

AN ESSENTIAL LOCAL GEOMETRIC MORPHISM WHICH IS NOT LOCALLY CONNECTED THOUGH ITS INVERSE IMAGE PART DEFINES AN EXPONENTIAL IDEAL

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In [BP80] the authors introduced and studied a property of geometric morphisms between toposes which they called “molecular” and nowadays is usually referred to as “locally connected”. One of the various characterizations of this property is that the inverse image part F of the geometric morphism $F \dashv U: \mathcal{E} \rightarrow \mathcal{S}$ preserves dependent products, i.e. right adjoints to pullback functors. As described below this requirement is tantamount to F having a fibered or “indexed” left adjoint. In *loc. cit.* the authors also consider the property that F has an enriched left adjoint, which amounts to F preserving (ordinary) exponentials and having a left adjoint.

In this note, we exhibit a geometric morphism $F \dashv U: \mathcal{E} \rightarrow \mathcal{S}$ which satisfies the weaker of these two sets of conditions, but not the stronger: thus, it is not locally connected although it is essential (i.e. F has a left adjoint) and the inverse image F preserves exponentials. In fact, F will be full and faithful and U will have a right adjoint, i.e. our geometric morphism is *local*; moreover, the left adjoint of F will preserve finite products, so that F even exhibits \mathcal{S} as an *exponential ideal* in \mathcal{E} .

1 Preliminaries

A geometric morphism $F \dashv U: \mathcal{E} \rightarrow \mathcal{S}$ is called *locally connected* just when F has a left adjoint L which is fibered or indexed over \mathcal{S} . A succinct way of expressing this is as the requirement that

$$\begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 b \downarrow & & \downarrow a \\
 FJ & \xrightarrow{Fu} & FI
 \end{array}
 \quad \text{a pullback} \quad \text{implies} \quad
 \begin{array}{ccc}
 LB & \xrightarrow{Lf} & LA \\
 \hat{b} \downarrow & & \downarrow \hat{a} \\
 J & \xrightarrow{u} & I
 \end{array}
 \quad \text{a pullback} \quad (1)$$

as discussed, for example, in [BP80, Jo02, Str20].

When I (and thus also FI) is a terminal object, this condition boils down to the requirement that $L \dashv F$ validates *Frobenius reciprocity*, i.e. that for all $I \in \mathcal{S}$ and $A \in \mathcal{E}$ the canonical map $\langle \widehat{\pi}_1, L\pi_2 \rangle: L(FI \times A) \rightarrow I \times LA$ is an isomorphism. One easily checks that local connectedness is equivalent to Frobenius reciprocity holding not only for the adjunction $L \dashv F$ itself, but also for each adjunction on slices $L_I = \Sigma_{\varepsilon_I} \circ L_{/FI} \dashv F_{/I} = F_I$ (where ε_I is the counit of $L \dashv F$ at I). Furthermore, by [Jo02, Lemma A.1.5.8] an adjunction between cartesian closed categories validates Frobenius reciprocity if and only if its right adjoint preserves exponentials.

In fact, for a geometric morphism $F \dashv U: \mathcal{E} \rightarrow \mathcal{S}$ the following conditions are equivalent:

- (i) $F \dashv U$ is locally connected;
- (ii) F preserves dependent products (i.e. right adjoints to pullback functors);
- (iii) $F_{/I}$ preserves exponentials for all $I \in \mathcal{S}$,

as formulated in [Jo02, Proposition C3.3.1].

Some of the notions and remarks above extend to finitely complete categories. We call an adjunction $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$ between finitely complete categories *stably Frobenius* when it validates the condition in diagram (1). Stably Frobenius adjunctions with full and faithful right adjoint F are sometimes called *semi-left-exact reflections* [CHK]; they are reflections whose left adjoint preserves pullbacks along morphisms in the image of F .

Further, if \mathbb{A} and \mathbb{B} are locally cartesian closed then by [Jo02, Lemma A1.5.8] an adjunction $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$ is stably Frobenius if and only if condition (iii) above holds, i.e. all slices $F_{/I}$ preserve exponentials. In fact, in this case we can say slightly more; F not only preserves exponentials but *creates* them:

Lemma 1.1. *If $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$ is a semi-left-exact reflection between finitely complete categories, and \mathbb{B} is locally cartesian closed, then so too is \mathbb{A} .*

Proof. This is [GL12, Lemma 4.3]. □

For a reflection $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$, a stronger condition than Frobenius reciprocity is the requirement that L preserve all finite products. If \mathbb{B} is cartesian closed then, by [Jo02, Proposition A4.3.1], L preserve all finite products just when the subcategory determined by F is (not only exponential-closed but) an exponential *ideal*. The “stable” version of this condition is that L preserves all pullbacks over objects in the image of F ; in [CHK] this condition was called *having stable units*.

It has been shown in Prop. 10.3 of [LM15] that for essential connected geometric morphisms $F \dashv U: \mathcal{E} \rightarrow \mathcal{S}$ the left adjoint L of F has stable units if and only if $F \dashv U$ is locally connected and L preserves binary products.

As shown in [Jo11, Proposition 2.7] if $L \dashv F \dashv U \dashv R: \mathcal{S} \rightarrow \mathcal{E}$ with F (and thus also R) full and faithful then \mathcal{S} is an exponential ideal in \mathcal{E} (via F) whenever F preserves exponentials and all components of the canonical transformation $\theta: U \rightarrow L$ are epimorphic. Here, as in [Jo11, LM15] $\theta_A: UA \rightarrow LA$ is the

unique map whose image under F is $\eta_A \circ \varepsilon_A: FUA \rightarrow A \rightarrow FLA$. As also shown in [Jo11], for a locally connected, hyperconnected and local geometric morphism $F \dashv U: \mathcal{E} \rightarrow \mathcal{S}$, the left adjoint L of F necessarily preserves finite products.

2 The counterexample

Any functor $L: \mathbb{B} \rightarrow \mathbb{A}$ between small categories induces a geometric morphism $L^* \dashv L_*: \widehat{\mathbb{B}} \rightarrow \widehat{\mathbb{A}}$ between presheaf categories, where L^* and L_* are restriction and right Kan extension along L . This geometric morphism is always essential, since we have $L_! \dashv L^*$, where $L_!$ is left Kan extension along L . If L has a fully faithful right adjoint $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$, then L_* has the fully faithful right adjoint F_* , so our geometric morphism is in fact *local*.

Lemma 2.1. *For reflections $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$ between small finitely complete categories the following assertions hold.*

- (i) L^* is fully faithful;
- (ii) If L preserves finite products, then so does $L_!$, so that L^* exhibits $\widehat{\mathbb{A}}$ as an exponential ideal in $\widehat{\mathbb{B}}$;
- (iii) If $L^* \dashv L_*$ is locally connected then $L \dashv F$ is semi-left-exact.

Proof. For (i) note that L^* is naturally isomorphic to $F_!: \widehat{\mathbb{A}} \rightarrow \widehat{\mathbb{B}}$, and left Kan extension along a fully faithful functor is always fully faithful.

For (ii), since L preserves finite products, $L_!$ preserves finite products of representables. Since every presheaf is a colimit of representables, and \times preserves colimits in each variable, it follows that $L_!$ preserves finite products.

Finally, for (iii), to say that $L^* \dashv L_*$ is locally connected is to say that $L_! \dashv L^*$ is stably Frobenius. Since $L_!$ and $L^* \cong F_!$ preserve representable objects, the adjunction $L \dashv F$ is also stably Frobenius; and since F by assumption is full and faithful, the adjunction $L \dashv F$ is in fact semi-left-exact. \square

From Lemmas 1.1 and 2.1 we immediately obtain:

Theorem 2.2. *Consider a reflection between small finitely complete categories $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$ for which:*

- L preserves finite products;
- \mathbb{B} is locally cartesian closed;
- \mathbb{A} is not locally cartesian closed.

The geometric morphism $L^ \dashv L_*: \widehat{\mathbb{B}} \rightarrow \widehat{\mathbb{A}}$ is essential and local but not locally connected, i.e. L^* does not preserve dependent products, although the left adjoint $L_!$ of L^* preserves finite products, i.e. the full subcategory of $\widehat{\mathbb{B}}$ given by the image of $\widehat{\mathbb{A}}$ under L^* is an exponential ideal.*

To complete the construction of our counterexample, it thus suffices to find a reflection $L \dashv F: \mathbb{A} \rightarrow \mathbb{B}$ with the properties listed above. In fact, we give two examples of quite different flavour.

Example 2.3. In [GL12] the authors consider the following situation. Let \mathbb{B} be a small category equivalent to the category of finite reflexive graphs and morphisms between them, and let $F: \mathbb{A} \rightarrow \mathbb{B}$ be the inclusion of the full subcategory of finite preorders. Example 3.9 of *loc. cit.* shows that \mathbb{A} is an exponential ideal in \mathbb{B} , so that L preserves finite products. Of course, as a category of finite presheaves, the category \mathbb{B} is locally cartesian closed; however, as also noted in *loc. cit.*, the category \mathbb{A} is not so. So by the corollary above, the geometric morphism $L^* \dashv L_*$ is essential and local, with inverse image giving rise to an exponential ideal, but is not locally connected.

Example 2.4. A further counterexample arises from realizability models of dependent type theories as studied in [Str91]. For our purposes we may restrict to the case where the underlying partial combinatory algebra is the *First Kleene Algebra* corresponding to the standard notion of computation on natural numbers.

For \mathbb{B} one takes the category of modest sets and for \mathbb{A} the full reflective subcategory on $\neg\neg$ -closed subobjects of powers of N , the natural numbers object of \mathbb{B} . The category \mathbb{B} is locally cartesian closed and \mathbb{A} forms a full reflective subcategory of \mathbb{B} which is an exponential ideal since it is a model of the Calculus of Constructions as verified in Chapter 2 of [Str91]. Moreover, as follows from (the proof of) Theorem 2 in the Appendix of [Str91], the category \mathbb{A} is not a sub-locally-cartesian-closed-category of \mathbb{B} .

For sake of completeness we describe the counterexample explicitly but refer the reader for its verification to *loc.cit.* Let $I = N^N$ and A be the $\neg\neg$ -closed subobject of N^N consisting of those f with $f(0) \leq 1$ and $f(0) = 0$ if and only if $f(n+1) = 0$ for all n . Let $g: A \rightarrow I$ send f to $\lambda n.f(n+1)$ and $h: B \rightarrow A$ be the projection $\pi: A \times N \rightarrow A$. Obviously, both g and h are maps in \mathbb{A} but the domain of the dependent products $\Pi_g h$ taken in \mathbb{B} is not isomorphic to an object in \mathbb{A} as can be shown using the Kreisel-Lacombe-Shoenfield Theorem well known from computability theory.

Note that in both examples the geometric morphism $L^* \dashv L_*$ is not hyperconnected since L^* is not full on subobjects. The reason is that there are sieves S in \mathbb{B} on some FA such that some $u: B \rightarrow FA$ is in S but its reflection FLu is not in S .

In [HR20] one finds an example of an essential hyperconnected and local geometric morphism which, however, fails to be locally connected. Though their counterexample and ours are quite different in nature, together they seem to suggest that an essential hyperconnected local geometric morphism need not be locally connected even if its inverse image part is an exponential ideal. This would provide a (negative) answer to the question raised immediately after [LM15, Corollary 10.4].

A positive answer to this question would mean that the inverse image part of a hyperconnected and local geometric morphism necessarily preserves dependent

products whenever it preserves exponentials. This, however, is most unlikely since it would mean that dependent functions spaces are definable in first order intuitionistic logic from ordinary function spaces.

References

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