Separable differential operators and applications

Veli. B. Shakhmurov
Department of Mechanical Engineering, Okan University, Akfirat, Tuzla 34959
Istanbul, Turkey,
E-mail: veli.sahmurov@okan.edu.tr
Institute of Mathematics and Mechanics,
Azerbaijan National Academy of Sciences
AMS: 34Gxx, 35Jxx, 35Kxx, 47Hxx, 34Bxx

ABSTRACT

The nonlocal boundary value problems for degenerate differential-operator equations with variable coefficients are studied. The L_p separability properties of elliptic problems and well-posedeness of parabolic problems in mixed $L_{\mathbf{p}}$ spaces are derived. Then by using the regularity properties of linear problems, the existence and uniqueness of solution of nonlinear elliptic problem is obtained. Note that, applications of these problems can be models of different physics process.

Key Words: Abstract harmonic analysis**,** differential-operator equations, degenerate PDE, semigroups of operators, Sobolev-Lions spaces, separable differential operators

0. Introduction

In this work, the boundary value problems (BVPs) for parameter dependent degenerate differential-operator equations (DOEs) are considered. Namely, the equations contain a small parameter. These problems have numerous applications in PDE, pseudo DE, mechanics and environmental engineering. The BVP for DOEs have been studied extensively by many researchers (see e.g. [2-5, 7-21, 24-26] and the references therein). A comprehensive introduction to the DOEs and historical references may be found in [12] and [25] . The maximal regularity properties for DOEs have been studied e.g. in [2, 6-7, 15-21, 24]. Degenerate DOEs in abstract function spaces are investigated e.g. in [3, 11, 16, 19, 20, 22]. The maximal regularity properties of BVP for elliptic equations are studied e.g. in [1, 8, 23]. The main objective of the present paper is to discuss maximal regularity properties of the following degenerate elliptic DOE

$$-\varepsilon a(x)\, u^{(2)}(x)+A(x)\, u(x)+\varepsilon^{\frac{1}{2}} A_1(x)\, u^{(1)}(x)+A_0(x)\, u(x)+\lambda u = f(x)\,, \quad (0.1)$$

where ε is a small positive parameter, λ is a complex parameter, $a(x)$ is a complex valued function and A, A_0, A_1 are linear operators in a Banach space E. Since the above equation depends on parameter ε, then the solution u also depend of ε, i.e., $u(x) = u(x, \varepsilon)$. Note that, the principal part of the above problem is nonselfadjoint and also have the variable coefficients. The regularity properties for the problem of type (0.1) was studied in [16] for $\varepsilon = 1$. Here, several conditions for the separability and sharp resolvent estimates uniformly with respect to parameter ε are given. Especially, it is shown that the corresponding differential operator is R-positive and also generates an analytic semigroup. In first section, we introduce some notations, definitions and background. In section 2, we consider nonlocal nonhomogenous BVP for the degenerate DOE with constant coefficients. We prove that this problem is isomorphism from $W_p^2(0,1;E(A),E)$ onto $L_p(0,1;E) \times E_1 \times E_2$, where E_k are interpolation spaces between $E(A)$ and E (see section 1 for definition of these spaces). In section 3, we show that the problem (0.1) is $L_p(0,1;E)$ separabile, i.e., we prove that problem (0.1) for $f \in L_p(0,1;E)$ has a unique solution $u \in W_p^2(0,1;E(A),E)$ and the following uniform coercive estimate holds

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\| u^{(i)} \right\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \leq C \|f\|_{L_p(0,1;E)}$$

for $|\arg \lambda| \leq \varphi$, $\varphi < \pi$ with sufficiently large $|\lambda|$, where the constant C depend only on p and A.

The section 4 devoted to R-positivity of the corresponding differential operator. In section 5, the uniform well-posedeness of initial and BVP for the degenerate abstract parabolic equation

$$\frac{\partial u}{\partial t} + \varepsilon a(x) \frac{\partial^2 u}{\partial x^2} + A(x) u(x,t) = f(x,t)$$

is established in E-valued mixed $L_{\mathbf{p}}$ space. In section 6, nonlocal BVP for degenerate abstract elliptic equation considered in the moving domain. By using the maximal regularity properties of linear problem (0.1), in section 7 we derive the existence and uniqueness of nonlocal BVP for the following nonlinear degenerate abstract equation

$$-q(x) u^{(2)}(x) + B\left(x, u, u^{(1)}\right) u(x) = F\left(x, u, u^{(1)}\right),$$

where q is a real valued function, B and F are nonlinear operator in a Banach space E. In application, the separability properties of the system of degenerate parabolic equations is obtained.

Modern analysis methods, particularly abstract harmonic analysis, the operator theory, interpolation of Banach Spaces, theory of semigroups of linear operators, microlocal analysis, embedding and trace theorems in vector-valued Sobolev-Lions spaces are the main tools implemented to carry out the analysis.

1. Notations, definitions and background

Let $L_p(\Omega; E)$ denote the space of strongly measurable E-valued functions defined on Ω with the norm

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega;E)} = \left(\int \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, 1 \leq p < \infty.$$

Let $\mathbf{p}=(p_1,p_2,...,p_n)$. $L_{\mathbf{p}}(G;E)$, $G=\prod\limits_{k=1}^{n}(0,b_k)$ will denote the space of all measurable E-valued $\mathbf{p}$-summable functions with mixed norm

$$\|f\|_{L_{\mathbf{p}}(G;E)}=\left(\left(\int\limits_0^{b_n}\left(...\int\limits_0^{b_2}\left(\int\limits_0^{b_1}\|f(x)\|_E^{p_1}dx_1\right)^{\frac{p_2}{p_1}}dx_2\right)^{\frac{p_3}{p_2}}...\right)^{\frac{p_n}{p_{n-1}}}dx_n\right)^{\frac{1}{p_n}}<\infty.$$

The Banach space E is called an UMD-space if the Hilbert operator $(Hf)(x)=\lim\limits_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy$ is bounded in $L_p(R,E)$, $p\in(1,\infty)$ (see. e.g. [6]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, p, $q\in(1,\infty)$.

Let $\mathbb{C}$ be the set of the complex numbers and

$$S_\varphi=\{\lambda;\ \ \lambda\in\mathbb{C},\ |\arg\lambda|\leq\varphi\}\cup\{0\},0\leq\varphi<\pi.$$

A linear operator A is said to be φ-positive in a Banach space E with bound $M>0$ if $D(A)$ is dense on E and $\left\|(A+\lambda I)^{-1}\right\|_{B(E)}\leq M(1+|\lambda|)^{-1}$ for any $\lambda\in S_\varphi$, $0\leq\varphi<\pi$, where I is the identity operator in E and $B(E)$ denotes the space of bounded linear operators in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and denoted by A_λ. It is known [23, §1.15.1] that a positive operator A has well-defined fractional powers A^θ. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with norm

$$\|u\|_{E(A^\theta)}=\left(\|u\|^p+\left\|A^\theta u\right\|^p\right)^{\frac{1}{p}},1\leq p<\infty,\ 0<\theta<\infty.$$

Let E_1 and E_2 be two Banach spaces. By $(E_1,E_2)_{\theta,p}$, $0<\theta<1,1\leq p\leq\infty$ we will denote the interpolation spaces obtained from $\{E_1,E_2\}$ by the K-method [23, §1.3.2].

Weight function γ satisfies A_p condition (i.e. $\gamma\in A_p$) if there is a constant C such that

$$\left(\frac{1}{|Q|}\int\limits_Q\gamma(x)\,dx\right)\left(\frac{1}{|Q|}\int\limits_Q\gamma^{-\frac{1}{p-1}}(x)\,dx\right)^{p-1}\leq C,\ p\in(1,\infty)$$

for all cubes $Q\subset R^n$.

Let $S(R^n;E)$ denote the Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on R^n. Let F denote the Fourier transformation. A function $\Psi\in C(R^n;B(E))$ is called Fourier multiplier in $L_p(R^n;E)$ if the map

$$u\to\Phi u=F^{-1}\Psi(\xi)Fu,u\in S(R^n;E)$$

is well defined and extends to a bounded linear operator in $L_p(R^n;E)$. The set of all multipliers in $L_p(R^n;E)$ denotes by $M_p(E)$.

Let $W_h = \{\Psi_h \in M_p(E),\ h \in \mathbb{C}\}$ be a collection of multipliers in $M_p(E)$. We say W_h is a uniform collection of multipliers if there exists a positive constant M independent of h such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_p(R^n;E)} \leq M \|u\|_{L_p(R^n;E)}$$

for all $h \in Q$ and $u \in S(R^n;E)$.

Let $\mathbb{N}$ denote the set of natural numbers and $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. A set $K \subset B(E_1,E_2)$ is called R-bounded (see e.g. [8]) if there is a constant C such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\|\sum_{j=1}^m r_j(y) T_j u_j\right\|_{E_2} dy \leq C \int_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy.$$

The smallest $C > 0$ for which the above estimate holds is called a R-bound of the collection K and denoted by $R(K)$.

A set $W_h \subset L(E_1,E_2)$ is called uniform R-bounded in $h \in \mathbb{C}$ if there is a constant C independent of h such that for all $T_1(h), T_2(h), ..., T_m(h) \in W_h$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\|\sum_{j=1}^m r_j(y) T_j(h) u_j\right\|_{E_2} dy \leq C \int_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy.$$

Definition 1.1. A Banach space E is said to be the space satisfying multiplier condition, if for any $\Psi \in C^{(1)}(R; B(E))$ the R-boundedness of the set

$$\left\{\xi^k D^k \Psi(\xi) : \xi \in \mathbb{R} \setminus \{0\},\ k = 0,1\right\}$$

implies $\Psi \in M_{p,\gamma}(E)$.

An operator $A(t)$ is said to be uniformly φ-positive in E if $D(A(t))$ is independent of t and dense in E and $\left\|(A(t)+\lambda)^{-1}\right\| \leq \frac{M}{1+|\lambda|}$ for $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, where M is independent of t.

Definition 1.2.The φ-positive operator $A(x)$, $x \in G$ is said to be uniformly R-positive in a Banach space E if there exists $\varphi \in [0\ ,\ \pi)$ such that the set

$$\left\{A(x)(A(x)+\xi I)^{-1} : \xi \in S_\varphi\right\}$$

is uniformly R-bounded, that is

$$\sup_{x \in G} R\left(\left\{\left[A(x)(A(x)+\xi I)^{-1}\right] :\ \xi \in S_\varphi\right\}\right) \leq M.$$

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embeds into E. Let us consider the space $W_p^m(a,b;E_0,E)$, consisting of all functions $u \in L_p(a,b;E_0)$ that have generalized derivatives $u^{(m)} \in L_p(a,b;E)$ with the norm

$$\|u\|_{W_p^m} = \|u\|_{W_p^m(a,b;E_0,E)} = \|u\|_{L_p(a,b;E_0)} + \left\|u^{(m)}\right\|_{L_p(a,b;E)} < \infty.$$

Let $\varepsilon > 0$ be a parameter. We define the following parameterized norm in $W_p^m(a,b;E_0,E)$

$$\|u\|_{W_{p,\varepsilon}^m} = \|u\|_{W_{p,\varepsilon}^m(a,b;E_0,E)} = \|u\|_{L_p(a,b;E_0)} + \left\|\varepsilon u^{(m)}\right\|_{L_p(a,b;E)} < \infty.$$

$BMO(E)$ is the space of all E-valued local integrable functions with the norm

$$\sup_B \oint_B \|f(x) - f_B\|_E \, dx = \|f\|_{*,E} < \infty,$$

where B ranges in the class of the balls in R^n and f_B is the average $\frac{1}{|B|}\int_B f(x)\,dx$.

For $f \in BMO(E)$ and $r > 0$ we set

$$\sup_{\rho \le r} \oint_B \|f(x) - f_B\|_E \, dx = \eta(r),$$

where B ranges in the class of the balls with radius ρ.

We will say that a function $f \in BMO(E)$ is in the space $VMO(E)$ if $\lim_{r \to +0} \eta(r) = 0$. We will call $\eta(r)$ the VMO modulus of f.

If $E = \mathbb{C}$, then $BMO(E)$ and $VMO(E)$ coincide with John-Nirenberg class BMO and Sarason class VMO respectively.

From [23, §1.8.2] we obtain the following:

Theorem A. Assume m and j are integers $0 \le j \le m-1$, $\theta_j = \frac{pj+1}{pm}$, $p \in (1,\infty)$; $\varepsilon \in (0,1)$ is a parameter, $x_0 \in [0,b]$. Then, the linear transformation $u \to u^{(j)}(x_0)$ is bounded from $W_p^m(0,b;E_0,E)$ onto $(E_0,E)_{\theta_j,p}$ and the following inequality holds

$$\varepsilon^{\theta_j}\left\|u^{(j)}(x_0)\right\|_{(E_0,E)_{\theta_j,p}} \le C\left(\left\|tu^{(m)}\right\|_{L_p(0,b;E)} + \|u\|_{L_p\ (0,b;E_0)}\right).$$

As a corollary of [20, Theorems 2.3, 2.4] we have the following result:

Theorem B. Assume the following conditions are satisfied:

(1) $b = b(s)$ is a continuous function on $[c,d]$;

(2) E is a Banach space satisfying the multiplier condition with respect to p and γ;

(3) A is a R-positive operator in E, $0 \le \mu \le 1 - \frac{j}{m}$, $p \in (1,\infty)$;

(4) $\varepsilon \in (0,1)$ and $h \in (0,h_0)$, are some parameters, where $h_0 < \infty$;

(5) there exists a bounded linear extension operator from $W_p^m(0,b;E(A),E)$ to $W_p^m(R;E(A),E)$.

Then, the embedding

$$D^j W_p^m(0,b;E(A),E) \subset L_p\left(0,b;E\left(A^{1-\frac{j}{m}-\mu}\right)\right)$$

is continuous and the following uniform estimate holds

$$\left\|\varepsilon^{\frac{j}{m}} u^{(j)}\right\|_{L_p\left(0,b;E\left(A^{1-\frac{j}{m}-\mu}\right)\right)} \leq h^{\mu}\|u\|_{W_{p,\varepsilon}^m(0,b;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(0,b;E)}$$

for all $u \in W_p^m(0,b;E(A),E)$.

Consider the following parameter dependent degenerate DOE on $\mathbb{R} = (-\infty, \infty)$

$$(L+\lambda)u = -\varepsilon u^{[2]}(x) + (A+\lambda)u(x) = f(x), \tag{1.1}$$

where A is a linear operator in a Banach space E.

Let

$$X = L_p((-\infty,\infty);E), Y = W_p^{[2]}((-\infty,\infty);E(A),E).$$

From [17, Theorem 4.1] we obtain:

Theorem C. Assume:

(1) $\varepsilon \in (0,1)$ is a small parameter;

(2) E is the Banach space satisfying the multiplier condition with respect to p;

(3) A is a R positive operator in E.

Then, problem (1.1) has a unique solution $u \in Y$ for $f \in X$ and $|\arg\lambda| \leq \varphi$ with sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\|u^{[i]}\right\|_X + \|Au\|_X \leq C\|f\|_X.$$

2. Degenerate DOEs with constant coefficients

Consider the nonlocal BVP for degenerate DOE

$$(L_\varepsilon+\lambda)u = -\varepsilon u^{(2)}(x) + (A+\lambda)u(x) = f(x),\ x \in (0,1),$$
$$L_{k\varepsilon}u = \sum_{i=0}^{m_k} \varepsilon^{\sigma_i}\left[\alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1)\right] = f_k,\ k=1,2, \tag{2.1}$$

where $m_k \in \{0,1\}; \sigma_i = \frac{i}{2} + \frac{1}{2p(1-\gamma)}$, α_{ki}, β_{ki} are complex numbers, A is a possible unbounded operator in a Banach space E and $f_j \in X_j = (E(A),E)_{\theta_j,p}$, $\theta_j = \frac{m_j}{2} + \frac{1}{2p}$, $j=1,2$. Let $\alpha_k = \alpha_{km_k}$, $\beta_k = \beta_{km_k}$. For the sake of simplicity $L_\varepsilon, L_{1\varepsilon}, L_{2\varepsilon}$ will be denoted by L, L_1, L_2, respectively. Nonlocal BVP for PDE studied e.g. in $[18-22]$.

Remark 2.0. Generally the operator A is non selfadjoint (only in particularly case it can be selfedjoint), so in general case $\left((E(A),E)_{\theta,p}\right) \neq E\left(A^{1-\theta}\right)$.

Really, if we chooce $E=H$, where H is a Hilbert space and A to be selfadjoint than we can take complex interpolation $\left[H\left(A\right),H\right]_{\theta}=H\left(A^{1-\theta}\right)$.

Any function $u\in W_{p}^{2}\left(0,1;E\left(A\right),E\right)$ satisfying the equation (2.1) a.e. on $(0,1)$ will be called the solution of (2.1).

Condition 2.1. Assume the following conditions are satisfied:

(1) $\varepsilon\in(0,1)$ is a small parameter and $\eta=\left(-1\right)^{m_{1}}\alpha_{1}\beta_{2}-\left(-1\right)^{m_{2}}\alpha_{2}\beta_{1}\neq 0$;

(2) E is a Banach space satisfying the multiplier condition with respect to p;

(3) A is a R positive operator in E.

The main result of this section is the following

Theorem 2.1. Let the Condition 2.1hold. Then, problem (2.1) has a unique solution $u\in W_{p}^{2}\left(0,1;E\left(A\right),E\right)$ for $f\in L_{p}\left(0,1;E\right)$, $f_{j}\in X_{j}$ and for sufficiently large $|\lambda|$ with $|\arg\lambda|\leq\varphi$. Moreover, the following uniform coercive estimate holds

$$\sum_{i=0}^{2}\left|\lambda\right|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|u^{(i)}\right\|_{L_{p}(0,1;E)}+\left\|Au\right\|_{L_{p}(0,1;E)}\leq C\left(\left\|f\right\|_{L_{p}(0,1;E)}+\sum_{j=1}^{2}\left\|f_{j}\right\|_{X_{j}}\right). \tag{2.3}$$

First, we consider the homogenous problem

$$\left(L+\lambda\right)u=0,\ \ L_{k}u=f_{k},\ \ k=1,2, \tag{2.4}$$

where L and L_{k} are defined as in (2.2).

Let

$$X=L_{p}\left(\left(0,b\right);E\right),Y=W_{p}^{2}\left(0,b;E\left(A\right),E\right).$$

In a similar way as [18, Theorem 3.2] and [20, Theorem 5.1] we obtain the following result:

Proposition 2.1. Assume E is a Banach space satisfying the multiplier condition with respect to $p\in(1,\infty)$ and A is a R-positive operator in E. Let

$$0<\varepsilon\leq 1,\ \ \eta=\left(-1\right)^{m_{1}}\alpha_{1}\beta_{2}-\left(-1\right)^{m_{2}}\alpha_{2}\beta_{1}\neq 0,\ \theta_{k}=\frac{m_{k}}{2}+\frac{1}{2p\left(1-\gamma\right)}.$$

Then, problem (2.4) has a unique solution $u\in Y$ for $f_{k}\in E_{k}$ and $\lambda\in S_{\varphi}$ with sufficiently large $|\lambda|$. Moreover, the following coercive uniform estimate holds

$$\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\left|\lambda\right|^{1-\frac{i}{2}}\left\|u^{(i)}\right\|_{X}+\left\|Au\right\|_{X} \tag{2.5}$$

$$\leq M\sum_{k=1}^{2}\left(\left\|f_{k}\right\|_{E_{k}}+\left|\lambda\right|^{1-\theta_{k}}\left\|f_{k}\right\|_{E}\right).$$

Now, consider the problem (2.2).

Theorem 2.2. Let the Condition 2.1 hold. Then, the operator $u \to \{(L_\varepsilon + \lambda) u, L_1 u, L_2 u\}$ is an isomorphism from Y onto $X \times E_1 \times E_2$ for $|\arg \lambda| \leq \varphi$, $0 \leq \varphi < \pi$ and sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds:

$$\sum_{j=0}^{2} \varepsilon^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \left\| u^{(j)} \right\|_X + \|Au\|_X \tag{2.9}$$

$$\leq C \left[\|f\|_X + \sum_{k=1}^{2} \left(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E \right) \right].$$

We have proved the uniqueness of solution for (2.3) in Proposition 2.1. Let us define

$$\bar{f}(x) = \left\{ \begin{array}{ll} f(x) & \text{if } x \in [0,b] \\ 0 & \text{if } x \notin [0,b] \end{array} \right\}.$$

Now we have to show that the problem (2.3) has a solution $u \in Y$ and that $u = u_1 + u_2$, where u_1 is the restriction of solution of the problem

$$(L_\varepsilon + \lambda) u = \bar{f}(x), \; x \in R = (-\infty, \infty) \tag{2.10}$$

and u_2 is a solution for

$$(L_\varepsilon + \lambda) u = 0, \; L_k u = f_k - L_k u_1. \tag{2.11}$$

A solution to (2.10) should be in the following form

$$u(y) = F^{-1} L_\varepsilon^{-1}(\lambda, \xi) F \bar{f} = \frac{1}{2\pi} \int_{\infty}^{\infty} e^{i\xi y} L_\varepsilon^{-1}(\lambda, \xi) \left(F \bar{f} \right)(\xi) \, d\xi,$$

where $L_\varepsilon(\lambda, \xi) = \varepsilon \xi^2 + A + \lambda$. In a similar way as [18, Theorem 3.2] we obtain that the operator-valued functions $\Psi_{\varepsilon\lambda}(\xi) = A L_\varepsilon^{-1}(\lambda, \xi)$ and $\sigma_{\varepsilon\lambda}(\xi) = \sum_{j=0}^{2} \varepsilon^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \xi^j L_\varepsilon^{-1}(\lambda, \xi)$ are uniform Fourier multipliers in $L_p(R; E)$. Then, we get that (2.10) has a solution $u \in Y$ and

$$\sum_{j=0}^{2} \varepsilon^{\frac{j}{2}} |\lambda|^{1-\frac{j}{2}} \left\| u^{(j)} \right\|_{L_p(R;E)} + \|Au\|_{L_{p,\tilde{\gamma}}(R;E)} \leq C \left\| \bar{f} \right\|_{L_p(R;E)}. \tag{2.12}$$

Let u_1 be the restriction of u on $(0, b)$. Then (2.12) implies $u_1 \in Y$. By Theorem A, we get

$$u_1^{(m_k)}(.) \in (E(A); E)_{\theta_k, p}, \; k = 1, 2,$$

which implies $L_k u_1 \in E_k$. Thus, by using trace theorems and Proposition 2.1, problem (2.11) has a unique solution $u_2 \in Y$ and for sufficiently large $|\lambda|$ and

$$\sum_{j=0}^{2}\varepsilon^{\frac{j}{2}}\left|\lambda\right|^{1-\frac{j}{2}}\left\|u_1^{(j)}\right\|_X+\left\|Au_1\right\|_X\leq C\left\|f\right\|_X. \tag{2.14}$$

From (2.12) and (2.14) we get

$$\sum_{j=0}^{2}\varepsilon^{\frac{j}{2}}\left|\lambda\right|^{1-\frac{j}{2}}\left\|u_2^{(j)}\right\|_X+\left\|Au_2\right\|_X\leq$$

$$C\left(\left\|f\right\|_X+\sum_{k=1}^{2}\left(\left\|f_k\right\|_{E_k}+\left|\lambda\right|^{1-\theta_k}\left\|f_k\right\|_E\right)\right)$$

which together with (2.14) concludes the proof.

Theorem 2.2 implies that problem (2.3) has a unique solution $u\in Y$ for $f\in X$, $f_j\in X_j$ and $\left|\arg\lambda\right|\leq\varphi$ with sufficiently large $\left|\lambda\right|$. Moreover, the uniform coercive estimate (2.9) holds. By virtue of Theorem 2.2 we obtain the assertion of Theorem 2.1.

Let B_ε denote the operator generated by (2.1) with $f_k=0$ and $\lambda=0$ i.e.,

$$D\left(B_\varepsilon\right)=W_p^2\left(0,1;E\left(A\right),E,L_k\right)=\left\{u\in W_p^2\left(0,1;E\left(A\right),E\right),\ L_ku=0\right\},$$

$$B_\varepsilon u=-\varepsilon u^{(2)}\left(x\right)+Au.$$

3. DOEs with variable coefficients

Consider the following BVP for DOE with parameter

$$\left(L+\lambda\right)u=-\varepsilon a\left(x\right)u^{(2)}\left(x\right)+A_\lambda\left(x\right)u\left(x\right)+\sum_{i=0}^{1}\varepsilon^{\frac{i}{2}}A_i\left(x\right)u^{(i)}\left(x\right)=f\left(x\right), \tag{3.1}$$

$$L_ku=\sum_{i=0}^{m_k}\varepsilon^{\sigma_i}\alpha_{ki}u^{(i)}\left(0\right)+\beta_{ki}u^{(i)}\left(1\right)=0,\ k=1,2,\ x\in\left(0,1\right), \tag{3.2}$$

where $\sigma_i=\frac{i}{2}+\frac{1}{2p}$, $m_k\in\{0,1\}$, α_{ki}, β_{ki} are complex numbers; A and $A_i\left(x\right)$ are linear operators in a Banach space E, $A_\lambda\left(x\right)=A\left(x\right)+\lambda$, ε is a small positive and λ is a complex parameter.

Let us consider the boundary value problem $(3.4)-(3.5)$.

Condition 3.1. Assume the following conditions are satisfied:

(1) $\alpha_{km_k},\beta_{km_k}\neq 0$, $a\left(y\right)$ is a positive continuous function on $[0,b]$, $a\left(0\right)=a\left(b\right)$;

(2) E is the Banach space satisfying the multiplier condition with respect to p and the weight function

$\eta\neq 0,1<p<\infty$, $\varepsilon\in\left(0,1\right)$;

(3) $A\left(y\right)$ is a R positive operator in E uniformly with respect to $y\in[0,b]$ and $A\left(y\right)A^{-1}\left(y_0\right)\in C\left([0,b];B\left(E\right)\right)$, $y_0\in(0,b)$, $A\left(0\right)A^{-1}\left(y_0\right)=A\left(b\right)A^{-1}\left(y_0\right)$;

(4) for any $\delta > 0$ there is a positive $C(\delta)$ such that

$$\|A_1(y)u\| \leq \delta \|u\|_{(E(A),E)_{\frac{1}{2},\infty}} + C(\delta)\|u\|$$

for $u \in (E(A), E)_{\frac{1}{2},\infty}$ and $\|A_0(y)u\| \leq \delta \|A(y)u\| + C(\delta)\|u\|$ for $u \in D(A)$.

Theorem 3.1. Let the Condition 3.1 hold. Then, problem (3.4) − (3.5) has a unique solution $u \in Y$ for $f \in X$ and $|\arg \lambda| \leq \varphi$ with $|\lambda|$ large enough. Moreover, the following uniform coercive estimate holds

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\|u^{(i)}\right\|_X + \|Au\|_X \leq C\|f\|_X. \tag{3.6}$$

By using microlocal analysis, first we will show the uniqueness of solution. Let $G_1, G_2, ..., G_N$ be intervals in $\mathbb{R}$ covering $(0, b)$ and $\{\varphi_j\}$, $j = 1, 2, ..., N$ be a corresponding partition of unity, i.e. φ_j are sufficiently smooth functions, $supp\varphi_j \subset G_j$ and $\sum_{j=1}^{N} \varphi_j(y) = 1$. Assume $u \in Y$ is a solution of the problem (3.4) − (3.5). We have $u(y) = \sum_{j=1}^{N} u_j(y)$, where $u_j(y) = u(y)\varphi_j(y)$. Then, from the equalities (3.4) and (3.5) we obtain

$$(L+\lambda)u_j = -\varepsilon a(y) u_j^{(2)}(y) + [A(y)+\lambda] u_j(y) = f_j(y), \tag{3.7}$$
$$L_k u_j = 0, \ k = 1, 2, \ j = 1, 2, ...N,$$

where

$$f_j = f\varphi_j - a\varepsilon \left[2u^{(1)}\varphi_j^{(1)} + u\varphi_j^{(2)}\right] + \varepsilon^{\frac{1}{2}}\varphi_j^{(1)} A_1 u - A_0 u_j. \tag{3.8}$$

By freezing the coefficients in (3.7) we obtain

$$-a(y_{0j})\varepsilon u_j^{(2)}(y) + A_\lambda(y_{0j}) u_j(y) = F_j(y), \ y \in \left(\text{supp } \varphi_j\right) \cap (0, b), \tag{3.9}$$
$$L_k u_j = 0, \ k = 1, 2, \ j = 1, 2, ...N,$$

where

$$F_j = F_j(u) = f_j + [A(y_{0j}) - A(y)] u_j + [a(y) - a(y_{0j})] u_j^{(2)}. \tag{3.10}$$

Since functions $u_j(x)$ have compact supports, extending $u_j(x)$ in outside of supp φ_j,we obtain BVPs with constant coefficients

$$-\varepsilon a(y_{0j}) u_j^{(2)}(y) + A_\lambda(y_{0j}) u_j(y) = F_j(y), \ y \in G_j, \tag{3.11}$$
$$L_k u_j = 0, \ k = 1, 2.$$

By using Theorem 2.1 and embedding theorem B, we get

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|u_j^{(i)}\right\|_{G_j,p}+\|Au_j\|_{G_j,p}\leq C\left[\|f\|_{G_j,p}+\|u_j\|_{G_j,p}\right]. \tag{3.17}$$

Then, by using (3.17) and the fact that $u(y)=\sum_{j=1}^{N}u_j(y)$ we obtain

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|u^{(i)}\right\|_{p}+\|Au\|_{p}\leq C\|(L+\lambda)u\|_{p} \tag{3.18}$$

Consider the operator O in $L_p(0,b;E)$ generated by $(3.4)-(3.5)$, i.e.,

$$D(O_\varepsilon)=W_p^2(0,b;E(A),E,L_k), O_\varepsilon u=-\varepsilon au^{(2)}+Au+\varepsilon^{\frac{1}{2}}A_1u^{(1)}+A_0u.$$

The estimate (3.20) implies that $(3.4)-(3.5)$ has a unique solution and the operator $O_\varepsilon+\lambda$ has an inverse in its rank space. We need to show that this rank space coincides with the space $L_p(0,b;E)$. Whence, we obtain that the BVP $(3.4)-(3.5)$ for $f\in X$ has a unique solution

$$u(y)=(O_\varepsilon+\lambda)^{-1}f=(U_\varepsilon+\lambda)\left(I+\sum_{j=1}^{N}\Phi_{j\lambda\varepsilon}\right)^{-1}f= \tag{3.25}$$

$$\sum_{j=1}^{N}\varphi_j(y)O_{j\lambda\varepsilon}^{-1}[I-K_{j\lambda\varepsilon}]^{-1}\left(I+\sum_{j=1}^{N}\Phi_{j\lambda\varepsilon}\right)^{-1}f.$$

Thus, by (3.25), Theorem 2.1 and Theorem C we get the desired result.

Let $\mathbf{G}_\varepsilon$ denote the operator in $L_p(0,1;E)$ generated by problem $(3.1)-(3.2)$, i.e.,

$$D(G_\varepsilon)=W_p^2(0,1;E(A),E,L_k),\ G_\varepsilon u=-\varepsilon au^{(2)}+Au+\varepsilon^{\frac{1}{2}}A_1u^{(1)}+A_0u.$$

By virtue of Theorem 3.1 and Remarks 2.1, 3.1 we obtain:

Result 3.2. Let all conditions of Theorem 3.1 be satisfied. Then, problem $(3.1)-(3.2)$ has a unique solution $u\in W_p^2(0,1;E(A),E)$ for $f\in L_p(0,1;E)$ and $|\arg\lambda|\leq\varphi$ with sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|u^{(i)}\right\|_{L_p(0,1;E)}+\|Au\|_{L_p(0,1;E)}\leq C\|f\|_{L_p(0,1;E)}.$$

5. Abstract Cauchy problem for parabolic equation with parameter

Result 3.2 implies that $\mathbf{G}_{\varepsilon}$ is positive in $F = L_p(0,1;E)$. In the following theorem we will prove that the operator $\mathbf{G}_{\varepsilon}$ is also R-positive in F.

Theorem 5.0. Let the Condition 2.1 hold. Then, $\mathbf{G}_{\varepsilon}$ is R-positive in F.

Consider the following initial-value problem with parameter

$$\frac{\partial u}{\partial t} - \varepsilon a(x) \frac{\partial^2 u}{\partial x^2} + A(x) u + \sum_{i=0}^{1} A_i(x) \frac{\partial^i u}{\partial x^i} = f(x,t), \tag{5.1}$$

$$L_k u = \sum_{i=0}^{m_k} \varepsilon^{\sigma_i} \alpha_{ki} \left(D_x^{(i)} u\right)(0,t) + \varepsilon^{\sigma_i} \beta_{ki} \left(D_x^{(i)} u\right)(b,t) = 0,\ k = 1,2,$$

$$u(x,0) = 0,\ t \in (0,\infty),\ x \in (0,b). \tag{5.2}$$

where $u = u(x,t)$ is a solution $\alpha_{ki},\ \beta_{ki}$ are complex numbers, ε is a positive parameter, $a(x)$ is a complex-valued function on $(0,b)$, $A(x)$ and $A_k(x)$ are linear operators in a Banach space E, $d > 0$, σ_i are positive numbers defined in section 2.

Let $\mathbf{p} = (p, p_1)$ and $\Delta_+ = (0,b) \times (0,\infty)$.

Theorem 5.1. Assume Condition 3.1 hold for $\varphi > \frac{\pi}{2}$. Then, for $f \in L_{\mathbf{p}}(\Delta_+;E)$ and sufficiently large $d > 0$ problem (5.1) $-$ (5.2) has a unique solution belonging to $W_{\mathbf{p},\alpha}^{1,[2]}(\Delta_+;E(A),E)$ and the following coercive estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p}}(\Delta_+;E)} + \varepsilon \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_{\mathbf{p}}(\Delta_+;E)} + \|Au\|_{L_{\mathbf{p}}(\Delta_+;E)} \leq C \|f\|_{L_{\mathbf{p}}(\Delta_+;E)}.$$

The problem (5.1) can be expressed as the following abstract Cauchy problem

$$\frac{du}{dt} + (\mathbf{G}_{\varepsilon} + d) u(t) = f(t),\ u(0) = 0. \tag{5.3}$$

From Theorems 4.1 we get that $\mathbf{G}_{\varepsilon}$ is R-positive in $F = L_p(0,b;E)$. By [23, §1.14], G_{ε} is a generator of an analytic semigroup in F. Then, by virtue of [24, Theorem 4.2] problem (6.2) has a unique solution $u \in W_{p_1}^1((0,\infty); D(\mathbf{G}_{\varepsilon}), F)$ for $f \in L_{p_1}((0,\infty);F)$ and sufficiently large $d > 0$. Moreover, the following uniform estimate holds

$$\left\|\frac{du}{dt}\right\|_{L_{p_1}((0,\infty);F)} + \|\mathbf{G}_{\varepsilon} u\|_{L_{p_1}((0,\infty);F)} \leq C \|f\|_{L_{p_1}(R_+;F)}.$$

Since $L_{p_1}(0,\infty;F) = L_{\mathbf{p}}(\Delta_+;E)$, by Theorem 3.1 we have

$$\|(\mathbf{G}_{\varepsilon} + d) u\|_{L_{p_1}(R_+;F)} = \|u\|_{D(\mathbf{G}_{\varepsilon})}.$$

Hence, the assertion follows from the above estimate.

Remark 5.1. Conditions $a(0) = a(b)$, $A(0) A^{-1}(y_0) = A(b) A^{-1}(y_0)$ arise due to nonlocality of the boundary conditions (3.1) and (5.1) . If boundary conditions are local then conditions mentioned above are not required any more.

6. Elliptic DOE on the moving domain

Consider the degenerate problem (3.1)−(3.2) on the moving domain $(0, b(s))$:

$$-a(x) u^{(2)} + A(x) u + \sum_{i=0}^{1} A_i(x) u^{(i)}(x) + du = f(x), \tag{6.1}$$

$$L_k u = \sum_{i=0}^{m_k} \alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(b(s)) = 0,\ k = 1, 2,$$

where α_{ki}, β_{ki} are complex numbers, a is a positive function; A and $A_i(x)$ are linear operators in a Banach space E, the end point $b(s)$ depend on the parameter s, $x \in (0, b(s))$ and $b(s)$ is a positive continues function on compact domain $\sigma \subset \mathbb{R}$. Theorem 3.1 implies the following:

Proposition 6.1. Assume the Condition 3.1 hold for $b = b(s)$. Then, problem (6.1) has a unique solution $u \in W_p^{(2)}((0, b); E(A), E)$ for $f \in L_p(0, b; E)$ and sufficiently $d > 0$. Moreover, the following coercive uniform estimate holds

$$\left\| u^{(2)} \right\|_{L_p(0,b;E)} + \|Au\|_{L_p(0,b;E)} \leq C \|f\|_{L_p(0,b;E)}. \tag{6.2}$$

Proof. Under the substitution $\tau = xb^{-1}(s)$ the problem (6.1) reduced to the following BVP in fixed domain $(0, 1)$:

$$b^{-2}(s) \tilde{a}(\tau) u^{(2)} + \tilde{A}(\tau) u + \sum_{i=0}^{1} b^{-i}(s) \tilde{A}_i(\tau) u^{(i)}(\tau) = \tilde{f}(\tau),\ \tau \in (0, 1),$$

$$\sum_{i=0}^{1} b^{-i}(s) \left[\alpha_{kji} u^{(i)}(0) + \beta_{kji} u^{(i)}(1) \right] = 0,\ k = 1, 2,$$

where

$$\tilde{a}_k(\tau) = a_k\left(\tau b^{-1}\right),\ \tilde{A}(\tau) = A\left(\left(\tau b^{-1}\right)\right),\ \tilde{A}_i(\tau) = A_i\left(\tau b^{-1}\right),\ \tilde{f}(\tau) = f\left(\left(\tau b^{-1}\right)\right).$$

Then, by virtue of Theorem 3.1 we obtain the required assertion.

7. Nonlinear abstract elliptic problem

Consider the following nonlinear parabolic problem

$$-q(x) u^{(2)}(x) + B\left(x, u, u^{(1)}\right) u = F\left(x, u, u^{(1)}\right), \tag{7.1}$$

$$L_k u = \sum_{i=0}^{m_k} \alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(a) = 0,\ k = 1, 2, \tag{7.2}$$

where q is a real valued function, α_{ki}, β_{ki} are complex numbers, $m_k \in \{0, 1\}$.

In this section we will prove the existence and uniqueness of maximal regular solution for the nonlinear problem (7.1) – (7.2). Let

$$U=(u_0,u_1)\,,\ X=L_p\left(0,a;E\right),\ Y=W_p^2\left(0,a;E\left(A\right),E\right),$$

$$E_i=\left(E\left(A\right),E\right)_{\theta_i,p}\,,\ \theta_i=\frac{i+\frac{1}{p}}{2},\ X_0=\prod_{i=0}^{1}E_i,$$

Remark 7.1. By using J.Lions-I. Petree result (see e.g [23, § 1.8.]) and Remarks 2.1 we obtain that the embedding $D^iY\in E_i$ is continuous and there is a constant C_1 such that for $w\in Y$, $W=\{w_i\}$, $w_i=D^iw\left(\cdot\right)$, $i=0,1$,

$$\left\|u\right\|_{\infty,X_0}=\prod_{i=0}^{1}\left\|D^iw\right\|_{C([0,a],E_j)}=\sup_{x\in[0,a]}\prod_{i=0}^{1}\left\|D^iw\left(x\right)\right\|_{E_j}\leq C_1\left\|w\right\|_Y.$$

For $r>0$ denote by O_r the closed ball in X_0 of radios r, i.e.

$$O_r=\left\{u\in X_0,\ \left\|u\right\|_{X_0}\leq r\right\}.$$

Consider the linear problem,

$$Lu=-q\left(x\right)w^{(2)}\left(x\right)+\left(A\left(x\right)+d\right)w\left(x\right)=f, \tag{7.3}$$

$$L_kw=0,\ k=1,2,$$

where $A\left(x\right)$ is a linear operator in a Banach space E for $x\in(0,a)$, L_k are boundary conditions defined by (7.1) and $d>0$.

Assume E is a Banach space satisfying the multiplier condition with respect to $p\in(1,\infty)$, $q\left(0\right)=q\left(a\right)$ and $A\left(x\right)$ is uniformly R-positive in E, $A\left(0\right)A^{-1}\left(y_0\right)=A\left(a\right)A^{-1}\left(y_0\right)$. By virtue Theorem 3.1 and Proposition 6.1, problem (7.3) has a unique solution $w\in Y$ for all $f\in X$ and for sufficiently large $d>0$. Moreover, the following coercive estimate holds

$$\left\|w\right\|_Y\leq C_0\left\|f\right\|_X,$$

where the constant C_0 do not depend on $f\in X$ and $a\in(0\ a_0]$.

Condition 7.1. Assume the following satisfied:

(1) $\alpha_{km_k},\beta_{km_k}\neq0$, $q\left(x\right)$ is a positive continuous function on $[0,a]$, $q\left(0\right)=q\left(a\right)$;

(2) E is a Banach space satisfying the multiplier condition with respect to $p\in(1,\infty)$;

(3) $F:[0,a]\times X_0\rightarrow E$ is a measurable function for each $\upsilon_i\in E_i$, $i=0,1$; $F\left(x,.,.\right)$ is continuous with respect to $x\in[0,a]$ and $f\left(x\right)=F\left(x,0\right)\in X$. Moreover, for each $r>0$ there exists the positive functions $h_k\left(x\right)$ such that

$$\left\|F\left(x,U\right)\right\|_E\leq h_1\left(x\right)\left\|U\right\|_{X_0},$$

$$\left\|F(x,U)-F\left(x,\bar{U}\right)\right\|_E \leq h_2(x)\left\|U-\bar{U}\right\|_{X_0},$$

where $h_k \in L_p(0,a)$ with

$$\left\|h_k\right\|_{L_p(0,a)} < C_0^{-1},\ k=1,2;$$

and $U=\{u_0,u_1\}$, $\bar{U}=\{\bar{u}_0,\bar{u}_1\}$, $u_i,\ \bar{u}_i \in E_i$ and $U,\ \bar{U} \in O_r$.

(4) there exist $\Phi_i \in E_i$, such that the operator $B(x,\Phi)$ for $\Phi=\{\Phi_i\}$ is R-positive in E uniformly with respect to $x \in [0,a]$; $B(x,\Phi)B^{-1}\left(x^0,\Phi\right) \in C([0,a];L(E))$; $B(x,0)=A(x)$, $A(0)A^{-1}(y_0)=A(a)A^{-1}(y_0)$;

(5) $B(x,U)$ for $x \in (0,a)$ is a uniform positive operator in a Banach space E, where domain definition $D(B(x,U))$ does not depend on x,U and B: $(0,a)\times X_0 \to L(E(A),E)$ is continuous. Moreover, for each $r>0$ there is a positive constant $L(r)$ such that

$\left\|\left[B(x,U)-B\left(x,\bar{U}\right)\right]\upsilon\right\|_E \leq L(r)\left\|U-\bar{U}\right\|_{X_0}\|A\upsilon\|_E$ for $x \in (0,a)$, $U,\bar{U} \in O_r$ and $\upsilon \in D(B(x,U))$.

Theorem 7.1. Assume the Condition 7.1 holds. Then there is $a \in (0\ a_0]$ such that problem (7.1)−(7.2) has a unique solution belongs to $W_p^2((0,a;E(A),E)$.

Proof. We want to solve the problem (7.1) − (7.2) locally by means of maximal regularity of the linear problem (7.3) via the contraction mapping theorem. For this purpose, let w be a solution of the linear problem (7.3). Consider a ball

$$B_r=\left\{\upsilon \in Y,\ L_k(\upsilon-w)=0,\ \|\upsilon-w\|_Y \leq r\right\}.$$

Let $w \in Y$ be a solution of the problem (7.3) and

$$W=w(0),w^{[1]}(0).$$

Given $\upsilon \in B_r$ solve the linear problem

$$-q(x)u^{(2)}(x)+A(x)u(x)+du=F(x,\upsilon)+$$
$$[B(x,0)-B(x,\upsilon)]\upsilon(x),\ L_ku=0,\ k=1,2, \tag{7.4}$$

where

$$V=\left(\upsilon,\upsilon^{(1)}\right),\ \upsilon \in Y.$$

Consider the function

$$\Phi(x)=F(x,\upsilon)+[B(x,0)-B(x,\upsilon)]\upsilon(x).$$

Let first, we show that $\Phi \in X$ and $\|\Phi\|_X \leq C_0^{-1}r$ for $\upsilon \in Y$, $\|\upsilon\|_Y \leq r$. Indeed, by Remark 7.1, $\upsilon \in C([0,a];E_0)$, one has

$$B(x,0)-B(x,\upsilon) \in C([0,a];L(E(A),E)).$$

Hence, by assumption (3), Φ is measurable and

$$\|\Phi\|_X \leq L(r)\|\upsilon\|_{X_0}\|A\upsilon\|_X+h(t)\|\upsilon\|_{X_0}.$$

Then, by using the Remark 7.1 we obtain

$$\|\Phi\|_X \leq rL(r)\|v\|_X + r\|h_1\|_{L_p} \leq r^2 L(r) + r\|h_1\|_{L_p} \leq r.$$

Define a map Q on B_r by $Qv = u$, where u is a solution of the problem (7.4). We want to show that $Q(B_r) \subset B_r$ and that Q is a contraction operator provided a is sufficiently small, and r is chosen properly. For this aim, by using maximal regularity properties of the problem (7.3) we have

$$\|Qv - w\|_Y = \|u - w\|_Y \leq C_0 \{\|F(x,v) - F(x,0)\|_X +$$
$$\|[B(x,0) - B(x,V)]v\|_X\}.$$

By assumption (3) for $v \in O_r$ we get

$$\|F(x,v) - F(x,0)\|_X \leq \|h_2\|_{L_p(0,a)}\|v\|_{X_0}.$$

By assumptions (4), (5) and Remark 7.2, for $v \in O_r$ we have

$$\|[B(x,0)v - B(x,V)]v\|_X \leq \sup_{x\in[0,a]}\Big\{\|[B(x,0) - B(x,W)]v\|_{L(X_0,X)}$$
$$+ \|[B(x,W) - B(x,V)]v\|_{L(X_0,X)}\|v\|_Y\Big\} \leq$$
$$L(r)\Big[\|W\|_{X_0}\|Av\|_X + \|v - w\|_{\infty,X_0}\Big][\|v - w\|_Y + \|w\|_Y] \leq$$
$$rL(r)\left\{\left[\|W\|_{X_0}\|v\|_Y + C_1\|v - w\|_Y\right] + L(r)\|w\|_Y\right\}.$$

By chousing r and $a \in (0\ a_0]$ so that $\|w\|_Y < \delta_a$ by assumptions (3)-(5) we obtain from the above inequalities

$$\|Qv - w\|_Y \leq r + r^2 L(r)\|W\|_{X_0} + r^2 L(r) C_1 + rL(r)\|w\|_Y < r.$$

That is the operator Q maps B_r into itself, i.e.

$$Q(B_r) \subset B_r.$$

Let $u_1 = Q(v_1)$ and $u_2 = Q(v_2)$. Then $u_1 - u_2$ is a solution of the problem

$$-q(x)u^{[2]}(x) + A(x)u(x) + du = F(x,v_1) -$$
$$F(x,v_1) + [B(x,v_2) - B(x,0)][v_1(x) - v_2(x)] -$$
$$[B(x,v_1) - B(x,v_2)]v_1(x),\ L_k u = 0,\ k = 1,2.$$

In a similar way, by using the assumption (5) we obtain

$$\|u_1 - u_2\|_Y \leq C_0\{rL(r)\|v_1 - v_2\|_X + L(r)\|v_1 - v_2\|_Y\|v_1\|_X$$
$$+ \|h_2\|_{L_p}\|v_1 - v_2\|_Y\Big\} \leq C_0\Big[2rL(r) + \|h_2\|_{L_p}\Big]\|v_1 - v_2\|_Y.$$

Thus Q is a strict contraction. Eventually, the contraction mapping principle implies a unique fixed point of Q in B_r which is the unique strong solution

$$u \in Y = W_{p,\gamma}^{[2]}(0,a;E(A),E).$$

8. Degenerate abstract elliptic and parabolic equations

The main objective of the present section is to discuss maximal regularity properties of the BVP degenerate elliptic DOE

$$-\varepsilon a(x) u^{[2]}(x)+A(x) u(x)+\varepsilon^{\frac{1}{2}} A_1(x) u^{[1]}(x)+A_0(x) u(x)+\lambda u=f(x), \quad (8.1)$$

$$L_{k\varepsilon} u=\sum_{i=0}^{m_k} \varepsilon^{\sigma_i}\left[\alpha_{ki} u^{[i]}(0)+\beta_{ki} u^{[i]}(1)\right]=f_k,\ k=1,2, \quad (8.2)$$

where

$$D_x^{[i]} u=u^{[i]}(x)=\left(x^\gamma \frac{d}{dx}\right)^i u(x),\ \gamma \geq 0,\ x \in(0,1),$$

ε is a small positive parameter, λ is a complex parameter, $a(x)$ is a complex valued function and A, A_0, A_1 are linear operators in a Banach space E. Since the above equation depends on parameter ε, then the solution u also depend of ε, i.e., $u(x)=u(x,\varepsilon)$. Note that, the principal part of the above problem is nonselfadjoint and also have the variable coefficients. The regularity properties for the problem of type (0.1) was studied in [16] for $\varepsilon=1$. Here, several conditions for the separability and sharp resolvent estimates uniformly with respect to parameter ε are given. Especially, it is shown that the corresponding differential operator is R-positive and also generates an analytic semigroup. In first section, we introduce some notations, definitions and background. In section 2, we consider nonlocal nonhomogenous BVP for the degenerate DOE with constant coefficients. We prove that this problem is isomorphism from $W_{p,\gamma}^{[2]}(0,1;E(A),E)$ onto $L_p(0,1;E)\times E_1\times E_2$, where E_k are interpolation spaces between $E(A)$ and E (see section 1 for definition of these spaces). In section 3, we show that the problem (0.1) is $L_p(0,1;E)$ separabile, i.e., we prove that problem (0.1) for $f\in L_p(0,1;E)$ has a unique solution $u\in W_{p,\gamma}^{[2]}(0,1;E(A),E)$ and the following uniform coercive estimate holds

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}}\left\|u^{[i]}\right\|_{L_p(0,1;E)}+\|Au\|_{L_p(0,1;E)} \leq C\|f\|_{L_p(0,1;E)}$$

for $|\arg\lambda|\leq\varphi$, $\varphi<\pi$ with sufficiently large $|\lambda|$, where the constant C depend only on p and A.

Then we established the uniform well-posedeness of initial and BVP for the degenerate abstract parabolic equation

$$\frac{\partial u}{\partial t}+\varepsilon a(x) \frac{\partial^{[2]} u}{\partial x^2}+A(x) u(x,t)=f(x,t),\ x\in(0,b),\ t\in(0,\infty),$$

$$\sum_{i=0}^{m_k} \varepsilon^{\sigma_i}\left[\alpha_{ki} u^{[i]}(0,t)+\beta_{ki} u^{[i]}(1,t)\right]=f_k,\ k=1,2, \quad (8.3)$$

$$u(x.0)=0,$$

where $u=u(x,t)$ is a solution, $D_x^{[i]}u=\frac{\partial^{[i]}u}{\partial x^i}=\left[x^\gamma\frac{\partial}{\partial x}\right]^i u(x,t)$, α_{ki}, β_{ki} are complex numbers, ε is a positive parameter, $a(x)$ is a complex-valued function on $(0,b)$, $A(x)$ and $A_k(x)$ are linear operators in a Banach space E, $d>0$, $\sigma_i=\sigma_i=\frac{i}{2}+\frac{1}{2p(1-\gamma)}$.

Let $\mathbf{p}=(p,p_1)$ and $\Delta_+=(0,b)\times(0,\infty)$.

In this secton, we established the uniform well-posedeness of initial and BVP for the degenerate abstract parabolic equation:

Theorem 8.1. Assume Condition 3.1 hold for $\varphi>\frac{\pi}{2}$. Then, for $f\in L_{\mathbf{p}}(\Delta_+;E)$ and sufficiently large $d>0$ problem $(5.1)-(5.2)$ has a unique solution belonging to $W_{\mathbf{p},\alpha}^{1,[2]}(\Delta_+;E(A),E)$ and the following coercive estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p}}(\Delta_+;E)}+\varepsilon\left\|\frac{\partial^{[2]}u}{\partial x^2}\right\|_{L_{\mathbf{p}}(\Delta_+;E)}+\|Au\|_{L_{\mathbf{p}}(\Delta_+;E)}\leq C\|f\|_{L_{\mathbf{p}}(\Delta_+;E)}.$$

9. The mixed value problem for system of parabolic equations

Consider the initial and BVP for the system of parabolic equations

$$\frac{\partial u_m}{\partial t}-\varepsilon a(x)\frac{\partial^{[2]}u_m}{\partial x^2}+\sum_{j=1}^N d_{mj}(x)u_j(x,t)$$

$$+\sum_{i=0}^1\sum_{j=1}^N\varepsilon^{\frac{i}{2}}b_{imj}(x)\frac{\partial^{[i]}u_j}{\partial x}=f_m(x,t), \tag{9.1}$$

$$\sum_{i=0}^{m_k}\varepsilon^{\sigma_i}\alpha_{ki}\left(D_x^{[i]}u_m\right)(0,t)+\varepsilon^{\sigma_i}\beta_{ki}\left(D_x^{[i]}u_m\right)(b,t)=0,\ k=1,2,$$

$$u_m(x,0)=0,\ t\in(0,\infty),\ x\in(0,b),$$

$$m=1,2,...,N,\ N\in\mathbb{N},$$

where $u=(u_1,u_2,...,u_N)$, $m_{kj}\in\{0,1\}$, α_{ki}, β_{ki} are complex numbers, a is a complex valued functions, σ_i are positive numbers defined in section 2 and

$$s_j=s(1-\theta_j),\ s>0,\ B_j=l_q^{s_j},\ j=1,2,$$

Let A be the operator in $l_q(N)$ defined by

$$D(A)=l_q^s(N),\ A=[d_{mj}(x)],\ d_{mj}(x)=g_m(x)2^{sj},\ m,j=1,2,...,N,$$

where

$$l_q(N)=\left\{u=\{u_j\},\ j=1,2,...N,\|u\|_{l_q(N)}=\left(\sum_{j=1}^N|u_j|^q\right)^{\frac{1}{q}}<\infty\right\},$$

$$l_q(A) = \left\{ u \in l_q(N), \|u\|_{l_q(A)} = \|Au\|_{l_q(N)} = \left(\sum_{j=1}^{N} \left| 2^{sj} u_j \right|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in (0, b), \ 1 < q < \infty, \ N = 1, 2, ..., \infty.$$

Let

$$\Delta_+ = (0, b) \times (0, \infty), \ B = L\left(L_p\left(G; l_q(N)\right)\right).$$

By applying the Theorem 5.1 we obtain the following result.

Theorem 9.1. Assume a, $d_{mj} \in C([0, b])$, $a(x) > 0$, $d_{ii}(x) > 0$ and eigenvalues of the matrix $[d_{mi}(x)]$ are positive for all $x \in (0, b)$, $m, j = 1, 2, ..., N$. Moreover, $b_{imj} \in L_\infty(0, b)$ and there exist $\nu \in (0, 1)$ and $\mu \in \left(0, \frac{1}{2}\right)$ such that

$$\sup_m \sum_{j=1}^{N} b_{0mj}(x)\, d_{jm}^{-(1-\nu)}(x) < M, \ \sup_m \sum_{j=1}^{N} b_{1mj}(x)\, d_{jm}^{-\left(\frac{1}{2}-\mu\right)}(x) < M \text{ for } x \in (0, b).$$

Then for $f(t, x) = \{f_m(t, x)\}_1^\infty \in L_p(\Delta_+; l_q)$, $p, q \in (1, \infty)$ and for sufficiently large $d > 0$, problem (9.1) has a unique solution $u = \{u_m(t, x)\}_1^\infty$ that belongs to the space $W_{\mathbf{p},\gamma}^{1,[2]}(\Delta_+, l_q(D), l_q)$ and the following coercive uniform estimate holds

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p}}(\Delta_+; l_q)} + \varepsilon \left\| D_x^{[2]} u \right\|_{L_{\mathbf{p}}(\Delta_+; l_q)} + \|Au\|_{L_{\mathbf{p}}(\Delta_+; l_q)} \le C \|f\|_{L_{\mathbf{p}}(\Delta_+; l_q)}.$$

References

1. S. Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, Comm. Pure Appl. Math.,15 (1962), 119-147.

2. H. Amann, Linear and quasi-linear equations,1, Birkhauser, Basel 1995.

3. R. Agarwal R, D. O' Regan, V. B. Shakhmurov, Separable anisotropic differential operators in weighted abstract spaces and applications, J. Math. Anal. Appl. 2008, 338, 970-983.

4. Ashyralyev A., Claudio Cuevas and Piskarev S., "On well-posedness of difference schemes for abstract elliptic problems in spaces", Numer. Func. Anal.Opt., v. 29, No. 1-2, Jan. 2008, 43-65.

5. O. V. Besov, V. P. Ilin, S. M. Nikolskii, Integral representations of functions and embedding theorems, Nauka, Moscow, 1975.

6. D. L. Burkholder, A geometrical conditions that implies the existence certain singular integral of Banach space-valued functions, Proc. conf. Harmonic analysis in honor of Antoni Zygmund, Chicago, 1981,Wads Worth, Belmont, (1983), 270-286.

7. C. Dore and S. Yakubov S, Semigroup estimates and non coercive boundary value problems, Semigroup Forum 60 (2000), 93-121.

8. R. Denk, M. Hieber, J. Prüss J, R-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (2003), n.788.

9. P. Guidotti, Optimal regularity for a class of singular abstract parabolic equations, J. Differ. Equations, v. 232, 2007, 468–486.

10. A. Favini, V. B. Shakhmurov, Ya. Yakubov, Regular boundary value problems for complete second order elliptic differential-operator equations in UMD Banach spaces, Semigroup form, v. 79 (1), 2009.

11. A. Favini, A. Yagi, Degenerate Differential Equations in Banach Spaces, Taylor & Francis, Dekker, New-York, 1999.

12. S. G. Krein, "Linear differential equations in Banach space", American Mathematical Society, Providence, 1971.

13. A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhauser, 2003.

14. P. E. Sobolevskii, Coerciveness inequalities for abstract parabolic equations, Doklady Akademii Nauk SSSR, 57(1), (1964), 27-40.

15. J. Prüss, Maximal regularity for evolution equations in L_p-spaces, Lectures given at the summer School, Positivity and semigroups, Monopoli, 2002.

16. V. B. Shakhmurov V. B, Regular degenerate separable differential operators and applications, Potential Analysis, 2011, v.35 (3), 201-212

17. V. B. Shakhmurov, Embedding theorems and maximal regular differential operator equations in Banach-valued function spaces, Journal of Inequalities and Applications, (4) 2005, 605-620.

18. V. B. Shakhmurov, Linear and nonlinear abstract equations with parameters, Nonlinear Analysis, Method and Applications, 2010, v. 73, 2383-2397.

19. V. B. Shakhmurov, The embedding theorems in abstract function spaces and their applications, Mathematics of the USSR-Sbornik, 134(1-2), 1987, 261-276.

20. V. B. Shakhmurov, Degenerate differential operators with parameters, Abstract and Applied Analysis, 2007, v. 2006, 1-27.

21. V. B. Shakhmurov, Separable anisotropic differential operators and applications, J. Math. Anal. Appl. 2006, 327(2), 1182-1201.

22. A. L. Skubachevskii, Nonlocal boundary value problems, Journal of Mathematical Sciences, V. 155, No. 2, 2008.

23. H. Triebel, "Interpolation theory, Function spaces, Differential operators.", North-Holland, Amsterdam, 1978.

24. L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p regularity, Math. Ann. 319, (2001), 735-758.

25. S. Yakubov and Ya. Yakubov, "Differential-operator Equations. Ordinary and Partial Differential Equations ", Chapman and Hall /CRC, Boca Raton, 2000.

26. Ti-Jun Xiaoa, Jin Liang, Complete second order differential equations in Banach spaces with dynamic boundary conditions, J. Differential Equations 200 (2004) 105–136.