LINEAR REPRESENTATIONS OF REGULAR RINGS AND COMPLEMENTED MODULAR LATTICES WITH INVOLUTION

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Abstract. Faithful representations of regular $*$-rings and modular complemented lattices with involution within orthosymmetric sesquilinear spaces are studied within the framework of Universal Algebra. In particular, the correspondence between classes of spaces and classes of representables is analyzed; for a class of spaces which is closed under ultraproducts and non-degenerate finite dimensional subspaces, the latter are shown to be closed under complemented [regular] subalgebras, homomorphic images, and ultraproducts and being generated by those members which are associated with finite dimensional spaces. Under natural restrictions, this is refined to a 1-1-correspondence between the two types of classes.

1. Introduction

For $*$-rings, there is a natural and well established concept of representation in a vector space V_F endowed with an orthosymmetric sesquilinear form: a homomorphism ε into the endomorphism ring of V_F such that $\varepsilon(r^*)$ is the adjoint of $\varepsilon(r)$. Famous examples of (faithful) representations are due to Gel'fand-Naimark-Segal (C^*-algebras in Hilbert space) and Kaplansky (primitive $*$-rings with minimal right ideal) [26, Theorem 1.2.2].

(Faithful) representability of $*$-regular rings within anisotropic inner product spaces has been studied by Micol [40] and used to derive results in the universal algebraic theory of these structures. For the $*$-regular rings of classical quotients of finite Rickart C^*-algebras (cf. Ara and Menal [1]), representations have been established in [25]. For complemented modular lattices L (CML for short) with involution $a \mapsto a'$, an analogue of the concept of representation is a $(0,1)$-lattice homomorphism ε into the lattice of all subspaces such that $\varepsilon(a')$ is the orthogonal subspace to $\varepsilon(a)$ (cf. Niemann [42]). The latter has been considered in the context of synthetic orthogeometries in [18], continuing earlier work on anisotropic geometries and modular ortholattices [20, 21, 22]. Primary examples are atomic CML with associated irreducible desarguean orthogeometry and those CML which arise as lattices of principal right ideals of representable regular $*$-rings.

The (proofs of the) main results of these studies relate closure properties of a class $\mathcal{S}$ of spaces with closure properties of the class $\mathcal{R}$ of algebraic structures (faithfully) representable within spaces from $\mathcal{S}$. In particular, for $\mathcal{S}$ closed under ultraproducts and non-degenerate finite dimensional subspaces, one has $\mathcal{R}$ closed under ultraproducts, homomorphic images, and $*$-regular [complemented, respectively] subalgebras. Moreover, with an approach due to Tyukavkin [46], it has been shown that $\mathcal{R}$ is generated, under these operators, by the endomorphism $*$-rings [by the subspace lattices with involution $U \mapsto U^\perp$, respectively] associated with finite dimensional

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spaces from $\mathcal{S}$ (cf. Theorem 9.4). Conversely, any class $\mathcal{R}$ of structures generated in this way has its members representable within $\mathcal{S}$.

The first purpose of the present paper is to extend these results to regular $*$-rings on one hand, to representations within orthosymmetric sesquilinear spaces on the other. The second one is to give a more transparent presentation by dealing with types of classes naturally associated with representations in linear spaces. We call a class of structures $\mathcal{R}$ as above an $\exists$-*semivariety* of regular $*$-rings [complemented modular lattices with involution or CMIL for short] and we call $\mathcal{S}$ a *semivariety* of spaces. The quantifier '$\exists$' refers to the required existence of quasi-inverses [complements, respectively]. In this setting, the above-mentioned relationship between classes of structures $\mathcal{R}$ and classes of spaces $\mathcal{S}$ can be refined to a 1-1-correspondence (cf. Theorem 9.7). Also, we observe that $\mathcal{R}$ remains unchanged if $\mathcal{S}$ is enlarged by forming two-sorted substructures, corresponding to the subgeometries in the sense of [18], (cf. Theorem 9.4). We also provide a useful condition on $\mathcal{S}$ such that $\mathcal{R}$ is an $\exists$-*variety*, i.e. $\mathcal{R}$ also closed under direct products (see Proposition 10.2).

In the context of synthetic orthogeometries, the class $\mathcal{R}$ of representables is an $\exists$-variety if $\mathcal{S}$ is also closed under orthogonal disjoint unions. No such natural construction is available for sesquilinear spaces. The alternative, chosen by Micol [40], was to generalize the concept of faithful representation to a family of representations with kernels intersecting to 0 [the identical congruence, respectively]; thus, associating with any semivariety of spaces an $\exists$-variety of generalized representables. We derive these results in our more general setting (cf. Proposition 10.4).

For reference in later applications, e.g. to decidability results refining those of [19], we consider $*$-rings which are also algebras over a fixed commutative $*$-ring. Once the definitions are adapted, only a minimum of additional effort is needed in proofs.

2. Lattices with involution

We assume familiarity with the basic constructions of Universal Algebra as presented e.g. in [7, 16], see also [39]. First, we recall some notation. For a class $\mathcal{C}$ of algebraic structures of a fixed similarity type, by $\mathsf{H}(\mathcal{C})$, $\mathsf{S}(\mathcal{C})$, $\mathsf{P}(\mathcal{C})$, $\mathsf{P}_s(\mathcal{C})$, $\mathsf{P}_\omega(\mathcal{C})$, and $\mathsf{P}_u(\mathcal{C})$, we denote the class of all homomorphic images, of structures isomorphic to substructures, direct products, subdirect products, direct products of finitely many factors, and ultraproducts of members of $\mathcal{C}$, respectively. Elements of a reduced product $\prod_{i\in I} A_i/\mathcal{F}$ are denoted as $[a_i \mid i \in I]$. An algebraic structure A is *subdirectly irreducible* if it has a least non-trivial congruence. In particular, if A is subdirectly irreducible and $A \in \mathsf{SP}(\mathcal{C})$, then $A \in \mathsf{S}(\mathcal{C})$. By Birkhoff's Theorem, any algebraic structure is a subdirect product of its subdirectly irreducible homomorphic images.

A class $\mathcal{C}$ of algebraic structures of the same type is a *universal class* if it is closed under S and P_u; a *positive universal class*, shortly a *semivariety*, if it is closed also under H; a *variety* if, in addition, it is closed under P. The following statement is well known and easily verified.

Fact 2.1. A class $\mathcal{K}$ is universal [a semivariety, a variety] if and only if it can be defined by universal sentences [positive universal sentences, identities, respectively].

For the following concepts, we refer to [5, 6]. We consider *lattices* with $0,1$ as algebraic structures $\langle L; \cdot, +, 0, 1\rangle$ such that, for a suitable partial order, $ab = a \cdot b = \inf\{a, b\}$ and $a+b = \sup\{a, b\}$. We write $a \oplus b$ instead of $a+b$, when $ab = 0$. Lattices form a well known equationally defined class. The same applies to the subclass of *modular* lattices; that is, lattices satisfying

$$a \geq c \text{ implies } a(b+c) = ab + c.$$

A modular lattice L has *height* $n < \omega$ (which is also called the *dimension* of L and denoted by $\dim L$), if L has $(n+1)$-element maximal chains; we write $\dim a = \dim[0,a]$. An *atom* is an element $a \in L$ with $\dim a = 1$. A lattice L is *complemented* if for all $a \in L$, there is $b \in L$ such that $a \oplus b = 1$. In a CML [i.e., a complemented modular lattice] L, any interval $[u,v]$ is complemented, too; L is *atomic* if for any $a > 0$ there is an atom $p \leq a$. It follows that $a \not\leq b$ in L if and only if there is an atom $p \in L$ such that $p \leq a$ and $p \not\leq b$.

If a lattice L is endowed with an additional operation $x \mapsto x'$ such that

$$\begin{aligned} &x \leq y' \text{ if and only if } y \leq x'; \\ &(x+y)' = x'y'; \\ &1' = 0, \quad 0' = 1, \end{aligned}$$

then we speak of a *Galois lattice*. Observe that $x \leq y$ implies $y' \leq x'$, that $x''' = x'$, and that $x \mapsto x''$ defines a closure operator on L. We call such a lattice a lattice with *involution* if, in addition, $x'' = x$ for all $x \in L$; equivalently, if $x \mapsto x'$ is a dual automorphism of order 2 of the lattice L. Thus lattices with involution form an equationally defined class. The following statement is straightforward to prove. L:1

Lemma 2.2. *Let L_0, L_1 be modular lattices with involution.*

(i) *A map $\varphi\colon L_0 \to L_1$ is a homomorphism, if $\varphi(x+y) = \varphi(x)+\varphi(y)$, $\varphi(x') = \varphi(x)'$ for all x, $y \in L_0$, and $\varphi(0) = 0$.*
(ii) *A subset $X \subseteq L_0$ is a subalgebra of L_0, if $0 \in X$ and X is closed under operations $+$ and $'$.*

The subclass of *ortholattices* consists of lattices with involution satisfying the identity $xx' = 0$ (or equivalently, $x + x' = 1$). We write MIL [CMIL] shortly for [complemented] modular lattices with involution and MOL for modular ortholattices. We use each abbreviation also to denote the class of *all* lattices with the corresponding property. Observe that $\dim u = \dim[u',1]$ in any MIL. For a modular Galois lattice L, let $L_f = \{u, u' \mid u \in L, \dim u < \omega\}$.

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Fact 2.3. If L is a Galois CML then $L_f \in \mathsf{S}(L)$ and is an atomic CMIL which is the directed union of its subalgebras $[0,u] \cup [u',1]$, where $\dim u < \omega$ and $u \oplus u' = 1$ (which are all CMILs). If L is a CMIL, then $L_f = \{a \in L \mid \dim a < \omega \text{ or } \dim[a,1] < \omega\}$.

Proof. If p is an atom of L then p' is a coatom. Indeed, if $p' \leq x < 1$ then $0 < x' \leq p$, whence $x' = p$ and $x \leq x'' = p'$.

It follows that $\dim[v',u'] \leqslant \dim[u,v]$ if the latter is finite. Namely, if $\dim[u,v] = 1$ then $v = u + p$, where p is a complement of u in $[0,v]$, whence an atom and so $v' = u'p'$ is a lower cover of u' unless $v' = u'$. Thus, if $\dim u < \omega$, then $\dim u'' \leq \dim[u',1] \leq \dim u$, $u'' = u$, and $x \mapsto x'$ provides a pair of mutually inverse lattice anti-isomorphisms between the intervals $[0,u]$ and $[u',1]$ of L. Therefore, $[u',1] \subseteq L_f$. Since $\{u \in L \mid \dim u < \omega\}$ is closed under joins and $0 \in L_f$, $L_f \in \mathsf{S}(L)$ by Lemma 2.2(ii) and, in particular, L_f is an MIL.

If $X \subseteq L_f$ is finite, then there is $u \in L$ such that $\dim u < \omega$ and $X = Y \cup Z$, where y, $z' \in [0,u]$ for all $y \in Y$, $z \in Z$. Choose v as a complement of $u + u'$ in $[u,1]$. Then u, u', $v \in L_f$ whence $u = v(u+u')$ implies $u' = v' + uu'$. It follows that $v+v'=v+u+uu'+v'=v+u+u'=1$ and $vv'=(v+v')'=1'=0$ whence $X \subseteq [0,v] \cup [v',1]$. This proves the first statement.

If L is a CMIL and $\dim[a,1] < \omega$ then $\dim a' = \dim[a,1] < \omega$, thus $a' \in L_f$ and $a = a'' \in L_f$. □

For a lattice congruence θ on an MIL L, we put $\theta' = \{(a',b') \mid (a,b) \in \theta\}$. Then θ' is also a lattice congruence on L and the congruences on L are exactly joins $\theta \vee \theta'$, where θ is a lattice congruence on L. We call L *strictly subdirectly irreducible* if its lattice reduct is subdirectly irreducible; i.e., L has a unique minimal (nontrivial) lattice congruence θ (whence $\theta = \theta'$). Similarly, L is *strictly simple* if it is simple as a lattice. In the case of MOLs one has $\theta = \theta'$ for all θ; thus subdirectly irreducible MOLs [simple MOLs] are strictly subdirectly irreducible [strictly simple, respectively].

atom

Fact 2.4. A strictly subdirectly irreducible CMIL L is atomic provided it contains an atom. For the minimal nontrivial congruence θ, one has $a \in L_f$ iff $a\theta 0$ or $a\theta 1$. In particular, L_f is strictly subdirectly irreducible and atomic, too.

Proof. Let p be an atom in L. By modularity, the lattice congruence generated by $(0,p)$ is minimal. Thus given $a > 0$, one has $(0,p)$ in the lattice congruence generated by $(0,a)$, whence by modularity, $p/0$ is projective to some subquotient x/y of $a/0$. Then any complement q of y in $[0,x]$ is an atom. Thus L is atomic and it follows that $x\theta y$ iff $\dim[xy, x+y] < \omega$. In view of Fact 2.3, we are done. □

3. Projective spaces and orthogeometries

D:proj

Definition 3.1. A *projective space* P is a set, whose elements are called *points*, endowed with a ternary relation $\Delta \subseteq P^3$ of *collinearity* satisfying the following conditions:

(i) if $\Delta(p_0,p_1,p_2)$, then $\Delta(p_{\sigma(0)},p_{\sigma(1)},p_{\sigma(2)})$ and $p_{\sigma(0)} \neq p_{\sigma(1)}$ for any permutation σ on the set $\{0,1,2\}$;
(ii) if $\Delta(p_0,p_1,a)$ and $\Delta(p_0,p_1,b)$, then $\Delta(p_0,a,b)$;
(iii) if $\Delta(p,a,b)$ and $\Delta(p,c,d)$, then $\Delta(q,a,c)$ and $\Delta(q,b,d)$ for some $q \in P$.

The space P is *irreducible* if for any $p \neq q$ in P there is $r \in P$ such that $\Delta(p,q,r)$. A set $X \subseteq P$ is a *subspace* of P if p, $q \in X$ and $\Delta(p,q,r)$ together imply that $r \in X$.

Any projective space P is the disjoint union of its irreducible subspaces P_i, $i \in I$, which are called its *components*. The set $L(P)$ of all subspaces of an [irreducible] projective space P is a [subdirectly irreducible] atomic CML, in which all atoms are compact. Moreover, $L(P) \cong \prod_{i\in I} L(P_i)$ via the map $X \mapsto (X \cap P_i \mid i \in I)$. Conversely, any atomic CML L with compact atoms is isomorphic to $L(P)$ via the map $a \mapsto \{p \in P \mid p \leq a\}$, where P is the set of atoms of L and p, q, $r \in P$ are collinear if and only if $r < p+q$. Recall that Jónsson's *Arguesian* lattice identity [29] holds in $L(P)$ if and only if P is desarguean.

For a vector space V_F, let $L(V_F)$ denote the lattice of all linear subspaces of V_F.

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Fact 3.2. (i) For any vector space V_F, $L(V_F)$ is a CML. Moreover, there exists an irreducible desarguean projective space P such that $L(V_F) \cong L(P)$.
(ii) For any irreducible desarguean projective space P with $\dim L(P) > 2$, there is a vector space V_F which is unique up to (2-sorted) isomorphism, such that $L(P) \cong L(V_F)$.
(iii) If P is irreducible and $\dim L(P) > 3$, then P is desarguean.

(iv) Any subdirectly irreducible CML of height at least 4 is Arguesian.

Proof. Claim (i) is the content of [10, Proposition 2.4.15]. For (ii), see [10, Proposition 2.5.6] and [10, Chapter 9]. For (iii), see [9, Chapter 11]. As to claim (iv), according to Frink [11], any CML L embeds into $L(P)$ for some projective space P. Since L is subdirectly irreducible as a lattice, it embeds into $L(P_i)$ for some irreducible component P_i of P, which is desarguean since $\dim L(P_i) > 3$, whence statement (iv) follows. □

D:ortho

Definition 3.3. An *orthogeometry* is a pair $\langle P; \perp \rangle$, where P is a projective space P endowed with a symmetric binary relation $\perp$ of *orthogonality* such that for any p, q, r, $s \in P$, the following hold:

(i) if $p \perp q$, $p \perp r$, and $\Delta(q, r, s)$, then $p \perp s$;
(ii) if $p \neq q$ and $r \not\perp p$, q, then $r \perp t$ for some $t \in P$ such that $\Delta(p, q, t)$;
(iii) there is $t \in P$ such that $p \not\perp t$.

Then the subspace lattice $L(P)$ together with the map

$$X \mapsto X^{\perp} = \{q \in P \mid q \perp p \text{ for all } p \in X\}$$

is a Galois CML which we denote by $\mathbb{L}(P, \perp)$. Observe that conditions (ii) and (iii) amount to $p^{\perp}$ being a coatom of $L(P)$ for any $p \in P$. For an MIL lattice L with a least element 0, let $P_L = \{p \in L \mid 0 \prec p\}$ be the set of *atoms* of L. We define a collinearity on P_L by putting $\Delta(p, q, r)$ for distinct atoms p, q, $r \in P_L$ such that $p \leq q + r$ in L. Furthermore, $p \perp q$, if $p \leq q'$.

Fact 3.4. [18, Lemma 4.2] For any MIL L, $\mathbb{G}(L) = \langle P_L, \perp \rangle$ is an orthogeometry.

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Fact 3.5. The lattice $L = \mathbb{L}(P, \perp)_f = \{X, X^{\perp} \mid X \in L(P),\ \dim X < \omega\}$ is a CMIL with $L = L_f$. Conversely, for any CMIL L with $L = L_f$, one has $L \cong \mathbb{L}\mathbb{G}(L)_f$.

Proof. See [18, Theorem 1.1] and Fact 2.3. □

4. Rings

When mentioning rings, we always mean associative rings, possibly without unit; in the latter case, the principal right ideal generated by a equals $\{za \mid z \in \mathbb{Z}\} \cup \{ar \mid r \in R\}$; also in this case, we denote it by aR. For a ring R, let $L(R)$ denote the set of all principal right ideals; which is a poset with respect to inclusion.

A $*$-*ring* is a ring R endowed with an *involution*; that is, an anti-automorphism $x \mapsto x^*$ of order 2, such that

$$(r + s)^* = r^* + s^*, \quad (rs)^* = s^* r^*, \quad (r^*)^* = r \quad \text{for all } r, s \in R,$$

cf. [26] and [43, Chapter 2.13].

An element e of a $*$-ring R is a *projection*, if $e = e^2 = e^*$. A $*$-ring R is *proper* if $r^* r = 0$ implies $r = 0$ for all $r \in R$. Throughout this paper, let Λ be a commutative $*$-ring with unit. A $*$-Λ-*algebra* R is an associative (left) unital Λ-algebra which is a $*$-ring such that

$$(\lambda r)^* = \lambda^* r^* \text{ for all } r \in R,\ \lambda \in \Lambda.$$

By $\mathcal{A}_\Lambda$, we denote the class of all $*$-Λ-algebras. Here, unless stated otherwise, we consider the scalars $\lambda \in \Lambda$ as unary operations $r \mapsto \lambda r$ on R; in other words, we consider $*$-Λ-algebras as 1-sorted algebraic structures. In view of the equality $\lambda^* 1 = (\lambda 1)^*$, the action of Λ does not require particular attention.

Congruence relations on R are in 1-1-correspondence with $*$-*ideals*; that is, ideals I with $I = I^*$, where $I^* = \{r^* \mid r \in I\}$. We call R *strictly subdirectly irreducible* if its ring reduct is subdirectly irreducible, i.e. has a unique minimal non-zero ideal I; in this case, $I = I^*$. Similarly, R is *strictly simple* if 0 and R are the only ideals. In the $*$-ring literature, such $*$-rings are called 'simple', while simple $*$-rings are called '$*$-simple'. We say that an algebra $R \in \mathcal{A}_\Lambda$ is *atomic* if any non-zero right (equivalently, left) ideal contains a minimal one.

A ring R is [*von Neumann*] *regular* if for any $a \in R$, there is an element $x \in R$ such that $axa = a$; such an element is called a *quasi-inverse* of a. A $*$-ring R is $*$-*regular* if it is regular and proper. By $\mathcal{R}_\Lambda$ and $\mathcal{R}^*_\Lambda$, we denote the classes of all regular and of all $*$-regular members of $\mathcal{A}_\Lambda$. Observe that regular [$*$-regular] $*$-rings with unit can be dealt with as members of $\mathcal{R}_\mathbb{Z}$ [$\mathcal{R}^*_\mathbb{Z}$, respectively]. We refer to [41, 3, 4, 36, 14, 45] for more details.

For any subset X of a ring R, we call the set

$$\mathsf{Ann}^l(X) = \{s \in R \mid sx = 0 \text{ for all } x \in X\}$$

the *left annihilator* of X. The *right annihilator* $\mathsf{Ann}^r(X)$ is defined symmetrically. For a vector space V_F over a division ring F, let $\mathsf{End}(V_F)$ denote the set of all endomorphisms of V_F.

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Fact 4.1. (i) For any vector space V_F, $\mathsf{End}(V_F)$ is a regular simple ring.

(ii) A ring R is regular if it admits a regular ideal I such that R/I is regular. Any ideal of a regular ring is regular.

(iii) A ring R is regular [$*$-ring R is $*$-regular] if and only if for any $a \in R$ there is an idempotent [a (unique) projection, respectively] $e \in R$ such that $aR = eR$.

(iv) For any a, b in a regular ring R, there is an idempotent $e \in aR + bR$ such that $ea = a$ and $eb = b$.

(v) For idempotents e, f in a regular ring R, one has $\mathsf{Ann}^l(eR) = \{s \in R \mid se = 0\} = R(1-e)$ and $eR \subseteq fR$ if and only if $\mathsf{Ann}^l(fR) \subseteq \mathsf{Ann}^l(eR)$.

(vi) The classes $\mathcal{R}_\Lambda$ and $\mathcal{R}^*_\Lambda$ are closed under operators H and P.

Proof. Statements (i)-(v) are well known, cf. [4, 1.26], [14, Lemma 1.3], [14, Theorem 1.7]. In (vi), closure under P is obvious, closure under H follows by (iii). $\square$

In particular, in the $*$-regular case, any ideal is a $*$-ideal by Fact 4.1(iii); thus subdirectly irreducibles [simples] are strictly subdirectly irreducible [strictly simple, respectively].

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Fact 4.2. (i) The principal right ideals of a regular ring R form a sublattice $L(R)$ of the lattice of all right ideals of R; $L(R)$ is sectionally complemented and modular.

(ii) For $R \in \mathcal{R}_\Lambda$ [$R \in \mathcal{R}^*_\Lambda$], $L(R)$ is a CMIL [MOL, respectively] endowed with the involution $eR \mapsto (1-e^*)R$, where e is an idempotent [a projection, respectively]; we denote it by $\mathbb{L}(R)$.

(iii) If $R_i \in \mathcal{R}_\Lambda$, $i \in I$, and $R = \prod_{i\in I} R_i$ then $\mathbb{L}(R) \cong \prod_{i\in I} \mathbb{L}(R_i)$.

(iv) If $\varepsilon\colon R \to S$ is a homomorphism and R, S are regular rings, then $\overline{\varepsilon}\colon L(R) \to L(S)$, $\overline{\varepsilon}\colon aR \mapsto \varepsilon(a)S$ is a 0-preserving homomorphism. If ε is injective, then so is $\overline{\varepsilon}$; if ε is surjective, then so is $\overline{\varepsilon}$. If R, $S \in \mathcal{R}_\Lambda$, then $\overline{\varepsilon}\colon \mathbb{L}(R) \to \mathbb{L}(S)$ is a homomorphism.

In Fact 4.2(ii), one can consider the preorder $e \leq f$ iff $fe = e$ on the set of idempotents of R and obtain the lattice $\mathbb{L}(R)$ factoring by the equivalence relation $e \sim f$ iff $e \leq f \leq e$; the involution is given by $e \mapsto 1 - e^*$. For $R \in \mathcal{R}^*_\Lambda$, any of the equivalence classes contains a unique projection so that $\mathbb{L}(R)$ is also called the *projection* [*ortho*]*lattice* of R.

Proof. (i) By Fact 4.1(ii), for any $a \in R$, there is an idempotent $e \in R$ such that $aR = eR$. For any idempotents $e, f \in R$, there is $x \in R$ such that $(f-ef)x(f-ef) = f - ef$. Therefore, $(f - ef)R = g_0R$, $R(f - ef) = Rg_1$, where $g_0 = (f - ef)x$ and $g_1 = x(f - ef)$ are idempotents. According to the proof of [45, §2, Theorem 1], cf. also the proof of [14, Theorem 1.1],

$$eR + fR = (e + g_0)R; \quad eR \cap fR = (f - fg_1)R.$$

Furthermore, for any idempotent e, $f \in R$ such that $e \leq f$, $(f - e)R$ is obviously a complement of eR in $[0, fR]$, whence $L(R)$ is a sectionally complemented modular lattice.

(ii) According to (i), $L(R)$ is a CML. For $R \in \mathcal{R}_\Lambda$, the map $eR \mapsto \mathsf{Ann}^l(eR) = R(1 - e) \mapsto (1 - e^*)R$ combines a dual isomorphism of $L(R)$ onto the lattice of left principal ideals with an isomorphism of the latter onto $L(R)$.

(iii) The idempotents of R are $(e_i \mid i \in I)$, where $e_i \in R_i$ is an idempotent. Thus the map

$$\varphi\colon \prod_{i\in I} \mathbb{L}(R_i) \to \mathbb{L}(R); \quad \varphi\colon (e_iR_i \mid i \in I) \mapsto (e_i \mid i \in I)R,$$

where $e_i \in R_i$ is an idempotent for all $i \in I$, is well-defined, injective and onto. Moreover, φ preserves the involution and the ordering. As φ^{-1} also preserves the ordering, φ is a lattice homomorphism. See also [23, Lemma 30].

(iv) The fact that $\overline{\varepsilon}\colon L(R) \to L(S)$ is a 0-preserving homomorphism follows from the proof of (i). If ε is onto, then $\overline{\varepsilon}$ is also obviously onto. Suppose that ε is injective and e_0, $e_1 \in R$ are idempotents such that $\varepsilon(e_0)S = \varepsilon(e_1)S$. Then $\varepsilon(e_0) = \varepsilon(e_1)\varepsilon(e_0) = \varepsilon(e_1e_0)$, whence $e_0 = e_1e_0$. Similarly, $e_1 = e_0e_1$ and thus $e_0R = e_1R$. Moreover, if R, $S \in \mathcal{R}_\Lambda$, then φ preserves also 1 and the involution. See also [40]. □

5. Classes

Dealing with a class $\mathcal{C}$ of $*$-Λ-algebras or MILs, let $\mathsf{S}_\exists(\mathcal{C})$ [$\mathsf{P}_{s\exists}(\mathcal{C})$] consist of all regular or complemented members of the class $\mathsf{S}(\mathcal{C})$ [of the class $\mathsf{P}_s(\mathcal{C})$, respectively]. Call $\mathcal{C}$ an $\exists$-*semivariety* if it is closed under operators H, $\mathsf{S}_\exists$, P_u and an $\exists$-*variety* if it is also closed under P, cf. [23], also [30] for an analogue within semigroup theory. Let $\mathsf{W}_\exists(\mathcal{C})$ [$\mathsf{V}_\exists(\mathcal{C})$] denote the least $\exists$-semivariety [$\exists$-variety, respectively] which contains the class $\mathcal{C}$.

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Fact 5.1. Let $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ or $\mathcal{C} \subseteq$ CMIL.

(i) $\mathsf{OS}_\exists(\mathcal{C}) \subseteq \mathsf{S}_\exists\mathsf{O}(\mathcal{C})$ for any class operator $\mathsf{O} \in \{\mathsf{P}_u, \mathsf{P}, \mathsf{P}_\omega\}$.
(ii) $\mathsf{S}_\exists\mathsf{H}(\mathcal{C}) \subseteq \mathsf{HS}_\exists(\mathcal{C})$.
(iii) $\mathsf{W}_\exists(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{P}_u(\mathcal{C})$.
(iv) $\mathsf{V}_\exists(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{P}(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{P}_u\mathsf{P}_\omega(\mathcal{C}) = \mathsf{P}_{s\exists}\mathsf{W}_\exists(\mathcal{C})$.
(v) $\mathsf{W}_\exists(\mathcal{C})$ and $\mathsf{V}_\exists(\mathcal{C})$ are axiomatic classes.
(vi) $A \in \mathsf{W}_\exists(\mathcal{C})$ if $B \in \mathsf{W}_\exists(\mathcal{C})$ for all finitely generated $B \in \mathsf{S}_\exists(A)$.

These statements are well known for arbitrary algebraic structures if suffix $\exists$ is omitted. For the proof of Fact 5.1, we refer to Appendix.

6. ε-Hermitean spaces and associated structures

Let Λ be a commutative $*$-ring and let F be a division ring which is a Λ-algebra endowed with an anti-automorphism $x \mapsto x^*$ such that $(\lambda r)^* = \lambda^* r^*$ for all $\lambda \in \Lambda$ and $r \in F$. The class of all such division rings will be denoted by $\mathcal{F}_\Lambda$. Again, the action of Λ is not essential. For better readability, we will denote elements of F by λ, μ, etc.

For $F \in \mathcal{F}_\Lambda$, we consider *sesquilinear spaces* which are [right] vector spaces V_F endowed with a *scalar product* or a *sesquilinear form* $\langle\ \mid\ \rangle\colon V \times V \to F$; that is, for all u, v, $w \in V$ and all λ, $\mu \in F$, one has

$$\langle u \mid v + w\rangle = \langle u \mid v\rangle + \langle u \mid w\rangle, \quad \langle u\lambda \mid v\mu\rangle = \lambda^*\langle u \mid v\rangle\mu,$$

cf. [35, 17, 10]. Since we consider only one scalar product on V_F at a time, we use V_F to denote the space endowed with the scalar product. Unless stated otherwise, such spaces are dealt with as 2-sorted structures with sorts V and F. In particular, this applies to the concepts of isomorphism, ultraproduct, and substructure. In contrast, *subspace* will always mean an F-linear subspace; i.e. here we follow the 1-sorted view on the vector space V_F.

A sesquilinear space $V_F \neq 0$ is *non-degenerate* if $\langle u \mid v\rangle = 0$ for all $v \in V$ implies $u = 0$. For $\varepsilon \in F$, V_F is *ε-hermitean* if $\langle v \mid u\rangle = \varepsilon \cdot \langle u \mid v\rangle^*$ for all u, $v \in V$; V_F is *hermitean* if it is 1-hermitean and $\lambda \mapsto \lambda^*$ is an involution on F; V_F is *skew symmetric* if it is (-1)-hermitean and $\lambda^* = \lambda$ for all $\lambda \in F$; V_F is *anisotropic* if $\langle v \mid v\rangle \neq 0$ for all $v \in V$.

We say that $V_{F'}$ arises from V_F by *scaling* with $0 \neq \mu \in F$ if $F' = F$ as Λ-algebra, is endowed with the involution $r \mapsto \mu r^* \mu^{-1}$, and $V_{F'}$ is considered with the scalar product $(u, v) \mapsto \mu\langle u \mid v\rangle$.

For vectors u, $v \in V$, we say that v is *orthogonal* to u and write $u \perp v$, if $\langle u \mid v\rangle = 0$. The space V_F is *orthosymmetric* or *reflexive* if $\perp$ is a symmetric relation. The *orthogonal* of a subset X is the subspace $X^\perp = \{v \in V \mid x \perp v \text{ for all } x \in X\}$.

For φ, $\psi \in \mathsf{End}(V_F)$, we say that ψ is an *adjoint* of φ if $\langle\varphi(u) \mid v\rangle = \langle u \mid \psi(v)\rangle$ for all u, $v \in V$. If V_F is non-degenerate then any $\varphi \in \mathsf{End}(V_F)$ has at most one adjoint $\psi \in \mathsf{End}(V_F)$; if such exists, we write $\psi = \varphi^*$.

Fact 6.1. The relations of orthogonality and adjointness are left unchanged under scaling; in particular, orthosymmetry is preserved under scaling. The following are equivalent for any space V_F with $\dim V/V^\perp > 1$

(i) V_F is orthosymmetric.
(ii) V_F is ε-hermitean for some (unique) $\varepsilon \in F \setminus \{0\}$.
(iii) Up to scaling, V_F is either hermitean or skew-symmetric.

If V_F is orthosymmetric, then adjointness is a symmetric relation on $\mathsf{End}(V_F)$. If V_F is non-degenerate, then any $\varphi \in \mathsf{End}(V_F)$ has at most one adjoint $\psi \in \mathsf{End}(V_F)$.

Proof. We refer to [17, I §1.3, §1.5]. Observe that any right vector space V_F becomes a left F^{op}-vector space, where $\lambda v = v\lambda$. Also, from a scalar product $\langle\ \mid\ \rangle$ in our sense, one obtains a form Φ, which is linear in the left hand and semilinear in the right hand argument, putting $\Phi(v, w) = \langle w \mid v\rangle^*$. $\square$

A subspace $U \in L(V_F)$ is *closed* if $U = U^{\perp\perp}$. A sesquilinear space V_F which is orthosymmetric and non-degenerate will be called *pre-hermitean*. In the sequel, we consider only pre-hermitean spaces. If V_F is, in addition, anisotropic, we also speak of an *inner product space*. The subspace lattice $L(V_F)$ with the additional unary operation $X \mapsto X^\perp$ is denoted by $\mathbb{L}(V_F)$. The lattice of all closed subspaces of V_F is denoted by $\mathbb{L}_c(V_F)$.

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Fact 6.2. Let V_F be a pre-hermitean space.

(i) $\mathbb{L}(V_F)$ is a Galois CML. In particular, $W = W^{\perp\perp}$ if $\dim W < \omega$.
(ii) $\mathbb{G}(V_F) = (P, \perp)$, where P is the irreducible projective space of one-dimensional linear subspaces of V_F and $vF \perp wF$ iff $v \perp w$, is an orthogeometry.
(iii) The map $U \mapsto \{p \in P \mid p \subseteq U\}$ defines an isomorphism from $\mathbb{L}(V_F)$ onto $\mathbb{L}(\mathbb{G}(V_F))$.
(iv) $\mathbb{L}\big(\mathbb{G}(V_F)\big)_f \cong \mathbb{L}(V_F)_f$ and it is a strictly subdirectly irreducible Arguesian CMIL; if V_F is anisotropic, then $\mathbb{L}(V_F)_f$ is an MOL.
(v) For any strictly subdirectly irreducible Arguesian CMIL L of height at least 3 such that $L = L_f$, there is a (unique up to isomorphism and scaling, that is, *similitude*) pre-hermitean space V_F such that $L \cong \mathbb{L}(V_F)_f$; if L is an MOL, then V_F is anisotropic.
(vi) If $\mathsf{char}\, F \neq 2$, then $\dim V_F < \omega$ if and only if $\mathbb{L}(V_F)$ is an MIL.

Proof. Statement (i) follows from (ii)-(iii) and the fact that $\mathbb{L}\big(\mathbb{G}(V_F)\big)$ is a Galois CML. Statement (ii) is the content of [10, Proposition 14.1.6]. Statement (iii) follows from Fact 3.2(i) and [10, Proposition 14.1.6].

To prove (iv), we notice first that $\mathbb{L}\big(\mathbb{G}(V_F)\big)_f$ is a CMIL by Fact 3.5. Moreover, $\mathbb{L}(V_F)_f \in \mathsf{S}\big(\mathbb{L}(V_F)\big)$ is an Arguesian lattice. Strict subdirect irreducibility of $\mathbb{L}\big(\mathbb{G}(V_F)\big)_f$ follows from (i), [10, Example 2.7.2], and [18, Corollary 1.5]. Furthermore, if V_F is anisotropic, then $X^\perp$ is an orthocomplement of X for any $X \in \mathbb{L}(V_F)$ with $\dim X < \omega$.

We prove now (v). By [18, Corollary 1.5], there is an irreducible orthogeometry $(P, \perp)$ such that $L \cong \mathbb{L}(P, \perp)_f$. By Fact 3.2(ii), there is a vector space U_K such that $L(P) \cong L(U_K)$. By [10, Theorem 14.1.8], there is a sesquilinear form Φ on U_K such that U_K is pre-hermitean and $L \cong \mathbb{L}(P, \perp)_f \cong \mathbb{L}(U_K)_f$. For uniqueness see [17, p. 33]. If L is an MOL, then V_F is obviously anisotropic.

Finally, if $\dim V_F < \omega$, then $\mathbb{L}(V_F) = \mathbb{L}(V_F)_f$ is an MIL by (iv). Conversely, if $\mathbb{L}(V_F)$ is an MIL, then the lattice $\mathbb{L}_c(V_F)$ is a sublattice of $\mathbb{L}(V_F)$, whence is modular. Thus $\dim V_F < \omega$ by [33, Theorem], and statement (vi) follows. □

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Fact 6.3. Any subalgebra L of $\mathbb{L}(V_F)$ which is a MIL extends to a subalgebra $\hat{L}$ of $\mathbb{L}(V_F)$ which is a MIL and such that $\hat{L}_f = \mathbb{L}(V_F)_f$. In particular, $\hat{L}$ is a strictly subdirectly irreducible atomic MIL. Moreover, if L is a CMIL then $\hat{L}$ is a CMIL.

Proof. Existence of $\hat{L}$ with required properties follows from the proof of [18, Theorem 2.1]. In particular, $\hat{L}$ is atomic. Strict subdirect irreducibility of $\hat{L}$ follows from [18, Corollary 1.5], see also Fact 6.2(iv). □

On any subspace U_F of V_F, we have the induced scalar product. When U_F is non-degenerate, U_F is pre-hermitean, too. A finite dimensional subspace U of V_F is non-degenerate if and only if $U \cap U^\perp = 0$, if and only if $V = U \oplus U^\perp$ (as $\dim V/U^\perp = \dim U$). We write in this case $U \in \mathbb{O}(V_F)$ and say that U is a *finite dimensional orthogonal summand.*

du

Fact 6.4. A pre-hermitean space V_F [the lattice $\mathbb{L}(V_F)_f$] is directed union of the subspaces $U \in \mathbb{O}(V_F)$ [of subalgebras $[0, U] \cup [U^\perp, V]$, $U \in \mathbb{O}(V_F)$, respectively].

Moreover, $[0,U]\cup[U^\perp,V]\cong\mathbb{L}(U_F)\times\mathbf{2}$ for any $U\in\mathbb{O}(V_F)$ such that $U\neq V$. In particular, $\mathbb{L}(U_F)\in\mathsf{HS}_\exists\big(\mathbb{L}(V_F)_f\big)$ for any $U\in\mathbb{O}(V_F)$.

Proof. The first claim concerning spaces has been proved in [18, Theorem 1.2] for $\mathbb{G}(V_F)$, whence the claim about spaces follows by Fact 6.2. The claim for lattices follows by Facts 2.3 and 6.2(iii). □

For a subspace U of V_F, the subspace $\mathsf{rad}\,U=U\cap U^\perp$ is the *radical* of U. The F-vector space $U_F/\mathsf{rad}\,U$ is endowed with the scalar product $\langle v+\mathsf{rad}\,U\mid w+\mathsf{rad}\,U\rangle=\langle v\mid w\rangle$ with respect to the given anti-automorphism of F. We call $U_F/\mathsf{rad}\,U$ a *subquotient space* and denote it also by $U/\mathsf{rad}\,U$.

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Fact 6.5. Let V_F be a pre-hermitean space and let U_F be a subspace of V_F. Then $U_F/\mathsf{rad}\,U$ is non-degenerate; it is ε-hermitean if V_F is. The space $U_F/\mathsf{rad}\,U$ is isomorphic to any subspace W_F of V_F such that $U=W\oplus\mathsf{rad}\,U$.

Proof. The map $w\mapsto w+\mathsf{rad}\,U$ establishes an isomorphism (of sesquilinear spaces) from W onto $U/\mathsf{rad}\,U$. □

Recall that for a pre-hermitean space V_F, $\mathsf{End}(V_F)$ denotes the endomorphism Λ-algebra of V_F. Observe that $\ker\varphi=(\mathsf{im}\,\varphi^*)^\perp$ if φ^* exists. The endomorphisms of V_F having an adjoint form a Λ-subalgebra of $\mathsf{End}(V_F)$, denoted by $\mathsf{End}^*(V_F)$, which is closed under adjoints and forms a $*$-ring under this involution; thus $\mathsf{End}^*(V_F)\in\mathcal{A}_\Lambda$. We also observe that for $v\in V$, $\lambda\in\Lambda$, and $\varphi\in\mathsf{End}^*(V_F)$, one has

$$(\lambda\varphi)(v)=\varphi(v)\lambda,\quad (\lambda\varphi)^*=\lambda^*\varphi^*.$$

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Fact 6.6. For $U\in\mathbb{L}(V_F)$, one has $V=U\oplus U^\perp$ if and only if there is a projection $\pi_U\in\mathsf{End}^*(V_F)$ such that $U=\mathsf{im}\,\pi_U$. Such a projection π_U is unique.

Projection π_U in terms of Fact 6.6 is called the *orthogonal projection* onto U. Par abus de langage, π_U also denotes the induced epimorphism $V\to U$, while ε_U denotes the identical embedding $U\to V$. Observe that π_U and ε_U are adjoints of each other in the sense that

$$\langle\varepsilon_U(u)\mid v\rangle=\langle u\mid\pi_U(v)\rangle\quad\text{for all } u\in U,\ v\in V.$$

Moreover, the computational rules of $\mathsf{End}^*(V_F)$ yield, in particular, $(\varepsilon_U\varphi\pi_U)^*=\varepsilon_U\varphi^*\pi_U$ for any $\varphi\in\mathsf{End}^*(U_F)$. Finally, $\pi_U\varepsilon_U=\mathsf{id}_U$, while $\pi_U\varepsilon_U\pi_U=\pi_U$ and $U^\perp=\ker(\varepsilon_U\pi_U)$.

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Fact 6.7. Let V_F a pre-hermitean space and let $\dim V_F=n<\omega$.

(i) There is a *dual* pair of bases $\{v_1,\dots,v_n\}$ and $\{w_1,\dots,w_n\}$ of V_F; that is, $\langle v_i\mid w_i\rangle=1$ for all $i\in\{1,\dots,n\}$ and $\langle v_i\mid w_j\rangle=0$ for all $i\neq j$. Given such a dual pair of bases and $\varphi\in\mathsf{End}(V_F)$ with $\varphi(v_j)=\sum_i w_ia_{ij}$, $\varphi^*\in\mathsf{End}(V_F)$ exists and $\varphi^*(v_i)=\sum_j w_ja_{ij}^*$. In particular, $\mathsf{End}^*(V_F)$ contains all endomorphisms of V_F and $\mathsf{End}^*(V_F)\in\mathcal{R}_\Lambda$.

(ii) $\mathsf{End}^*(V_F)\in\mathcal{R}_\Lambda^*$ if V_F is anisotropic.

(iii) If $U_F\in\mathbb{O}(V_F)$ then $\mathsf{End}^*(U_F)\in\mathsf{HS}_\exists\big(\mathsf{End}^*(V_F)\big)$.

Proof. For existence of dual bases, see [31, §II.6]. Straightforward and well known calculations prove (i) and (ii); in particular, regularity of $\mathsf{End}^*(V_F)$ follows from Fact 4.1(i). In (iii), let R consist of all $\varphi\in\mathsf{End}^*(V_F)$ which leave both U and $U^\perp$ invariant. As $R\cong\mathsf{End}^*(U_F)\times\mathsf{End}^*(U_F^\perp)$, we get (iii). □

We put $J(V_F) = \{\varphi \in \mathsf{End}^*(V_F) \mid \dim \mathsf{im}\, \varphi < \omega\}$.

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Fact 6.8. Let V_F be a pre-hermitian space.

(i) $J(V_F)$ is an ideal and a strictly simple regular subalgebra of $\mathsf{End}^*(V_F)$ (without unit).

(ii) The principal right ideals of $J(V_F)$ form an atomic sectionally complemented sublattice of the lattice of all right ideals of $J(V_F)$, which is isomorphic to the lattice of finite dimensional subspaces of V_F via the map $\varphi J(V_F) \mapsto \mathsf{im}\, \varphi$.

(iii) For any $\varphi_1, \ldots, \varphi_n \in J(V_F)$, there is $U \in \mathbb{O}(V_F)$ such that $\pi_U \varphi_i = \varphi_i = \varphi_i \pi_U$ for all $i \in \{1, \ldots, n\}$.

Proof. (i) If $\dim V_F < \omega$ then $J(V_F) = \mathsf{End}(V_F)$ by Fact 6.7(i). Let $\dim V_F \geqslant \omega$. Clearly, $J(V_F)$ is an ideal and a Λ-subalgebra of $\mathsf{End}^*(V_F)$ (without unit). Observe that $\pi_U \in J(V_F)$ for any $U \in \mathbb{O}(V_F)$ by Fact 6.6. Moreover by Fact 6.4, for any W with $\dim W < \omega$, there exists $U \in \mathbb{O}(V_F)$ such that $W \subseteq U$.

Consider $\varphi \in J(V_F)$ and recall that subspaces $\mathsf{ker}\, \varphi = (\mathsf{im}\, \varphi^*)^\perp$ and $\mathsf{ker}\, \varphi^* = (\mathsf{im}\, \varphi)^\perp$ are closed. To prove that $\varphi^* \in J(V_F)$, choose $W \in \mathbb{O}(V_F)$ such that $W \supseteq \mathsf{im}\, \varphi = (\mathsf{ker}\, \varphi^*)^\perp$. Then $W^\perp \subseteq \mathsf{ker}\, \varphi^*$, whence $\mathsf{im}\, \varphi^* = \varphi^*(W)$ is finite-dimensional. It follows that

($*$) For any $\varphi_1, \ldots, \varphi_n \in J(V_F)$, there is $U \in \mathbb{O}(V_F)$ such that $U \supseteq \mathsf{im}\, \varphi_i + \mathsf{im}\, \varphi_i^*$ for all $i \in \{1, \ldots, n\}$ and $\varphi_i(U) = \mathsf{im}\, \varphi_i$ and $\varphi_i^*(U) = \mathsf{im}\, \varphi_1^*$. In particular,
  (a) U is a finite-dimensional pre-hermitean space;
  (b) $V = U \oplus U^\perp$;
  (c) $U^\perp \subseteq \bigcap_i \mathsf{ker}\, \varphi_i \cap \mathsf{ker}\, \varphi_i^*$;
  (d) $\pi_U \in J(V_F)$;
  (e) $\varepsilon_U \psi \pi_U \in J(V_F)$ and $(\varepsilon_U \psi \pi_U)^* = \varepsilon_U \psi^* \pi_U$ for any $\psi \in \mathsf{End}(U_F)$.

To prove that φ has a quasi-inverse in $J(V_F)$, choose for φ a subspace $U \in \mathbb{O}(V_F)$ according to ($*$). By Fact 6.7(i), $\pi_U \varphi \varepsilon_U \in \mathsf{End}^*(U_F)$ has a quasi-inverse $\psi \in \mathsf{End}^*(U_F)$. We claim that $\chi = \varepsilon_U \psi \pi_U$ is a quasi-inverse of φ in $J(V_F)$. Indeed, $\chi \in J(V_F)$ by (e) and $\varphi(v) = 0 = \chi(v)$ for any $v \in U^\perp$ by (c) and $\varphi \chi \varphi(v) = \pi_U \varphi \varepsilon_U \psi \pi_U \varphi \varepsilon_U(v) = \pi_U \varphi \varepsilon_U(v) = \varphi(v)$ for any $v \in U$.

To prove that $J(V_F)$ is strictly simple, it suffices to show that for any $0 \neq \varphi$, $\psi \in J(V_F)$, ψ belongs to the ideal generated by φ. Again, choose for φ and ψ a subspace $U \in \mathbb{O}(V_F)$ according to ($*$). Applying Fact 4.1(i) to $\pi_U \varphi \varepsilon_U$, $\pi_U \psi \varepsilon_U \in \mathsf{End}(U_F)$, we get that there are $m < \omega$ and $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_m \in \mathsf{End}(U_F)$ such that $\pi_U \psi \varepsilon_U = \sum_{i=1}^m \tau_i \pi_U \varphi \varepsilon_U \sigma_i$. Then according to ($*$), $\psi = \sum_{i=1}^m \varepsilon_U \tau_i \pi_U \varphi \varepsilon_U \sigma_i \pi_U$ and $\varepsilon_U \sigma_i \pi_U$, $\varepsilon_U \tau_i \pi_U \in J(V_F)$ for all $i \in \{1, \ldots, m\}$ by Fact 6.7(i).

(ii) We prove first that $\varphi_0 J(V_F) \subseteq \varphi_1 J(V_F)$ is equivalent to $\mathsf{im}\, \varphi_0 \subseteq \mathsf{im}\, \varphi_1$ for any $\varphi_0, \varphi_1 \in \mathsf{End}^*(V_F)$. Suppose first that $\mathsf{im}\, \varphi_0 \subseteq \mathsf{im}\, \varphi_1$ and take an arbitrary $\psi \in J(V_F)$; then $\varphi_0 \psi$, $\varphi_1 \psi \in J(V_F)$. Choose for $\varphi_0 \psi$ and $\varphi_1 \psi$ a subspace $U \in \mathbb{O}(V_F)$ according to ($*$). Then $\xi_i = \pi_U \varphi_i \psi \varepsilon_U \in \mathsf{End}(U_F)$ for any $i < 2$ and $\mathsf{im}\, \xi_0 \subseteq \mathsf{im}\, \xi_1$. As $\dim U_F < \omega$, $\xi_0 = \xi_1 \chi$ for some $\chi \in \mathsf{End}(U_F)$. According to (c), $\varphi_0 \psi(v) = \varphi_1 \psi(v) = 0$ for any $v \in U^\perp$, whence

$$\varphi_0 \psi = \pi_U \varphi_0 \psi \varepsilon_U \pi_U = \xi_0 \pi_U = \xi_1 \chi \pi_U = \pi_U \varphi_1 \psi \varepsilon_U \chi \pi_U = \varphi_1 \psi \varepsilon_U \chi \pi_U \in \varphi_1 J(V_F),$$

as $\psi \varepsilon_U \chi \pi_U \in J(V_F)$. The reverse implication is trivial by Fact 6.4.

Besides that, for any finite-dimensional $W \in L(V_F)$, there is $\varphi \in J(V_F)$ such that $W = \mathsf{im}\, \varphi$. Indeed by Fact 6.4, there is $U \in \mathbb{O}(V_F)$ such that $W \subseteq U$, whence $W = \mathsf{im}\, \psi$ for some $\psi \in \mathsf{End}(U_F)$. Then $W = \mathsf{im}\, \chi$ with $\chi = \varepsilon_U \psi \pi_U \in J(V_F)$ by Fact 6.6. This establishes the claimed lattice isomorphism.

(iii) Finally, given $\varphi_1, \ldots, \varphi_n \in J(V_F)$, choose a subspace $U \in \mathbb{O}(V_F)$ according to $(*)$. Then $\mathsf{im}\,\varphi_i + \mathsf{im}\,\varphi_i^* \subseteq U$, whence $\pi_U\varphi_i = \varphi_i$ and $\pi_U\varphi_i^* = \varphi_i^*$. □

atex2

Fact 6.9. Any subalgebra R of $\mathsf{End}^*(V_F)$ extends to a subalgebra $\hat{R}$ of $\mathsf{End}^*(V_F)$ such that $J(V_F)$ is a unique minimal ideal of $\hat{R}$. In particular, $\hat{R}$ is strictly subdirectly irreducible and atomic with the left [right] minimal ideals being those of $J(V_F)$. Moreover, if R is regular then $\hat{R}$ is also regular.

Proof. We refer to [40]. Let $\hat{R} = R + J(V_F)$. Clearly, $\hat{R}$ is a subalgebra of $\mathsf{End}^*(V_F)$ and $J(V_F)$ is an ideal of $\hat{R}$ by Fact 6.8(i). If $I \neq 0$ is a left ideal of $\hat{R}$ then choose $\varphi \in I$ such that $I \neq 0$. Then by Fact 6.4, $0 \neq \pi_U\varphi \in J(V_F) \cap I$ for some $U \in \mathbb{O}(V_F)$. By Fact 6.8(ii), there is a minimal left ideal $M \subseteq J(V_F)\pi_U\varphi \subseteq I$ of $J(V_F)$. Then M is also a minimal left ideal of $\hat{R}$. If I is an ideal of $\hat{R}$, then arguing as above and applying simplicity of $J(V_F)$, which follows from Fact 6.8(i), we get that $J(V_F) \subseteq I$.

Finally, Facts 4.1(ii) and 6.8(ii) imply regularity of $\hat{R}$ when R is regular. □

In particular, Fact 6.9 applies to $R = \{\lambda \mathsf{id}_V \mid \lambda \in F\}$; in this case, we denote the corresponding subalgebra $\hat{R}$ by $\mathsf{End}^*_{\mathrm{f}}(V_F)$.

7. Representations

A *representation* of an MIL (or CMIL) L in V_F is a homomorphism $\varepsilon\colon L \to \mathbb{L}(V_F)$. It is *faithful* if it is injective, i.e. an embedding; in this case, we usually identify L with its image in $\mathbb{L}(V_F)$. A map $\varepsilon\colon L \to \mathbb{L}(V_F)$ is a representation if it preserves joins, involution, and the least element.

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Lemma 7.1. *Let ε be a representation of an MIL L in a pre-hermitean space V_F.*

(i) *Any element in the image of ε is closed.*
(ii) *If ε is faithful and V_F is anisotropic, then L is an MOL.*

Proof. Let $x \in L$ be arbitrary.

(i) We have $\varepsilon(x) = \varepsilon(x'') = \varepsilon(x')^\perp = \varepsilon(x)^{\perp\perp}$.

(ii) If V_F is anisotropic, then we have $\varepsilon(xx') = \varepsilon(x) \cap \varepsilon(x)^\perp = 0$. As ε is faithful, we conclude that $xx' = 0$. Hence $'$ is an orthocomplement. □

The following is as obvious as crucial. A representation of an MIL $\varepsilon\colon L \to \mathbb{L}(V_F)$ can be viewed as a 3-sorted structure with sorts L, V, and F and with the map ε being captured by the binary relation (cf. [38, 37, 44] for this method)

$$\{(a, v) \mid v \in \varepsilon(a)\} \subseteq L \times V,$$

which we denote by ε again.

axiom2

Fact 7.2. There is a recursive first order axiomatization of the class of all 3-sorted structures associated with [faithful] representations of MILs in pre-hermitean spaces.

A *representation* of an MIL L within an orthogeometry $(P, \perp)$ is a homomorphism $\eta\colon L \to \mathbb{L}(P, \perp)$. The following obvious fact relates the two concepts of a representation.

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Fact 7.3. For an MIl L, ε is a [faithful] representation in V_F if and only if the mapping $\eta\colon a \mapsto \{p \in P \mid p \subseteq \varepsilon(a)\}$ is a [faithful] representation of L in the orthogeometry $(P, \perp) = \mathbb{G}(V_F)$.

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Theorem 7.4. *Let L be an Arguesian strictly subdirectly irreducible CMIL [MOL] such that $\dim L > 2$ and L has an atom. Then L admits a faithful representation ε within some [anisotropic] pre-hermitean space V_F such that ε induces a bijection between the sets of atoms of L and of $\mathbb{L}(V_F)$. In particular, ε restricts to an isomorphism from L_f onto $\mathbb{L}(V_F)_f$. The space V_F is unique up to isomorphism and scaling.*

Proof. By Fact 2.4 L is atomic, whence L_f is strictly subdirectly irreducible and atomic. Moreover by Fact 6.2(v), $L_f \cong \mathbb{L}(V_F)_f$ for some pre-hermitean space V_F which is unique up to isomorphism and scaling. By definition and Fact 3.5, $\mathbb{G}(L) = \mathbb{G}(L_f) = \mathbb{G}(V_F)$. By [18, Lemma 10.4], L has a faithful representation within the orthogeometry $\mathbb{G}(L)$, whence in the orthogeometry $\mathbb{G}(V_F)$. The desired conclusion follows from Fact 7.3. □

The following fact is a corollary of Theorem 7.4 which is in principle already in [6].

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Fact 7.5. Up to isomorphism, the strictly simple Arguesian CMILs L of finite height $n > 2$ are the lattices $\mathbb{L}(V_F)$, where V_F is a pre-hermitean space with $\dim V_F = n$. The space V_F is determined by L up to isomorphism and scaling; V_F is anisotropic, if L is an MOL.

A *representation* of $R \in \mathcal{A}_\Lambda$ within a pre-hermitean space V_F is a $\mathcal{A}_\Lambda$-homomorphism $\varepsilon\colon R \to \mathsf{End}^*(V_F)$. It is convenient to consider representations as unitary R-F-bimodules. More precisely, one has an action $(r, v) \mapsto rv = \varepsilon(r)(v)$ of R on the left and an action $(v, \lambda) \mapsto v\lambda$ of F on the right satisfying the laws of unitary left and right modules and such that

$$(\lambda r)v = (rv)\lambda = r(v\lambda) \quad \text{for all } v \in V,\ r \in R,\ \lambda \in \Lambda,$$

where $v\lambda = v(\lambda 1_F)$. Moreover,

$$\langle rx \mid y\rangle = \langle x \mid r^*y\rangle \quad \text{for all } r \in R,\ x, y \in V$$
$$(\lambda r)^*v = (\lambda^* r^*)v = (r^*v)\lambda^* \quad \text{for all } v \in V,\ r \in R,\ \lambda \in \Lambda.$$

We denote a representation of $R \in \mathcal{A}_\Lambda$ in V_F by ${}_RV_F$. The R-F-bimodule ${}_RV_F$ will be considered as a 3-sorted structure with sorts V, R, and F; R, $F \in \mathcal{A}_\Lambda$ are considered as 1-sorted structures, where $\lambda \in \Lambda$ serves to denote the unary operation $x \mapsto \lambda x$. Our main concern will be *faithful* representations; that is, representations ${}_RV_F$ such that $rv = 0$ for all $v \in V$ if only if $r = 0$. Observe that a regular algebra R is ∗-regular, if it admits a faithful representation in an anisotropic space.

axiom

Fact 7.6. Given a recursive commutative ∗-ring Λ with unit, there is a recursive first order axiomatization of the class of all 3-sorted structures ${}_RV_F$ where R, $F \in \mathcal{A}_\Lambda$, V_F is a hermitean space, and $\varepsilon(r)(v) = w$ iff $rv = w$ defines a faithful representation of R in V_F.

Lrep

Proposition 7.7. (i) *If ε is a faithful representation of $R \in \mathcal{R}_\Lambda$ in a pre-hermitean space V_F, then the map $\eta\colon aR \mapsto \mathsf{im}\,\varepsilon(a)$ defines a faithful representation of $\mathbb{L}(R)$ in V_F.*
(ii) *If $\dim V_F < \omega$ then $\mathbb{L}(V_F) \cong \mathbb{L}(\mathsf{End}^*(V_F))$.*

Proof. (i) We refer to [12]. We may assume that $R \subseteq \mathsf{End}^*(V_F)$; that is, $\varepsilon = \mathsf{id}$. By Facts 4.1(i) and 4.2(iv), η is a $(0,1)$-lattice embedding of $L(R)$ into $L(V_F)$. Moreover, for any $v \in V$ and an idempotent $\varphi \in R$, one has $v \in \big(\eta(\varphi R)\big)^\perp = (\mathsf{im}\,\varphi)^\perp$ iff

$\langle\varphi^*(v) \mid w\rangle = \langle v \mid \varphi(w)\rangle = 0$ for all $w \in V$, iff $\varphi^*(v) = 0$, iff $v = (\mathsf{id} - \varphi^*)(v)$, iff $v \in \mathsf{im}(\mathsf{id} - \varphi^*) = \eta\big((\varphi R)'\big)$, whence η preserves the involution.

(ii) By (i) and Fact 6.7(ii), the identical map ε on $\mathsf{End}^*(V_F)$ defines a faithful representation of $\mathbb{L}(V_F)$. It is surjective since any subspace is the image of some endomorphism $\varphi \in \mathsf{End}^*(V_F)$. □

atrep2

Theorem 7.8. *Let $R \in \mathcal{A}_\Lambda$ be a primitive ring having a minimal right ideal.*

(i) *The algebra R admits a faithful representation ε within some pre-hermitean space V_F such that $\varepsilon^{-1}(J(V_F)) \subseteq R$.*
(ii) *The space V_F can be chosen hermitean if and only if there is a projection $e \in R$ such that Re is a faithful simple R-module (otherwise, V_F is skew symmetric).*
(iii) *R is as in (ii) if and only if R is atomic and strictly subdirectly irreducible; in this case, the minimal ideal is $\varepsilon^{-1}(J(V_F))$.*
(iv) *The space V_F is unique up to isomorphism and scaling.*

Proof. Statements (i)-(iii) are due to Kaplansky, cf. [26, Theorem 1.2.2] and [40, 42]. We prove (iv). By (i), Fact 6.8(ii), and Proposition 7.7(i), $\mathbb{L}(R)_f$ has a representation in V_F, which is an isomorphism onto $\mathbb{L}(V_F)_f$. Uniqueness of V_F follows from Theorem 7.4. □

P:iso

Fact 7.9. Up to isomorphism, the strictly simple artinian members R of $\mathcal{R}_\Lambda$ are exactly the endomorphism algebras $\mathsf{End}^*(V_F)$, where V_F is a pre-hermitean space and $\dim V_F < \omega$. Moreover, V_F is uniquely determined by R up to isomorphism and scaling; V_F is anisotropic if $R \in \mathcal{R}^*_\Lambda$.

Proof. Let $R \in \mathcal{R}_\Lambda$ be strictly simple and artinian. By Theorem 7.8(i), R has a faithful representation ε in a pre-hermitean space V_F. According to Proposition 7.7(i), $\mathbb{L}(R) = \mathbb{L}(R)_f$ has a faithful representation in V_F. As $\mathbb{L}(R)$ is a strictly simple lattice of finite height, V_F is finite-dimensional by Fact 7.5 and Theorem 7.4, whence ε is an isomorphism by Theorem 7.8(i), as $J(V_F) = \mathsf{End}^*(V_F)$. We also refer to Jacobson [28, Chapter IV, §12]. □

ultra

8. Preservation theorems

Lemma 8.1. *Let $\mathcal{U}$ be an ultrafilter over a set I. Let also V_{iF_i} be a pre-hermitean space over $F_i \in \mathcal{A}_\Lambda$ for all $i \in I$. Then $F = \prod_{i\in I} F_i/\mathcal{U} \in \mathcal{A}_\Lambda$ and $V = \prod_{i\in I} V_i/\mathcal{U}$ is a pre-hermitean space over F.*

(i) *If L_i is an MIL and $(L_i, V_i, F_i; \varepsilon_i)$ is a faithful representation for all $i \in I$, then the associated ultraproduct $(L, V_F, F; \varepsilon)$ is a faithful representation of $L = \prod_{i\in I} L_i/\mathcal{U}$.*
(ii) *If $R_i \in \mathcal{A}_\Lambda$ and ${}_{R_i}V_{iF_i}$ is a faithful representation for all $i \in I$, then the associated ultraproduct ${}_RV_F$ is a faithful representation of $R = \prod_{i\in I} R_i/\mathcal{U}$.*
(iii) *Let U be an n-dimensional subspace of V_F, $n < \omega$. Then there are $J \in \mathcal{U}$ and n-dimensional subspaces U_i of V_{iF_i}, $i \in J$, such that $U \cong \prod_{i\in J} U_i/\mathcal{U}_J$, where $\mathcal{U}_J = \{X \in \mathcal{U} \mid X \subseteq J\}$, and*

$$\mathbb{L}(U_F) \cong \prod_{i\in J} \mathbb{L}(U_{iF_i})/\mathcal{U}_J, \quad \mathsf{End}^*(U_F) \cong \prod_{i\in J} \mathsf{End}^*(U_{iF_i})/\mathcal{U}_J$$

Proof. Statements (i) and (ii) are immediate by Facts 7.2 and 7.6. In (iii) observe that for a fixed positive integer n, there is a set of first order formulas expressing that a set of vectors $\{v_1, \dots, v_n\}$ is independent [is a basis], as well as a set of first order formulas expressing that a vector v in the span of $\{v_1, \dots, v_n\}$. Thus, by the Łoś Theorem, a basis of U determines J and bases of spaces U_i, $i \in J$. Now, apply (i) to lattices $L_i = \mathbb{L}(U_{iF_i})$, $i \in J$, to get an embedding of $\prod_{i \in J} L_i/\mathcal{U}_J$ into $\mathbb{L}(U_F)$. Surjectivity of this embedding is granted by the sentence stating that for any v_1, …, v_n, there is a such that $v \in \varepsilon(a)$ if and only if v in the span of v_1, …, v_n. Similarly, we apply (ii) in the ring case and use the sentence stating that for any basis v_1, …, v_n and any w_1, …, w_n, there is r such that $rv_i = w_i$ for all $i \in \{1, \dots, n\}$. ☐

Inheritance of existence of representations under homomorphic images has been dealt with, in different contexts, in [20, 18] for CMILs and by Micol in [40] for $*$-rings. Apparently, it needs saturation properties of ultrapowers. Considering a fixed structure A, add a new constant symbol $\underline{a}$, called a *parameter*, for each $a \in A$. In what follows, $\Sigma(x_1, \dots, x_n)$ is a set of formulas with free variables x_1, …, x_n in this extended language. Given an embedding $h\colon A \to B$, we call B *modestly saturated* [*ω-saturated*] *over A via h*, if any set of formulas $\Sigma(x_1, \dots, x_n)$, with parameters from A [and finitely many parameters from B, respectively], which is finitely realized in A [in B, respectively] is realized in B (where $\underline{a}$ is interpreted as $\underline{a}^B = h(a)$). The following is a particular case of [8, Corollary 4.3.14].

sat

Fact 8.2. Every structure A admits an elementary embedding h into some structure B which is [ω-]saturated over A via h. One can choose B to be an ultrapower of A and h to be the canonical embedding. Identifying a with $h(a)$, one may assume B to be an elementary extension of A.

homlat

Theorem 8.3. *Let a CMIL L [a $*$-Λ-algebra R] have a faithful representation within a pre-hermitean space V_F. There is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L [such that for any regular ideal $I = I^*$, the algebra R/I] admits a faithful representation within $(U/\operatorname{rad} U)_{\hat{F}}$, where $U = U^{\perp\perp}$ is a subspace of $\hat{V}_{\hat{F}}$.*

Proof. For $R \in \mathcal{A}_\Lambda$, we use the same idea as in the proof of [25, Proposition 25]. Though here, the scalar product induced on U, as defined below, might be degenerated. According to Fact 8.2, there is an ultrapower ${}_{\hat{R}}\hat{V}_{\hat{F}}$ of the faithful representation ${}_RV_F$ which is modestly saturated over ${}_RV_F$ via the canonical embedding. Then $\hat{V}$ is an R-module via the canonical embedding of R into $\hat{R}$ and

$$U = \{v \in \hat{V} \mid av = 0 \text{ for all } a \in I\} = \bigcap_{a \in I} (a^* \hat{V})^\perp$$

is a closed subspace of $\hat{V}_{\hat{F}}$ and a left (R/I)-module. Moreover as $I = I^*$, one has

$$\langle (r + I)v \mid w \rangle = \langle v \mid (r^* + I)w \rangle \text{ for all } v, w \in U,$$

We observe that $U^\perp$ is also an (R/I)-module. Indeed, if $v \in U^\perp$ then $\langle (r+I)v \mid u \rangle = \langle v \mid (r^* + I)u \rangle = 0$ for all $u \in U$. Thus with $W = \operatorname{rad} U$, one obtains an (R/I)-$\hat{F}$-bimodule U/W, where $(r + I)(v + W) = rv + W$ for all $r \in R$ and all $v \in U$, which is also a subquotient of V_F.

We show that ${}_{R/I}(U/W)_{\hat{F}}$ is a faithful representation of R/I; that is, for any $a \in R \backslash I$, there has to be $u \in U$ such that $au \notin W$. It suffices to show that for any $a \in R \backslash I$, there are u, $v \in U$ such that $\langle au \mid v \rangle \neq 0$. Since $u \in U$ means $bu = 0$ for all $b \in I$, we have to show that the set

$$\Sigma(x, y) = \{\langle \underline{a}x \mid y \rangle \neq 0\} \cup \{\underline{b}x = 0 = \underline{b}y \mid b \in I\}$$

of formulas with parameters from $\{a\} \cup I$ and variables x, y of type V is satisfiable in ${}_{\hat{R}}\hat{V}_{\hat{F}}$. Due to saturation, it suffices to show that for any $b_1, \ldots, b_n \in I$, there are u, $v \in V$ such that $\langle au \mid v\rangle \neq 0$ and $b_i u = b_i v = 0$ for all $i \in \{1, \ldots, n\}$. In view of Fact 4.1(iv) and regularity of I, there is an idempotent $e \in I$ such that $b_i e = b_i$ for all $i \in \{1, \ldots, n\}$; in particular $b_i u = b_i v = 0$ whenever $eu = ev = 0$. Thus it suffices to show that there are u, $v \in V$ such that $eu = ev = 0$ but $\langle au \mid v\rangle \neq 0$.

Assume the contrary; namely, let $eu = ev = 0$ imply $\langle au \mid v\rangle = 0$ for all u, $v \in V$. For arbitrary u', $v' \in V$, let $u = (1-e)u'$ and $v = (1-e)v'$. As $eu = ev = 0$, we get by our assumption that $\langle au \mid v\rangle = \langle (1-e^*)au \mid v'\rangle = 0$. This holds for all $v' \in V$, whence $(1-e^*)au = 0$ since V_F is non-degenerated. Thus $(1-e^*)a(1-e)u' = 0$ for all $u' \in V$, whence $(1-e^*)a(1-e) = 0$, as ${}_R V_F$ is a faithful representation. But then $a = e^*a + ae - e^*ae \in I$, a contradiction.

In the case of CMILs, given a representation $\varepsilon\colon L \to \mathbb{L}(V_F)$, let $G = \mathbb{G}(V_F)$, cf. Fact 6.2, and let $\pi(v) = vF$ for $v \in V$. We consider the 4-sorted structure $(L, V, F, G; \varepsilon, \pi)$. According to Fact 8.2, there is an ultrapower $(\hat{L}, \hat{V}, \hat{F}, \hat{G}; \hat{\varepsilon}, \hat{\pi})$ of $(L, V, F, G; \varepsilon, \pi)$ which is modestly saturated over $(L, V, F, G; \varepsilon)$ via the canonical embedding. By Lemma 8.1(i), $(\hat{L}, \hat{V}, \hat{F}; \hat{\varepsilon})$ is a faithful representation. In view of Fact 6.2(ii), $\hat{G} \cong \mathbb{G}(\hat{V}_{\hat{F}})$ via $\hat{\pi}$; and $\hat{\rho}\colon W \mapsto \{v \in \hat{V} \mid \hat{\pi}(v) \in W\}$ defines an isomorphism from $\mathbb{L}(\hat{G})$ onto $\mathbb{L}(\hat{V}_{\hat{F}})$ by Fact 6.2(iii).

Now, let θ be a congruence of L. According to the proof of [18, Theorem 13.1], there is a faithful representation $\eta\colon L/\theta \to \mathbb{L}(W/W')$ in a subquotient W/W' of G, where the subspace W is closed and $W' = W \cap W^\perp$. Then $\hat{\rho}(W)/\hat{\rho}(W')$ is a subquotient of $\hat{V}_{\hat{F}}$, $\hat{\rho}(W)$ is a closed subspace of $\hat{V}$, and $\hat{\rho}\hat{\eta}$ is a faithful representation of L/θ in $\hat{\rho}(W)/\hat{\rho}(W')$ by Fact 7.3. The proof is complete. □

C:83 **Corollary 8.4.** *Let a MOL L have a faithful representation within a pre-hermitean space V_F. There is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L admits a faithful representation within an pre-hermitean closed subspace $U_{\hat{F}}$ of $\hat{V}_{\hat{F}}$.*

Proof. According to the proof of [18, Theorem 13.1] and the proof of Theorem 8.3, there is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L admits a faithful representation within a subquotient W/W' of the orthogonal geometry $\mathbb{G}(\hat{V}_{\hat{F}})$. As L is an MOL, according to the definition of W' (given in [18, page 355] and denoted by U there), one has $W' = \varnothing$. Hence in the proof of Theorem 8.3, $\mathsf{rad}\, U = \hat{\rho}(W') = 0$. □

Importance of representations for the universal algebraic theory of CMILs and regular $*$-rings derives from the following

findim **Theorem 8.5.** *Let V_F be a pre-hermitean space and let $L \in$ MIL [$R \in \mathcal{A}_\Lambda$] have a faithful representation within V_F. Then $L \in \mathsf{W}\big(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)\big)$ [$R \in \mathsf{W}\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$, respectively]. If $L \in$ CMIL [$R \in \mathcal{R}_\Lambda$], then $L \in \mathsf{W}_\exists\big(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)\big)$ [$R \in \mathsf{W}_\exists\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$, respectively].*

Proof. We may assume that $\dim V_F \geqslant \omega$. In view of Fact 6.3, we may also assume that L is an atomic subalgebra of $\mathbb{L}(V_F)$ such that $L_f = \mathbb{L}(V_F)_f$. Therefore, Fact 6.4 yields that L_f is the directed union of its subalgebras $[0, U] \cup [U^\perp, V] \cong \mathbb{L}(U_F) \times \mathbf{2}$, $U \in \mathbb{O}(V_F)$. Moreover for any $U \in \mathbb{O}(V_F)$, the algebra $\mathbb{L}(U_F) \times \mathbf{2}$ embeds into $\mathbb{L}(W_F)$, where $X \in \mathbb{L}(U^\perp)$, $\dim X = 1$, and $W = U + X \in \mathbb{O}(V_F)$. Therefore,

$$L_f \in \mathsf{W}\big(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)\big).$$

Finally, the proof of [18, Theorem 16.3] yields $L \in \mathsf{W}(L_f)$ and $L \in \mathsf{W}_\exists(L_f)$ if L is complemented.

Dealing with an algebra $R \in \mathcal{A}_\Lambda$, we first show that $\mathsf{End}^*_\mathsf{f}(V_F) \in \mathsf{W}_\exists\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$. By Fact 5.1(vi), it suffices to prove this inclusion for finitely generated algebras $B \in \mathsf{S}_\exists(\mathsf{End}^*_\mathsf{f}(V_F))$. By Fact 6.8(iii), we may assume that B is of the form $\{\varepsilon_U \varphi \pi_U + \lambda\, \mathsf{id} \mid \varphi \in \mathsf{End}^*(V_F),\ \lambda \in F\}$ for some $U \in \mathbb{O}(V_F)$. Thus $B \cong \mathsf{End}^*(U_F) \times F$ and the latter embeds into $\mathsf{End}^*(W_F)$, where $X \in \mathbb{L}(U^\perp)$, $\dim X = 1$, and $W = U + X \in \mathbb{O}(V_F)$.

In view of Fact 6.9, we may assume that R is a subalgebra of $\mathsf{End}^*(V_F)$ containing $A = \mathsf{End}^*_\mathsf{f}(V_F)$. Let $J = J(V_F)$ and let J_0 denote the set of projections in J. By Fact 8.2, there is an ultrapower $({}_{\hat{R}}\hat{V}_{\hat{F}}; \hat{A})$ of $({}_R V_F; A)$ which is ω-saturated over $({}_R V_F; A)$. We may assume that R is a subalgebra of $\hat{R}$ and $\hat{A}$ is an ultrapower of A; in particular, $\hat{A} \in \mathsf{W}_\exists\big(\mathsf{End}^*(U_F) \mid U \in \mathbb{O}(V_F)\big)$. For $a \in \hat{A}$ and $r \in R$, we put

$$a \sim r, \quad \text{if} \quad ae = re \text{ and } a^*e = r^*e \text{ for all } e \in J_0.$$

Claim 1. *For any $a \in \hat{A}$ and any r, $s \in R$, $a \sim r$ and $a \sim s$ imply $r = s$.* c-1

Proof of Claim. For any $U \in \mathbb{O}(V_F)$, we have $\pi_U \in J_0$, whence $r\pi_U = a\pi_U = s\pi_U$. Considering r and s as endomorphisms of V_F, we get that they coincide on any $U \in \mathbb{O}(V_F)$, whence they coincide on V_F by Fact 6.4. □

Claim 2. *$S = \{a \in \hat{A} \mid a \sim r$ for some $r \in R\}$ is a subalgebra of $\hat{A}$ and the map* c-2

$$g\colon \hat{A} \to R, \quad g\colon a \mapsto r, \ \textit{where } a \sim r$$

is a homomorphism.

Proof of Claim. It follows from Claim 1 that g is well-defined. Let a, $b \in \hat{A}$ and r, $s \in R$ be such that $a \sim r$ and $b \sim s$. Then, obviously, $a + b \sim r + s$, $\lambda a \sim \lambda r$ for any $\lambda \in \Lambda$, and $a^* \sim r^*$. Let $e \in J_0$, then $be \in J$. By Fact 6.8(iii), there is $f \in J_0$ such that $fbe = be$. Therefore, we get $abe = afbe = rfbe = rbe = rse$, whence $ab \sim rs$.

Obviously, $0_{\hat{V}}$, $\mathsf{id}_{\hat{V}} \in \hat{A}$. For any $U \in \mathbb{O}(V_F)$ we have $\pi_U \in J_0$. Therefore, $0_{\hat{V}}\pi_U = 0_U$ and $\mathsf{id}_{\hat{V}}\,\pi_U = \pi_U$ imply in view of Fact 6.4 that $0_{\hat{V}} \sim 0_R$ and $\mathsf{id}_{\hat{V}} \sim 1_R$. □

Claim 3. *The homomorphism g is surjective.* c-3

Proof of Claim. Surjectivity of g is shown via the supposed saturation property. Given $r \in R$, consider a finite set $E \subseteq J_0$. According to Fact 6.8(iii), there is $e \in J_0$ such that $ef = f$ for all $f \in E$ and $er^*f = r^*f$ for all $f \in E$. Take $a = re$ and observe that $af = ref = rf$ and $a^*f = er^*f = r^*f$ for all $f \in E$. Thus the set of formulas

$$\Sigma(x) = \big\{[xe = re] \;\&\; [x^*e = r^*e] \;\&\; [e = e^2 = e^*] \mid e \in J\big\}$$

with a free variable x of type A is finitely realized in $({}_R V_F; A)$. As $({}_{\hat{R}}\hat{V}_{\hat{F}}; \hat{A})$ is ω-saturated over $({}_R V_F; A)$, we get that there is $a \in \hat{A}$ with $a \sim r$. □

Claim 4. *If R is regular, then S is also regular.* c-4

Proof of Claim. In view of Fact 4.1(ii), it suffices to prove that $\mathsf{ker}\, g = \{a \in S \mid a \sim 0\}$ is regular. Observe that $a \sim 0$ means that $ae = 0 = a^*e$ for any $e \in J_0$, equivalently $(1-e)a = a = a(1-e)$. Again, let $E \subseteq J_0$ be finite. By Fact 6.8(iii), there is $e \in J_0$ such that $ef = f$ for any $f \in E$. The ring A is regular by Facts 6.8(i) and 6.9, whence $\hat{A}$ is also regular. Therefore, the ring $(1-e)\hat{A}(1-e)$ is regular by [4, 2.4]. Thus there is $b \in \hat{A}$ such that $aba = a$ and $(1-e)b = b = b(1-e)$; in particular, $be = 0 = eb$ whence $b^*e = 0$. This implies that $bf = bef = 0$ and $b^*f = b^*ef = 0$ for all $f \in E$. Therefore, the set of formulas

$$\Sigma(x) = \{axa = a\} \cup \bigl\{[xe = 0] \ \&\ [x^*e = 0] \ \&\ [e = e^2 = e^*] \mid e \in J\bigr\}$$

with a variable x of type A is finitely realized in $({}_{\hat{R}}\hat{V}_{\hat{F}}; \hat{A})$. Thus $\Sigma(x)$ is realized in $({}_{\hat{R}}\hat{V}_{\hat{F}}; \hat{A})$, and we obtain $b \in \hat{A}$ such that $aba = a$ and $b \sim 0$; that is, $b \in \ker g$. □

The desired statements concerning $*$-Λ-algebras follow from Claims 2-4. □

Remark 8.6. The statements of Theorem 8.5 concerning $*$-Λ-algebras were proved in case of representability in inner product spaces in [25, Theorem 16]. Requiring semivariety generation, only, a more direct approach is possible. For $R \in \mathcal{A}_\Lambda$, one chooses in the proof of [25, Theorem 16] $I = \mathbb{O}(V_F)$. By Fact 6.4, any finite dimensional subspace of V_F is contained in some $U \in I$. Moreover, with the induced scalar product, U_F is a pre-hermitean space. A similar approach works for MILs.

9. (∃-)SEMIVARIETIES OF REPRESENTABLE STRUCTURES

Let $\mathcal{S}$ be a class of pre-hermitean spaces V_F, where $F \in \mathcal{A}_\Lambda$ and Λ is a fixed commutative $*$-ring. In such a case, we also speak of a class of spaces *over* Λ. The class $\mathcal{S}$ will always be assumed to be closed under isomorphisms and all class operators include isomorphic copies. We denote by $\mathcal{L}(\mathcal{S})$ [$\mathcal{R}(\mathcal{S})$, respectively] the class of all CMILs [all $R \in \mathcal{R}_\Lambda$ respectively] having a faithful representation within some member of $\mathcal{S}$ (we also say that these structures are *representable* within $\mathcal{S}$). We consider here conditions on $\mathcal{S}$ which assure that classes $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are ∃-(semi)varieties.

Introducing class operators for spaces, let $\mathsf{S}(\mathcal{S})$ and $\mathsf{P}_u(\mathcal{S})$ denote the classes of all non-degenerate 2-sorted substructures and all ultraproducts of members of $\mathcal{S}$ respectively. In contrast to that, following the one-sorted view, let $\mathsf{S}_{1f}(\mathcal{S})$ [$\mathsf{S}_{1q}(\mathcal{S})$] denote the class of (isomorphic copies of) non-degenerate finite dimensional subspaces [of all subquotients $U/\operatorname{\mathsf{rad}} U$ with $U = U^{\perp\perp}$, respectively] of members of $\mathcal{S}$. The following statement follows from Facts 6.2(i) and 6.5.

L:S

Lemma 9.1. *For any class $\mathcal{S}$ of spaces over Λ, $\mathsf{S}_{1f}(\mathcal{S}) \subseteq \mathsf{S}_{1q}(\mathcal{S})$ and $\mathsf{S}_{1f}\mathsf{S}_{1q}(\mathcal{S}) = \mathsf{S}_{1f}(\mathcal{S})$.*

Let also $\mathsf{I}_s(\mathcal{S})$ denote the class of spaces which arise from $\mathcal{S}$ by scaling and observe that $\mathsf{I}_s\mathsf{O}(\mathcal{S}) \subseteq \mathsf{O}\mathsf{I}_s(\mathcal{S})$ for any of the mentioned class operators. Moreover,

$$\mathcal{L}\bigl(\mathsf{I}_s(\mathcal{S})\bigr) = \mathcal{L}(\mathcal{S}) \quad \text{and} \quad \mathcal{R}\bigl(\mathsf{I}_s(\mathcal{S})\bigr) = \mathcal{R}(\mathcal{S}).$$

Call $\mathcal{S}$ a *universal class*, if it is closed under P_u, S, and I_s. Observe that $\mathsf{S}\mathsf{P}_u\mathsf{I}_s(\mathcal{S})$ is the smallest universal class containing a class $\mathcal{S}$. Call $\mathcal{S}$ a *semivariety* if it is closed under P_u and S_{1f}. Of course, any universal class is a semivariety, and the smallest semivariety containing a class $\mathcal{S}$ is contained in $\mathsf{S}\mathsf{P}_u(\mathcal{S})$.

malcev

Proposition 9.2. *Let $\mathcal{S}$ be a [recursively] axiomatized class of pre-hermitean spaces over a [recursive] commutative $*$-ring Λ. Then $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are [recursively] axiomatizable.*

Proof. Let Γ_r denote the set of first order axioms defining representations ${}_RV_F$ within $V_F \in \mathcal{S}$ (cf. Fact 7.6) and let Σ_r denote the set of all universal sentences in the signature of $*$-Λ-algebras which are consequences of Γ_r. Then Σ_r defines the class of all $*$-Λ-algebras representable in $\mathcal{S}$. Adding to Σ_r the $\forall\exists$-axiom of regularity defines the subclass $\mathcal{R}(\mathcal{S})$. If Λ is recursive and $\mathcal{S}$ is recursively axiomatizable, then Γ_r is recursive. By Gödel's Completeness Theorem, Σ_r is recursively enumerable. By Craig's trick [27, Exercise 6.1.3], Σ_r is also recursive.

Similarly, taking Γ_l to be the set of first order axioms defining representations of CMILs within spaces from $\mathcal{S}$, and denoting by Σ_l the set of all universal sentences in the signature of CMILs which are consequences of Γ_l, we get that Σ_l defines the class $\mathcal{L}(\mathcal{S})$ of all CMILs representable in $\mathcal{S}$. Moreover, if Γ_l is recursive, then Σ_l is also recursive. We also refer to [38, 44]. $\square$

A *tensorial embedding* of a pre-hermitean space V_F into another one W_K is given by a $*$-Λ-algebra embedding $\alpha\colon F \to K$ and an injective α-semilinear map $\varepsilon\colon V_F \to W_K$ such that W_K is spanned by $\mathsf{im}\,\varepsilon$ as a K-vector space and $\langle\varepsilon(v) \mid \varepsilon(w)\rangle = \alpha\big(\langle v \mid w\rangle\big)$ for all v, $w \in V$; in particular, ε is an isomorphism of V_F onto a two-sorted substructure of W_K. A *joint tensorial extension* of spaces V_{iF_i}, $i \in \{0,1\}$, is given by a pre-hermitean space $W_F = U_0 \oplus^\perp U_1$ and tensorial embeddings of V_{iF_i} into U_{iF} for $i \in \{0,1\}$.

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Lemma 9.3. *Let F, F_0, $F_1 \in \mathcal{A}_\Lambda$, let V_F be a pre-hermitean space, and let V_{0F_0} and V_{1F_1} be finite dimensional pre-hermitean spaces.*

(i) *If α_i and ε_i define a tensorial embedding of V_{iF_i} into V_F, $i < 2$, then $\mathsf{End}^*(V_{iF_i})$ embeds into $\mathsf{End}^*(V_F)$ and $\mathbb{L}(V_{iF_i})$ embeds into $\mathbb{L}(V_F)$.*

(ii) *If V_F is a joint tensorial extension of V_{0F_0} and V_{1F_1}, then $\mathsf{End}^*(V_{0F_0}) \times \mathsf{End}^*(V_{1F_1})$ embeds into $\mathsf{End}^*(V_F)$ and $\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1})$ embeds into $\mathbb{L}(V_F)$.*

Proof. (i) In view of Fact 6.7(i), V_{iF_i} has a dual pair $\{v_1,\dots,v_n\}$, $\{w_1,\dots,w_n\}$ of bases; applying ε_i, one obtains such a pair for V_F. Indeed, V_F is obviously spanned by both, $\{\varepsilon_i(v_1),\dots,\varepsilon_i(v_n)\}$ and $\{\varepsilon_i(w_1),\dots,\varepsilon_i(w_n)\}$. Suppose that $\Sigma_{j=1}^n\varepsilon_i(v_j)\lambda_j = 0$ for some λ_1, ..., $\lambda_n \in F$. Then for any $k \in \{1,\dots,n\}$, one gets

$$0 = \langle 0 \mid w_k\rangle = \big\langle \Sigma_{j=1}^n\varepsilon_i(v_j)\lambda_j \mid \varepsilon_i(w_k)\big\rangle = \Sigma_{j=1}^n\alpha_i\big(\langle v_j \mid w_k\rangle\big)\lambda_j = \lambda_k,$$

whence $\{\varepsilon_i(v_1),\dots,\varepsilon_i(v_n)\}$ is a basis of V_F. Similarly, $\{\varepsilon_i(w_1),\dots,\varepsilon_i(w_n)\}$ is a basis of V_F.

For $\varphi \in \mathsf{End}^*(V_{iF_i})$, let $\xi_i(\varphi)$ be the F-linear map on V defined by $\xi_i(\varphi)\colon \varepsilon_i(v_j) \mapsto \varepsilon_i\big(\varphi(v_j)\big)$ for all $j \in \{1,\dots,n\}$. Clearly, ξ_i is a Λ-algebra embedding of $\mathsf{End}^*(V_{iF_i})$ into $\mathsf{End}^*(V_F)$. Moreover by Fact 6.7(ii), $\xi(\varphi^*) = \xi(\varphi)^*$. For the claim about Galois lattices, apply Facts 4.1(i), 4.2(iv), and 6.8(ii).

(ii) As $V_F = U_0 \oplus^\perp U_1$, by (i), there are $*$-Λ-algebra embeddings

$$\xi_i\colon\ \mathsf{End}^*(V_{iF_i}) \to \mathsf{End}^*(U_{iF}),\quad i \in \{0,1\}.$$

Thus there is a unique embedding

$$\xi\colon\ \mathsf{End}^*(V_{0F_0}) \times \mathsf{End}^*(V_{1F_1}) \to \mathsf{End}^*(V_F)$$

such that $\xi(\varphi_0,\varphi_1)|_{U_i} = \xi_i(\varphi_i)$ for $i \in \{0,1\}$. By Facts 4.2(iii),6.7(i), and 7.7(ii),

$$\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1}) \cong \mathbb{L}\big(\mathsf{End}^*(V_{0F0})\big) \times \mathbb{L}\big(\mathsf{End}^*(V_{1F_1})\big) \cong \mathbb{L}\big(\mathsf{End}^*(V_{0F_0}) \times \mathsf{End}^*(V_{1F_1})\big).$$

By Proposition 7.7(i), the latter admits a faithful representation in V_F. $\square$

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Theorem 9.4. *Let $\mathcal{S}$ be a class of pre-hermitean spaces over Λ. Then*

(i) $\mathcal{L}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big) = \mathcal{L}\big(\mathsf{SP}_u\mathsf{I}_s(\mathcal{S})\big) = \mathsf{W}_\exists\big(\mathcal{L}(\mathcal{S})\big) = \mathsf{W}_\exists\big(\mathbb{L}(V_F) \mid V_F \in \mathsf{S}_{1f}(\mathcal{S})\big)$;

(ii) $\mathcal{R}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big) = \mathcal{R}\big(\mathsf{SP}_u\mathsf{I}_s(\mathcal{S})\big) = \mathsf{W}_\exists\big(\mathcal{R}(\mathcal{S})\big) = \mathsf{W}_\exists\big(\mathsf{End}^*(V_F) \mid V_F \in \mathsf{S}_{1f}(\mathcal{S})\big)$.

In particular, if the class $\mathcal{S}$ is a semivariety then the classes $\mathcal{L}(\mathcal{S}) = \mathcal{L}\big(\mathsf{SP}_u\mathsf{I}_s(\mathcal{S})\big)$ and $\mathcal{R}(\mathcal{S}) = \mathcal{R}\big(\mathsf{S}P_u\mathsf{I}_s(\mathcal{S})\big)$ are $\exists$-semivarieties generated by their strictly simple finite height or artinian members, respectively.

Proof. The proofs of (i) and (ii) follow the same lines. We prove (ii).

Inclusion $\mathsf{S}_\exists\mathsf{P}_u\big(\mathcal{R}(\mathcal{S})\big) \subseteq \mathcal{R}\big(\mathsf{P}_u(\mathcal{S})\big)$ follows immediately from Lemma 8.1. Then $\mathsf{W}_\exists\big(\mathcal{R}(\mathcal{S})\big) \subseteq \mathcal{R}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big)$ by Theorem 8.3. By Theorem 8.5, $\mathcal{R}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big) \subseteq \mathsf{W}_\exists\big(\mathsf{End}^*(V_F) \mid V_F \in \mathsf{S}_{1f}\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big)$. By Lemmas 8.1(iii) and 9.1, for any $V_F \in \mathsf{S}_{1f}\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S}) = \mathsf{S}_{1f}\mathsf{P}_u(\mathcal{S})$, we have $V_F \in \mathsf{P}_u\mathsf{S}_{1f}(\mathcal{S})$ and $\mathsf{End}^*(V_F) \in \mathsf{P}_u\big(\mathsf{End}^*(W_K) \mid W_K \in \mathsf{S}_{1f}(\mathcal{S})\big)$. It follows that

$$\mathsf{W}_\exists\big(\mathcal{R}(\mathcal{S})\big) \subseteq \mathcal{R}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big) \subseteq \mathsf{W}_\exists\big(\mathsf{End}^*(W_K) \mid W_K \in \mathsf{S}_{1f}(\mathcal{S})\big) \subseteq \mathsf{W}_\exists\big(\mathcal{R}(\mathcal{S})\big).$$

Now, consider $R \in \mathcal{R}\big(\mathsf{SP}_u(\mathcal{S})\big)$; that is, R is represented in a 2-sorted substructure W_K of some $V_F \in \mathsf{P}_u(\mathcal{S})$. By Theorem 8.5, we have $R \in \mathsf{W}_\exists\big(\mathsf{End}^*(U_K) \mid U_K \in \mathsf{S}_{1f}(W_K)\big)$. Let U'_F denote the F-subspace of V_F spanned by U. By Lemma 9.3(i), $\mathsf{End}^*(U_K) \in \mathsf{S}_\exists\big(\mathsf{End}^*(U'_F)\big)$. Thus, $R \in \mathsf{W}_\exists\big(\mathcal{R}(\mathcal{S})\big)$. Hence

$$\mathcal{R}\big(\mathsf{SP}_u\mathsf{I}_s(\mathcal{S})\big) \subseteq \mathcal{R}\big(\mathsf{I}_s\mathsf{SP}_u(\mathcal{S})\big) = \mathcal{R}\big(\mathsf{SP}_u(\mathcal{S})\big) \subseteq \mathsf{W}_\exists\big(\mathcal{R}(\mathcal{S})\big) = \mathcal{R}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big).$$

Inclusion $\mathcal{R}\big(\mathsf{S}_{1q}\mathsf{P}_u(\mathcal{S})\big) \subseteq \mathcal{R}\big(\mathsf{SP}_u\mathsf{I}_s(\mathcal{S})\big)$ is trivial by Lemma 9.1. □

More closure properties on $\mathcal{S}$ are needed if one intends to get a one-to-one correspondence between classes of spaces and classes of structures in Theorem 9.4.

spread

Definition 9.5. Let V_F, W_K be pre-hermitean spaces over Λ, $\dim V_F < \omega$, and let $\mathcal{S}$ be a class of pre-hermitean spaces over Λ.

(i) V_F is an *L-spread* of W_K if $\dim V_F > 2$ and $\mathbb{L}(V_F) \in \mathcal{L}(W_K)$ The class $\mathcal{S}$ is *L-spread closed*, if it contains all L-spreads of its members.

(ii) V_F is an *R-spread* of W_K if $\mathsf{End}^*(V_F) \in \mathcal{R}(W_K)$. The class $\mathcal{S}$ is *R-spread closed*, if it contains all R-spreads of its members.

(iii) An R-[L-]spread closed universal class or a semivariety $\mathcal{S}$ is *small*, if $\mathcal{S}$ coincides with the smallest R-[L-]spread closed universal class or a semivariety which contains all members of $\mathcal{S}$ of dimension $n < \omega$ [of dimension $2 < n < \omega$, respectively].

Example 9.6. Consider the class $\mathcal{S}$ of all anisotropic hermitean spaces, where $F \in \mathsf{SP}_u(\mathbb{Q})$; in particular, $F \models \forall x\, x^2 \neq 2$ and $\mathcal{S}$ is a universal class which does not contain K^3_K with the canonical scalar product, where $K = \mathbb{Q}(\sqrt{2})$. Though, $K^{3\times 3}$ whence $\mathbb{L}(K^{3\times 3})$ are representable within $\mathbb{Q}^6_{\mathbb{Q}} \in \mathcal{S}$ by

$$a + b\sqrt{2} \mapsto a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad \text{where } a, b \in \mathbb{Q},$$

which yields a $*$-ring embedding of K into $\mathbb{Q}^{2\times 2}$ thus giving rise to an embedding of $K^{3\times 3}$ into $(\mathbb{Q}^{2\times 2})^{3\times 3}$. In the sense of Definition 9.5, K^3_K is an L-spread and an R-spread of $\mathbb{Q}^6_{\mathbb{Q}}$.

11 **Theorem 9.7.** (i) *For any $\exists$-semivariety $\mathcal{V}$ of Arguesian CMILs generated by its strictly simple members of finite height at least 3, there is a small L-spread closed semivariety [universal class] $\mathcal{S}$ of pre-hermitean spaces over $\mathbb{Z}$ such that $\mathcal{V} = \mathcal{L}(\mathcal{S})$. Moreover, the class of members of $\mathcal{S}$ of dimension at least 3 is unique.*

(ii) *For any $\exists$-semivariety $\mathcal{V} \subseteq \mathcal{R}_\Lambda$ generated by its strictly simple artinian members, there is a small R-spread closed semivariety [universal class] $\mathcal{S}$ of pre-hermitean spaces over Λ such that $\mathcal{V} = \mathcal{R}(\mathcal{S})$. Moreover, such a class $\mathcal{S}$ is unique.*

The class $\mathcal{S}$ above is anisotropic, if $\mathcal{V}$ consists of MOLs or $\mathcal{V} \subseteq \mathcal{R}^*_\Lambda$.

Remark 9.8. In the case of MOLs, it suffices to require in Theorem 9.7 that $\mathcal{V}$ is generated by its strictly simple members of finite height and that $\mathcal{V}$ is not 2-distributive. In this case, $\mathcal{V}$ contains all MOLs of height 2.

Proof. (i) Given an $\exists$-semivariety $\mathcal{V}$ of CMILs with all required properties, let $\mathcal{K}_\mathcal{V}$ denote the class of strictly simple members of $\mathcal{V}$ of finite height at least 3. By Fact 7.5, for any $L \in \mathcal{K}_\mathcal{V}$, there is a pre-hermitean space V_F over $\mathbb{Z}$ such that $\dim V_F = \dim L$ and $L \cong \mathbb{L}(V_F)$. By $\mathcal{S}_\mathcal{V}$, we denote the class of spaces V_F over $\mathbb{Z}$ such that $\mathbb{L}(V_F) \in \mathcal{K}_\mathcal{V}$.

We put $\mathcal{G}_0 = \mathsf{S}_{1f}(\mathcal{S}_\mathcal{V})$. For any ordinal $\alpha \geqslant 0$, let $\mathcal{G}_{\alpha+1}$ be the union of two classes: $\mathsf{P}_u(\mathcal{G}_\alpha)$ and the class of all $V_F \in \mathsf{S}_{1f}(V'_F)$, where V'_F is an L-spread of some $W_K \in \mathcal{G}_\alpha$. Let also $\mathcal{G}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\beta$, if α is a limit ordinal.

Claim 1. $\mathsf{S}_{1f}(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ *and* $\mathbb{L}(V_F) \in \mathcal{V}$ *for any* α *and* $V_F \in \mathcal{G}_\alpha$ *with* $\dim V_F < \omega$. c-13

Proof of Claim. We argue by induction on α. For $\alpha = 0$, the first claim follows from the definition of $\mathcal{G}_0$. Moreover, if $U_F \in \mathsf{S}_{1f}(V_F)$ and $\mathbb{L}(V_F) \in \mathcal{V}$ then $\mathbb{L}(U_F) \in \mathsf{HS}_\exists(\mathbb{L}(V_F)) \subseteq \mathcal{V}$ by Fact 6.4. The limit step is trivial. In the step from α to $\alpha+1$, we assume first that V_F is isomorphic to an ultraproduct of spaces $V_{iF_i} \in \mathcal{G}_\alpha$, $i \in I$. If $U_F \in \mathsf{S}_{1f}(V_F)$ and $n = \dim U_F$ then, by Lemma 8.1(iii), U_F is isomorphic to an ultraproduct of some $U_{iF_i} \in \mathsf{S}_{1f}(V_{iF_i})$ with $\dim U_{iF_i} = n$, $i \in J$, for some $J \subseteq I$. By the inductive hypothesis, $U_{iF_i} \in \mathcal{G}_\alpha$ and $\mathbb{L}(U_{iF_i}) \in \mathcal{V}$. Thus $U_F \in \mathcal{G}_{\alpha+1}$ and $\mathbb{L}(U_F) \in \mathcal{V}$ by Lemma 8.1(iii).

Now, let V'_F be an L-spread of $W_K \in \mathcal{G}_\alpha$ and let $V_F \in \mathsf{S}_{1f}(W_K)$. If $U_F \in \mathsf{S}_{1f}(V_F)$ then $U_F \in \mathsf{S}_{1f}(V'_F)$, whence $U_F \in \mathcal{G}_{\alpha+1}$ by definition. By Theorem 8.5 and the inductive hypothesis,

$$\mathbb{L}(V'_F) \in \mathsf{W}_\exists\big(\mathbb{L}(W'_K) \mid W'_K \in \mathbb{O}(W_K)\big) \subseteq \mathcal{V}.$$

By Fact 6.4, $\mathbb{L}(U_F) \in \mathsf{HS}_\exists(\mathbb{L}(V'_F)) \subseteq \mathcal{V}$. □

It follows that the L-spread closed semivariety $\mathbb{K}(\mathcal{V})$ of pre-hermitean spaces over $\mathbb{Z}$ generated by $\mathcal{S}_\mathcal{V}$ is the union of the classes $\mathcal{G}_\alpha$, where α ranges over all ordinals. Thus in view of the assumption $\mathcal{V} = \mathsf{W}_\exists(\mathcal{K}_\mathcal{V})$ and Claim 1, one gets by Theorem 9.4(i)

$$\mathcal{V} \subseteq \mathcal{L}\big(\mathbb{K}(\mathcal{V})\big) = \mathsf{W}_\exists\big(\mathbb{L}(V_F) \mid V_F \in \mathbb{K}(\mathcal{V}),\, \dim V_F < \omega\big) \subseteq \mathcal{V}.$$

To prove uniqueness, let $\mathcal{S}$ and $\mathcal{S}'$ be small L-spread closed semivarieties of pre-hermitean spaces over $\mathbb{Z}$ such that $\mathcal{L}(\mathcal{S}) = \mathcal{V} = \mathcal{L}(\mathcal{S}')$. For any $V_F \in \mathcal{S}$ with $2 < \dim V_F < \omega$, we have $\mathbb{L}(V_F) \in \mathcal{L}(\mathcal{S}) = \mathcal{L}(\mathcal{S}')$, whence V_F is an L-spread of $\mathcal{S}'$ and $V_F \in \mathcal{S}'$. Similarly, interchanging the roles of $\mathcal{S}$ and $\mathcal{S}'$, we get that $\mathcal{S}$ and $\mathcal{S}'$ have the same members of finite dimension at least 3.

To deal with the case of universal classes, one includes into the union $\mathcal{G}_\alpha$ a third class, namely $\mathsf{S}(\mathcal{G}_\alpha)$. Claim 1 and its proof remains valid, only the case of the third class remains to be considered. Indeed, assume that $V_F \in \mathcal{G}_{\alpha+1}$ is a 2-sorted substructure of $W_K \in \mathcal{G}_\alpha$ and let $U_F \in \mathsf{S}_{1f}(V_F)$. Then $U_F \in \mathsf{S}(W_K)$ and $U_F \in \mathcal{G}_{\alpha+1}$ by definition. Moreover, U_F is a 2-sorted substructure of the K-subspace U'_K of W_K

spanned by U. In particular, $U'_K \in \mathsf{S}_{1f}(W_K)$ and the inductive hypothesis yields $U'_K \in \mathcal{G}_\alpha$ and $\mathbb{L}(U'_K) \in \mathcal{V}$. As $\mathbb{L}(U_F)$ embeds into $\mathbb{L}(U'_K)$ by Lemma 9.3(i), it follows that $\mathbb{L}(U_F) \in \mathcal{V}$.

(ii) The proof follows the same lines as the one of (i) replacing Fact 7.5 by Fact 7.9. Fact 6.4 by Fact 6.7(iii), and Theorem 9.4(i) by Theorem 9.4(ii). □

For results of the same type as Theorem 9.7, see also [24, Theorems 4.4-5.4].

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10. ∃-varieties and representations

We first consider a condition on $\mathcal{S}$ under which the class of representables is an ∃-variety. Then we review the approach of Micol [40] to capture ∃-varieties via the concept of generalized representation.

S-var

Definition 10.1. A semivariety $\mathcal{S}$ of pre-hermitean spaces over Λ is a *variety* if for any finite dimensional V_{0F_0}, $V_{1F_1} \in \mathcal{S}$, there is a joint tensorial extension $V_F \in \mathcal{S}$.

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Proposition 10.2. *If $\mathcal{S}$ is a variety of pre-hermitean spaces over Λ, then $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are ∃-varieties.*

Proof. In view of Proposition 5.1(iv) and Theorem 9.4, it suffices to notice that for any finite-dimensional spaces V_{0F_0}, $V_{1F_1} \in \mathcal{S}$, the structures $\mathsf{End}^*(V_{0F_0}) \times \mathsf{End}^*(V_{1F_1})$ and $\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1})$ have a faithful representation within some member of $\mathcal{S}$ by Lemma 9.3(ii). □

Classes $\mathcal{L}(\mathcal{S})$ of CMILs having a faithful representation within some member of a class $\mathcal{S}$ of orthogeometries have been considered in [18]. The closure properties of Theorem 9.4(i) hold also in this case with $\mathsf{S}(\mathcal{S})$ denoting formation of non-degenerate subgeometries of members of $\mathcal{S}$, $\mathsf{S}_{1f}(\mathcal{S})$ and $\mathsf{S}_{1q}(\mathcal{S})$ — formation of non-degenerate finite dimensional subspaces and of subquotients $U/\operatorname{\mathsf{rad}} U$ with $U = U^{\perp\perp}$. In addition, one has the class $\mathsf{U}(\mathcal{S})$ of all disjoint orthogonal unions of members of $\mathcal{S}$ and thus $\mathsf{P}\big(\mathcal{L}(\mathcal{S})\big) \subseteq \mathcal{L}\big(\mathsf{U}(\mathcal{S})\big)$, cf. [18, Theorem 2.2]. Moreover, mimicking the concept of an L-spread and the proof of Theorem 9.7, one obtains

Theorem 10.3. *For any ∃-variety $\mathcal{V}$ of CMILs generated by its finite height members, there is a small L-spread and U-closed semivariety [universal class] $\mathcal{S}$ of orthogeometries such that $\mathcal{V} = \mathcal{L}(\mathcal{S})$. Moreover, such a class $\mathcal{S}$ is unique.*

The objective of Micol [40] was to derive results for $*$-regular rings, analogous to those above. Of course, representation requires some structure of the type of sesquilinear spaces. Apparently, in general there is no axiomatic class of such spaces which would serve for representing direct products of representable structures. Micol solved this problem by introducing the concept of a *generalized representation.* This concept was transferred to MOLs by Niemann [42].

A *g-representation* of $A \in$ CMIL [$A \in \mathcal{R}_\Lambda$] within a class $\mathcal{S}$ of pre-hermitean spaces is a family $\{\varepsilon_i \mid i \in I\}$ of representations ε_i of A in $V_{iF_i} \in \mathcal{S}$, $i \in I$. It is *faithful* if $\bigcap_{i\in I} \ker \varepsilon_i = 0$. Let $\mathcal{L}_g(\mathcal{S})$ [$\mathcal{R}_g(\mathcal{S})$] denote the class of all $A \in$ CMIL [$A \in \mathcal{R}_\Lambda$] having a faithful g-representation within $\mathcal{S}$; equivalently, the class of structures A having a subdirect decomposition into factors $\varepsilon_i(A)$, $i \in I$, which have a faithful representation within $\mathcal{S}$.

Call an artinian algebra $R \in \mathcal{R}_\Lambda$ *strictly artinian* if $I = I^*$ for any ideal I of R. By the Wedderburn-Artin Theorem, this is equivalent to the fact that R is isomorphic to a direct product of strictly simple factors (cf. [34, §3.4]). Similarly, call a finite height CMIL L *strictly finite height* if $\theta = \theta'$ for any lattice congruence θ of L. By [5, Theorem IV.7.10]), this is equivalent to the fact that L is a direct product of strictly simple factors.

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Proposition 10.4. *The following statements are true.*

(i) *For any semivariety $\mathcal{S}$ of pre-hermitean spaces, the class $\mathcal{L}_g(\mathcal{S}) = \mathsf{P}_{s\exists}\big(\mathcal{L}(\mathcal{S})\big)$ $[\mathcal{R}_g(\mathcal{S}) = \mathsf{P}_{s\exists}\big(\mathcal{R}(\mathcal{S})\big)]$ is an $\exists$-variety generated by its strictly simple finite height [artinian] members, which are of the form $\mathbb{L}(V_F)$ $[\mathsf{End}^*(V_F)]$ with $V_F \in \mathcal{S}$, $\dim V_F < \omega$.*

(ii) *For any $\exists$-variety $\mathcal{V} \subseteq$ CMIL $[\mathcal{V} \subseteq \mathcal{R}_\Lambda]$ which is generated by its strictly finite height at least 3 [artinian] members, there is a semivariety $\mathcal{S}$ of pre-hermitean spaces such that $\mathcal{V} = \mathcal{L}_g(\mathcal{S})$ $[\mathcal{V} = \mathcal{R}_g(\mathcal{S})]$.*

(iii) *$A \in \mathcal{L}_g(\mathcal{S})$ $[A \in \mathcal{R}_g(\mathcal{S})]$ if and only if A has an atomic extension $\hat{A}$ which is a subdirect product of atomic strictly subdirectly irreducible structures A_i such that $\mathbb{L}(A_i)_f \cong \mathbb{L}(V_{iF_i})$ [the minimal ideal of A_i is isomorphic to $J(V_{iF_i})$] with $V_{iF_i} \in \mathcal{S}$.*

Proof. Statement (i) follows from Facts 5.1(iii)-(iv), 7.5, 7.9, and Theorem 9.4. Statement (ii) follows from Facts 5.1(iv), 7.5, 7.9, and Theorem 9.7. Finally, statement (iii) follows from Facts 6.3, 6.9 and Theorems 7.4, 7.8. □

For $*$-regular rings, the result of Proposition 10.4 is in essence due to Micol [40]. To prove that g-representability is preserved under homomorphic images, she axiomatized families of inner product spaces as 3-sorted structures, where the third sort mimics the index set I. Again, a saturation property is needed for the proof and regularity is crucial. The fact that the $\exists$-variety of g-representable structures is generated by its artinian members was shown by her reducing to countable subdirectly irreducible structures R, deriving countably based representation spaces (and forming 2-sorted subspaces), and using the approach of Tyukavkin [46] with respect to a countable orthogonal basis. Conversely, a substantial part of Theorem 9.4 follows from Proposition 10.4.

Appendix A. Existence semivarieties

We characterize $\exists$-(semi)varieties contained in CMIL or in $\mathcal{R}_\Lambda$ as model classes, proving at the same time the operator identities of Fact 5.1. With no additional effort, this can be done to include other classes of algebraic structures.

Given a set Σ of first order axioms, by $\mathrm{Mod}\,\Sigma$ we denote the model class $\{A \mid A \models \Sigma\}$ of Σ. By $\mathrm{Th}\,\mathcal{C}$ $[\mathrm{Th}_L\,\mathcal{C}]$, we denote the set of sentences [from the fragment L] of first order language which are valid in $\mathcal{C}$. As usual, let $\overline{x}$ denote a sequence of variables of length being given by context.

RegC

Definition A.1. A class $\mathcal{C}_0$ of algebraic structures of the same similarity type is *regular* if there is a (possibly empty) set Ψ_0 of conjunctions $\alpha(\overline{x}, y)$ of atomic formulas $\big($i.e. formulas of the form $\bigwedge_{i=1}^k s_i(\overline{x}, y) = t_i(\overline{x}, y)\big)$ and a class $\mathcal{S}$ such that

(i) $\mathcal{C}_0 = \mathcal{S} \cap \mathrm{Mod}\{\forall \overline{x} \exists y\ \alpha(\overline{x}, y) \mid \alpha(\overline{x}, y) \in \Psi_0\}$;

(ii) $\mathcal{S}$ is closed under S and $\mathcal{C}_0$ is closed under H and P;

(iii) For any structures A, $B \in \mathcal{C}_0$, for any surjective homomorphism $\varphi\colon A \to B$, for any formula $\alpha(\overline{x}, y) \in \Psi_0$, and for any $\overline{a}$, $b \in B$ such that $B \models \alpha(\overline{a}, b)$, there are $\overline{c}$, $d \in A$ such that $\varphi(\overline{c}) = \overline{a}$, $\varphi(d) = b$, and $A \models \alpha(\overline{c}, d)$.

Without loss of generality, one may consider also the case when $\alpha(\overline{x}, y)$ is of the form $\bigwedge_{i=1}^{k} p_i\big(t_1(\overline{x}, y), \dots, t_m(\overline{x}, y)\big)$, where p_i is a relation symbol of arity m or the symbol $=$ with $m = 2$.

From Definition A.1(ii) it follows immediately that any regular class is closed under P_u. In the sequel, we shall fix a regular class $\mathcal{C}_0$ and write for any $\mathcal{C} \subseteq \mathcal{C}_0$:

$$\mathsf{S}_{\exists}(\mathcal{C}) = \mathcal{C}_0 \cap \mathsf{S}(\mathcal{C}) \quad \text{and} \quad \mathsf{P}_{s\exists}(\mathcal{C}) = \mathcal{C}_0 \cap \mathsf{P}_s(\mathcal{C}).$$

Let $\mathcal{C}_0$ be a regular class. A *Skolem expansion* A^* of $A \in \mathcal{C}_0$ adds for each $\alpha(\overline{x}, y) \in \Psi_0$ an operation f_α on A such that $A \models \alpha(\overline{a}, f_\alpha(\overline{a}))$ for all $\overline{a} \in A$.

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Definition A.2. A class $\mathcal{C}_0$ is *strongly regular* if it is regular and

(iii′) For any structures A, $B \in \mathcal{C}_0$, for any surjective homomorphism $\varphi \colon A \to B$, for any formula $\alpha(\overline{x}, y) \in \Psi_0$, for any $\overline{a}$, $b \in B$ such that $B \models \alpha(\overline{a}, b)$, and for any $\overline{c} \in A$ such that $\varphi(\overline{c}) = \overline{a}$ there is $d \in A$ such that $\varphi(d) = b$ and $A \models \alpha(\overline{c}, d)$.

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Remark A.3. It is obvious that if a class $\mathcal{C}_0$ satisfies (iii′) of Definition A.2, then $\mathcal{C}_0$ satisfies (iii) of Definition A.1. For any strongly regular class $\mathcal{C}_0$, for any A, $B \in \mathcal{C}_0$, and for any surjective homomorphism $\varphi \colon A \to B$, if B^* is a Skolem expansion of B, then there is a Skolem expansion A^* of A such that $\varphi \colon A^* \to B^*$ is a homomorphism. Clearly, $\mathcal{C}_0$ is strongly regular if it satisfies (i)-(ii) of Definition A.1 and for any $\alpha \in \Psi_0$ and for any $\overline{a} \in A \in \mathcal{C}_0$, there is unique b such that $\alpha(\overline{a}, b)$. This applies, in particular, to completely regular [inverse] semigroups.

In what follows, when we speak of a [strongly] regular class $\mathcal{C}$, we always assume that the set of formulas Ψ_0 and the classes $\mathcal{C}_0$ and $\mathcal{S}$ are given according to Definition A.1 [Definition A.2, respectively].

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Proposition A.4. *For any variety $\mathcal{V}$ with a $*$-ring reduct, the class of structures $A \in \mathcal{V}$ having $*$-regular reducts forms a strongly regular class. In particular, the class $\mathcal{R}^*_\Lambda$ of all $*$-regular $*$-Λ-algebras is strongly regular.*

Proof. Let $\Psi_0 = \{xyx = y\}$ and let $\mathcal{S} = \mathcal{V} \cap \mathrm{Mod}(\forall x \; xx^* = 0 \to x = 0)$. Then $\mathcal{C}_0$ defined as in Definition A.1(i) consists of the $*$-regular members of $\mathcal{V}$. Closure of $\mathcal{C}_0$ under H and P follows from the fact that $*$-regularity can be defined by the sentence:

$$\forall x \exists y \; (y = y^2 = y^*) \;\&\; (\exists u \; x = uy) \;\&\; (\exists u \; y = ux).$$

The proof of (iii′) essentially goes as in [15, Lemma 1.4], cf. [23, Lemma 9]. Indeed, the two-sided ideal $I = \ker \varphi$ is regular. Let $c \in A$ be such that $a = \varphi(c)$, and let $aba = a$ in B. There is $y \in A$ such that $\varphi(y) = b$. Then $c - cyc \in I$. Since I is regular, there is $u \in I$ such that $(c - cyc)u(c - cyc) = c - cyc$. It follows from the latter that $cuc - cycuc - cucyc + cycucyc + cyc = c$. Taking $d = u - ucy - ycu + ycucy + y$, we get $cdc = cuc - cucyc - cycuc + cycucyc + cyc = c$ and $d - y = u - ucy - ycu + ycucy \in I$, whence $\varphi(d) = b$. $\square$

Further examples of strongly regular classes are the class of all regular [complemented] members of any variety having ring [bounded modular lattice, respectively] reducts, see [23, Lemma 9]. The latter can be easily modified to the class of all relatively complemented lattices; here $\alpha(x_1, x_2, x_3, y)$ is given by $y\big((x_1 + x_2)x_3 + x_1x_2\big) = x_1x_2 \;\&\; y + \big((x_1 + x_2)x_3 + x_1x_2\big) = x_1 + x_2$.

We consider fragments of the first order language associated with a given regular class $\mathcal{C}_0$. Let L_u consist of all quantifier free formulas; up to equivalence, we may assume that L_u consists of conjunctions of formulas $\bigwedge_{i=1}^n \beta_i \rightarrow \bigvee_{j=1}^m \gamma_j$, where β_i, γ_j are atomic formulas and n, $m \geqslant 0$. The set $L_q \subseteq L_u$ of all *quasi-identities* is defined by $m = 1$. The set L_p consists of all formulas of the form

$$\bigwedge_{i=1}^{n} \alpha_i(\overline{x}_i, y_i) \ \rightarrow \ \bigvee_{j=1}^{m} \gamma_j,$$

where $n \geqslant 0$, $m \geqslant 1$, and $\alpha_i(\overline{x}_i, y_i) \in \Psi_0$. Then $L_e \subseteq L_p$ is defined by $m = 1$; its members are called *conditional identities*, while those of L_p are *conditional disjunctions of equations*. As usual, validity of a formula means validity of its universal closure. We write Th_x instead of Th_{L_x}.

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Theorem A.5. *Let $\mathcal{C}_0$ be a regular class and let $\mathcal{C} \subseteq \mathcal{C}_0$. Then*

(i) $\mathcal{C}_0 \cap \operatorname{Mod} \mathrm{Th}_u \mathcal{C} = \mathsf{S}_\exists \mathsf{P}_u(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by universal sentences relatively to $\mathcal{C}_0$ if and only if it is closed under $\mathsf{S}_\exists$ and P_u.*

(ii) $\mathcal{C}_0 \cap \operatorname{Mod} \mathrm{Th}_q \mathcal{C} = \mathsf{S}_\exists \mathsf{P}_u \mathsf{P}_\omega(\mathcal{C}) = \mathsf{S}_\exists \mathsf{P}\mathsf{P}_u(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by quasi-identities relatively to $\mathcal{C}_0$ if and only if it is closed under $\mathsf{S}_\exists$, P_u, and P_ω [under $\mathsf{S}_\exists$, P_u, and P, respectively].*

(iii) $\mathcal{C}_0 \cap \operatorname{Mod} \mathrm{Th}_p \mathcal{C} = \mathsf{H}\mathsf{S}_\exists \mathsf{P}_u(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by conditional disjunctions of equations relatively to $\mathcal{C}_0$ if and only if it is closed under H, $\mathsf{S}_\exists$, and P_u.*

(iv) $\mathcal{C}_0 \cap \operatorname{Mod} \mathrm{Th}_e \mathcal{C} = \mathsf{H}\mathsf{S}_\exists \mathsf{P}_u \mathsf{P}_\omega(\mathcal{C}) = \mathsf{H}\mathsf{S}_\exists \mathsf{P}\mathsf{P}_u(\mathcal{C})$. *In particular, $\mathcal{C}$ is definable by conditional identities relatively to $\mathcal{C}_0$ if and only if it is closed under H, $\mathsf{S}_\exists$, P_u, and P_ω [under H, $\mathsf{S}_\exists$, P, and P_u, respectively].*

Classes as in (iii) and (iv) will be called $\exists$-*semivarieties* and $\exists$-*varieties*, respectively. If Ψ_0 is empty, one has *semivarieties* and *varieties*. By $\mathsf{W}_\exists(\mathcal{C})$ [by $\mathsf{V}_\exists(\mathcal{C})$, $\mathsf{W}(\mathcal{C})$, $\mathsf{V}(\mathcal{C})$, respectively], we denote the smallest $\exists$-semivariety [$\exists$-variety, semivariety, variety, respectively] containing $\mathcal{C}$, cf. Theorem A.5(iii)-(iv).

Of course, the statements of Theorem A.5 are well known results in the case of empty Ψ_0. Proofs of (i) and (ii) are included since they can be seen as a preparation for proofs of (iii)-(iv); the latter are our primary interest.

Proof. Inclusion in the model class is well known and easy to verify in any of the cases (i)-(iv) using Definition A.1. In particular in cases (iii)-(iv), inclusion $\mathsf{H}(\mathcal{C}) \subseteq \operatorname{Mod} \mathrm{Th}_x \mathcal{C}$ follows directly from Definition A.1(iii).

The proof of the reverse inclusion relies on adapting the method of diagrams. Given a structure A, let $a \mapsto x_a$ be a bijection onto a set of variables and let $\bar{x} = (x_a \mid a \in A)$. We consider quantifier free formulas $\chi(\bar{x})$ in these variables; evaluations $\bar{x}$ in a structure B are given as $\bar{b} = (b_a \mid a \in A) \in B^A$, and we write $B \models \chi(\bar{b})$ if $\chi(\bar{x})$ is valid under evaluation $\bar{b}$. For a set $\Phi = \Phi(\bar{x})$ of formulas, $B \models \Phi(\bar{b})$ if $B \models \chi(\bar{b})$ for all $\chi(\bar{x}) \in \Phi$. Let At denote the set of atomic formulas and let

$$\begin{aligned}
\Delta^+(A) =&\{\chi(\bar{x}) \in At \mid A \models \chi(\bar{a})\};\\
\Delta^-(A) =&\{\neg\chi(\bar{x}) \mid \chi(\bar{x}) \in At,\ A \not\models \chi(\bar{a})\};\\
\Delta^0(A) =&\Big\{\alpha\big(t_1(\bar{x}), \dots, t_n(\bar{x}), x_a\big) \mid \\
&\qquad t_1, \dots, t_n \text{ are terms, } \alpha(x_1, \dots, x_n, y) \in \Psi_0,\ A \models \alpha\big(t_1(\bar{a}), \dots, t_n(\bar{a}), a\big)\Big\};
\end{aligned}$$

$$\begin{aligned}
\Delta_u(A) &= \Delta_q(A) = \Delta^+(A) \cup \Delta^-(A);\\
\Delta_p(A) &= \Delta_e(A) = \Delta^0(A) \cup \Delta^-(A).
\end{aligned}$$

For $x \in \{u, q, p, e\}$ and a finite subset Φ of $\Delta_x(A)$, let $\Phi^- = \Phi \cap \Delta^-(A)$, $\Phi^+ = \Phi \backslash \Phi^-$, and let $\Phi^\dagger$ denote the formula

$$\bigwedge_{\phi \in \Phi^+} \phi \ \rightarrow \ \bigvee_{\neg\chi \in \Phi^-} \chi;$$

while for $\neg\chi \in \Phi^-$, let $\Phi^\dagger_\chi$ denote the quasi-identity

$$\bigwedge_{\phi \in \Phi^+} \phi \ \rightarrow \ \chi.$$

Thus for any finite $\Phi \subseteq \Delta_u(A)$ and for $\chi \in \Phi^-$, we have $\Phi^\dagger \in L_u$ and $\Phi^\dagger_\chi \in L_q$, while for any finite $\Phi \subseteq \Delta_p(A)$ and for $\chi \in \Phi^-$, we have $\Phi^\dagger \in L_p$ and $\Phi^\dagger_\chi \in L_e$. Observe that $A \not\models \Phi^\dagger$ and $A \not\models \Phi^\dagger_\chi$ in any case (verified by substituting x_a with a). Let $A \in \mathcal{C}_0 \cap \operatorname{Mod}\operatorname{Th}_x \mathcal{C}$. We have to obtain A from $\mathcal{C}$ by means of operators.

First, we consider the case $x \in \{u, p\}$. Let $\Phi \subseteq \Delta_x(A)$ be finite. As $A \not\models \Phi^\dagger$, we have that $\Phi^\dagger \notin \operatorname{Th}_x \mathcal{C}$. Thus there are a structure $B_\Phi \in \mathcal{C}$ and $\bar{b}_\Phi = (b_{\Phi a} \mid a \in A) \in B_\Phi^A$ such that $B_\Phi \not\models \Phi^\dagger(\bar{b}_\Phi)$, i.e. $B_\Phi \models \Phi(\bar{b}_\Phi)$.

As in the proof of the Compactness Theorem, let I be the set of all finite subsets of $\Delta_x(A)$ and let $\mathcal{U}$ be an ultrafilter containing all sets $\{\Psi \in I \mid \Psi \supseteq \Phi\}$, where $\Phi \in I$. Let $B = \prod_{\Phi \in I} B_\Phi/\mathcal{U}$, $b_a = (b_{\Phi a} \mid \Phi \in I)/\mathcal{U}$ and $\bar{b} = (b_a \mid a \in A)$. By (the quantifier free part of) the Łoś Theorem, we have $B \models \Delta_x(A)(\bar{b})$. Moreover, $B \in \mathsf{P}_u(\mathcal{C}) \subseteq \mathcal{C}_0$.

Let C be the subalgebra of B generated by the set $\{b_a \mid a \in A\}$. We claim that $C \in \mathcal{C}_0$, i.e. $C \in \mathsf{S}_\exists(B)$. Indeed, let $\alpha(x_1, \dots, x_n, y) \in \Psi_0$ and let $c_1, \dots, c_n \in C$. As C is generated by the set $\{b_a \mid a \in A\}$, there are terms $t_1(\bar{x}), \dots, t_n(\bar{x})$ such that $c_i = t_i(\bar{b})$ for all $i \in \{1, \dots, n\}$. Since $A \in \mathcal{C}_0$, by Definition A.1(i) there is $a \in A$ such that

$$A \models \alpha\big(t_1(\bar{a}), \dots, t_n(\bar{a}), a\big).$$

Therefore,

$$\alpha\big(t_1(\bar{x}), \dots, t_n(\bar{x}), x_a\big) \in \Delta^+(A) \cap \Delta^0(A).$$

Since $B \models \Delta_x(A)(\bar{b})$, we conclude that $B \models \alpha\big(t_1(\bar{b}), \dots, t_n(\bar{b}), b_a\big)$. This implies that $C \models \alpha(c_1, \dots, c_n, b_a)$. On the other hand, $B \in \mathsf{P}_u(\mathcal{C}) \subseteq \mathcal{C}_0 \subseteq \mathcal{S}$, as $\mathcal{C}_0$ is closed under P_u by Definition A.1(ii). Therefore, $C \in \mathsf{S}(B) \subseteq \mathsf{S}(\mathcal{S}) \subseteq \mathcal{S}$ again by Definition A.1(ii). This implies by Definition A.1(i) that $C \in \mathcal{C}_0$ which is our desired conclusion. Furthermore, the map

$$\varphi\colon C \to A; \quad t(\bar{b}) \mapsto t(\bar{a})$$

is well-defined (since $B \models \Delta^-(A)(\bar{b})$), a homomorphism (in view of term composition), and surjective (since $\varphi(b_a) = a$). Moreover, in case $x = u$, φ is an isomorphism, as $B \models \Delta^+(A)(\bar{b})$. This proves (i) and (iii).

Let $x \in \{q, e\}$. Given a finite subset $\Phi \subseteq \Delta_x(A)$ and $\neg\chi \in \Phi^-$, one has $A \not\models \Phi^\dagger_\chi$, whence $\Phi^\dagger_\chi \notin \mathrm{Th}_x\,\mathcal{C}$. Thus there are a structure $B_{\Phi,\chi} \in \mathcal{C}$ and $\bar{b}_{\Phi\chi} = (b_{\Phi\chi a} \mid a \in A) \in B^A_{\Phi,\chi}$ such that

$$B_{\Phi,\chi} \models \Phi^+(\bar{b}_{\Phi\chi}) \quad \text{and} \quad B_{\Phi,\chi} \models \neg\chi(\bar{b}_{\Phi\chi}).$$

Taking $B_\Phi = \prod_{\neg\chi\in\Phi^-} B_{\Phi,\chi} \in \mathsf{P}_\omega(\mathcal{C})$ and $b_{\Phi a} = (b_{\Phi\chi a} \mid \neg\chi \in \Phi^-)$, we get that $B_\Phi \models \Phi(\bar{b}_\Phi)$. As above, let $B = \prod_{\Phi\in I} B_\Phi/\mathcal{U}$, $b_a = (b_{\Phi a} \mid \Phi \in I)/\mathcal{U}$, so that $B \models \Delta_x(A)(\bar{b})$. Let C be again the subalgebra of B generated by the set $\{b_a \mid a \in A\}$. We get as above that $C \in \mathsf{S}_\exists\mathsf{P}_u\mathsf{P}_\omega(\mathcal{C})$. Thus $A \in \mathsf{H}(C)$ for $x = e$ and $A \cong C$ for $x = q$ follow exactly as above.

It remains to show that $A \in \mathsf{HS}_\exists\mathsf{PP}_u(\mathcal{C})$ if $x = e$ and $A \in \mathsf{S}_\exists\mathsf{PP}_u(\mathcal{C})$ if $x = q$. Here, we fix $\neg\chi \in \Delta(A)^-$ and consider the set $I_\chi = \{\Phi \in I \mid \neg\chi \in \Phi^-\}$. Then there is a non-principal ultrafilter $\mathcal{U}_\chi$ on I which contains all sets $\{\Psi \in I_\chi \mid \Psi \supseteq \Phi\}$ with $\Phi \in I_\chi$. Take

$$B_\chi = \prod_{\Phi\in I_\chi} B_{\Phi,\chi}/\mathcal{U}_\chi; \quad b_{\chi a} = (b_{\Phi\chi a} \mid \Phi \in I_\chi)/\mathcal{U}_\chi; \quad \bar{b}_\chi = (b_{\chi a} \mid a \in A),$$

so that $B_\chi \models \neg\chi(\bar{b}_\chi)$ and $B_\chi \models \Delta^+(A)(\bar{b}_\chi)$ if $x = q$, $B_\chi \models \Delta^0(A)(\bar{b}_\chi)$ if $x = e$. Then

$$B' = \prod_{\neg\chi\in\Delta^-(A)} B_\chi \in \mathsf{PP}_u(\mathcal{C}); \quad B' \models \Delta_x(A)(\bar{b}'), \text{ where } b'_a = (b_{\chi a} \mid \chi \in \Delta^-(A)).$$

Let C' be the subalgebra of B' generated by the set $\{b'_a \mid a \in A\}$. As above, $C' \in \mathsf{S}_\exists(B')$ and $A \in \mathsf{H}(C')$ (if $x = e$) or $A \cong C'$ (if $x = q$) via the map $\varphi'\big(t(\bar{b}')\big) = t(\bar{a})$. The proof is now complete. □

The following recaptures [23, Proposition 10]. For convenience, we include proofs.

Proposition A.6. *Let $\mathcal{C}_0$ be a strongly regular class and let $\mathcal{C} \subseteq \mathcal{C}_0$.* hs1b

- (i) $\mathsf{S}_\exists\mathsf{H}(\mathcal{C}) \subseteq \mathsf{HS}_\exists(\mathcal{C})$;
- (ii) $\mathsf{V}_\exists(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{P}(\mathcal{C})$;
- (iii) *If all members of $\mathcal{C}_0$ have a distributive congruence lattice, then $A \in \mathsf{W}_\exists(\mathcal{C})$ for any subdirectly irreducible structure $A \in \mathsf{V}_\exists(\mathcal{C})$.*

Proof. (i) Let structures A, B and C be such that $A \in \mathcal{C}$, $C \in \mathsf{S}_\exists(B)$, and let $\varphi\colon A \to B$ be a surjective homomorphism. Then $B, C \in \mathcal{C}_0$ by Definition A.1(ii). Choose a Skolem expansion C^* of C and extend it to a Skolem expansion B^* of B. According to Remark A.3, there is a Skolem expansion A^* of A such that $\varphi\colon A^* \to B^*$ is a homomorphism. Then $C^* \in \mathsf{S}(B^*) \subseteq \mathsf{SH}(A^*) \subseteq \mathsf{HS}(A^*)$, whence $C^* \in \mathsf{H}(D^*)$ for some $D^* \in \mathsf{S}(A^*)$ and $C \in \mathsf{H}(D)$ with $D \in \mathsf{S}_\exists(A)$.

(ii) According to Theorem A.5(iv), $\mathsf{V}_\exists(\mathcal{C}) = \mathsf{HS}_\exists\mathsf{PP}_u(\mathcal{C})$. Straightforward inclusions $\mathsf{P}_u(\mathcal{C}) \subseteq \mathsf{HP}(\mathcal{C})$ and $\mathsf{PH}(\mathcal{C}) \subseteq \mathsf{HP}(\mathcal{C})$ together with (i) imply:

$$\mathsf{V}_\exists(\mathcal{C}) \subseteq \mathsf{HS}_\exists\mathsf{PHP}(\mathcal{C}) \subseteq \mathsf{HS}_\exists\mathsf{HP}(\mathcal{C}) \subseteq \mathsf{HS}_\exists\mathsf{P}(\mathcal{C}).$$

The reverse inclusion is obvious.

(iii) Let $A \in \mathsf{V}_\exists(\mathcal{C})$ be subdirectly irreducible. Then by (ii), there is $B \in \mathsf{S}_\exists\mathsf{P}(\mathcal{C})$ such that $A \in \mathsf{H}(B)$. By Jónsson's Lemma, there is $C \in \mathsf{SP}_u(\mathcal{C})$ such that $A \in \mathsf{H}(C)$ and $C \in \mathsf{H}(B)$. The latter inclusion implies by Definition A.1(ii) that $C \in \mathcal{C}_0$, whence $C \in \mathsf{S}_\exists\mathsf{P}_u(\mathcal{C})$. □

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