Semantics of Type Theory
Formulated in Terms of Representability

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1 Giraud Splitting of $P_{\widehat{\mathbb{C}}}$

Let $\mathbb{C}$ be a category with finite limits. Then, as is well known, the category $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ of presheaves over $\mathbb{C}$ is a topos and, accordingly, a model of type theory. As introduced in Giraud's book from 1971 and used by Bénabou in unpublished work (around 2001/2) the fundamental fibration $P_{\widehat{\mathbb{C}}} = \partial_1 : \widehat{\mathbb{C}}^{\mathbf{2}} \to \widehat{\mathbb{C}}$ is equivalent to the split fibration $\mathbf{S}(\mathbb{C})$ over $\widehat{\mathbb{C}}$ as given by

$$\mathbf{S}(\mathbb{C})(X) = \mathbf{Set}^{\mathsf{Elts}(X)^{\mathsf{op}}} \qquad \mathbf{S}(\mathbb{C})(f : Y \to X) = \mathbf{Set}^{\mathsf{Elts}(f)^{\mathsf{op}}}$$

where $\mathsf{Elts}(X) = \mathsf{Y}_{\mathbb{C}}{\downarrow}X$ and $\mathsf{Elts}(f) = \mathsf{Y}_{\mathbb{C}}{\downarrow}f$, i.e. postcomposition with f.

As emphasized by Bénabou one may think of $\widehat{\mathbb{C}}$ as a ("higher order") *category of classes* and the full subcategory of *representable* presheaves as *category of sets* which, as well known, is equivalent to $\mathbb{C}$. A morphism $f : Y \to X$ is thought of as a family of sets indexed by a class iff it is *representable* in the sense of Grothendieck, i.e. for every $x : \mathsf{Y}_{\mathbb{C}}(I) \to X$ we have

$$\begin{tikzcd}
\mathsf{Y}_{\mathbb{C}}(J) \arrow[r, "y"] \arrow[d, "\mathsf{Y}_{\mathbb{C}}(u)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
\mathsf{Y}_{\mathbb{C}}(I) \arrow[r, "x"'] & X
\end{tikzcd}$$

for some $u : J \to I$ unique up to isomorphism in $\mathbb{C}/I$. Let us write $\mathbf{Rep}(\mathbb{C})$ for the full subcategory of $\widehat{\mathbb{C}}^{\mathbf{2}}$ on representable morphisms. Consider the diagram

$$\begin{tikzcd}
\mathbf{Rep}(\mathbb{C}) \arrow[r, hook] \arrow[dr] & \widehat{\mathbb{C}}^{\mathbf{2}} \arrow[d, "P_{\widehat{\mathbb{C}}}"] \\
& \widehat{\mathbb{C}}
\end{tikzcd}$$

which exhibits $\mathbf{Rep}(\mathbb{C}) \to \widehat{\mathbb{C}}$ as a subfibration of $P_{\widehat{\mathbb{C}}}$ since representable morphisms are stable under pullbacks in $\widehat{\mathbb{C}}$. Obviously, the (Giraud) splitting $\mathbf{S}(\mathbb{C})$ of $P_{\widehat{\mathbb{C}}}$ restricts to a splitting $\mathbf{s}(\mathbb{C})$ of $\mathbf{Rep}(\mathbb{C})$ where $\mathbf{s}(\mathbb{C})(X)$ is the full subcategory of $\mathbf{S}(\mathbb{C})(X)$ on those presheaves over $\mathsf{Elts}(X)$ whose corresponding morphism to X in $\widehat{\mathbb{C}}$ is representable. The situation is summarized in the following diagram in $\mathbf{Fib}(\widehat{\mathbb{C}})$

$$\begin{tikzcd} \mathbf{Rep}(\mathbb{C}) \arrow[r, hook] \arrow[d, "\simeq"'] & P_{\widehat{\mathbb{C}}} \arrow[d, "\simeq"] \\ \mathbf{s}(\mathbb{C}) \arrow[r, hook] & \mathbf{S}(\mathbb{C}) \end{tikzcd}$$

where the bottom arrow is split cartesian and the vertical arrows are (non-split) equivalences of fibrations.

It is straightforward to extend the equivalence between $P_{\widehat{\mathbb{C}}}$ and $\mathbf{S}(\mathbb{C})$ to an equivalence between a non-split and a split model of extensional Martin-Löf type theory with dependent sums and products, identity types and natural numbers (see section 2).

1.1 Giraud-Bénabou Splitting of $P_{\mathbb{C}}$

Using Axiom of Choice for classes, i.e. "global choice", one easily sees that the split fibration $\mathsf{Y}_{\mathbb{C}}^{*}\mathbf{s}(\mathbb{C})$ is equivalent to $P_{\mathbb{C}}$, the fundamental fibration of $\mathbb{C}$, because every morphism between representable presheaves is a representable morphism in $\widehat{\mathbb{C}}$ since $\mathbb{C}$ is assumed to have finite limits and $\mathsf{Y}_{\mathbb{C}}$ preserves them. Notice that this splitting of $P_{\mathbb{C}}$ is different from the "right adjoint splitting" $\prod_{\mathrm{Id}_{\mathbb{C}}} P_{\mathbb{C}}$ (as used for proving the "fibered Yoneda lemma") and the "left adjoint splitting" $\coprod_{\mathrm{Id}_{\mathbb{C}}} P_{\mathbb{C}}$ which is inconvenient to work with since maps in its total category are isomorphism classes of certain spans (see Appendix C). I think that the Giraud-Bénabou splitting as given by $\mathsf{Y}_{\mathbb{C}}^{*}\mathbf{s}(\mathbb{C})$ is much more convenient than the two other ones which are obtained via methods of splitting that are "uniform" in the sense that they apply to arbitrary fibrations. It is not unexpected that one may obtain better results when exploiting the specific nature of the fibration one wants to split.

But on the other hand there is a split cartesian functor E from the left adjoint splitting $\mathcal{S} \to \mathbb{C}$ of $P_{\mathbb{C}}$ as described in Appendix C to the Giraud-Bénabou splitting $\mathsf{Y}_{\mathbb{C}}^{*}\mathbf{s}(\mathbb{C})$ of $P_{\mathbb{C}}$ which suggest an isomorphic reformulation of $\mathcal{S} \to \mathbb{C}$ making it quite similar to $\mathsf{Y}_{\mathbb{C}}^{*}\mathbf{s}(\mathbb{C})$. For $u : I \to I_0$ and $a : A \to I_0$ in $\mathbb{C}$ let $E(u,a)$ be the presheaf over $\mathbb{C}/I$ sending $v : J \to I$ to the set $E(u,a)(v) = \{g : J \to A \mid a \circ g = u \circ v\}$ and whose morphism part is given by $E(u,a)(vw \xrightarrow{w} v)(g) = g \circ w$. A morphism $(w,f) : (v,b) \to (u,a)$ in $\mathcal{S}$ is sent by E to the natural transformation $E(w,f) : E(v,b) \Rightarrow \Sigma_w^{*}E(u,a)$ sending $g \in E(v,b)(v')$ to $q(u,a) \circ f \circ h_g$ where h_g is the unique map in $\mathbb{C}$ with $p(v,b) \circ h_g = v'$ and $q(v,b) \circ h_g = g$. One easily checks that E is a cartesian equivalence but not a split cartesian equivalence.

Thus, the left adjoint splitting of P is isomorphic to the split fibration over $\mathbb{C}$ whose fibre over I has as objects pairs (u, a) of maps in $\mathbb{C}$ where $u : I \to I_0$ and $a : A \to I_0$ and whose morphism from (v, b) to (u, a) are natural transformations from $E(v, b)$ to $E(u, a)$ and whose reindexing along $w : J \to I$ is given on objects by $w^*(u, a) = (uw, a)$ and on morphisms by $w^*\tau = (\Sigma_w^{\mathsf{op}})^*\tau$. Notice that this split fibration is very close in spirit to the Bénabou-Giraud splitting of $P_{\mathbb{C}}$ from which it is obtained by restricting to presheaves of the form $E(u, a)$ and refining them by the information (u, a) describing how they arise.

2 Reformulation of Awodey's *Natural Semantics* in the Spirit of Bénabou's Work on Representable Morphisms

As is well known the Yoneda functor $\mathsf{Y}_{\mathbb{C}} : \mathbb{C} \to \widehat{\mathbb{C}}$ preserves dependent products (to the extent they exist in $\mathbb{C}$). It trivially preserves dependent sums since they are given by composition. As observed in the previous section the presheaf topos $\widehat{\mathbb{C}}$ can be organized into a split model of type theory. For ease of exposition we prefer to formulate things in terms of the non-split model since splitting is taken care of by the methods of the previous section.

The basic idea of Awodey's natural semantics is to fix a **representable** morphism $p = p_U : E_U \to U$ in $\widehat{\mathbb{C}}$ thought of as a family of small types indexed by a type which may be big or small. Let us write $\mathcal{S} = \mathcal{S}_p$ for the class of morphisms in $\widehat{\mathbb{C}}$ which arise as pullbacks of p. We think of p as a *universe* giving rise to a class $\mathcal{S}_p$ of maps **small** in the sense of p. Now for such universes we may require various closure properties

Definition 2.1 *A universe $p = p_U : E_U \to U$ is closed under dependent sums iff $\mathcal{S}_p$ is closed under composition and it is closed under dependent products if $\mathcal{S}_p$ is closed under Π, i.e. $\Pi_a b$ is in $\mathcal{S}_p$ whenever a and b are in $\mathcal{S}_p$.*

These notions coincide with the ones given by Awodey. But our formulations are much simpler than his ones because he follows Voevodsky's habit of stating requirements in a "Logical Framework" style which are difficult to parse. What's perfectly clear when formulated syntactically in a Logical Framework get's somewhat obscure when formulated in terms of the categorical semantics of the Logical Framework in $\widehat{\mathbb{C}}$.

Notice that a universe $p_U : E_U \to U$ may be called small iff U is representable, i.e. indexed by a "set" and not a "class", i.e. p_U is isomorphic to $\mathsf{Y}(u)$ for some morphism u in $\mathbb{C}$.

But, more importantly, there is a representable morphism $p_{\mathbf{set}(\mathbb{C})} : E_{\mathbf{set}(\mathbb{C})} \to \mathbf{set}(\mathbb{C})$ from which all representable morphisms in $\widehat{\mathbb{C}}$ can be obtained via pullback as we have learned from J. Bénabou in late 2002 and will briefly explain now.

First consider the **big** presheaf $\mathbf{Set}(\mathbb{C})$ over $\mathbb{C}$ where $\mathbf{Set}(\mathbb{C})(I) = \widehat{\mathbb{C}/I} = \mathbf{Set}^{(\mathbb{C}/I)^{\mathsf{op}}}$ and for $u : J \to I$ in $\mathbb{C}$ reindexing along u is given by $\mathbf{Set}(\mathcal{C})(u) = \mathbf{Set}^{(\Sigma_u)^{\mathsf{op}}}$ (where $\Sigma_u : \mathbb{C}/J \to \mathbb{C}/I$ is postcomposition with u). We describe $E_{\mathbf{Set}(\mathbb{C})}$ by specifying the corresponding presheaf over $\mathsf{Elts}(\mathbf{Set}(\mathbb{C}))$. For $A \in \mathbf{Set}(\mathbb{C})(I)$ let $E_{\mathbf{Set}(\mathbb{C})}(I, A) = A(\mathrm{id}_I)$ and for $u : (J, u^*A) \to (I, A)$ in $\mathsf{Elts}(\mathbf{Set}(\mathbb{C}))$ the map $E_{\mathbf{Set}(\mathbb{C})}(u) : E_{\mathbf{Set}(\mathbb{C})}(I, A) \to E_{\mathbf{Set}(\mathbb{C})}(J, u^*A)$ sends $a \in A(\mathrm{id}_I)$ to $A(u : u \to \mathrm{id}_I)(a) \in u^*A(\mathrm{id}_I) = A(u)$. The map $p_{\mathbf{Set}(\mathbb{C})} : E_{\mathbf{Set}(\mathbb{C})} \to \mathbf{Set}(\mathbb{C})$ is given by first projection. Let $\mathbf{set}(\mathbb{C})$ be the subpresheaf of $\mathbf{Set}(\mathbb{C})$ where $\mathbf{set}(\mathbb{C})(I)$ consist of all **representable** presheaves over $\mathbb{C}/I$. The representable map $p_{\mathbf{set}(\mathbb{C})} : E_{\mathbf{set}(\mathbb{C})} \to \mathbf{set}(\mathbb{C})$ is obtained by pulling back $p_{\mathbf{Set}(\mathbb{C})}$ along the inclusion of $\mathbf{set}(\mathbb{C})$ into $\mathbf{Set}(\mathbb{C})$

$$\begin{tikzcd}
E_{\mathbf{set}(\mathbb{C})} \arrow[r, hook] \arrow[d, "p_{\mathbf{set}(\mathbb{C})}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\mathbf{Set}(\mathbb{C})} \arrow[d, "p_{\mathbf{Set}(\mathbb{C})}"] \\
\mathbf{set}(\mathbb{C}) \arrow[r, hook] & \mathbf{Set}(\mathbb{C})
\end{tikzcd}$$

Notice that $\mathbf{Set}(\mathbb{C})$ is the presheaf of objects of the split fibration $\mathsf{Y}_{\mathbb{C}}^*\mathbf{S}(\mathbb{C})$ and $\mathbf{set}(\mathbb{C})$ is the presheaf of objects of its split subfibration $\mathbf{s}(\mathbb{C})$.

Of course, the presheaves $\mathbf{Set}(\mathbb{C})$ and $\mathbf{set}(\mathbb{C})$ are too large to live in $\mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ and, therefore, in their definition one better replaces $\mathbf{Set}$ by a Grothendieck universe $\mathcal{U}$ chosen sufficiently large for containing $\mathbb{C}$ as an internal category. When understanding $p_{\mathbf{set}(\mathbb{C})}$ this way it is a representable morphism such that up to isomorphism its pullbacks are precisely the representable morphisms in $\mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$. However, when understanding $p_{\mathbf{Set}(\mathbb{C})}$ this way its pullbacks in $\mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ are precisely the morphisms in $\mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ with $\mathcal{U}$-small fibres.

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I am grateful to Jean Bénabou for explaining his work to me. The current note was triggered by some email discussion with S. Awodey, P. Lumsdaine and M. Shulman in early June 2014. In particular, the observation formulated in the last paragraph of subsection 1.1 is a result of this discussion.

A Natural Numbers with Large and Small Elimination

One says that the universe $p_U : E_U \to U$ contains the type of natural numbers iff U has a global element $\widetilde{n} : 1 \to U$ such that

$$\begin{tikzcd}
\mathsf{Y}_{\mathbb{C}}(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_U \arrow[d, "p_U"] \\
1 \arrow[r, "\widetilde{n}"'] & U
\end{tikzcd}$$

where N is a natural numbers object in $\mathbb{C}$. In this case $\mathsf{Y}_{\mathbb{C}}(N)$ has elimination w.r.t. families of types as given by representable morphisms. Notice that it has "large" elimination (w.r.t. all families of types in $\widehat{\mathbb{C}}$) iff $\mathsf{Y}_{\mathbb{C}}(N)$ is the natural numbers object in $\widehat{\mathbb{C}}$, i.e. isomorphic to $\coprod_{\mathbb{N}} 1$.

B Comprehension for $\mathbf{S}(\mathbb{C})$

Suppose $A \in \widehat{\mathbb{C}}$ and $B : \mathsf{Elts}(A)^{\mathsf{op}} \to \mathbf{Set}$ then this gives rise to the object $A.B \in \widehat{\mathbb{C}}$ which is constructed as follows. For $I \in \mathbb{C}$ the set $A.B(I) = \{\langle x, y\rangle \mid x \in A(I) \text{ and } y \in B(I, x)\}$ and for $u : J \to I$ in $\mathbb{C}$ and $\langle x, y\rangle \in A.B(I)$

$$A.B(u)(\langle x, y\rangle) = \langle A(u)(x), B(\langle u, x\rangle)(y)\rangle$$

where $\langle u, x\rangle : (J, A(u)(x)) \to (I, x)$ in $\mathsf{Elts}(A)$. There is an obvious natural transformation $A.B \to A$ given by first projection.

Notice that this comprehension functor from $\mathbf{Set}^{\mathsf{Elts}(A)^{\mathsf{op}}}$ to $\widehat{\mathbb{C}}/A$ is an equivalence.[1]

C The method of "local universes"

As observed by V. Voevodsky (and myself) around 2006 when given a category $\mathbb{C}$ with finite limits and a "universe" $p = p_U : E_U \to U$ the full subfibration $\mathcal{D}_p$ of $P_{\mathbb{C}}$

$$\begin{tikzcd}
\mathcal{D}_p \arrow[r, hook] \arrow[dr, "\mathcal{D}_p"'] & \mathbb{C}^{\mathbf{2}} \arrow[d] \\
& \mathbb{C}
\end{tikzcd}$$

[1] It is an easy exercise in the book by MacLane and Moerdijk to show that this functor is full, faithful and essentially surjective.

on pullbacks of p admits the following splitting. For every $a : I \to U$ **choose** a pullback

$$\begin{tikzcd} E_a \arrow[r, "q_a"] \arrow[d, "p_a"'] \arrow[dr, phantom, "\lrcorner" very near start] & E_U \arrow[d, "p_u"] \\ I \arrow[r, "a"'] & U \end{tikzcd}$$

This gives rise to the following split fibration $\mathcal{S}_p$ over $\mathbb{C}$. Objects of the total category $\int \mathcal{S}_p$ are morphism in $\mathbb{C}$ to U. For objects $a : I \to U$ and $b : J \to U$ a morphism from b to a in $\mathcal{S}_p$ is a pair (u, f) making the diagram

$$\begin{tikzcd} E_b \arrow[r, "f"] \arrow[d, "p_b"'] & E_a \arrow[d, "p_a"] \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

commute. Such a morphism is **split cartesian** iff $a \circ u = b$ and $q_a \circ f = q_b$ which implies that the square is a pullback. Of course, the map (u, f) is just cartesian iff the above square is a pullback. It is easy to check that split cartesian maps are closed under composition and that $(\mathrm{id}_I, \mathrm{id}_{E_a})$ is also split cartesian. Accordingly, the split cartesian morphisms give rise to a split cleavage of the fibration $\mathcal{S}_p \to \mathbb{C}$ sending $a : I \to U$ to I and (u, f) to u.

As shown by Voevodsky

(1) $\mathcal{S}_p$ has *split internal sums* iff $\mathcal{D}_p$ is closed under composition

(2) $\mathcal{S}_p$ has *split internal products* iff $\mathcal{D}_p$ is closed under Π

P. Lumsdaine and M. Warren have generalized this to a splitting of $P_{\mathbb{C}}$ via a method which may be called **"local universes"** and can be described as follows. For maps $u : J \to I$ and $a : A \to I$ in $\mathbb{C}$ choose a pullback

$$\begin{tikzcd} u^*A \arrow[r, "{q(u,a)}"] \arrow[d, "{p(u,a)}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "a"] \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

without assuming any coherence conditions for this choice of pullbacks, i.e. one chooses a **cleavage** of the fundamental fibration $P_{\mathbb{C}}$. Given such a choice we may construct a split fibration $\mathcal{S} \to \mathbb{C}$ as follows. The objects of $\mathcal{S}$ are cospans (u, a) in $\mathbb{C}$, i.e. $u : I \to I_0$ and $a : A \to I_0$. A morphism from (v, b) to (u, a) is a pair (w, f) making the diagram

$$\begin{tikzcd} v^*B \arrow[r, "f"] \arrow[d, "p(v,b)"'] & u^*A \arrow[d, "p(u,a)"] \\ J \arrow[r, "w"'] & I \end{tikzcd}$$

commute and composition is componentwise. A morphism (w, f) is cartesian iff the square is a pullback and it is *split cartesian* iff $a = b$, $v = u \circ w$ and $q(u, a) \circ f = q(v, a)$. It is easy to check that this choice of split cartesian arrows gives rise to a split cleavage of the fibration $\mathcal{S} \to \mathbb{C}$ sending an object (u, a) to the domain of u and a morphism (w, f) to w. Obviously, this fibration is equivalent to the fundamental fibration $P_{\mathbb{C}}$. Moreover, one can again show that the split fibration $\mathcal{S} \to \mathbb{C}$ has split internal sums and that it has split internal products iff $\mathbb{C}$ is locally cartesian closed.

Obviously, for maps $p : E \to U$ the fibration $\mathcal{S}_p \to \mathbb{C}$ appears as a full split subfibration of $\mathcal{S} \to \mathcal{C}$ which justifies the name "local universes". Obviously, the fibration $\mathcal{S} \to \mathbb{C}$ is isomorphic to the split fibration $\coprod_{\mathrm{Id}_{\mathbb{C}}} P_{\mathbb{C}}$. In the latter morphisms of the total category are equivalence classes from which one can choose representatives using the chosen cleavage of $P_{\mathbb{C}}$.

Lumsdaine and Warren's motivation for working with this "local universes" variant of $\coprod_{\mathrm{Id}_{\mathbb{C}}} P_{\mathbb{C}}$ is that it allows one to come up with a split version of intensional identity types arising from some model category structure on $\mathbb{C}$.