VARIETIES OF $*$-REGULAR RINGS. ERRATUM

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Abstract. Given a subdirectly irreducible $*$-regular ring R, we show that R is a homomorphic image of a regular $*$-subring of an ultraproduct of the (simple) eRe, e in the minimal ideal of R; moreover, R (with unit) is directly finite if all eRe are unit-regular. For any subdirect product of artinian $*$-regular rings we construct a unit-regular and $*$-clean extension within its variety.

0. Erratum

Lemma 5 below is incorrect and so is Thm.4. A conterexample will be giben, elsewhere.

1. Introduction

Studying (von Neumann) regular and $*$-regular rings from an Universal Algebra perspective was introduced by Goodearl, Menal, and Moncasi [7] (free regular rings) and Tyukavkin [15], who showed unit-regularity for subdirect products of countably many $*$-regular rings which are algebras over a given uncountable field. These also relate to the open question, raised by Handelman [6, Problem 48], whether all $*$-regular rings are directly finite or even unit-regular.

Tyukavkin's method of matrix limits has been elaborated by Micol [13], Niemann [14], Semenova, and this author [10, 11], to deal with $*$-regular rings which are representable as $*$-rings of endomorphisms of anisotropic inner product spaces (given by ε-hermitean forms). In particular, a variety (pseudo-inversion being included into the signature) of $*$-regular rings is generated by artinians if and only if its subdirectly irreducible members are representable [13, Theorem 3.18, Proposition 3.6]. A variant of the method constructs a $*$-regular preimage of R within a saturated extension of given $*$-regular R. This was used in [8] to prove that representable $*$-regular rings are directly finite. It follows, that within the variety generated by artinian $*$-regular rings all members are directly finite.

The present note continues to study representability, unit-regualrity, and direct finiteness from an Universal Algebra point of view. Any variety of $*$-regular rings is shown to be generated by its simple members - the analogous result for modular ortholattices is due to [9]. It follows, by [11, Theorem 10.1], that a variety has all subdirectly irreducibles representable if all simple members are representable. We also use the approach of [8] to show direct finiteness for subdirectly irreducible $*$-regular rings R with unit-regular eRe for all projections e in the minimal ideal. Finally, any subdirect product of artinian $*$-regular rings is shown to have a unit-regular and $*$-clean extension within its variety.

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2. Preliminaries

Recall some basic concepts and results from [1, 2, 6]. The signature of $*$-rings includes $*, +, -, \cdot, 0$ with or without unit; in the former case, unit 1 is included into the signature. Sections 5 and 6 require rings with unit. A *$*$-ring* R is a ring having as additional operation an involution $a \mapsto a^*$.

An element e of R is a *projection*, if $e = e^2 = e^*$. R is *proper*, if $aa^* = 0$ only for $a = 0$. A proper $*$-ring R is *$*$-regular* if it is (von Neumann) regular, that is for any $a \in R$, there is $x \in R$ such that $axa = a$; in particular, for any $a, b \in R$ there is c such that $aR + bR = cR$ and for any $a \in R$ there are (unique) projections e, f such that $aR = eR$ and $Ra = Rf$. The set $P(R)$ of projections is partially ordered by

$$e \leq f \Leftrightarrow fe = e \Leftrightarrow ef = e.$$

If $e \in P(R)$ then eRe is a $*$-regular ring with unit e, a $*$-subring of R if unit is not considered. In case of $*$-rings with unit, eRe is a homomorphic image of the regular $*$-subring $eRe+(1-e)R(1-e)$ of R, isomorphic to $eRe \times (1-e)R(1-e)$.

Considering $*$-rings R with unit, R is *unit-regular* if for any $a \in R$ there is invertible $u \in R$ (a *unit quasi-inverse*) such that $aua = a$. R is *$*$-clean* if for any $a \in R$ there are an invertible u and a projection e in R such that $a = u + e$. R is *directly finite* if $xy = 1$ implies $yx = 1$; this is implied by unit-regularity.

For the remainder of this Section and Sections 1 and 4, R will be a $*$-regular ring with or without unit; otherwise, unit is included into the signature. Recall that for any $a \in R$ there is a [Moore-Penrose] *pseudo-inverse* (or Rickart relative inverse) $a^+ \in R$, that is

$$a = aa^+a, \quad a^+ = a^+aa^+, \quad (aa^+)^* = aa^+, \quad (a^+a)^* = a^+a$$

cf. [12, Lemma 4]. In this case, a^+ is uniquely determined by a (and it follows $(a^*)^+ = (a^+)^*$). Thus, a sub-$*$-ring S of a $*$-regular ring R is $*$-regular if and only if it is closed under the operation $a \mapsto a^+$. Obviously, pseudo-inversion is compatible with surjective homomorphisms and direct products. Thus, including pseudo-inversion into the signature, $*$-regular rings form a variety and we speak of varieties of $*$-regular rings. $\exists$-varieties of $*$-regular rings (cf [10, 11]) are just the $*$-ring reducts of the latter. From [10, Proposition 9] one has the following useful property of pseudo-inversion.

(1) For any $a, e \in R$, e a projection, if $ae = 0$ then $(a^+)^*e = 0$ and if $a^*e = 0$ then $a^+e = 0$.

Observe that any ideal I of R is generated by the set $P(I)$ of projections in I, more precisely

$$(2) \quad I = \bigcup_{e \in P(I)} eR = \bigcup_{e \in P(I)} Re = \{a^* \mid a \in I\}.$$

Therefore, the congruence lattice of R is isomorphic to the lattice of ideals of the ring reduct of R. In particular, R is subdirectly irreducible if and only if it admits a unique minimal ideal, to be denoted by J.

Since in a regular ring the sets of right (left) ideals are closed under sums, it follows from (2)

(3) For any ideal I, given finite $A \subseteq I$, there is $e \in P(I)$ such that $fa = a = af$ for all $a \in A$ and $f \geq e$.

A $*$-regular ring R is *artinian* if there is a finite bound on the length of chains of projections; equivalently the ring R is right (left) artinian; in particular, if S is a $*$-subring, closed under $a \mapsto a^+$, of $\prod_{i=1}^n R_i$ with artinian R_i then S is artinian. Artinian $*$-regular rings are semisimple whence both unit-regular and $*$-clean (cf. [6, Theorem 4.1] and [16, Proposition 4]).

3. Tyukavkin approximation revisited

In order to have the method available in various contexts, we define the notion of approximation setup (both for the case with and without unit) and also for $*$-Λ-algebras. Recall from [11] that the latter are Λ-algebras R where Λ is a commutative $*$-ring and $(\lambda a)^* = \lambda^* a^*$ for all $\lambda \in \Lambda$ and $a \in R$. Any $*$-ring is a $*$-$\mathbb{Z}$-algebra.

An *approximation setup* now consists of $*$-regular $*$-Λ-algebras R and T, R a $*$-Λ-subalgebra of T, a set P of projections of T, and a filter $\mathcal{F}$ on P such that $\emptyset \notin \mathcal{F}$ and $\{p \in P \mid p \geq e\} \in \mathcal{F}$ for all $e \in P$. Moreover, we require the following

(a) For all $a \in R$, if $ae = 0 = ea$ for all $e \in P$ then $a = 0$.
(b) For all $a, b \in R$ and $e \in P$ there is $f \in P$, $f \geq e$, such that $ea = eaf$ and $be = fbe$.

Now, define $A_e = eTe$ and $A = \prod_{e \in P} A_e$, which are $*$-regular $*$-Λ-algebras. For $a \in R$ and $\alpha = (a_p \mid p \in P) \in A$ define

$$a \sim \alpha \Leftrightarrow \forall e \in P. \exists X \in \mathcal{F}. \forall p \in X.\ ea = ea_p \ \&\ ae = a_p e.$$

Lemma 1. *$S = \{\alpha \in A \mid \exists a \in R.\ a \sim \alpha\}$ is a $*$-regular $*$-Λ-subalgebra of A and there is a surjective homomorphism $\varphi : S \to R$ such that $\varphi(\alpha) = a$ if and only if $a \sim \alpha$. Moreover, for the canonical homomorphism ψ from A onto the reduced product $A/\mathcal{F}$ there is a surjective homomorphism $\chi : A/\mathcal{F} \to R$ such that $\varphi = \chi \circ \psi$.*

Proof. Consider $a \sim \alpha = (a_p \mid p \in P)$ and $b \sim \beta = (b_p \mid p \in P)$ and $e \in P$. Choose f according to (b). Choose X_1, Y_1 in $\mathcal{F}$ witnessing $a \sim \alpha$ for e respectively $b \sim \beta$ for f; choose X_2, Y_2 in $\mathcal{F}$ witnessing $b \sim \beta$ for e respectively $a \sim \alpha$ for f. Put $Z = X_1 \cap Y_1 \cap X_2 \cap Y_2$. Then one has for all $p \in Z$

$$\begin{array}{ccccccccc} eab & = & eafb & = & eafb_p & = & eab_p & = & ea_p b_p \\ abe & = & afbe & = & a_p fbe & = & a_p be & = & a_p b_p e. \end{array}$$

This shows $ab \sim \alpha\beta$. Closure of S under the other fundamental $*$-Λ-algebra operations is even more obvious. In case of unit (as constant) one has $1 \sim \alpha$, $\alpha = (p \mid p \in P)$ the unit of A, witnessed by $X = \{p \in P \mid p \geq e\}$ for $e \in P$. In view of (a), φ is a well defined map. To prove surjectivity of φ, given $a \in R$ put $a_p = pap$ and $\alpha = (a_p \mid p \in P)$. Now given e, choose f according to (b) with $a = b$, and $X = \{p \in P \mid p \geq f\}$; then $ea = eap = epap = ea_p$ and, similarly, $ae = a_p e$ for all $p \in X$. This proves $a \sim \alpha$.

To prove regularity of S, having $\operatorname{im}\varphi = R$ regular, it suffices to show that $\mathsf{ker}\varphi$ is regular (see [6, Lemma 1.3]). As in the proof of Assertion 22 in [10] we will show that $\mathsf{ker}\varphi$ is closed under pseudo-inversion in A. Consider $0 \sim \alpha = (a_p \mid p \in P)$. For any $e \in P$ there is $X \in \mathcal{F}$ such that $a_p e = 0 = a_p^* e$ for all $p \in X$ and by (1) it follows $(a_p^+)^* e = 0 = a_p^+ e$ for all $p \in X$. Thus, $0 \sim \alpha^+$. The last claim folllows from $\mathsf{ker}\psi \subseteq \mathsf{ker}\varphi$. □

4. Simple generators

Lemma 2. *For subdirectly irreducible R and any $0 \neq a \in R$ there is $e \in P(J)$, J the minimal ideal of R, such that $eae \neq 0$; also, for any nonzero $e \in P(J)$, eRe is a simple $*$-regular ring with unit e.*

Proof. As R is subdirectly irreducible, for any $a \neq 0$ the ideal I generated by a contains J; that is, for any $f \in P(J)$ one has $f = \sum_i r_i a s_i$ whence $f = \sum_i f r_i a s_i f$ for suitable $r_i, s_i \in R$. Now, choosing $f \neq 0$, by (3) there is $e \in P(J)$ such that $f r_i e = f r_i$ and $e s_i f = s_i f$ for all i whence $eae \neq 0$. By the same token, for $0 \neq a \in eRe$, where $0 \neq e \in P(J)$, one has $e = \sum_i r_i a s_i$ whence $e = \sum_i e r_i e a e s_i e$ is in the ideal of eRe generated by a. □

Theorem 3. *Any subdirectly irreducible $*$-regular ring R (with or without unit) is a homomorphic image of a $*$-regular $*$-subring of an ultraproduct of $*$-rings eRe, where e is a nonzero projection in the minimal ideal J. In particular, any variety of $*$-regular rings is generated by its simple members.*

Proof. We apply Lemma 1 with $A = \prod_{e \in P}$ where P is a cofinal subset of the set $P(J)$ of projections in the minimal ideal J such that $0 \notin P$. Since the sets $\{p \in P \mid p \geq e\}$, $e \in P(J)$, form a filter base, there is an ultrafilter $\mathcal{F}$ on P containing all these. This provides an approximation setup since conditions (a) and (b) are satisfied in view of Lemma 2 and (3). □

5. Direct finiteness

Theorem 4. *A subdirectly irreducible $*$-regular ring R is directly finite provided that the eRe, $e \in P(J)$, are unit-regular.*

Proof. Let C be the center of R. By hypothesis, the gRg, $g \in P(J)$, are directly finite and so are the $C(1-g)$, obviously. Hence, the direct products $gRg + C(1-g)$ and their directed union A are directly finite, too. Consider R with designated subset A, denoted as $(R; A)$.

In analogy to the proof of [11, Theorem 10.1] (where $A = \hat{J}(V_F)$) choose an ω-saturated elementary extension $(\hat{R}; \hat{A})$ of $(R; A)$ (such an extension exists, cf. [3, Corollary 4.3.1.4]). Also, choose $S = \{a \in \hat{A} \mid \exists r \in R.\, a \sim r\}$ where $a \sim r$ if $ae = re$ and $a^*e = r^*e$ for all $e \in P(J) =: J_0$. Then S is a $*$-subring of $\hat{A}$ and $a \mapsto r$ for $a \sim r$ a homomorphism φ of S onto R. This follows as Claims 1–3 in the proof of [11, Theorem 10.1], if reference to Propositions 2 and 4.4(iv) in [11] is replaced by that to Lemma 2 and (3). Moreover, in view of (1), $\ker \varphi$ is closed under pseudo-inversion in $\hat{R}$. By [6, Lemma 1.3] it follows that S is regular, whence $*$-regular.

Now, the proof of [8, Theorem 3] (with $\hat{A} = \tilde{A}$) carries due to the following. Indeed, a given finite subset of the set $\Sigma(x)$ of formulas, as mentioned there, can be satisfied choosing e as there, g, t, u according to Lemma 5, and $x = t + 1 - g$, $y = u + 1 - g$. □

Lemma 5. *Consider a regular ring R with ideal I such that each eRe, $e \in I$, is unit-regular. Then for any $r, s \in R$ with $sr = 1$ and idempotent $e \in I$ there are an idempotent $g \in I$, $e \in gRg$, and $t, u \in gRg$ such that $ut = g$, $te = re$, and $et = er$.*

Proof. Following [4] we consider R the endomorphism ring of a (right) R-module, namely $M_R = R_R$. Observe that r is an injective endomorphism of M_R. Let $U = \operatorname{im} e$, $W_1 = U + r^{-1}(U)$, $W_2 = r(W_1)$; in a particular, these are submodules of M_R and $r|W_1$ is an isomorphism of W_1 onto W_2. By (the proof of) [5, Lemma 2] there is an idempotent $g \in I$ such that $e, re, se \in S := gRg$. Put $W = \operatorname{im} g$ which is a submodule of M_R, and an S-module under the induced action of S, so that $S = \operatorname{End}(W_S) = \operatorname{End}(W_R)$.

By hypothesis, S is unit-regular whence, in particular, directly finite. Due to regularity of S, for any $h \in S$ and S-linear map $\varphi : hS \to W$ there is an extension $\bar{\varphi} \in S$, namely $\bar{\varphi}|(g-h)S = 0$. Due to direct finiteness, any injective such φ has an inverse in S. Also, by regularity, the submodules $W_1 = \operatorname{im} e + \operatorname{im} se$ and $W_2 = \operatorname{im} e + \operatorname{im} re$ are of the form $W_i = \operatorname{im} g_i$ with idempotents $g_i \in S$.

Let $X_i = \operatorname{im}(g - g_i)$ whence $W = W_i \oplus X_i$. Since $r|W_1 : W_1 \to W_2$ is an S-linear isomorphism, according to [4, Theorem 3] there is an S-linear isomorphism $\varepsilon : X_1 \to X_2$. Put $\delta(v) = \varepsilon(v) + g_2(r(v))$ for $v \in X_1$. If $\delta(v) = w \in W_2$ then $\varepsilon(v) \in W_2 \cap X_2$ whence $\varepsilon(v) = 0$ and $v = 0$; it follows that δ is an S-linear isomorphism of X_1 onto $Y \subseteq W$ where $Y \cap W_2 = 0$. Also, $g_2(\delta(v)) = g_2(r(v))$ since $g_2(X_2) = 0$. Define $t \in S$ as $t(v + w) = r(v) + \delta(w)$ for $v \in W_1$ and $w \in X_1$. t is injective whence it has inverse u in S. □

Corollary 6. *A variety of $*$-regular rings has all members directly finite if all its simple members are unit-regular.*

6. Unit-regular extensions

Theorem 7. *Every subdirect product R of artinian $*$-regular rings has a unit-regular and $*$-clean extension within its variety.*

The same holds for each first order Π_1- sentence (that is, prenex with quantification of the form $\forall \dots \forall \exists \dots \exists$) which is valid in all artinian $*$-regular rings.

Proof. By hypothesis, there are ideals I_k of R, $k \in K$, such that each R/I_k is artinian and $\bigcap_{k \in K} I_k = 0$. The first order structure $(R; I_k(k \in K))$ has a ω-saturated elementary extension $(\tilde{R}, \tilde{I}_k(k \in K))$. In particular, each $*$-regular ring $\tilde{R}/\tilde{I}_k$ is artinian (having the same finite bound on the length of chains of projections as does R/I_k). For $F \subseteq K$ put $\tilde{I}_F = \bigcap_{k \in F} \tilde{I}_k$ and $\tilde{R}_F := \tilde{R}/\tilde{I}_F$. It follows that, for any finite $F \subseteq K$ the subdirect product $\tilde{R}_F$ is also artinian, whence unit-regular. On the other hand, $\tilde{R}_K$ is a subdirect product of artinians and R embeds into $\tilde{R}_K$, canonically, since $\tilde{I}_k \cap R = I_k$ for all $k \in K$.

Now, fix $r \in R$ and consider the following set Σ of formulas with parameter r and variables x, y

$$\{I_k(xy - 1) \;\&\; I_k(yx - 1) \;\&\; I_k(rxr - r) \mid k \in K\}.$$

Given finite $\Sigma_0 \subseteq \Sigma$ there is finite $F \subseteq K$ such that predicate symbols occurring in Σ_0 have $k \in F$. Thus, Σ_0 is satisfied substituting, for x, y, elements $u, s \in \tilde{R}$ witnessing that $r + \tilde{I}_F$ has unit quasi-inverse u in $\tilde{R}_F$. By saturation there are $u, s \in \tilde{R}$ satisfying Σ; that is, u is a unit quasi-inverse of r in $\tilde{R}_K$. By the same approach one obtains unit v and projection e in $\tilde{R}_K$ such that $r = v + e$.

To summarize, any subdirect product R_n of artinians has an extension R_{n+1}, within the variety of R_n, which is a subdirect product of artinians and such that for every $r \in R_n$ there are units u, v and projection e in R_{n+1} with $r = rur$ and $r = v + e$. Thus, starting with $R = R_0$, the directed union $\bigcup_{n<\omega} R_n$ is the required extension. □

Corollary 8. *Within any variety $\mathcal{V}$ generated by artinian $*$-regular rings, every member is a homomorphic image of a $*$-regular $*$-subring of some unit-regular and $*$-clean member of $\mathcal{V}$.*

Proof. For any variety, given a class $\mathcal{G}$ of generators, free algebras are subdirect products of subalgebras of members of $\mathcal{G}$. □

7. Open problems

Let $\mathcal{A}$ denote the variety generated by artinian $*$-regular rings. Recall that subdirectly irreducible members of $\mathcal{A}$ are representable, whence all members of $\mathcal{A}$ are directly finite.

(i) Is every $*$-regular ring directly finite?
(ii) Is every unit-regular $*$-regular ring a member of $\mathcal{A}$?
(iii) Is every subdirectly irreducible member of $\mathcal{A}$ unit-regular?
(iv) Is a subdirectly irreducible $*$-regular ring R unit-regular provided so are all eRe, e a projection in the minimal ideal?

Observe that (iv) implies (iii).

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