k'.*

▷ Proof. By induction over the elements $a_n, \dots, a_{n+k-1}$ and Lemma 1.17. ◁

Corollary 1.22. *Let L be a simple MOL of height at least n. Then L contains a stable orthogonal (n,k)-frame.*

1.2.2 Rings and Frames

Combining Corollary 1.22 with Lemma 1.4 and Lemma 1.8, we get the following results.

Corollary 1.23. *Let R be a simple $*$-regular ring with unit and $h(R) \geq n$. Then the MOL $\overline{L}(R_R)$ contains a stable orthogonal (n,k)-frame.*

Corollary 1.24. *If R is a simple $*$-regular ring and e a projection in R, then the MOL $\overline{L}(R_{eR_e})$ of principal right ideals of R_e contains a stable orthogonal frame.*

Remark 1.1. Clearly, the format of the frame depends on the height of R_e.

1.2.3 Projectivity of Frames

It is well-known that global frames are projective. In this section, we state similar results for the above introduced frames. We begin with large partial (n,k)-frames.

Lemma 1.25. *Let K, L be CMLs, $f : K \twoheadrightarrow L$ a surjective 0-1-lattice homomorphism and $\Phi \subseteq L$ a large partial (n,k)-frame in L. Then there exists a section $M \leq K$ and a set $\Psi \subseteq M$ such that*

1. *$f_{|M} : M \to L$ is a surjective lattice homomorphism,*
2. *Ψ is a large partial frame in M of the same format as Φ,*
3. *$f[\Psi] = \Phi$.*

▷ Proof. Inductive process and appropriate choices of preimages. ◁

Similarly, we have the following.

Lemma 1.26. *Let K, L be CMLs, $f : K \twoheadrightarrow L$ a surjective* 0-1-*lattice homomorphism and $\Phi \subseteq L$ a stable (n,k)-frame in L. Then there exists a section $M \leq K$ and a set $\Psi \subseteq M$ such that*

1. *$f_{|_M} : M \to L$ is a surjective lattice homomorphism,*
2. *Ψ is a stable frame in M of the same format as Φ,*
3. *$f[\Psi] = \Phi$.*

$\triangleright$ Proof. Incorporate the choice of the necessary relative complements in the inductive procedure. To accomplish this, it is enough to show that if a, b are in K such that $b \leq a$ and $f(b) \oplus c = f(a)$ for some $c \in L$, then there exists $d \in K$ such that $b \oplus d = a$ and $f(d) = c$. $\triangleleft$

Lemma 1.27. *Let K, L be MOLs, $f : K \twoheadrightarrow L$ a surjective* 0-1-*lattice homomorphism and $\Phi \subseteq L$ a stable orthogonal (n,k)-frame in L. Then there exists a section $M \leq K$ and a set $\Psi \subseteq M$ such that*

1. *$f_{|_M} : M \to L$ is a surjective lattice homomorphism,*
2. *Ψ is a stable orthogonal frame in M of the same format as Φ,*
3. *$f[\Psi] = \Phi$.*

$\triangleright$ Proof. Incorporate the orthogonality into the inductive process. $\triangleleft$

1.3 Concepts of Representability

Definition 1.28. Linear representation
As in [Mic03] and [Nie03], a *linear positive representation* of a $*$-regular ring R is a tuple

$$\sigma = (D, V_D, \phi, \rho)$$

where D is an (involutive) skew field, V_D a right vector space over D, ϕ a scalar product on V_D, and

$$\rho : R \to End(V_D)$$

a ring homomorphism such that

$$\forall r \in R. \qquad \rho(r^*) = \rho(r)^{*_\phi}$$

If the morphism $\rho : R \to End(V_D)$ is injective, we call σ a *faithful* representation.

Definition 1.29. Generalised representation
Let R be an involutive ring, I an arbitrary non-empty index set and σ a tuple

$$\sigma = (I, \{D_i\}_{i\in I}, \{V_i\}_{i\in I}, \{\phi_i\}_{i\in I}, \rho)$$

consisting of an indexed family of (involutive) skew fields, an indexed family of vector spaces and an indexed family of scalar products, such that for each $i \in I$, V_i is a right vector space over D_i with scalar product ϕ_i, and a map

$$\rho : R \to \prod_{i\in I} End(V_{iD_i}).$$

If ρ is a $*$-ring morphism, i.e., for all $r \in R$ and all $i \in I$ the condition

$$\pi_i(\rho(r^*)) = \big(\pi_i(\rho(r))\big)^{*\phi_i}$$

holds, we call σ a *positive generalised representation* of R. For short, we speak of a *positive g-representation*, or just a *g-representation*.

If ρ is injective, we call σ a *faithful* g-representation.

Remark 1.2. Since this paper deals with $*$-regular rings only, we suppress the adjective *positive* when speaking of a linear or a generalised representation of a $*$-regular ring. We use the term *representation* for a linear representation as well as for a generalised representation, if the context leaves no ambiguity or both concepts are considered simultaneously.

Remark 1.3. Note that the properties of a structure to be a *(faithful linear) representation of a ring* can be expressed in first-order logic [Mic03].

Definition 1.30. Representation of a (sectional) MOL
A *representation* of a (sectional) MOL L consists of a tuple

$$\varsigma = (D, V_D, \langle\cdot,\cdot\rangle, \iota)$$

with $D, V_D, \langle\cdot,\cdot\rangle$, as above and a morphism

$$\iota : L \to L(V_D, \langle\cdot,\cdot\rangle)$$

of (bounded) lattices between L and the subspace lattice of V_D such that the (sectional) orthocomplementation on L corresponds to the (sectional) orthocomplementation on V_D given by the scalar product, that is, for all $x \in L$, we have $\iota(x') = \iota(x)^\perp$ ($\iota(x^{\prime}_{b}) = \iota(x)^\perp \cap \iota(b)$ for a sectional MOL).

We call a representation ς *faithful* if the morphism ι is injective. g-representations are defined, analoguously.

A *representation* of an MOL L in an inner product space (V_F, Φ) is an $(0, 1)$-lattice homomorphism $\varepsilon\colon L \to \mathrm{Sub}(V_F, \Phi)$ such that $\varepsilon(x') = \varepsilon(x)^\perp$ for all $x \in L$. Observe that, by modularity, $\varepsilon(x) = \varepsilon(x)^{\perp\perp}$ for all $x \in L$. A representation ε is *faithful*, if it is one-to-one. Both $*$-regular rings and MOLs will be called *representable* if they admit som faithful representation.

2 The Variety of $*$-Regular Rings

The term variety is used in the usual sense: A variety is a class of algebraic structures of the same type that is closed under products, homomorphic images and substructures. Obviously, an arbitrary product of $*$-regular rings, where the operations are as usual defined componentwise, is itself a $*$-regular ring. For homomorphic images, the following holds.

Proposition 2.1. *A homomorphic image of a $*$-regular ring is $*$-regular.*

▶ **Proof.** Due to [Good91], Lemma 1.3 and [Mic03], Proposition 1.7, every two-sided ideal of a $*$-regular ring is $*$-regular. ◀

For substructures, we recall the notion of the *Rickart relative inverse* of an element of a $*$-regular ring. Some preliminary work is needed.

Definition 2.2. Left and right projection
Let R be a $*$-regular ring. For an element $a \in R$, we call the unique projection e in R that generates the principal right ideal aR the *left projection of* a and the unique projection f in R that generates the principal left ideal Ra the *right projection of* a.

Remark 2.1. This terminology can be found in [Kap68], p. 27–28 or [Kap55], p. 525. We denote the left and right projection of a by $l(a)$ and $r(a)$, respectively. Furthermore, if R has a unit, we have

$$ann^l_R(a) = R(1-e) \quad \text{and} \quad ann^r_R(a) = (1-f)R.$$

The following result holds.

Lemma 2.3. *The left and right projection of an element a can be constructed in the following way: For $x \in R$, we set*

$$l(x) := x(x^*x)'x^* \quad \textit{and} \quad r(x) := x^*(xx^*)'x,$$

where x' denotes any quasi-inverse of x.

▷ Proof. See [Mic03], p. 9–10. ◁

Lemma 2.4. *Let R be a $*$-regular ring. Then for each element $a \in R$ there exists a unique element $q(a)$ such that the following conditions hold.*

1. $e := l(a) = aq(a^*a)a^*$.

2. $f := r(a) = a^*q(aa^*)a$.

3. $fq(a) = q(a)$.

4. $aq(a) = e$.

Furthermore, $q(a)$ has the properties that $q(a)a = f$, the left projection of $q(a)$ is f and the right projection of $q(a)$ is e.

▷ Proof. See [Kap68] or [Kap55]). We have defined a function $q : R \to R$ that maps each $a \in R$ to the unique element y with the listed properties. ◁

Remark 2.2. We call $q(a)$ the *relative inverse of a*. We note that a is the relative inverse of $q(a)$, so $q^2 = id_R$.

We arrive at the following result.

Proposition 2.5. *Let R be a $*$-regular ring.*
The q-subrings of R are exactly the $*$-regular subrings of R. Consequently, we incorporate the unary map $q : R \to R$ into the signature of $*$-regular rings, that is, a $*$-regular ring R is an algebra of type $(R, +, \cdot, ^*, q, 0)$.

▶ **Proof.** Assume that S is closed under q. For an element $a \in S$, the map q gives a quasi-inverse $q(a)$ of a, so S is regular. Since S is a $*$-subring of the $*$-regular subring R, S is itself $*$-regular.
Conversely, assume that $S \leq R$ is a $*$-regular subring of R. Let $x \in S$. Due to Lemma 2.3, we can construct the left and the right projection of x within S, using the involution on S and any quasi-inverses of x, x^*, xx^*, x^*x in S. By Lemma 2.4, there exists an element y with the desired properties within the $*$-regular ring S. Since y is the unique element with these properties, we have $y = q(x)$. Hence, S is closed under q. ◀

Combining Propositions 2.1 and 2.5, we have proven the first result.

Theorem 2.6. *The class $\mathcal{R}$ of $*$-regular rings forms a variety.*

2.1 Directed Unions and Rings without Unit

Definition 2.7. Directed union of rings
Let R be a ring and $\mathcal{S} = \{S_i : i \in I\}$ be a directed family of subrings of R. We say that R is a *directed union* of the family $\mathcal{S}$ if for each $r \in R$ there exists $k \in I$ such that $r \in S_k$.

Remark 2.3. Casually, we speak of *R being the directed union of the S_i*, without giving the family of the S_i an extra name, and we write $R = \bigcup_{i \in I} S_i$, using the usual symbol for an ordinary union. Of course, an arbitrary union of rings is in general not a ring; hence, the lax notion does not lead to the risk of misunderstandings.

Lemma 2.8. *Let R be a $*$-regular ring and assume that R is the directed union of a family $\mathcal{S}$ of $*$-regular subrings S_i of R. Then R is a $*$-regular subring of an ultraproduct of the rings S_i, $i \in I$.*

$\triangleright$ Proof. Since the class of all $*$-regular rings forms a variety, this follows from [Gor98], Theorem 1.2.12 (1). $\triangleleft$

2.2 Representability and Universal Algebra

We finish the first section with a look on representability of $*$-regular rings under an universal-algebraic perspective.

Lemma 2.9. *Let R be a $*$-regular ring with a representation $\sigma = (D, V_D, \langle \cdot, \cdot \rangle, \rho)$. Then every $*$-regular subring S of R is representable. If the representation of R is faithful, so is the representation of S.*

$\triangleright$ Proof. Just take the restriction $\rho_{|S}$. $\triangleleft$

Proposition 2.10. *Let $\{S_i : i \in I\}$ be a family of $*$-regular rings, I an arbitrary index set. Assume that each S_i admits a linear positive representation. Let U be an ultrafilter on I.*

Then the ultraproduct

$$R := (\prod_{i \in I} S_i)/U$$

admits a linear positive representation. If every S_i has a faithful *linear positive representation, then so does R.*

▶ **Proof.** Consider the class of 2-sorted structures

$$\begin{aligned} \mathcal{K} := \{(R, V) : \quad & R \text{ a } *\text{-regular ring, } V \text{ a vector space such that} \\ & R \text{ has a linear positive representation in } V\} \end{aligned}$$

Since the relation that the $*$-regular ring R has a (faithful) linear positve representation in the vector V can be expressed in first-order logic, an ultraproduct of a family of structures $(R_i, V_i) \in \mathcal{K}$ lies again in $\mathcal{K}$. ◀

3 Representability of $*$-Regular Rings

This section is devided into the following parts: In the first part, we will introduce notation and convention. In the second part, we will develop the general framework that is needed to tackle the problem of representability. In the third part, we will present a proof that a $*$-regular ring R is g-representable if and only if so is $\overline{L}(R_R)$.

3.1 Convention and Notation

We consider right modules over rings, denoted by M_S, N_T. Submodules will be denoted by M_i, N_i, neglecting the respective underlying ring. If the contrary is not explicitly stated (or obvious from the context), we assume that the underlying ring is a $*$-regular ring (with or without unit).

For morphisms between submodules $M_i, M_j \leq M$, we write $\varphi_{ji} : M_i \to M_j$, where the indices should be read from right to left. If M_i, M_j have trivial intersection, we define the graph of a morphism $\varphi_{ji} : M_i \to M_j$ by

$$\Gamma(\varphi_{ji}) = \{x - \varphi_{ji}(x) : x \in M_i\}.$$

Observation 3.1. Let $M_i \cap M_j = \{0\}$. Note that $\Gamma(\varphi_{ji})$ is a relative complement of M_j in $[0, M_i + M_j]$ and, conversely, each such relative complement gives rise to a morphism $\psi_{ji} : M_i \to M_j$.

Observation 3.2. Let M be a module with a direct decomposition

$$M = \bigoplus_{i \in I} M_i$$

and denote the corresponding projections and embeddings by π_i and ε_j, respectively.

Consider a morphism $\varphi_{ji} : M_i \to M_j$. Then the composition of φ_{ji} with the projection $\pi_i : M \to M_i$ yields a morphism $\varphi_{ji} \circ \pi_i : M \to M_j$ defined on all of M, i.e.,, $\varphi_{ji} \circ \pi_i \in Hom(M, M_i)$. Since $M_i \leq M$, we can consider $\varphi_{ji} \circ \pi_i$ as an element of $End(M)$, too. For the latter point of view, the formally correct approach would be to consider $\varepsilon_j \circ \varphi_{ji} \circ \pi_i$. To avoid technical and notational overload, we will treat $\varphi_{ji} \circ \pi_i$ itself as an element of $End(M)$. Note that the composition $\varphi_{ji} \circ \pi_i$ is nothing else than the extension of the map $\varphi_{ji} : M_i \to M_j$ to the module M, by defining the action of the extension to be trivial on the other summands of M. Very rarely, we write $\overline{\varphi_{ji}}$ for this extension: We just use overlined symbols if we want to distinguish between a partial map and its extension.

Conversely, consider a morphism $\varphi \in M$. We define

$$\varphi_i := \varphi \circ \epsilon_i : M_i \to M \qquad \varphi_{ji} := \pi_j \circ \varphi_i = \pi_j \circ \varphi \circ \epsilon_i : M_i \to M_j$$

Then we have a 1-1-correspondence between a morphism $\varphi : M \to M$, and a family $\{\varphi_i : i \in I\}$, where $\varphi_i : M_i \to M$, and a family $\{\varphi_{ji} : i, j \in I\}$, where $\varphi_{ji} : M_i \to M_j$ since each $\varphi \in End(M)$ can be decomposed in the following ways:

$$\varphi = \bigoplus_{i \in I} \varphi_i = \bigoplus_{i \in I} \sum_{j \in I} \varphi_{ji}$$

We agree to write $\varphi = \sum_{i,j \in I} \varphi_{ji}$, with the convention stated above. We agree to not impose a rigorous notational strictness, but to understand the notation in the natural sense. Similar to the observation above, we note that $\varphi_i \circ \pi_i$ is nothing else that the extension of the map $\varphi_i = \varphi \circ \epsilon_i : M_i \to M$ to all of M.

We note that the conventions are compatible with addition and multiplication: We can form the sum $\varphi_{ji} + \psi_{ji}$ and the composition $\varphi_{jk} \circ \psi_{ki}$ in the natural sense, and for $\varphi, \psi \in End(M)$, we have

$$(\varphi + \psi)_{ji} = \varphi_{ji} + \psi_{ji} \quad \text{and} \quad (\varphi \circ \psi)_{ji} = \sum_{k \in I} \varphi_{jk} \circ \psi_{ki}.$$

We agree to write $1 = id_M : M \to M$. Then we have $1_{ii} = id_{M_i} : M_i \to M_i$ (that is, the corresponding extension 1_{ii} acts like the identity on M_i and trivially on every other summand M_j) and $1_{ji} = 0_{ji} : M_i \to M_j$ (that is, the extension 1_{ji} coincides with the zero map on M).

Observation 3.3. For cyclic modules M_S, N_S with generators x, y, a morphism $\varphi : M \to N$ is determined by its action on the generator x of M. If $M = xS$, we have $f(xs) = f(x)s$ for every $xs \in M$. Conversely, each choice $z \in yS$ defines a morphism $g : M \to N$ via $xs \mapsto zs$.

In particular, let R be a regular ring and consider the module R_R. Assume that $I = eR, J = fR$ are principal right ideals in R (that is, cyclic submodules of R_R). Since R is regular, the generators e, f can be taken to be idempotent.

Let $r \in R$ such that $re \in J$, that is, $re = fc$ for some $c \in R$ (or, equivalently, $f(re) = re$). Then the left multiplication with r defines a right-R-module-homomorphism $\widehat{r}$ between I and J

$$\widehat{r} : I \to J \qquad es \mapsto r(es).$$

Remark 3.1. From now on, if possible, we denote the action defined by left multiplication with an element r by $\widehat{r}$. We will speak of the *left multiplication morphism* (or *left multiplication map* or *left multiplication*) $\widehat{r}$.

3.2 General Framework

In this section, we will develop the necessary machinery for the proof of the desired result. In order to simplify the lines of argument and to clarify the applied technique, we have chosen to separate the ring-theoretical aspects, the lattice- and frame-theoretical aspects and the general module-theoretical mechanisms as far as possible.

3.2.1 Decomposition Systems & Abstract Matrix Rings

Definition 3.1. Decomposition system of a module
Let M_S be a right S-module over S and $I = \{i : 0 \leq i < n+k\}$ an index set, where $n < \omega$ and $k \leq \omega$. A *decomposition system* ε *of* M *of format* (n,k) consists of

1. a decomposition $M = \bigoplus_{i \in I} M_i$ of M into a direct sum of submodules,
2. corresponding projections $\pi_i : M \twoheadrightarrow M_i$ and embeddings $\epsilon_i : M_i \hookrightarrow M$,
3. a family $\{\epsilon_{ij} : i,j \in I\}$ of maps ϵ_{ij},
4. submodules z_{ij} of M for $i \in I, j < n$, and
5. a 1-subring $C \leq End(M_0)$

such that the following conditions are satisfied:

1. For $i = j$, we have $\epsilon_{ii} = id_{M_i}$.
2. For $i,j < n$, $\epsilon_{ij}, \epsilon_{ji}$ are mutually inverse morphisms, i.e., $\epsilon_{ij} \circ \epsilon_{ji} = id_{M_i}$.
3. For $i \in I$, we have $\epsilon_{i0} \circ \epsilon_{0i} = id_{M_i}$ (in particular, ϵ_{0i} is injective).
4. For distinct indices i,j,k such that $k,j < n$, we have $\epsilon_{ki} = \epsilon_{kj} \circ \epsilon_{ji}$
5. For $j < n$, z_{ij} is a relative complement of $im(\epsilon_{ji})$ in $[0, M_j]$.
6. For $i \in I$, $\epsilon_{0i} \circ \epsilon_{i0} \in C$.

In other words, for $i,j < n$, the submodules M_i, M_j are isomorphic, while for $i \in I, j < n$, M_i is isomorphic to a submodule of M_j – and the morphisms ϵ_{ji} are the corresponding isomorphisms and embeddings.

The relative complements z_{ij} are integrated into the notion of a decomposition systems for the following reason: For i,j with $j < n$, the injective morphism $\epsilon_{ji} : M_i \hookrightarrow M_j$ has a left inverse $\epsilon_{ij} :: M_j \to M_i$, defined only on $im(\epsilon_{ji}) \leq M_j$. Taking the relative complement $z_{ij} \leq M_j$ of $im(\epsilon_{ji})$ in $[0, M_j]$, we can extend the partial morphism $\epsilon_{ij} :: M_j \to M_i$ to a morphism $\epsilon_{ij} : M_j \to M_i$ by setting $\epsilon_{ij}(x) := 0$ for all $x \in z_{ij}$ (i.e., the extension $\epsilon_{ij} : M_j \to M_i$ acts trivially on z_{ij}).

Remark 3.2. For the ease of notation, we stated that a decomposition system contains a family of maps ϵ_{ij} for $i,j \in I$. The required conditions should have made clear that only particular maps have to exist. Of course, the maps that do exist are (partial) morphisms satisfying the desired relations. (One might take the view that the *other* maps are partial maps with trivial domain.)

We write $\varepsilon = \varepsilon(C, M)$ to indicate the ring C and the module M under consideration.

We recall Observation 3.2 for the natural identifications and conventions.

Definition 3.2. Morphisms between decomposition systems
Let $M_S, M'_{S'}$ be modules over S, S' and $\varepsilon, \varepsilon'$ decomposition systems of M, M', respectively.[1] A morphism between the two decomposition systems $\varepsilon, \varepsilon'$ is a map $\eta : \varepsilon \to \varepsilon'$ such that the components of ε get mapped onto the components of ε'. In particular, the following hold.

1. $\eta(M_i) = M'_i$ and $\eta(\pi_i) = \pi'_i$, $\eta(\epsilon_i) = \epsilon'_i$ for all $i \in I$.
2. $\eta(z_{ij}) = z'_{ij}$ for all $i, j \in I$.
3. $\eta(\epsilon_{ij}) = \epsilon'_{ij}$ for all $i, j \in I$.
4. $\eta : C \to C'$ is a morphism of rings with units.

A morphism η between decomposition systems will be called *injective* or an *embedding of decomposition systems* if $\eta : C \to C'$ is injective.

Definition 3.3. Abstract matrix ring
Let M_S be a module and ε a decomposition system of M. The *abstract matrix ring* with respect to the decomposition system ε of M is

$$R(\varepsilon, C, M) := \{\varphi \in End(M_S) : \epsilon_{0j} \circ \varphi_{ji} \circ \epsilon_{i0} \in C \text{ for all } i, j\} \subseteq End(M_S)$$

where, as above, $\varphi_{ji} = \pi_j \circ \varphi \circ \epsilon_i$ and π_j, ϵ_i are the natural projections and embeddings belonging to decomposition system ε.

The following result justifies this definition.

Proposition 3.4. *The set $R(\varepsilon, C, M)$ is a 1-subring of $End(M_S)$.*

Proposition 3.5. *Let $M_S, M'_{S'}$ be two modules with decomposition systems $\varepsilon, \varepsilon'$ and $\eta : \varepsilon \to \varepsilon'$ a morphism of decomposition systems between ε and ε'. Declaring*

$$\eta(\varphi_{ji}) := \epsilon'_{j0} \circ \eta\Big(\epsilon_{0j} \circ \varphi_{ji} \circ \epsilon_{i0}\Big) \circ \epsilon'_{0i}$$

for morphisms $\varphi_{ji} : M_i \to M_j$, the map η can be extended to a map

$$\eta : R(\Phi, C, M) \to R(\Phi', C', M')$$

[1] Similarly, we denote the components of the two systems by the same letters, once with prime, once without.

in the following way. Since $\varphi \in End(M)$ decomposes into $\varphi = \bigoplus \varphi_i = \sum \varphi_{ji}$, we can define $\eta(\varphi_i) := \sum_{j\in I} \eta(\varphi_{ji})$ for a fixed index $i \in I$ and

$$\eta(\varphi) := \sum_{i\in I} \eta(\varphi_i) = \sum_{i,j\in I} \eta(\varphi_{ji}).$$

With this definition, $\eta : R(\Phi, C, M) \to R(\Phi', C', M')$ is a morphism of rings with unit. If the restriction $\eta_{|C} : C \to C'$ is injective, then so is the map $\eta : R(\Phi, C, M) \to R(\Phi', C', M')$.

3.2.2 Frames and Induced Structures

In this section, we approach the connection between the general framework and our particular setting. Starting with a frame in $L(M)$, we will develop the notion of the coefficient ring of a frame and the notion of an induced decomposition system.

Definition 3.6. Coefficient ring of a frame
Let M_S be a right module over S and Φ a stable (n, k)-frame in $L(M_S)$, contained in the sublattice $L \leq L(M_S)$ with $n \geq 3$. The *coefficient ring* of (Φ, L, M) is

$$C(\Phi, L, M) := \{\varphi \in End(M_0) : \Gamma(\epsilon_{10} \circ \varphi) \in L\} \subseteq End(M_0)$$

The following result justifies this definition.

Proposition 3.7. *The set $C(\Phi, L, M)$ is a 1-subring of $End(M_0)$.*

Remark 3.3. The lines of argument and the technique of this proof are well-known: It is possible to express the ring operations via lattice terms with constants in Φ. (See the works of von Neumann, Jónsson and Handelman.) These terms are uniform in the frame Φ. In particular, they are independent of the particular module M_S.

Proposition 3.8. The Decomposition System of a Frame
Let M_S be a right module over S and Φ a stable (n, k)-frame in $L(M_S)$ contained in the sublattice $L \leq L(M_S)$ with $n \geq 3$.

Then Φ induces a decomposition system in the following way. Since Φ is a frame, we have a decomposition of M into a direct sum $M = \bigoplus M_i$, together with corresponding projections and embeddings. As usual, the axes of perspectivity as well as the axes of subperspectivity are the graphs of morphisms between the summands. Since Φ is stable, we have relative complements z_{ij} as required.[2] *As ring C, we take the coefficient ring $C(\Phi, L, M)$.*

[2]That is, z_{ij} a relative complement of b_{ji} in $[0, a_j]$, where b_{ji} is the image of a_i under a_{ji} in $[0, a_j]$.

We denote the decomposition system induced by Φ by $\xi = \xi_{\Phi,L}(C, M)$.

▶ **Proof.** The only thing left to show is Property 5 in Definition 3.1. Consider $\epsilon_{0i}, \epsilon_{i0}$. Both graphs $\Gamma(\epsilon_{0i}), \Gamma(\epsilon_{i0})$ and of course $\Gamma(\epsilon_{10})$ are part of the frame Φ. Since we can express composition of maps by lattice terms with constants in Φ, we have $\Gamma(\epsilon_{10} \circ \epsilon_{0i} \circ \epsilon_{i0}) \in L$. ◀

Definition 3.9. Matrix ring of a frame
Let M_S be a right module over S and Φ a stable (n, k)-frame in $L(M_S)$ (with $n \geq 3$), contained in the sublattice $L \leq L(M_S)$, $C(\Phi, L, M)$ the coefficient ring as defined in Defintion 3.6 and $\xi = \xi_{\Phi,L}(C, M)$ the induced decomposition system as defined in Definition 3.8. The ring

$$R(\Phi, L, M) := R(\xi, C(\Phi, L, M), M)$$

will be called the *matrix ring (of Φ, L, M)*.

We consider the following situation: Let M and M' be modules over S and S', $L \leq L(M_S)$ a complemented 0-1-sublattice and Φ a stable frame in $L(M_S)$ contained in L of format (n, k) with $n \geq 3$. Assume that we are given a morphism $\iota : L \hookrightarrow L(M')$ of bounded complemented lattices.

Observation 3.4. The image $\Phi' := \iota[\Phi]$ is a stable frame in $L(M')$, contained in $L' := \iota[L] \leq L(M')$. In particular, we have $\iota(M_i) = M'_i$, $\iota(\pi_i) = \pi'_i$, $\iota(\epsilon_i) = \epsilon'_i$ and $\iota(z_{ij}) = z_{ij}$.

Proposition 3.10. *The morphism $\iota : L \to L'$ induces a morphism η between the induced decomposition systems*

$$\xi := \xi_{\Phi,L}(C(\Phi, L, M), L, M) \quad \text{and} \quad \xi' := \xi'_{\Phi',L'}(C(\Phi', L', M), L', M').$$

If $\iota : L \to L'$ is injective, then so is $\eta : \xi \to \xi'$.

▶ **Proof.** We want to define η via the lattice morphism $\iota : L \to L'$. For the first two properties of a morphism between two decomposition systems (see Definition 3.2), we define η to coincide with ι on the submodules M_i, z_{ij} of M and recall Observation 3.4.

Now consider the morphisms ϵ_{ji} given by the frame Φ, i.e., $\Gamma(\epsilon_{ji}) = a_{ji} \in \Phi$. Then

$$\iota\big(\Gamma(\epsilon_{ji})\big) = \iota(a_{ji}) = a'_{ji} = \Gamma(\epsilon'_{ji}) \in \Phi'$$

Setting

$$\eta(\epsilon_{ji}) := \epsilon'_{ji} \text{ for } i \neq j < n \quad \text{and} \quad \eta(\epsilon_{ii}) := \epsilon'_{ii} \text{ for arbitrary } i$$

we have guaranteed that η maps the morphism ϵ_{ji} to ϵ'_{ji}.

For appropriate indices i, j, k, the compatibility $\epsilon_{ki} = \epsilon_{kj} \circ \epsilon_{ji}$ is determined by the lattice-theoretical equation

$$[a_{kj} + a_{ji}] \cdot [a_k + a_i] = a_{ki}$$

of the elements of the frame Φ (and similarly for Φ'). Therefore, we have

$$\eta(\epsilon_{kj} \circ \epsilon_{ji}) = \eta(\epsilon_{ki}) = \epsilon'_{ki} = \epsilon'_{kj} \circ \epsilon'_{ji}$$

for appropriate indices i, j, k.

Secondly, we consider an element φ of the coefficient ring $C = C(\Phi, L, M)$, i.e., $\varphi : M_0 \to M_0$ with $\Gamma(\epsilon_{10} \circ \varphi) \in L$.

The property that $\epsilon_{10} \circ \varphi$ is a morphism between M_0 and M_1 is equivalent to the lattice-theoretical property that $\Gamma(\epsilon_{10} \circ \varphi)$ is a relative complement of M_1 in $[0, M_0 + M_1]$. Since $\iota : L \to L'$ is a lattice morphism mapping Φ to Φ', $\iota(\Gamma(\epsilon_{10} \circ \varphi))$ is a relative complement of M'_1 in $[0, M'_0 + M'_1]$, i.e., the graph of a morphism $\psi : M'_0 \to M'_1$. Composing ψ with ϵ'_{01}, we can define

$$\eta(\varphi) := \epsilon'_{01} \circ \psi : M'_0 \to M'_0$$

Thirdly, we can capture the ring operations on $C(\Phi, L, M)$ via lattice terms with constants in Φ. Hence, the ring operations are transferred via $\iota : L \to L'$ to Φ' and C'. Accordingly, the map $\eta : C(\Phi, L, M) \to C(\Phi', L', M')$ is a morphism of rings.

Finally, injectivity of ι implies injectivity of η. ◀

Corollary 3.11. *In the given situation, there exists a morphism*

$$\eta : R(\Phi, L, C) \to R(\iota[\Phi], \iota[L], C')$$

of rings with unit.

If $\iota : L \to L'$ is injective, then so is $\eta : R \to R'$. In particular, if L embedds into the subspace lattice $L(V)$ of a vector space V, we have a ring embedding

$$\eta : R(\Phi, L, C^M) \hookrightarrow End(V_D)$$

▷ Proof. Combine Proposition 3.10 and Proposition 3.5. ◁

3.3 Representability of $*$-Regular Rings

This section is dedicated to the desired result on representability of $*$-regular rings R such that $\overline{L}(R_R)$ is representable. First, we focus our attention on a $*$-regular ring R with unit such that the MOL $\overline{L}(R_R)$ contains a stable orthogonal frame. With that restriction, we aim at representability of simple $*$-regular rings with unit. Subsequently, we will deal with simple $*$-regular rings without unit and finally, with subdirectly irreducible $*$-regular rings (with and without unit). Due to the first main theorem that the class of all $*$-regular rings is a variety, with Theorem 3.31 , we reach the desired result that a $*$-regular ring R is g-representable if so is $\overline{L}(R_R)$.

3.3.1 $*$-Regular Rings with Frames

Remark 3.4. For this subsection, we assume that

1. R is a $*$-regular ring with unit,
2. $L := \overline{L}(R_R)$ is a MOL of height $h(L) \geq 3$,
3. Φ is a stable orthogonal (n, k)-frame in L with $n = 3$,
4. $M = R_R$, if not stated otherwise.

Remark 3.5. Moreover, we assume that there exists a faithful representation $\iota : L \hookrightarrow L(V_D, \langle\cdot,\cdot\rangle)$ of the MOL L. Consequently, Corollary 3.11 applies in its full strength.

Corollary 3.12. *Let e_i, e_j be projections in R and e_iR, e_jR the corresponding cyclic modules. Any morphism $\varphi_{ji} : e_iR \to e_jR$ is a left multiplication by a ring element $e_jse_i \in e_jRe_i$.*

Observation 3.5. By Proposition 3.8, the stable orthogonal frame Φ induces a decomposition system $\xi = \xi_{\Phi,L}(C, M) = \xi_{\Phi,L}(C, R_R)$. More exactly, we have the following correspondences.

1. The summands M_i correspond to principal right ideals e_iR generated by a projection e_i. Each projection π_i corresponds to a map $\widehat{e_i}$ given by left multiplication with e_i and coincides with the (extension of the) embedding $\epsilon_i = id_{M_i}$.
2. The morphisms $\epsilon_{ji} : e_iR \to e_jR$ are given by $\widehat{e}_{ji}$ with $e_{ji} := \epsilon_{ji}(e_i)$ an element of e_iRe_j.
3. For the coefficient ring of the frame, we have

$$C = C(\Phi, L, M) = C(\Phi, L, R_R) = \{\widehat{r} : r \in e_0Re_0\}.$$

Remark 3.6. From now on, we will denote the morphisms ϵ_{ji} given by the decomposition above by ϵ_{ji} and $\widehat{e_{ji}}$ interchangedly, as it suits the particular situation.

Corollary 3.13. *We have an isomorphism*

$$\theta : R \to R(\Phi, L, R_R)$$

of rings with unit.

▷ Proof. As stated in Remark 3.4 at the beginning of the section, we agree to write M for R_R. We recall the definition of the matrix ring of a frame:

$$R(\Phi, L, M) = \{\varphi \in End(M) : \forall i, j \in I.\ \epsilon_{0j} \circ \varphi_{ji} \circ \epsilon_{i0} \in C(\Phi, L, M)\}$$

Since R contains a unit, an endomorphism $\varphi \in End(M)$ is given by left multiplication $\widehat{r}$ for some $r \in R$. We notice that

$$(\widehat{r})_{ji} = \pi_j \circ \widehat{r} \circ \epsilon_i = \widehat{e_j} \circ \widehat{r} \circ \widehat{e_i} = \widehat{e_j r e_i} = \widehat{r_{ji}}, \quad \text{with } r_{ji} := e_j r e_i.$$

Then we have

$$\epsilon_{0j} \circ (\widehat{r})_{ji} \circ \epsilon_{i0} = \hat{e}_{0j} \circ \widehat{e_j r e_i} \circ \hat{e}_{i0} = \widehat{e_{0j} r e_{i0}}, \quad \text{since } e_{0j} e_j = e_{0j}, e_i e_{i0} = e_{i0}.$$

The equality

$$\epsilon_{0j} \circ (\widehat{r})_{ji} \circ \epsilon_{i0} = \widehat{e_{0j} r e_{i0}}$$

and Observation 3.5 lead to $\epsilon_{0j} \circ (\widehat{r})_{ji} \circ \epsilon_{i0} \in C(\Phi, L, M)$. Since the indices i, j were arbitrary, we have $\widehat{r} \in R(\Phi, L, M)$.

In particular, for an element $r \in R$, the left multiplication $\widehat{r} : M \to M$ decomposes into

$$\widehat{r} = \sum \widehat{r_{ji}} \quad \text{where } \widehat{r_{ji}} : e_i R \to e_j R, \quad \text{and } r_{ji} = e_j r e_i \in e_j R e_i.$$

that is, the isomorphism $\theta : R \to R(\Phi, L, M)$ is given by $\theta : r \mapsto \widehat{r}$.

Of course, we have

$$\begin{aligned} \Gamma(\epsilon_{10} \circ \widehat{e_{0j} r e_{i0}}) &= \Gamma(\widehat{e_{10}} \circ \widehat{e_{0j} r e_{i0}}) = \Gamma(\widehat{e_{1j} r e_{i0}}) \\ &= (e_0 - e_{1j} r e_{i0}) R \in L = \overline{L}(R_R) \end{aligned}$$

◁

Corollary 3.14. *We have an embedding*

$$\rho : R \hookrightarrow End(V_D)$$

of rings with unit.

▷ Proof. By Remark 3.5 the MOL $L = \overline{L}(R_R)$ is assumed to be representable in $L(V_D, \langle .,. \rangle)$. We combine that with Corollaries 3.11 and 3.13 and define $\rho := \eta \circ \theta$ to get the desired isomorphism. ◁

It is left to show that the isomorphism translates the involution on R into adjunction with respect to the scalar product. In the following, assume that in addition to the assumptions of ref..., we have the following.

1. $(V_D, \langle .,. \rangle)$ a vector with scalar product,
2. K a MOL represented in $L(V_D, \langle .,. \rangle)$, i.e., K is a modular sublattice of $L(V_D)$ and the orthocomplementation on K is induced by the scalar product $\langle .,. \rangle$ on V_D,
3. Ψ a stable orthogonal frame in the MOL K,
4. $U_i, U_j \leq V$ with $U_i, U_j \in \Psi$ and $f : U_i \to U_j, g : U_j \to U_i$ linear maps,

If we discuss both situations - that is, for the frame K with its frame Ψ or subspaces of V - simultaneously, we use the symbol Υ for the frame, N_i for submodules or subspaces and a, b for morphisms.

Definition 3.15. Adjointness on $End(V_D, \langle .,. \rangle)$
We call f *and* g *adjoint to each other* (with respect to $\langle .,. \rangle$) if

$$\forall v \in U_i, w \in U_j. \qquad \langle fv, w \rangle = \langle v, gw \rangle.$$

Remark 3.7. Notice that U_i, U_j are elements of the orthogonal frame Ψ, in particular, if $i \neq j$, then U_i and U_j are orthogonal to each other.

Lemma 3.16. *The following conditions are equivalent:*

1. *f and g are adjoint to each other in the sense of Definition 3.15.*
2. *The extensions $\overline{f}, \overline{g} : V \to V$ are adjoint to each other in the usual sense.*

If $i \neq j$, both these conditions are equivalent to $\Gamma(f) \perp \Gamma(-g)$.[3]

▷ Proof. The equivalence of the first two conditions is immediate. Now, if $i \neq j$, we have

$$\Gamma(f) = \{v - fv : v \in U_i\} \qquad \Gamma(-g) = \{w + gw : w \in U_j\}$$

[3]Notice that $\Gamma(f), \Gamma(g), \Gamma(-g)$ are contained in the MOL K.

and

$$
\begin{aligned}
& \Gamma(f) \perp \Gamma(-g) \Leftrightarrow \langle v - fv, w + gw \rangle = 0 \text{ for all } v \in U_i, w \in U_j \\
\Leftrightarrow \quad & \langle v, w \rangle + \langle v, gw \rangle - \langle fv, w \rangle - \langle fv, gw \rangle = 0 \text{ for all } v \in U_i, w \in U_j \\
\Leftrightarrow \quad & \langle v, gw \rangle - \langle fv, w \rangle = 0 \text{ for all } v \in U_i, w \in U_j \\
\Leftrightarrow \quad & \langle v, gw \rangle = \langle fv, w \rangle \text{ for all } v \in U_i, w \in U_j \\
\Leftrightarrow \quad & f \text{ and } g \text{ are adjoint to each other in the sense of Definition 3.15,}
\end{aligned}
$$

where the terms $\langle v, w \rangle, \langle fv, gw \rangle$ vanish since U_i, U_j are orthogonal to each other. ◁

Now, we derive a result similar to Lemma 3.16 for a $*$-regular ring R and the relation between the involution on R and the orthogonality on $L = \overline{L}(R_R)$.

Lemma 3.17. *The involution on R can be captured via the orthogonality on L, more exactly, for $a_{ij} \in e_i R e_j, b_{ji} \in e_j R e_i$ with $i \neq j$, the following conditions are equivalent:*

1. $a_{ij} = b_{ji}^*$
2. $\Gamma(\widehat{a_{ij}}) \perp \Gamma(-\widehat{b_{ji}})$

▷ Proof. Since $\widehat{a_{ij}} : e_j R \to e_i R$ and $-\widehat{b_{ji}} : e_i R \to e_j$, we have

$$\Gamma(\widehat{a_{ij}}) = (e_j - a_{ij} e_j) R \qquad \Gamma(-\widehat{b_{ji}}) = (e_i + b_{ji} e_i) R$$

The orthogonality on L is given by $pR \perp qR :\Leftrightarrow q^* p = 0$. Calculating yields

$$
\begin{aligned}
(e_i + b_{ji} e_i)^* \cdot (e_j - a_{ij} e_j) &= (e_i + e_i b_{ji}^*)(e_j - a_{ij} e_j) \\
&= e_i e_j - e_i a_{ij} e_j + e_i b_{ji}^* e_j - e_i b_{ji}^* a_{ij} e_j \\
&= e_i(-a_{ij} + b_{ji}^*) e_j = -a_{ij} + b_{ji}^*,
\end{aligned}
$$

since $a_{ij}, b_{ji}^* \in e_i R e_j$. Hence $a_{ij} = b_{ji}^*$ iff $\Gamma(\widehat{a_{ij}}) \perp \Gamma(-\widehat{b_{ji}})$. ◁

Corollary 3.18. Uniqueness
Let (i, j) be an arbitrary pair of indices.

A linear map $f : U_i \to U_j$ has at most one adjoint $g : U_j \to U_i$. Due to this uniqueness, it is legitimate to write $f^* = g$ if f, g are adjoint to each other.

Likewise, a map $\widehat{a_{ij}} : e_j R \to e_i R$ gives rise to a map $\widehat{b_{ji}} : e_i R \to e_j R$, namely $\widehat{b_{ji}} = \widehat{a_{ij}^*}$. If $i \neq j$, we have $\widehat{b_{ji}} = \widehat{a_{ij}^*}$ iff $\Gamma(\widehat{a_{ij}}) \perp \Gamma(-\widehat{b_{ji}})$.

Lemma 3.19. *For each morphism $\epsilon_{ki} : N_i \to N_k$, there exists an adjoint $\epsilon_{ki}^* : N_k \to N_i$ in Υ.*

Corollary 3.20. *For $i, k \in I$ with $k < n$ we have $\epsilon_{ki}^* \circ \epsilon_{ik}^* = id_{U_i}$.*

▷ Proof. We have

$$\epsilon_{ki}^* \circ \epsilon_{ik}^* = (\epsilon_{ik} \circ \epsilon_{ki})^* = (id_{N_i})^* = id_{N_i}.$$

◁

Remark 3.8. Obviously, Lemma 3.19 and Corollary 3.20 hold for arbitrary indices $i, k \in I$: Recall that for if $i = k$, we have $\epsilon_{ik} = \epsilon_{ii} = \epsilon_i = id_{N_i}$, which is an hermitian idempotent map.

Corollary 3.21. *Let $a : N_i \to N_j$ and $b : N_j \to N_i$ be as before.*

Then a and b are adjoint to each other iff $\epsilon_{i0}^* \circ b \circ \epsilon_{1j}^*$ and $\epsilon_{1j} \circ a \circ \epsilon_{i0}$ are adjoint to each other.

▷ Proof. Assume that $\epsilon_{i0}^* \circ b \circ \epsilon_{1j}^*$ and $\epsilon_{1j} \circ a \circ \epsilon_{i0}$ are adjoint to each other, where adjoint is either understood in the sense of Definition 3.15 or in the sense of Lemma 3.17. Then

$$\epsilon_{0i}^* \circ (\epsilon_{1j} \circ a \circ \epsilon_{i0})^* \circ \epsilon_{j1}^* = (\epsilon_{j1} \epsilon_{1j} a \epsilon_{i0} \epsilon_{0i})^* = a^*$$

and

$$\epsilon_{0i}^* \circ (\epsilon_{1j} \circ a \circ \epsilon_{i0})^* \circ \epsilon_{j1}^* = \epsilon_{0i}^* \circ (\epsilon_{i0}^* \circ b \circ \epsilon_{1j}^*) \circ \epsilon_{j1}^* = (\epsilon_{0i}^* \epsilon_{i0}^*) b (\epsilon_{1j}^* \epsilon_{j1}^*) = b,$$

so $a^* = b$.

Now, assume that a, b are adjoint to each other. Then

$$(\epsilon_{1j} \circ a \circ \epsilon_{i0})^* = \epsilon_{i0}^* \circ a^* \circ \epsilon_{1j}^* = \epsilon_{i0}^* \circ b \circ \epsilon_{1j}^*.$$

◁

Remark 3.9. Corollary 3.21 holds for arbitrary indices i, j, too. In particular, we can complete Lemma 3.17 and Corollary 3.18 by noting the following: If $a_{ii}, b_{ii} \in e_i R e_i$, we have

$$a_{ii}^* = b_{ii} \Leftrightarrow \Gamma(\epsilon_{i0}^* \circ \widehat{a_{ii}} \circ \epsilon_{1j}^*) \perp \Gamma(-\epsilon_{1j} \circ \widehat{b_{ii}} \circ \epsilon_{i0})$$

Proposition 3.22. *The map*

$$\rho = \eta \circ \theta : R \to End(V, \langle ., . \rangle)$$

defined in Corollary 3.14 is a $*$-ring-embedding of involutive rings with unit.

▶ **Proof.** First, we recall that the map $\eta : R(\Phi, L, M) \hookrightarrow End(V_D)$ defined in Corollary 3.11 was defined via the lattice embedding $\iota : L \hookrightarrow L(V_D)$. In this situation, we consider a MOL L and a MOL-embedding of L into $L(V_D, \langle .,. \rangle)$. As shown before, for morphisms $\epsilon_{ji} : M_i \to M_j \in \Phi$, we can define an adjoint operator $\epsilon_{ji}^* : M_j \to M_i$ via the orthogonality on L. For morphisms $\varphi_{10} : M_0 \to M_1$ and $\psi_{01} : M_1 \to M_0$, the relation of adjointness could be captured via orthogonality on graphs. In particular, we have $\rho(\epsilon_{ji}^*) = \big(\rho(\epsilon_{ji})\big)^*$ and $\rho(\varphi_{10}^*) = \big(\rho(\varphi_{10})\big)^*$.

Now, for $r \in R$, consider $e_j r e_i \in e_j R e_i$. Then

$$\begin{aligned}
\big(\rho(e_j r e_i)\big)^* &= \big(\rho(e_{j1}(e_{1j} e_j r e_i e_{i0}) e_{0i})\big)^* = \big(\rho(e_{j1})\rho(e_{1j} e_j r e_i e_{i0})\rho(e_{0i})\big)^* \\
&= \big(\rho(e_{0i})\big)^* \big(\rho(e_{1j} e_j r e_i e_{i0})\big)^* \big(\rho(e_{j1})\big)^* \\
&= \rho(e_{0i}^*)\rho\big((e_{1j} e_j r e_i e_{i0})^*\big)\rho(\epsilon_{j1}^*) \\
&= \rho(e_{0i}^*)\rho\big(e_{i0}^* e_i^* r^* e_j^* e_{1j}^*\big)\rho(\epsilon_{j1}^*) \\
&= \rho\big(e_{0i}^* e_{i0}^* e_i^* r^* e_j^* e_{1j}^* \epsilon_{j1}^*\big) = \rho(e_i^* r^* e_j^*) = \rho\big((e_j r e_i)^*\big).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\big(\rho(r)\big)^* &= \Big(\rho\big(\sum e_j r e_i\big)\Big)^* = \Big(\sum \rho(e_j r e_i)\Big)^* = \sum \big(\rho(e_j r e_i)\big)^* \\
&= \sum \rho\big((e_j r e_i)^*\big) = \sum \rho(e_i^* r^* e_j^*) = \rho\Big(\sum e_i^* r^* e_j^*\Big) \\
&= \rho\Big(\sum (e_j r e_i)^*\Big) = \rho\Big(\big(\sum e_j r e_i\big)^*\Big) = \rho(r^*).
\end{aligned}$$

◀

3.3.2 Simple $*$-Regular Rings

Corollary 3.23. *Every simple $*$-regular ring S with unit admits a faithful linear representation provided that $\overline{L}(S_S)$ does so.*

▷ Proof. We may assume that S is non-Artinian, hence, we can assume that S has height at least 3. By Corollary 1.23, the MOL $L = \overline{L}(S_S)$ contains a stable orthogonal frame of format (n, k) with $n \geq 3$. It follows by Proposition 3.22 that S is faithfully representable. ◁

Proposition 3.24. *Every simple $*$-regular ring R admits a faithful linear representation provided that $\overline{L}(R_R)$ does so.*

▶ **Proof.** Let R be a simple $*$-regular ring R without unit. Consider the set $P(R)$ of all projections in R. Since R is $*$-regular, $P(R)$ is a lattice. In particular, it is a directed set. By Lemma 1.9, we have that R is the directed union of its subrings R_e, $e \in P(R)$. By Lemma 2.8, R is a $*$-regular subring of an ultraproduct of the R_e, $e \in P(R)$. By Lemma 1.8, for each projection e, the ring R_e is a simple $*$-regular ring with unit e and $\bar{L}(R_{eR_e}) \cong [0, eR]$ is representable. By Corollary 3.23, each R_e is faithfully representable. Hence, we can conclude with Lemma 2.9 and Proposition 2.10 that R has a faithful representation. ◀

3.3.3 Representations of Ideals

Observation 3.6. Let I be a two-sided ideal of the $*$-regular ring R. We can consider I as a $*$-regular ring (without unit, if I is non-trivial) on its own; hence, we can consider representations of I.

Proposition 3.25. *Let R be a $*$-regular ring and I a two-sided ideal in R with a representation $\varrho : I \to End(V_D, \langle\cdot,\cdot\rangle)$. Denote the set of all projections in I by $P(I)$, abbreviate $V_p := \varrho(p)[V]$ for a projection $p \in P(I)$ and set*

$$\rho(r) := \bigcup_{p \in P(I)} \varrho(rp)_{|V_p}$$

Then ρ is a representation of R an appropriate subspace U of V, where the scalar product on U_D is given by restriction.

▶ **Proof.** First, we have to show that the given definition of ρ indeed defines a map $\rho : R \to End(V_D)$. Recalling that the set of all projections of a $*$-regular ring is directed, we consider two projections $e, f \in P(I)$ with $e \leq f$, that is, with $e = fe$. We have to show that the restrictions coincide on V_e, i.e., $\varrho(rf)_{|V_e} = \varrho(re)_{|V_e}$. Since $e = fe$, we have

$$\varrho(re)_{|V_e} = \varrho(rfe)_{|V_e} = (\varrho(rf) \circ \varrho(e))_{|V_e} = \varrho(rf)_{|V_e},$$

as desired.

Second, we have to show that the map $\rho : R \to End(V_D)$ is a $*$-ring-homomorphism. For 0 in R, we have

$$\rho(0) = \bigcup_{p \in P(I)} \varrho(0p)_{|V_p} = 0_V.$$

If $1 \in R$, we have

$$\rho(1) = \bigcup_{p \in P(I)} \varrho(1p)_{|V_p} = 1_U$$

with $U := \bigcup_{p \in P(I)} V_p$, that is, $\rho[R]$ acts on the subspace U of V_D.

For addition, let $r, s \in R$. We have

$$\begin{aligned}
\rho(r+s) &= \bigcup_{p\in P(I)} \varrho((r+s)p)_{|V_p} = \bigcup_{p\in P(I)} \varrho(rp+sp)_{|V_p} \\
&= \bigcup_{p\in P(I)} \varrho(rp)_{|V_p} + \varrho(sp)_{|V_p} = \bigcup_{p\in P(I)} \varrho(rp)_{|V_p} + \bigcup_{q\in P(I)} \varrho(sq)_{|V_q} \\
&= \rho(r) + \rho(s).
\end{aligned}$$

Therefore, $\rho(r+s) = \rho(r) + \rho(s)$ and $\rho(-r) = -\rho(r)$for all $r, s \in R$.

For multiplication, let r, s be in R. We note that for each $p \in P(I)$, there exists a $q_p \in P(I)$ such that $sp = q_p sp$. We claim that

$$\bigcup_{p\in P(I)} (\varrho(rq_p) \circ \varrho(sp))_{|V_p} = \bigcup_{q\in P(I)} \varrho(rq)_{|V_q} \circ \bigcup_{p\in P(I)} \varrho(sp)_{|V_p}.$$

▷ Proof. Take $v \in U$. Then there exists $p_v \in P(I)$ with $v \in V_{p_v}$, so on the one hand

$$\left(\bigcup_{p\in P(I)} \big(\varrho(rq_p) \circ \varrho(sp)\big)_{|V_p}\right)(v) = \big(\varrho(rq_{p_v})v \circ \varrho(sp_v)\big)(v),$$

while on the other hand

$$\begin{aligned}
&\left(\bigcup_{q\in P(I)} \varrho(rq)_{|V_q} \circ \bigcup_{p\in P(I)} \varrho(sp)_{|V_p}\right)(v) = \bigcup_{q\in P(I)} \varrho(rq)_{|V_q}\Big(\bigcup_{p\in P(I)} \varrho(sp)_{|V_p}(v)\Big) \\
= &\bigcup_{q\in P(I)} \varrho(rq)_{|V_q}\Big(\varrho(sp_v)(v)\Big) = \varrho(rq_{p_v})(\varrho(sp_v)(v)).
\end{aligned}$$

◁

Thus, we have

$$\begin{aligned}
\rho(rs) &= \bigcup_{p\in P(I)} \varrho((rs)p)_{|V_p} = \bigcup_{p\in P(I)} \varrho(rq_p sp)_{|V_p} \\
&= \bigcup_{p\in P(I)} (\varrho(rq_p) \circ \varrho(sp))_{|V_p} = \bigcup_{q\in P(I)} \varrho(rq)_{|V_q} \circ \bigcup_{p\in P(I)} \varrho(sp)_{|V_p} \\
&= \rho(r) \circ \rho(s).
\end{aligned}$$

Now we examine the involution on R. For $r \in R$, consider $v, w \in V$. Then take $e \in P(I)$ with $v \in V_e$. There exists $f_1 \in P(I)$ such that $f_1 re = re$ and

$f_2 \in P(I)$ such that $w \in V_{f_2}$. Choosing $f := f_1 \vee f_2$, we have

$$\begin{aligned}
\langle \rho(r)v, w\rangle &= \langle \bigcup_{p\in P(I)} \varrho(rp)_{|V_p}v, w\rangle = \langle \varrho(re)v, w\rangle = \langle \varrho(fre)v, w\rangle \\
&= \langle v, \varrho(er^*f)w\rangle = \langle v, \varrho(e)\varrho(r^*f)w\rangle = \langle \varrho(e)v, \varrho(r^*f)w\rangle \\
&= \langle v, \varrho(r^*f)w\rangle = \langle v, \rho(r^*)w\rangle,
\end{aligned}$$

where we have used that $\varrho : I \to End(V_D)$ is a $*$-ring-homomorphism and $v \in V_e, w \in V_f$. ◀

Lemma 3.26. *Let R be a $*$-regular ring and I a two-sided ideal in R with a representation $\varrho : L \to End(V_D, \langle ., .\rangle)$. Denote the action of R on the ideal I given by left multiplication by λ_I, that is*

$$\lambda_I : R \to End(I_I) \qquad \lambda_I(r)(x) := \widehat{r}(x) = rx.$$

If the representation $\varrho : I \to End(V_D, \langle\cdot,\cdot\rangle)$ is faithful and $\lambda_I : R \to End(I_I)$ is injective, then the representation $\rho : R \to End(V_D, \langle\cdot,\cdot\rangle)$ defined in Proposition 3.25 is faithful.

▷ Proof. Assume that $\rho(r) = 0$. This is equivalent to $\varrho(re) = 0$ for all $e \in P(I)$. As $\varrho : I \to End(V_D, \langle\cdot,\cdot\rangle)$ is faithful, this means that $re = 0$ for all $e \in P(I)$. Since I is a $*$-regular ring, for every element $x \in I$ there exists $e \in P(I)$ such that $ex = x$. Hence, we have that $rx = 0$ for all $x \in I$. Since we assumed the action of R on I given by left multiplication to be injective, we have that $r = 0$. This shows that ρ is injective. ◁

3.3.4 Subdirectly Irreducible $*$-Regular Rings

In this section, we will show that each subdirectly irreducible $*$-regular ring R has a faithful representation provided that $\overline{L}(R_R)$ does so.

Observation 3.7. We may assume that R is non-Artinian: Since every regular ring is semi-prime, a subdirectly irreducible $*$-regular ring which is Artinian is semi-simple, hence representable.

Furthermore, the minimal two-sided ideal of R is non-Artinian, too (see [**?**], Proposition 2).

Proposition 3.27. *Let R be a subdirectly irreducible $*$-regular ring and let J be the minimal two-sided ideal of R. Then the action $\lambda_J : R \to End(J_J)$ of R defined by*

$$\lambda_J(r)(x) = \hat{r}(x) = rx$$

is injective.

► **Proof.** Consider the left annihilator $A := ann_R^l(J)$ of J in R. While a priori A is only a left ideal, it can be shown that A is indeed closed under left and right multiplication by elements of R. Since A is closed under addition, A is a two-sided ideal in R. Since J does not annihilate itself, we can conclude that A is trivial. Therefore, the action of R on J defined by left multiplication is injective. ◄

Lemma 3.28. *The minimal ideal J of a subdirectly irreducible $*$-regular ring R is a simple $*$-regular ring.*

▷ Proof. For a non-vanishing ideal A in J, consider the ideal generated by A in R. ◁

Remark 3.10. Note that one does not need that R contains a unit.

Lemma 3.29. *Let R be a $*$-regular ring and I a minimal two-sided ideal in R, in particular simple as a ring. Let e be a projection in I.*

Then the ring $R_e = eRe$ is simple.

▷ Proof. For a non-vanishing ideal A in R_e, consider the ideal generated by A in I. ◁

Proposition 3.30. *Considered as simple $*$-regular ring, the minimal ideal J of a subdirectly irreducible $*$-regular ring R has a faithful representation provided that $\bar{L}(I_R)$ does so.*

▷ Proof. Proposition 3.24. ◁

Theorem 3.31. *Every subdirectly irreducible $*$-regular ring R has a faithful representation provided that $\overline{L}(R_R)$ does so.*

▷ Proof. Lemma 3.26. ◁

References

[Good91] K. R. Goodearl, *Von Neumann Regular Rings* Krieger Publishing Company, 1991.

[Gor98] V. A. Gorbunov, *Algebraic Theory of Quasivarieties* Siberian School of Logic and Algebra, Consultants Bureau, New York, 1998.

[Kap68] I. Kaplansky, *Rings of Operators*, W. A. Benjamin, Inc., 1968.

[Mae58] F. Maeda *Kontinuierliche Geometrien* Springer, Berlin, 1958.

[Skor64] L. A. Skornyakov, *Complemented Modular Lattices and Regular Rings* Oliver and Boyd Ltd., 1964.

[vN60] J. von Neumann, *Continous Geometries*, Princeton, 1960.

[Day82] A. Day, *Geometrical applications in modular*
Universal Algebra and Lattice Theory (R. S. Freese ans O. C. Garcia, eds.)
Lecture Notes in Mathematics, 1004, 1983, p. 111 – 141. Springer.

[Day84] A. Day, *Applications of coordinatization in modular lattice theory: The legacy of J. von Neumann* Order, 1, 1984, p. 295 – 300.
Reidel Publishing Company, Boston, U.S.A.

[Herr95] C. Herrmann, *Alan Day's work on modular and Arguesian lattices*
Algebra Universalis, 34, 1995, p. 35 – 60.
Dedicated to the memory of Alan Day.

[HR99] C. Herrmann, M. Roddy, *Proatomic modular ortholattices: Representation and equational theory* Note di matematica e fisica, 10, 1999, p. 57 – 88.

[Jón60] B. Jónsson, *Representations of complemented modular lattices*
Transactions of the AMS, 97, 1960, p. 64 – 94.

[Kap55] I. Kaplansky, *Any orthocomplemented complete modular lattice is a continuous geometry* Annals of Mathematics, 61, 1955, p. 524 – 542.

[Mic03] F. Micol, *On representability of $*$-regular rings and modular ortholattices*
FB 04 TUD, 2003.

[Nie03] N. Niemann, *Representations of involutive rings* FB 04 TUD, 2003.

[Weh98] F. Wehrung, *The dimension monoid of a lattice*
Algebra Universalis, 40 (3), 1998, p. 247 – 411.