PERSPECTIVITY IN COMPLEMENTED MODULAR LATTICES AND REGULAR RINGS

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Abstract. Based on an analogue for systems of partial isomorphisms between lower sections in a complemented modular lattice we prove that principal right ideals $aR \cong bR$ in a (von Neumann) regular ring R are perspective if $aR \cap bR$ is of finite height in $L(R)$. This is applied to derive, for existence-varieties $\mathcal{V}$ of regular rings, equivalence of unit-regularity and direct finiteness, both conceived as a property shared by all members of $\mathcal{V}$.

1. Introduction

(Von Neumann) regular rings R and complemented modular lattices are closely connected fields since the work of von Neumann cf. [19] - with R one associates its lattice $L(R)$ of principal right ideals. *Unit-quasi-inverses* u of elements a (i.e. $aua = a$) have been introduced by Ehrlich [4, 5], a ring being *unit-regular* if each element admits some unit-quasi-inverse (such rings are, in particular *directly finite*: $ab = 1$ implies $ba = 1$). Ehrlich also showed that a regular ring R is unit-regular if and only if for all idempotents e, f one has $eR \cong fR$ implying $(1-e)R \cong (1-f)R$. Handelman [11] added further equivalent conditions, one of them being that $eR \cong fR$ implies eR perspective to fR in $L(R)$. Perspectivity, regularity, and unit-regularity of elements in general rings have been intensively studied, see e.g. [6, 16, 17, 18].

The purpose of the present note is to give a sufficient condition on $aR \cong bR$ in a regular ring R granting that aR is perspective to bR (and thus a to have a unit-quasi-inverse) and to show that this applies if $aR \cap bR$ is of finite height in $L(R)$.

Here, establishing perspectivity relies on calculations in $L(R)$, for convenience done in abstract complemented modular lattices endowed with a system of isomorphisms between lower sections requiring properties present in the case of isomorphisms induced by isomorphisms between principal right ideals. The principal result is a reduction process associating $e_{n+1} \leq e_n$ and $f_{n+1} \leq f_n$ with given e_n, f_n such that e_n is perspective to f_n (and so e_0 perspective to f_0) if $e_{n+1} \cap f_{n+1} = e_n \cap f_n$

or if e_{n+1} is perspective to f_{n+1}. In $\aleph_0$-complete complemented modular lattices e_0 is perspective to f_0 if the meet of the e_n is perspective to the meet of the f_n.

If one considers regular rings endowed with an operation of quasi-inversion, termination of this reduction process after n-steps can be captured by an identity. This is applied to study unit-regularity in the context of *existence varieties* $\mathcal{V}$ of regular rings, that is, classes closed under homomorphic images, direct products, and regular subrings. It is shown that for such class unit-regularity is equivalent to direct finiteness, both considered as a property required for all members. (Compare this to the result of Baccella and Spinosa [2] that a semiartinian regular ring is unit-regular if and only if all its homomorphic images are directly finite.) Another property shown equivalent to unit-regularity is that $\mathcal{V}$ does not contain nonartinian subdirectly irreducibles. Though, having $\mathcal{V}$ generated by artinians is not sufficient for direct finiteness in view of the result, established by Goodearl, Menal, and Moncasi [8, Thm. 2.5], that free regular rings are residually artinian (and, according to Herrmann and Semenova [13, Cor. 14], even residually finite). Varieties of $*$-regular rings (with pseudo-inversion) generated by artinians have all members directly finite [14] (and may contain nonartinian simple members) but unit-regularity remains an open problem.

2. Complemented modular lattices

2.1. **Preliminaries.** We refer to Birkhoff [3] and von Neumann [19]. A *lattice* L is a set endowed with a partial order $\leq$ such that any two elements a, b have infimum and supremum written as *meet* $a \cap b$ and *join* $a + b$. We also write $ab = a \cap b$ and apply the usual preference rules. All lattices to be considered will have smallest element 0 and greatest element 1.

For $u \leq v$ in L, the *interval* $[u, v] = \{x \mid u \leq x \leq v\}$ is again a lattice with the inherited partial order and operations. A lattice L is *modular* if

$$b \leq a \Rightarrow a(b + c) = b + ac.$$

Then the maps $x \to x + b$ and $y \to ay$ are mutually inverse isomorphisms between $[ab, a]$ and $[b, b + a]$. An element u of a modular lattice L is *neutral* if $(u + x)(u + y) = u + xy$ for all $x, y \in L$; the set of neutral elements is a sublattice of L. An element a of a modular lattice L is of *height* d if some (equivalently: each) maximal chain in $[0, a]$ has *length* d, that is $d + 1$-elements. For the following see [3, Ch. III Thm. 15].

Fact 1. *In a modular lattice, the direct product* $[ab, a] \times [ab, b]$ *embeds into* $[ab, a+b]$ *via* $(x, y) \mapsto x + y = (x+b)(y+a)$.

If $ab = 0$ then we write $a + b = a \oplus b$. If $a \oplus b = 1$ then b is a *complement* of a. A lattice L is *complemented* if each element admits some complement. If L is, in addition, modular then we speak of a CML. In a CML each interval $[u, v]$ is again a CML (within $[u, v]$, a complement of x is given by $yv + u = (y + u)v$ where $x \oplus y = 1$ in L).

In a CML, elements a, b are *perspective*, written as $a \sim b$, if they have a common complement; equivalently, $a \sim_c b$ for some c, the latter meaning that $a + b = a + c = b + c$ and $ab = ac = bc$. We write $a \approx_c$ if $a \sim_c b$ and $ab = 0$; also $a \approx b$ if $a \approx_c b$ for some c. Applying Fact 1 one obtains the following.

Fact 2. *In a modular lattice, if* $a_i \sim_{c_i} b_i, i = 1, 2$ *and* $a_1b_1a_2b_2 \geq (a_1 + b_1)(a_2 + b_2)$ *then* $a_1 + b_1 \sim_{c_1+c_2} a_2 + b_2$.

Fact 3. *In a modular lattice one has* $a \sim b$ *if and only if* $x \sim y$ *for some (equivalently: all)* x, y *such that* $a = x \oplus ab$ *and* $b = y \oplus ab$, *Moreover, one has* $a \oplus y = a + b = b \oplus x$ *for such* x, y.

Proof. Observe that for such x, y one has $ay = aby = 0$ whence by modularity $a(x + y) = x$ and, similarly, $bx = 0$ and $b(x + y) = y$. By modularity it follows that $ab(x+y) = 0$. Now one has $ab+x+y = a+b$ so that the map $z \mapsto z+ab$ is an isomorphism of $[0, x+y]$ onto $[ab, a+b]$. Moreover, $a+y = a+ab+y = a+b$. Thus $a \oplus y = a+b$ and, similarly, $b \oplus x = a + b$. □

2.2. **Two lemmas on modular lattices.**

Lemma 4. *In a modular lattice, if* $z = x \oplus y$, $w = u \oplus v$, *and* $zw = xu$ *then* $yv = 0$. *If, in addition,* $x \sim u$ *and* $y \sim v$ *then also* $z \sim w$.

Proof. Clearly, $yv \leq yzw \leq yx = 0$. Moreover, by modularity one has $(x + w)z = x + wz = x$ whence $(x + w)y = (x + w)zy = xy = 0$ and, similarly, $(u + z)v = 0$. Now, by Fact 1 $(x + w)(u + z) = x + u$ it follows $(y + v)(x + u) = (y + v)(x + w)(u + z) = [y(x + w) + v)](u + z) = v(u + z) = 0$ by modularity. In particular, this implies $xu(y + v) = 0$ establishing the isomorphism $r \mapsto r + xu$ of $[0, y + v]$ onto $[xu, y + xu + v]$. Thus, if $y \sim v$ then one has also $y + xu \sim v + xu$. Assuming that, in addition, $x \sim u$ one derives $z \sim w$ by Fact 2 since $(x + u)(y + xu + v) = (x + u)(y + v) + xu = xu$. □

Lemma 5. *In modular lattice,* L, $a \sim b$ *and* $d = (a + d)(b + d)$ *jointly imply* $a + d \sim b + d$.

Proof. Assume $a \sim_c b$ in L and let S denote the sublattice S generated by a, b, c, d. Let D_2 and M_3 denote the 2-element lattice and the height 2 lattice with 3 atoms, respectively. Obviously. S is also generated by c, and the two chains $a \leq a+d$ and $b \leq b+d$. Thus, by [12] S is a sub direct product of lattices D_2, M_3, and lattices M where (the images of) $a, a{+}d, b, b{+}d$ generate a boolean sublattice with $(a{*}b)(a{+}d)(b{+}d) = 0$ and $a+d+b = 1$ and where (the image of) c is a common complement of the "atoms" $d = (a+d)(b+d)$, a, and b. Since also $a \sim_c b$, M is easily seen to be trivial in both cases..

Thus, S is a subdirect product of factors D_2 and M_3, only. Due to the given relations, in any of these factors the images of d and $a+b$ take value 0 or 1, only; this means that $a+b$ and d are neutral elements of S. It follows, that $u = d(a+b)$ is neutral, too, whence $a \sim_c b$ implies $a+u \sim_{c+u} b+u$ and this in turn $a+d \sim_{c+d} b+d$ via the isomorphism of $[u, a+b]$ onto $[d, 1]$. □

2.3. **Partial isomorphisms and perspectivity.** Motivated by the case where L is the lattice of principal right ideals of a regular ring, for any CML L we consider lattice isomorphisms $\alpha : [0, e] \to [0, f]$, shortly written as $\alpha : e \to f$. For $g = e \cap f$ one has both $\alpha^{-1}(g)$ and $\alpha(g)$ defined. If $e' \leq e$ let $\alpha_{|e'}$ denote the restriction of α to $[0, e']$, that is $\alpha_{|e'} : e' \to f'$, $f' = \alpha(e')$.

Observation 6. *$\alpha^{-1}(g) \cap \alpha(g) \leq g$ and equality holds if and only if $\alpha(g) = g$.*

Proof. Clearly, $\alpha^{-1}(g) \cap \alpha(g) \leq \alpha^{-1}(f) \cap \alpha(e) = e \cap f = g$. Also, if $\alpha(g) = g$ then $\alpha^{-1}(g) = g$ whence $\alpha^{-1}(g) \cap \alpha(g) = g$. Conversely, if the latter holds then $g \leq \alpha(g)$ and $\alpha(g) \leq \alpha(\alpha^{-1}(g)) = g$ and it follows $g = \alpha(g)$. □

We introduce a reduction process which yields perspectivity provided that it stops after finitely many steps. With $g = e \cap f$ one has $\alpha^{\#} : \alpha^{-1}(g) \to \alpha(g)$ defined by $\alpha^{\#}(x) = \alpha^2(x)$.

We consider *admissible* systems A of isomorphisms $\alpha : e \to f$ requiring the following axioms:

(A1) If $\alpha : e \to f$ is in A and $e' \leq e$ then $\alpha_{|e'}$ is in A.
(A2) If $\alpha : e \to f$ is in A and $e \cap f = 0$ then $e \approx f$.
(A3) If $\alpha : e \to f$ is in A and $g = e \cap f$ then $\alpha^{\#}$ is in A.

Lemma 7. *Consider $\alpha : e \to f$ in admissible A.*

(i) *If $\alpha(e \cap f) = e \cap f$ then $e \sim f$.*
(ii) *If $\alpha^{-1}(e \cap f) \sim \alpha(e \cap f)$ then $e \sim f$.*

Proof. Let $g = e \cap f$. In (i) choose y so that $y \oplus g = e$ and put $v = \alpha(y)$. Then $f = \alpha(e) = \alpha(y \oplus g) = \alpha(y) \oplus \alpha(g) = v \oplus g$. With $x = u = g$ in Lemma 4 it follows $yv = 0$ whence $y \sim v$ by axioms (A1) and (A2); moreover, $e \sim f$ by Lemma 4, again.

(ii): Put $x = g + \alpha^{-1}(g)$ and $u = g + \alpha(g)$, observe that $u = \alpha(x)$. Observe that $g \leq x \leq e$ and $g \leq u \leq f$ whence $x \cap u = g$. Thus, by hypothesis and Lemma 5 one has $x \sim u$. Choose y such that $y \oplus x = e$ and put $v := \alpha(y)$. Then $f = \alpha(x \oplus y) = \alpha(x) \oplus v$ and we conclude $y \cap v = 0$ by Lemma 4 whence $y \sim v$ by axioms (A1) and (A2); finally, $e \sim f$ by Lemma 4. $\square$

Given $\alpha : e \to f$ in admissible A define, by induction, $\alpha_0 = \alpha$, $e_0 = e$, $f_0 = f$ and $\alpha_{n+1} = \alpha_n^{\#} : e_{n+1} \to f_{n+1}$. Put $g_n = e_n \cap f_n$ and observe that $e_{n+1} \leq e_n$ and $f_{n+1} \leq f_n$ whence also $g_{n+1} \leq g_n$.

Theorem 8. *Given a CML, L, an admissible system A of partial isomorphisms in L, and $\alpha : e \to f$ in A, one has $e \sim f$ provided that $e_m \sim f_m$ for some m. In particular, this applies if $\alpha_m(g_m) = g_m$ respectively $g_{m+1} = g_m$ for some m or if $e \cap f$ is of finite height in L.*

Proof. If $e_m \sim f_m$ (which by (i) of Lemma 7 is the case if $\alpha_m(g_m) = g_m$) then $e_{m-1} \sim f_{m-1}$ by (ii) of Lemma 7 and it follows $e \sim f$ by induction. Now, assume $e \cap f$ of finite height in L. By Observation 6, $g_{m+1} < g_m$ unless $\alpha_m(g_m) = g_m$; thus, the latter has to occur for some m. $\square$

We now give for the case $g_{m+1} = g_m$ a proof with a single application of Lemma 5, only. A sequence $a_0, \ldots, a_m$ in a modular lattice is *independent*, if for all $n < m$, $a_n \sum_{n<k\leq m} a_k = 0$; equivalently if $(\sum_{n\in I} a_n)(\sum_{n\in J} a_n) = 0$ for all $I, J \subseteq \{0, \ldots, m\}$ such that $I \cap J = \emptyset$. Induction using Fact 2 yields the following.

Fact 9. *In a modular lattice, if $a_0 + b_0, a_1 + b_1, \ldots, a_m + b_m$ is independent and $a_n \approx b_n$ for all $n \leq m$ then $\sum_{n\leq m} a_n \approx \sum_{n\leq m} b_n$.*

Lemma 10. *Consider $\alpha_n : e_n \to f_n$ and g_n, $n \leq m+1$ as above and, for $n \leq m$, x_n such that $e_n = x_n \oplus (e_{n+1} + g_n)$ and $y_n = \alpha_n(x_n)$. Then the following hold.*

(i) *$f_n = y_n \oplus (g_n + f_{n+1})$ and $(e_k + f_k)(x_n + y_n) = 0$ for all $k > n$.*
(ii) *$x_n \approx y_n$ for all $n \leq m$.*
(iii) *$x_0, y_0, \ldots, x_m, y_m$ is an independent sequence.*
(iv) *$x := \sum_{k\leq m} x_k \approx y := \sum_{k\leq m} y_k$.*
(v) *If $g_m = g_{m+1}$ then $x + g_0 = e_0$, $y + g_o = f_0$, and $e_0 \sim f_0$.*

Proof. Recall that $e_{n+1} = \alpha_n^{-1}(g_n) \leq e_n$ and $f_{n+1} = \alpha_n(g_n) = \alpha_n^2(e_{n+1}) \leq f_n$ and observe that $e_n = x_n \oplus (\alpha_n^{-1}(g_n) + g_n)$. It follows that $f_n =$

$\alpha_n(e_n) = y_n \oplus (g_n + \alpha_n(g_n)) = y_n \oplus (g_n + f_{n+1})$ for $n \leq m$. Observe that $x_n y_n \leq e_n f_n = g_n$ whence $x_n y_n \leq x_n g_n = 0$. Thus $x_n \approx y_n$ by axiom (A2).

Moreover, modularity yields $(e_{n+1}+f_{n+1})(x_n+y_n) \leq (e_{n+1}+f_n)(x_n+y_n) = (e_{n+1}+f_n)x_n+y_n = (e_{n+1}+f_n)e_n x_n+y_n = (e_{n+1}+f_n e_n)x_n+y_n = (e_{n+1}+g_n)x_n+y_n = 0+y_n = y_n$ and so $(e_{n+1}+f_{n+1})(x_n+y_n) = (e_{n+1}+f_{n+1})y_n = (e_{n+1}+f_{n+1})f_n y_n = (e_{n+1}f_n+f_{n+1})y_n \leq (g_n+f_{n+1})y_n = 0$. It follows for all $n < m$

$$\left(\sum_{n<k\leq m} x_k + y_k\right)(x_n+y_n) \leq (e_{n+1}+f_{n+1})(x_n+y_n) = 0,$$

proving (iii); (iv) follows by Fact 9. Dealing with (v), observe that $e_{m+1} = g_m$ whence $g_0 + x_m \geq e_m$. Now, backward induction yields $g_0 + \sum_{m\geq k\geq n} x_k \geq e_n$ for all $n \leq m$, whence $g_0 + x = e_0$. Similarly, one has $g_0 + y = f_0$ and it follows $e_0 \sim f_0$ by Lemma 2.

□

2.4. **$\aleph_0$-complete complemented modular lattices.** A CML is $\aleph_0$-*complete* if supremum $\sum_n a_n$ and infimum $\prod_n a_n$ exist for all families $(a_n \mid n < \omega)$. According to Amemiya and Halperin [1, Thm.9.5] any such is also $\aleph_0$-*continuous*, i.e. $b\sum_n a_n = \sum_n b a_n$ (resp. $b + \prod_n a_n = \prod_n (b+a_n)$) if $(a_n \mid n < \omega)$ is upward (resp. downward) directed. A sequence $(a_n \mid n < \omega)$ is *independent* if each of its finite subsequences is independent.

Fact 11. *In an $\aleph_0$-complete CML, if $a_0+b_0, a_1+b_1, \ldots$ is independent and $a_n \approx b_n$ for all $n < \omega$ then $\sum_n a_n \approx \sum_n b_n$.*

Proof. Suppose $a_n \approx_{c_n} b_n$ for $n < \omega$ and write $x_n^+ = \sum_{m\leq n} x_m$ and $x_\omega = \sum_{n<\omega} x_n = \sum_{n<\omega} x_n^+$ for any sequence $x_0, x_1, \ldots$. By Fact 9 one has $a_n^+ \approx_{c_n^+} b_n^+$ for all n. Since sequences $x_0^+, x_1^+, \ldots$ are upward directed, $\aleph_0$-continuity yields $a_\omega b_\omega = (\sum_{n<\omega} a_n^+) b_\omega = \sum_{n<\omega} a_n^+ b_\omega = \sum_{n<\omega} a_n^+ \sum_{m<\omega} b_m^+ = \sum_{n<\omega}\sum_{m<\omega} a_n^+ b_m^+ \leq \sum_{n,m<\omega} a^+_{\max(n.m)} b^+_{\max(n,m)} = 0$. By symmetry one obtains $a_\omega c_\omega = b_\omega c_\omega = 0$ while $a_\omega + b_\omega = a_\omega + c_\omega = b_\omega + c_\omega$ is obvious. □

Theorem 12. *Assume that $\alpha : e \to f$ is member of an admissible system A of partial isomorphisms of an $\aleph_0$-complete CML, L, and that $\prod_{n<\omega} e_n \sim \prod_{n<\omega} f_n$ for e_n, f_n defined as in Subsection 2.3. Then it follows that $e \sim f$ in L.*

Proof. Given $\alpha : e \to f$ in A define $\alpha_n : e_n \to f_n$ as in Subsection 2.3 and $g_n = e_n f_n$. Put $e_\infty = \prod_n e_n$, $f_\infty = \prod_n f_n$, and $g_\infty = \prod_n g_n =$

$e_\infty f_\infty$ and recall that $e_{n+1} = \alpha_n^{-1}(g_n) \leq e_n$ and $f_{n+1} = \alpha_n(g_n) = \alpha_n^2(e_{n+1}) \leq f_n$.

Choose x_n such that $e_n = x_n \oplus (e_{n+1} + g_n)$ and $y_n = \alpha_n(x_n)$. By Lemma 10 one has $f_n = \alpha_n(e_n) = y_n \oplus (g_n + f_{n+1})$ and the sequence $x_0, y_0, x_1, y_1, \ldots$ is independent; thus, Fact 11 yields $x_\omega \approx y_\omega$ where $x_\omega = \sum_n x_n$ and $y_\omega = \sum_n y_n$. On the other hand, again by Lemma 10 and modularity,

$$\left(e_\infty + f_\infty + \sum_{n<k\leq m} x_k + y_k\right)(x_n + y_n) = 0 \text{ for all } m > n$$

that is $e_\infty + f_\infty, x_0 + y_0, x_1 + y_1, \ldots$ is an independent sequence, too. By hypothesis $e_\infty \sim f_\infty$ and in view of $(x_\omega + y_\omega)(e_\infty + f_\infty) = 0$ Fact 2 applies to yield $x_\omega + e_\infty \sim y_\omega + f_\infty$.

Now $g + e_{n+1} + x_\omega \geq e_{n+1} + g_n + x_n = e_n$ and by induction it follows $g + e_n + x_\omega = e$ for all n. Thus, by $\aleph_0$-continuity one has $g+x_\omega+e_\infty = \prod_n(g+x_\omega+e_n) = e$. Similarly, one obtains $g+y_\omega+f_\infty = f$. Finally, $e \sim f$ follows by Lemma 5. □

3. Regular rings

3.1. **Preliminaries.** A ring R (associative and with unit) is (von Neumann) *regular* if for each $a \in R$ there is a *quasi-inverse* or *inner inverse* $x \in R$ such that $axa = a$; equivalently, every right (left) principal ideal is generated by an idempotent, see Goodearl [7] and Wehrung [20]. For a regular ring, R, the principal right ideals form a complemented sublattice $L(R)$ of the lattice of all right ideals; in particular, $L(R)$ is modular. For artinian R, the height of $L(R)$ is the *length* of R.

An element a of R is *unit-regular* if there is a unit $u \in R$, a *unit-quasi-inverse*, such that $aua = a$. R is *unit-regular* if all its elements are unit-regular. Any such ring is *directly finite* (that is $ab = 1$ implies $ba = 1$), the converse not being true for regular rings, in general.

If e is an idempotent in a regular ring R, then the *corner* eRe is a regular ring with unit e, a homomorphic image of the regular subring $eRe + (1-e)R(1-e)$ of R.

Idempotents e, f are Murray von Neumann *equivalent* if $e = yx$ and $f = xy$ for some x, y. For the following see e.g. Handelman [11].

(1) For any a there is a *generalized* or *reflexive inverse* b such that $aba = a$ and $bab = b$, e.g. $b = xax$ where $axa = a$. Then ab and ba are idempotents.
(2) e, f are idempotents and equivalent if and only if $e = ba$ and $f = ab$ for some a, b as in (1). Moreover, in this case $\omega_{a,b}(r) = ar$

defines an isomorphism $\omega_{a,b} : bR = eR \to aR = fR$ of right R-modules with inverse $\omega_{b,a}$. It follows that $fae = a$ and $ebf = b$.

(3) For idempotents e, f, every R-module isomorphism $\omega : eR \to fR$ is as in (2) where $\omega(e) = a$ and $\omega^{-1}(f) = b$.

(4) If $eR \sim fR$ in $L(R)$ then $eR \cong fR$ as right R-modules. If $eR \cap fR = 0$ then the converse holds, too.

(5) If c is another generalized inverse of a then $bR \sim cR$ (being complements of $\{x \in R \mid ax = 0\}$) and $x \mapsto cax$ is an isomorphism of bR onto cR.

To prove (3), put $a = \omega(e)$ and $b = \omega^{-1}(f)$. Then one has $\omega(r) = \omega(er) = \omega(e)r = ar$ for $r \in eR$ and, similarly, $\omega^{-1}(s) = bs$ for $s \in fR$. Thus, $aba = \omega(ba) = \omega(\omega^{-1}(a)) = a$ and, similarly, $bab = b$.

A regular ring R is *perspective* if isomorphic direct summands of R_R are perspective in $L(R)$; equivalently, aR is perspective to bR for all $aR \cong bR$ - for a more general result see Mary [17, Thm.3.1].

Theorem 13. Handelman [11] *A regular ring is unit-regular if and only if it is perspective.*

An element a of R is *stronlgy π-regular* if there is n such that $a^n \in a^{n+1}R \cap Ra^{n+1}$. R is *stronlgy π-regular*if so are all its elements.

Theorem 14. Goodearl and Menal [10, Thm. 5.8]. *Strongly π-regular regular rings are unit-regular.*

In general rings, a strongly π-regular element is unit-regular provided all its powers are regular. For a detailed discussion and proofs see Khurana [15]

3.2. **Perspectivity.** The following "local version" of Handelman's Theorem should be well known.

Lemma 15. *Given a generalized inverse b of a in a regular ring R one has idemptents $e = ba$, $f = ab$, and g such that $gR = eR \cap fR$. Now, the following are equivalent.*

(i) *fR and eR are perspective in $L(R)$.*

(ii) *For some (all) idempotents e', f' such that $e'R \oplus gR = eR$ and $f'R \oplus gR = fR$ one has $e'R \cong f'R$.*

(iii) *For some (all) idempotents e', f' such that $e'R \oplus gR = eR$ and $f'R \oplus gR = fR$ there is a unit u of R such that $aua = a$ and $e'R \cong f'R$ via ue'.*

Proof. In view of (2) in the preceding subsection, $e = ba$ and $f = ab$ are idempotents. Consider e', f' as in (ii) and (iii) and observe that such exist since $L(R)$ is complemented.

By Fact 3 we have $e'R \cap f'R = 0$ and, moreover, $eR \sim fR$ if and only if $e'R \sim f'R$. By (4), the latter is equivalent to $e'R \cong f'R$. This proves that (i) is equivalent to (ii).

Now, assume (ii), in particular $f'R \cong e'R$ via some isomorphism ω'. Choose an idempotent h such that $eR + fR = hR$. Then

$$eR \oplus f'R \oplus (1-h)R = R = fR \oplus e'R \oplus (1-h)R,$$

again by Fact 3. In view of (2) define

$$\omega(r + s + t) = \omega_{b,a}(r) + \omega'(s) + t \text{ for } r \in fR, s \in e'R \text{ and } t \in (1-h)R$$

to obtain an automorphism of the right R-module $R = 1R$. By (3) there are u, v in R such that $\omega = \omega_{u,v}$; in particular, u is a unit and $v = u^{-1}$. Moreover, $ur = \omega_{b.a}(r) = br$ for $r \in fR$, in particular $ua = ba = e$. Thus $aua = ae = a$, proving that (ii) implies (iii).

Finally, assume (iii). Then $u^{-1}f' = e'$ whence $x \mapsto ux$ is an R-module isomorphism of $e'R$ onto $f'R$ with inverse $y \mapsto u^{-1}y$. Thus, (ii) and (iii) are equivalent, too. □

For $A, B \in L(R)$ and right module isomorphism $\omega : A \to B$ one has the induced lattice isomorphism $\omega_L : [0, A] \to [0, B]$. Let $A(R)$ denote the set of all these.

Lemma 16. *$A(R)$ is an admissible system of partial isomorphisms of $L(R)$.*

Proof. Consider $\omega : A \to B$ in $A(R)$ and observe that $\omega_L(X) = \omega(X)$ for all $X \leq A$. Thus, if $A' \leq A$ in $L(R)$ then $(\omega_{|A'})_L : [0, A'] \to [0, B']$ in $A(R)$ with $B' = \omega(A') \leq B$, proving axiom (A1). Similarly, for $C = A \cap B$, $A' = \omega^{-1}(C)$, and $B' = \omega(C)$ one has $\omega_L^{\#} = (\omega_{|C} \circ \omega_{|A'})_L$ in $A(R)$, proving axiom (A3). Finally, (A2) follows from (4). □

Corollary 17. *For A, B in the lattice $L(R)$ of principal right ideals of the regular ring R, if $A \cap B$ is of finite height in $L(R)$ then A, B are perspective in $L(R)$ if and only if they are isomorphic as R-modules.*

Proof. Assume $A \cap B$ of finite height. If $A \cong B$ then there is a lattice isomorphism $\alpha : [0, A] \to [0, B]$ in $A(R)$ and in view of Lemma 16 and Theorem 8 it follows that $A \sim B$. The converse follows from (4). □

Corollary 18. *An element a in a regular ring R is unit-regular provided that there is a reflexive inverse b of a such that $bR \cap aR$ is of finite height in $L(R)$.*

3.3. **Regular rings with operation of quasi-inversion.** A regular ring may be considered as an algebraic structure also endowed with an operation $a \mapsto a'$ of quasi-inversion. The class $\mathcal{R}$ of all these structures is then defined by the identities for rings with unit together with $xx'x = x$. As observed above, the term $x^+ = x'xx'$ then yields a generalized inverse a^+ of a and $\gamma(x) = xx^+$ yields idempotents $\gamma(a)$ such that $\gamma(a)R = aR$. For the following see Wehrung [20, Lemma 8-3.12].

Lemma 19. *There are binary terms $x \vee y$, $x \wedge y$, and $x \ominus y$ in the language of $\mathcal{R}$ such that, for all $R \in \mathcal{R}$ and $a, b \in R$, $a \vee b$, $a \wedge b$, and $a \ominus b$ are idempotent, $(a \vee b)R = aR + bR$, $(a \wedge b)R = aR \cap bR$, and $(a \ominus (a \wedge b))R \oplus (a \wedge b)R = aR$.*

Theorem 20. *For each natural number n there are binary terms $t_n(x, y)$, $u_n(x, y)$, and $p_n(x, y)$ in the language of $\mathcal{R}$ such that the following hold for all $R \in \mathcal{R}$ and mutually reflexive inverses $a, b \in R$: $t_n(a, b)$ is idempotent; moreover, if $t_{n+1}(a, b)t_n(a, b) = t_n(a, b)$ then*

(i) *bR and aR are perspective in $L(R)$: $bR \sim_{p_n(a,b)} aR$:*
(ii) *$u_n(a, b)$ is a unit such that $au_n(a, b)a = a$.*

If R is of length at most $n + 2$ then $t_{n+1}(a, b)t_n(a, b) = t_n(a.b)$ for all mutally reflexive inverses a, b in R.

Proof. With idempotents $e_0 = ba$ and $f_0 = ab$ one has the isomorphism $\omega_{ab} : e_0R \to f_0$ given by $x \mapsto ax$ inducing the isomorphism $\alpha = \alpha_0 : [0, eR] \to [0, fR]$ given by $\alpha(xR) = axR$ with inverse $\alpha^{-1}(xR) = bxR$. Recalling the construction in Subsection 2.3 put $g_0 = e_0 \wedge f_0$ and, recursively,

$$g_n = e_n \wedge f_n,\ e_{n+1} = \gamma(b^{2^n} g_n),\ f_{n+1} = \gamma(a^{2^n} g_n)$$

to obtain $\alpha_{n+1} : [0, e_{n+1}R] \to [0, f_{n+1}R]$ given by $\alpha_{n+1}(xR) = \alpha_n^2(xR) = a^{2^n} xR$. Accordingly, put $t_0(x, y) = yx \wedge xxy$, and, inductively,

$$t_{n+1}(x, y) = y^{2^n} t_n(x, y) \wedge x^{2^n} t_n(x, y).$$

Thus, for a, b as above one has $t_n(a, b)R = g_nR = e_nR \cap f_nR$ whence

$$t_{n+1}(a, b)t_n(a, b) = t_n(a.b) \ \Leftrightarrow\ g_{n+1}R = g_nR.$$

Thus, supposing $t_{n+1}(a, b)t_n(a, b) = t_n(a, b)$, bR and aR are perspective in $L(R)$ by Theorem 8 and Lemma 15 applies to provide the existence of a unit u in R such that $aua = a$ and idempotent $p \in R$ such that $bR \sim_{pR} aR$. To prove the existence of terms $u_n(x, y)$ and $p_n(x, y)$, as required, it suffices to observe that all this applies, in particular, to R being the free algebra in $\mathcal{R}$ with generators a, b and relations $aba = a$, $bab = b$, and $t_{n+1}(a, b)t_n(a, b) = t_n(a, b)$.

Now, assume that $g_k \neq g_{k+1}$ for all $k \leq m$. Then one obtains a chain $e_0R + f_0R > e_0R > g_0R > \ldots > g_{m+1}R$ of length $m+3$ in $L(R)$. Thus, if r is of length at most $n+2$ then $g_m = g_{m+1}$ for some $m \leq n+2$ and it follows $g_k = g_m$ for all $k \geq m$, in particular $g_n = g_m = g_{n+1}$. □

Example 21. (i) *There are a, b, c in some unit-regular ring R such that a, b and a, c are pairs of reflexive inverses, aR, bR, and cR pairwise perspective, $t_{n+1}(a,b)t_n(a,b) \neq t_n(a,b)$ for all n, and $t_0(a,c) = 0$.*

(ii) *There are a regular ring R and reflexive inverses a, b in R such that $t_0(a,b) = 0$ but both a and b are not strongly π-regular,*

Proof. Considering (i) let V a vector space of dimension $n+3$. We show by induction that $\mathsf{End}(V)$ contains some a, b with associated $g_n > g_{n+1}$. More precisely, we show that for any subspaces $V_1 \neq V_2$ of codimension 1 there is such a with generalized inverse a^+ and restricting to an isomorphism $V_1 \to V_2$ and such that $V_1 = \mathsf{im}\, a^+$ and $V_2 = \mathsf{im}\, a$. If $n = 0$ choose v_i such that $V_1 \cap V_2 = \mathsf{span}\, v_3$ and $V_i = \mathsf{span}\, v_i + V_1 \cap V_2$ for $i = 1, 2$. Define the endomorphism a by $a(v_1) = v_2$, $a(v_2) = 0$, and $a(v_1 + v_3) = v_3$ and a^+ by $a^+(v_1) = 0$, $a^+(v_2) = v_1$, and $a^+(v_3) = v_1 + v_3$. Proceeding from $n-1$ to n choose W of codimension 1 in V such that $V_1 \cap V_2 \not\subseteq W$ and put $W_i = W \cap V_i$. Choose endomorphisms a_0, a_0^+ of W connecting W_1 and W_2 according to hypothesis. Choose $v_3 \notin W$ and v_i such $V_i = \mathsf{span}\, v_i + W_i$ for $i = 1, 2$ and extend a_0 and a_0^+ to obtain a and a^+, defined for v_i as above.

By this construction there are finite dimensional $W_n = V_n \oplus U_n$ with reflexive inverses a_{0n}, b_{0n} in V_n such that $t_{n+1}(a_{0n}, b_{0n})t_n(a_{0n}, b_{0n}) \neq t_n(a_{0n}, b_{0n})$ and isomorphism $c_{0n} : V_n \to U_n$. Choose a_n extending a_{0n} and c_{0n}^{-1}, and b_n, c_n extending b_{0n} and c_{0n}, respectively, such that $b_n|U_n = 0$ and $c_n|U_n = 0$. Then the direct product of the $\mathsf{End}(W_n)$ provides R and a, b, c as required.

In (ii) consider a vector space V with basis $v_n, w_n (n \in \mathbb{N})$, $R = \mathsf{End}(V)$ and define $a(v_n) = w_n$, $a(w_n) = w_{n+1}$, $b(w_n) = v_n$, and $b(v_n) = v_{n+1}$. □

3.4. **(Existence-)varieties of unit-regular rings.** Observe that subrings of regular rings are not regular, in general, an obvious example being $\mathbb{Z} \subset \mathbb{Q}$. Thus, to deal with classes $\mathcal{C}$ of regular rings in the framework of Universal Algebra, without specifying operations of quasi-inversion, it is convenient to introduce the class operator $\mathsf{S}_\exists(\mathcal{C})$ associating with $\mathcal{C}$ the class of all regular rings which are subrings of members of $\mathcal{C}$. Referring to the usual operators H, P, and P_u for homomorphic images, direct products and ultraproducts (which preserve regularity), a class $\mathcal{V}$ of regular rings which is closed under under these operators is an *existence-variety* (cf. Hall [9] for this concept). For a class $\mathcal{C}$ of regular rings let $\mathsf{T}(\mathcal{C})$ consist of all regular rings endowed with an operation of quasi-inversion (that is, members of $\mathcal{R}$ as defined in the previous subsection) where the underlying ring is in $\mathcal{C}$. For the following see Propositions 7 and 10 and Theorem 16 of Herrmann and Semenova [13].

Fact 22. (i) *The smallest existence variety* $\mathsf{V}_\exists(\mathcal{C})$ *containing* $\mathcal{C}$ *is* $\mathsf{HS}_\exists\mathsf{P}(\mathcal{C})$.

(ii) $\mathcal{V} = \mathsf{S}_\exists\mathsf{P}\{R \in \mathcal{V} \mid R$ *subdirectly irreducible*$\}$ *for any existence variety* $\mathcal{V}$.

(iii) $R \in \mathsf{HS}_\exists\mathsf{P}_u(\mathcal{C})$ *for every subdirectly irreducible* $R \in \mathsf{V}_\exists(\mathcal{C})$.

(iv) *Any subdirectly irreducible regular ring* R *is an* F*-algebra for a suitable field* F. *Moreover, if such* R *is nonartinian then* $\mathsf{V}_\exists(R) = \mathsf{V}_\exists\{F^{n\times n} \mid n < \omega\}$.

(v) $\mathsf{TV}_\exists(\mathcal{C}) = \mathsf{HSPT}(\mathcal{C})$*; in particular, any identity in the language of* $\mathcal{R}$ *which is valid in* $\mathsf{T}(\mathcal{C})$ *is also valid in* $\mathsf{TV}_\exists(\mathcal{C})$.

Define $t_n(x) = t_n(x, x^+)$.

Theorem 23. *For an existence variety* $\mathcal{V}$ *of regular rings the following are equivalent*

(1) *All members of* $\mathcal{V}$ *are perspective.*
(2) *All members of* $\mathcal{V}$ *are unit-regular.*
(3) *All subdirectly irreducible members of* $\mathcal{V}$ *are directly finite.*
(4) *All subdirectly irreducible members of* $\mathcal{V}$ *are artinian.*
(5) *There is* $d < \omega$ *such that all artinian subdirectly irreducible members of* $\mathcal{V}$ *are of length* $\leq d$.
(6) *There are* $d < \omega$ *and and a class* $\mathcal{C}$ *of artinian regular rings of length* $\leq d$ *such that* $\mathcal{V} = \mathsf{V}_\exists(\mathcal{C})$.
(7) *There is* $n < \omega$ *such that* $t_{n+1}(x)t_n(x) = t_n(x)$ *is valid in* $\mathsf{T}(\mathcal{V})$.
(8) *There is* $m < \omega$ *such that the idenkities* $(x^{m+1})(x^{m+1})^+x^m = x^m$ *and* $x^m(x^{m+1})^+x^{m+1} = x^m$ *are valid in* $\mathsf{T}(\mathcal{V})$.

Actually, given $d \geq 2$ *one can choose* $n = d - 2$ *and* $m = d$.

Proof. (7) implies (1) by (i) of Theorem 8. (8) implies (1), too, in view of Theorem 14. (1) implies (2) by Theorem 13, and (2) implies (3).

Each of (3) and (4) implies (5): Indeed, assume that there are artinian subdirectly irreducibles $R_n \in \mathcal{V}$ with no bound on length. Renumbering and passing to corners and isomorphic copies, we may assume that $R_n \cong D_n^{n\times n}$ for some division ring D_n. Thus, for fixed m and all $n \geq m$, the ring R_n contains a subring $R_{mn} \cong D_n^{m\times m}$. Choose

$R_{mn} = 0$ for $n < m$. Thus, in particular $R_{mn} \in \mathcal{V}$ for all m, n. Recall that, for fixed m, the class of all rings isomorphic to $D^{m\times m}$ for some division ring d can be finitely first order axiomatized if one adds m^2 constants for a system of matrix units. Thus, choosing a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ one has for any fixed m the ultraproduct $(\prod_{n\in\mathbb{N}} R_{mn})/\mathcal{F}$ isomorphic to $D^{m\times m}$ where $D = (\prod_{n\in\mathbb{N}} D_n)/\mathcal{F}$. It follows $D^{m\times m} \in \mathcal{V}$ and thus $F^{m\times m} \in \mathcal{V}$ for all m where F is the prime subfield of D. Now, consider any infinite dimensional F-vector space W and $\mathsf{End}(W_F)$; the latter is subdirectly irreducible, nonartinian, and not directly finite. By Fact 22(iv) one has $\mathsf{End}(W_F) \in \mathcal{V}$ contradicting both (3) and (4).

(5) implies (4): Assume there is subdirectly irreducible $R \in \mathcal{V}$ which is not artinian. By Fact 22(iv), R is an F.algebra for some field F and $\mathcal{V} \supseteq \mathsf{V}_\exists(R) = \mathsf{V}_\exists\{F^{n\times n} \mid n < \omega\}$ so that the (subdirectly irreducible) $F^{n\times n} \in \mathcal{V}$ for all $n < \omega$, contradicting (5).

(5) implies (6): Since (5) implies (4), in view of Fact 22(ii) it follows that $\mathcal{V}$ is generated by members of length $\leq d$.

(6) implies (7) and (8): Let $\mathcal{R}_d$ consist of all artinian regular rings which are of length at most d. Thus, $\mathcal{V} \subseteq \mathsf{V}_\exists(\mathcal{R}_d)$. Now, consider subdirectly irreducible $R \in \mathsf{V}_\exists(\mathcal{R}_d)$. By Fact 22(iii) one has $R \in \mathsf{HS}_\exists\mathsf{P}_u(\mathcal{R}_d)$. Since the property of having length $\leq d$ can be expressed, easily, by a first order formula (in various ways), we have $\mathsf{P}_u(\mathcal{R}_d) \subseteq \mathcal{R}_d$ while $\mathsf{HS}_\exists(\mathcal{R}_d) \subseteq \mathcal{R}_d$ is obvious. This implies that $R \in \mathcal{R}_d$ whence $\mathcal{V} \subseteq \mathsf{V}_\exists(\mathcal{R}_d)$ by Fact 22(ii). By (ii) of Theorem 8 the identities $t_{n+1}(x)t_n(x) = t_n(x)$, $(x^{m+1})(x^{m+1})^+x^m = x^m$, and $x^m x^m (x^{m+1})^+ x^{m+1}$ (where $n = d-2$ and $m = d$) are valid in $\mathsf{T}(\mathcal{R}_d)$ and so in $\mathsf{T}(\mathcal{V})$ by Fact 22(v).

□

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