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Quotients of countably based spaces are not closed under sobrification

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In this note we show that *quotients of countably based spaces* (qcb spaces) and *topological predomains* as introduced by M. Schröder and A. Simpson are not closed under sobrification. As a consequence replete topological predomains need not be sober, i.e. in general repletion is not given by sobrification. Our counterexample also shows that a certain tentative "equalizer construction" of repletion fails for qcb spaces.
Our results extend also to the more general class of core compactly generated spaces.

1. Background

In (Schröder 2003; Simpson 2003) M. Schröder and A. Simpson introduced the categories **QCB** (quotients of countably based spaces) and **PreDom** (topological predomains) as frameworks for denotational semantics containing also most classical spaces such as e.g. separable Banach spaces. One easily shows that if a T_0 space is a quotient of a countably based space then it can also be obtained as quotient of a countably based T_0 space, i.e. as a quotient of a subspace of Scott's $\mathcal{P}\omega$. As from both the topological and the semantical point of view it is reasonable to restrict attention to T_0 spaces we accordingly do so in the rest of the paper. Thus **QCB** is defined as the category whose objects are T_0 quotients of countably based T_0 spaces and whose morphisms are the continuous maps. Subsequently we refer to the objects of **QCB** as *qcb spaces*.

In (Schröder 2003) qcb spaces have been characterised as those sequential T_0 spaces X for which there exists a *countable pseudobase*, i.e. a countable subset $\mathcal{B}$ of $\mathcal{P}(X)$ such

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that for every sequence (x_n) converging to x and open neighbourhood U of x there exists a $B \in \mathcal{B}$ with $x \in B \subseteq U$ and x_n eventually in B.

In (Menni and Simpson 2002; Schröder 2003) it has been shown that **QCB** is cartesian closed. As **QCB** contains the Sierpiński space Σ (with underlying set $\{\bot, \top\}$ and $\{\bot\}$ as its only non-open subset) the open subsets of X may be identified with **QCB**-morphisms from X to Σ. It has been shown by J. Lawson (see Theorem 4.7 of (Escardó et.al. 2004)) that the exponential Σ^X in **QCB** is homeomorphic to $(\mathcal{O}(X), \subseteq)$ endowed with its Scott topology. Accordingly,we will henceforth denote the covariant functor $\Sigma^{(-)}$ by $\mathcal{O}$. Notice that $\mathcal{O}(f) = f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ for continuous maps $f : X \to Y$.

In (Menni and Simpson 2002; Simpson 2003) it has been observed that **QCB** is equivalent (see (Battenfeld 2004) for a proof) to the category $\mathbf{ExPer}(\mathcal{P}\omega)$ of *extensional pers* over Scott's graph model $\mathcal{P}\omega$ (see (Phoa 1992) for definition and discussion of $\mathbf{ExPer}(\mathcal{P}\omega)$). This equivalence provides further evidence for the naturalness of the notion of qcb space and has the consequence that qcb spaces form a model of polymorphic λ-calculus.

Since qcb spaces in general do not enjoy the completeness properties required for denotational semantics A. Simpson in (Simpson 2003) introduced the notion of a *topological predomain*, i.e. a qcb space X which has suprema of ω-chains w.r.t. its information ordering $\sqsubseteq$ and where all open sets are also Scott open (w.r.t. $\sqsubseteq$). In (Simpson 2003) it has been stated that topological predomains have also arbitrary directed suprema w.r.t. $\sqsubseteq$ (see (Battenfeld et.al. 2006) for a proof). In a sense, however, this form of completeness is somewhat *ad hoc*. But already much earlier M. Hyland and P. Taylor introduced (independently in (Hyland 1991) and (Taylor 1991) respectively) the notion of *repleteness* which is formally quite similar to the notion of sobriety (see (Johnstone 1982)). The setting of (Hyland 1991) and (Taylor 1991) is much more general than qcb spaces. This generality, however, is not needed for our purposes and thus we recall the notion of repleteness for the particular case of qcb spaces.

As already mentioned the notion of repleteness looks very similar to the notion of sobriety. Recall that a space X is sober iff for every T_0 space Y a continuous map $e : X \to Y$ is a homeomorphism whenever $\mathcal{O}(e) = e^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an order isomorphism. Sober spaces form a full reflective subcategory of (T_0) spaces. We write $\eta_X : X \to \mathsf{Sob}(X)$ for the reflection map and notice that $\mathcal{O}(\eta_X) : \mathcal{O}(\mathsf{Sob}(X) \to \mathcal{O}(X)$ is an order isomorphism and that η_X is one-to-one if X is a T_0 space. Analogously, a qcb space X is called *replete* iff a map $e : X \to Y$ in **QCB** is a homeomorphism whenever $\mathcal{O}(e) = e^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an order isomorphism, i.e. an isomorphism in **QCB**. Replete qcb spaces form a full reflective subcategory of **QCB** and for every $X \in$ **QCB** the reflection map $r_X : X \to R(X)$ is one-to-one, $\mathcal{O}(r_X)$ an order isomorphism and (thus) $\mathsf{Sob}(r_X)$ is a homeomorphism. Despite the analogy the construction of repletion is much more complicated than that of sobrification (see (Hyland 1991; Taylor 1991; Streicher 1999)).

We now discuss the relation between sobriety and repleteness. From the above definitions it is clear that a sober qcb space is also replete. It can be shown that every replete qcb space is a topological predomain (see (Hyland 1991; Taylor 1991)). An example of a non-replete topological predomain is the non-sober dcpo (with its Scott topology) introduced in (Johnstone 1981) whose sobrification coincides with its repletion (since sobrification adds a single new point which can be obtained as limit of a sequence of point filters). Motivated by these observations one might hope that for qcb spaces repletion is given by sobrification. One easily sees that this is equivalent to qcb spaces being closed under sobrification which was raised as Question 6.1 in (Simpson 2003). In the next section we construct a qcb space whose sobrification is not qcb anymore and thus give a negative answer to Simpson's question.

2. A qcb space whose sobrification is not qcb

We will construct a relatively simple replete qcb space X whose sobrificaton $\mathsf{Sob}(X)$ is not sequential and thus *a fortiori* not qcb.

The underlying set of X is $\mathbb{N} \times \mathbb{N}$. We write π_0 and π_1 for first and second projection, respectively. For $p = (n, m) \in X$ and $f : \{i \in \mathbb{N} \mid i > n\} \to \mathbb{N}$ let $U(p, f) = \{p\} \cup \{(i, j) \in \mathbb{N}^2 \mid i > n \text{ and } j \geq f(i)\}$. Notice that $p \in U(p, f)$. A subset U of X is called **open** iff for every $p \in U$ there is an f with $U(p, f) \subseteq U$. Obviously, we have $U(p, \max(f, g)) = U(p, f) \cap U(p, g)$. Moreover, if $q \in U(p, f)$ then $U(q, g) \subseteq U(p, f)$ for some g. Thus, sets of the form $U(p, f)$ are open and for every $q \in U(p_1, f_1) \cap U(p_2, f_2)$ there is a g with $U(q, g) \subseteq U(p_1, f_1) \cap U(p_2, f_2)$. Thus, open sets are closed under finite intersections. Since open sets are closed also under arbitrary unions they form a topology on X. Moreover, sets of the form $U(p, f)$ provide a basis for this topology.

It is easy to see that X is a T_1 space. Thus, the specialization order on X is discrete. Accordingly, the space X is a topological predomain provided it is sequential and has a countable pseudobase and hence is a qcb space.

Lemma 2.1. For every $A \subseteq X$ and $p \in X \setminus A$ we have $p \in \bar{A}$ iff $A \cap (\{i\} \times \mathbb{N})$ is infinite for some $i > \pi_0(p)$.

Proof. Let $A \subseteq X$ and $p \in X \setminus A$.

For the forward direction suppose that $p \in \bar{A}$ and $A \cap (\{i\} \cap \mathbb{N})$ is finite for all $i > \pi_0(p)$. Then there exists an f with $A \cap U(p, f) = \emptyset$ contradicting $p \in \bar{A}$ since $U(p, f)$ is an open neighbourhood of p.

For the reverse direction suppose $A \cap (\{i\} \times \mathbb{N})$ is infinite for some $i > \pi_0(p)$. In order to show that $p \in \bar{A}$ suppose U is an open neighbourhood of p. Then there exists f with $U(p, f) \subseteq U$. Then, since $A \cap (\{i\} \times \mathbb{N})$ is infinite, there exists a $j \geq f(i)$ with $(i, j) \in A$. Thus $(i, j) \in A \cap U(p, f) \subseteq A \cap U$ as desired. $\square$

Lemma 2.2. X is a Fréchet space, i.e. for every $p \in \bar{A}$ there is a sequence (a_n) in A converging to p. Thus X is in particular also a sequential space.

Proof. Suppose $p \in \bar{A}$. W.l.o.g. assume that $p \notin A$. Then by Lemma 2.1 there exists an $i > \pi_0(p)$ such that $A \cap (\{i\} \times \mathbb{N})$ is infinite. Let $\phi : \mathbb{N} \to \mathbb{N}$ be strictly increasing with $\{(i, \phi(n)) \mid n \in \mathbb{N}\} = A \cap (\{i\} \times \mathbb{N})$. Let $a_n := (i, \phi(n)) \in A$. We show that $(a_n) \to p$.

Suppose U is an open neighbourhood of p. Then $U(p, f) \subseteq U$ for some f. Let $n_0 \in \mathbb{N}$ with $\phi(n_0) \geq f(i)$. Then for all $n \geq n_0$ we have $\phi(n) \geq f(i)$ and thus $a_n = (i, \phi(n)) \in U(p, f) \subseteq U$. $\square$

Let $\mathcal{B}_0$ be the collection of all $V_{i,n} := \{(i,j) \mid j \geq n\}$ with $i, n \in \mathbb{N}$. We define $\mathcal{B}$ as the set of all finite unions $B = B_1 \cup \ldots \cup B_n$ with $B_i \in \mathcal{B}_0 \cup \big\{\{x\} \mid x \in X\big\}$ and $B \cap \big(\{i_0\} \times \mathbb{N}\big)$ finite if $B \cap \big(\{i\} \times \mathbb{N}\big) = \emptyset$ for all $i < i_0$.

Lemma 2.3. $\mathcal{B}$ is a countable pseudobase for X.

Proof. Obviously $\mathcal{B}$ is countable since $\mathcal{B}_0$ and X are both countable.

For showing that $\mathcal{B}$ is a pseudobase for X suppose that (p_n) converges to p and U is an open neighbourhood of p. Then $U(p, f) \subseteq U$ for some f. For $i > \pi_0(p)$ let $I_i = \{n \in \mathbb{N} \mid p_n \in V_{i,f(i)}\}$.

Next we show that almost all I_i are empty. For sake of contradiction suppose this were not the case. Then there exists a subsequence (q_n) of (p_n) with $\pi_0(q_n) > \pi_0(p)$ and $\pi_0(q_n) < \pi_0(q_{n+1})$ for all $n \in \mathbb{N}$. Let $g : \{i \in \mathbb{N} \mid i > \pi_0(p)\} \to \mathbb{N}$ such that $q_n \notin U(p, g)$ for all $n \in \mathbb{N}$. Then $U(p, g)$ is an open neighbourhood of p containing no q_n which is impossible since (q_n) converges to p.

Let $i_0 \in \mathbb{N}$ with $I_i = \emptyset$ for $i \geq i_0$. Then $B = \{p\} \cup \bigcup_{\pi_0(p)<j<i_0} V_{j,f(j)} \in \mathcal{B}$ and it holds that $B \subseteq U(p, f) \subseteq U$ and (p_n) is eventually in B as desired. □

Thus, since X is a sequential space with a countable pseudobase the space X is qcb.

Lemma 2.4. The irreducible closed subsets of X are the singleton subsets and X itself.

Proof. As non-empty open subsets of X are closed under finite intersections they form a complete prime filter in $\mathcal{O}(X)$ and thus X is an irreducible closed subset of X. As X is a T_1 space the singleton sets are all closed and thus also irreducible closed.

Suppose C is an irreducible closed subset of X different from X. By Lemma 2.1 if $C \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite then for all $j < i$ and $n \in \mathbb{N}$ we have $(j, n) \in \overline{C \cap \big(\{i\} \times \mathbb{N}\big)} \subseteq \bar{C} = C$. Thus $C \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite for only finitely many i since otherwise $X = C$.

Thus, precisely one of the following two conditions holds

(1) $C \cap \big(\{i\} \times \mathbb{N}\big)$ is finite for all $i \in \mathbb{N}$

(2) there is a greatest $i \in \mathbb{N}$ with $C \cap \big(\{i\} \times \mathbb{N}\big)$ infinite.

In case (1) every point of C is isolated in the subspace C. Thus C cannot be irreducible closed unless C is a singleton.

In case (2) every point of the infinite set $C \cap \big(\{i\} \times \mathbb{N}\big)$ is isolated in the subspace C. But as irreducible closed sets contain at most one isolated point this is impossible. □

Thus $\mathsf{Sob}(X) = X \cup \{\infty\}$ where ∞ stands for the irreducible closed set X. The nonempty open sets of $\mathsf{Sob}(X)$ are those of the form $U \cup \{\infty\}$ where $U \in \mathcal{O}(X) \setminus \{\emptyset\}$.

Lemma 2.5. As a subset of $\mathsf{Sob}(X)$ the set X is sequentially closed but not closed w.r.t. the sober topology. Thus, the space $\mathsf{Sob}(X)$ is not sequential and hence not qcb.

Proof. Obviously X is not closed in $\mathsf{Sob}(X)$ since $\infty \in \bar{X} \setminus X$. Nevertheless X is a sequentially closed subset of $\mathsf{Sob}(X)$ which can be seen as follows.

For the sake of contradiction suppose (x_n) is a sequence in X converging to ∞ in $\mathsf{Sob}(X)$. As ∞ is in the closure of $S = \{x_n \mid n \in \mathbb{N}\}$ it is impossible that $S \cap \big(\{i\} \times \mathbb{N}\big)$ is finite for all $i \in \mathbb{N}$. Thus, there exists an $i \in \mathbb{N}$ with $S \cap \big(\{i\} \times \mathbb{N}\big)$ infinite. But then

$U = \{(j,k) \in \mathbb{N}^2 \mid i < j\} \cup \{\infty\}$ is an open neighbourhood of ∞ in $\mathsf{Sob}(X)$ such that infinitely many elements of S, namely those of $S \cap (\{i\} \times \mathbb{N})$, are not in U contradicting our assumption that (x_n) converges to ∞. □

Thus, we have verified that X is qcb but its sobrification $\mathsf{Sob}(X)$ is not.

Theorem 2.6. The space X is a replete qcb space but not sober.

Proof. Let $r_X : X \to R(X)$ be the reflection map from X to its repletion $R(X)$. Since $\mathsf{Sob}(r_X)$ is an isomorphism and $\eta_{R(X)} \circ r_X = \mathsf{Sob}(r_X) \circ \eta_X$ we have $\eta_X = i \circ r_X$ for $i = \mathsf{Sob}(r_X)^{-1} \circ \eta_{R(X)}$. Since $R(X)$ is a T_0 space the map $\eta_{R(X)}$ is one-to-one and thus i is one-to-one as well. Since $\mathcal{O}(r_X)$ and $\mathcal{O}(\eta_X)$ are both isomorphisms it follows that $\mathcal{O}(i)$ is an isomorphism, too. Thus, since i is one-to-one it follows that $i : R(X) \to \mathsf{Sob}(X)$ is a subspace embedding. As η_X factors through i we have either $R(X) = X$ or $R(X) = \mathsf{Sob}(X)$. Since $R(X)$ is sequential but $\mathsf{Sob}(X)$ is not it follows that $R(X) = X$, i.e. that X is replete.

Thus X is a replete qcb space which is not sober by Lemma 2.5. □

A. Simpson has pointed out to us that our counterexample can be used also for answering some open questions about the category **CCG** of *core compactly generated spaces* introduced in (Day 1972) and further investigated in (Escardó et.al. 2004). In Problem 9-5 of (Escardó et.al. 2004) it is asked whether

(a) core compactly generated spaces are closed under sobrification and
(b) the core compactly generated topology of a sober space is again sober.

For both questions the answer is negative.
ad (a) : In Theorem 6.10 of (Escardó et.al. 2004) it is shown that **CCG** contains **QCB** as the full subcategory of those core compactly generated spaces having a countable $\ll$-pseudobase (see (Escardó et.al. 2004) for definition). Obviously, the space X is in **CCG** since it is in **QCB**. If $\mathsf{Sob}(X)$ were in **CCG** as well then $\mathsf{Sob}(X)$ would be a subspace of $\mathcal{O}^2(\mathsf{Sob}(X)) \cong \mathcal{O}^2(X)$. From (Escardó et.al. 2004) we know that **QCB** is closed under subspaces in **CCG** and thus $\mathsf{Sob}(X)$ would be a qcb space in contradiction with Lemma 2.5.
ad (b) : One easily checks that $\{B \cup \{\infty\} \mid B \in \mathcal{B}\}$ is a countable $\ll$-pseudobase for $\mathsf{Sob}(X)$. By Corollary 6.6 of (Escardó et.al. 2004) the core compactly generated topology on $\mathsf{Sob}(X)$ coincides with its sequentialisation $\mathsf{Seq}(\mathsf{Sob}(X))$. Since by (the proof of) Lemma 2.5 the point ∞ is isolated in $\mathsf{Seq}(\mathsf{Sob}(X))$ the subset $X \subseteq \mathsf{Seq}(\mathsf{Sob}(X))$ is irreducible closed but, obviously, not the closure of a singleton set. Thus, the space $\mathsf{Seq}(\mathsf{Sob}(X))$ is not sober.

Obviously, these arguments apply to all cartesian closed categories of spaces considered in (Escardó et.al. 2004) as long as they contain all T_1 qcb spaces. Thus sobriety appears as fundamentally incompatible with cartesian closedness.

We conclude this section by showing that $\mathcal{O}(X)$ is sober although X is not. For this purpose we need the following lemma.

Lemma 2.7. Let $\mathcal{B}$ be the countable pseudobase for X as introduced before Lemma 2.3.

For every $B \in \mathcal{B}$ the set $[B] = \{U \in \mathcal{O}(X) \mid B \subseteq U\}$ is a Scott open filter in $\mathcal{O}(X)$. Moreover, every $\mathcal{U} \in \mathcal{O}^2(X)$ is the union of all $[B]$ with $B \in \mathcal{B}$ and $[B] \subseteq \mathcal{U}$ from which it follows that $\mathcal{O}(X)$ is countably based and thus qcb.

Proof. One easily sees that the elements of $\mathcal{B}$ are compact subsets of X and thus $[B] = \{U \in \mathcal{O}(X) \mid B \subseteq U\}$ is a Scott continuous filter in $\mathcal{O}(X)$. As $\mathcal{B}$ is closed under finite unions the set $\mathcal{B}_1 = \{[B] \mid B \in \mathcal{B}\}$ is closed under finite intersections and thus provides a countable basis for $\mathcal{O}(X)$ (since as shown in (Schröder 2003) if $\mathcal{B}$ is countable pseudobase for X then $\big\{[B_1] \cup \ldots \cup [B_n] \mid B_1, \ldots, B_n \in \mathcal{B}\big\}$ is a countable pseudobase for $\Sigma^X = \mathcal{O}(X)$). Thus, for every $\mathcal{U} \in \mathcal{O}^2(X)$ we have $\mathcal{U} = \bigcup\big\{[B] \mid B \in \mathcal{B}, [B] \subseteq \mathcal{U}\big\}$. □

3. Failure of the "equalizer construction" of repletion

In the first half of the 1990's several people suggested that for arbitrary X its repletion $R(X)$ might be given by the equalizer $E(X)$ of the maps $\eta_{\mathcal{O}^2(X)}, \mathcal{O}^2(\eta_X) : \mathcal{O}^2(X) \to \mathcal{O}^4(X)$ where for arbitrary Y the map $\eta_Y : Y \to \mathcal{O}^2(Y)$ sends $y \in Y$ to its neighbourhood filter $\eta_Y(y) = \{U \in \mathcal{O}(Y) \mid y \in U\}$. Since for $\mathcal{U} \in \mathcal{O}^2(X)$ and $\Phi \in \mathcal{O}^3(X)$ we have $\eta_{\mathcal{O}^2(X)}(\mathcal{U})(\Phi) = \Phi(\mathcal{U})$ and $\mathcal{O}^2(\eta_X)(\mathcal{U})(\Phi) = \mathcal{U}(\mathcal{O}(\eta_X)(\Phi)) = \mathcal{U}(\Phi \circ \eta_X)$ the equalizer $E(X)$ is the regular subject of $\mathcal{O}^2(X)$ consisting of those $\mathcal{U} \in \mathcal{O}^2(X)$ such that $\Phi(\mathcal{U}) = \mathcal{U}(\Phi \circ \eta_X)$ for all $\Phi \in \mathcal{O}^3(X)$.

We will show that for our space X from section 2 the equalizer $E(X)$ contains the element $\exists = \{U \in \mathcal{O}(X) \mid U \neq \emptyset\} \in \mathcal{O}^2(X)$ from which it will follow that $E(X)$ is different from $R(X) = X$. For this purpose we need the following lemma.

Lemma 3.1. The closure of $\{\eta_X(x) \mid x \in X\}$ in $\mathcal{O}^2(X)$ contains $\exists$ as an element.

Proof. For $n, i \in \mathbb{N}$ consider the sets

$$F_{i,n} = \{U \in \mathcal{O}(X) \mid (\exists p \in U.\, \pi_0(p) < i) \wedge (\forall j \geq n.\, (i,j) \in U)\}$$

which are easily shown to be elements of $\mathcal{O}^2(X)$ and satisfy $F_{i,n} \subseteq \eta_X(i,n)$. Thus, all $F_{i,n}$ lie in $\overline{\{\eta_X(x) \mid x \in X\}}$. Since for all $i \in \mathbb{N}$ the sequence $(F_{i,n})_{n\in\mathbb{N}}$ is increasing its union $G_i = \bigcup_{n\in\mathbb{N}} F_{i,n}$ is in $\overline{\{\eta_X(x) \mid x \in X\}}$ as well. One easily shows that we have $G_i = \{U \in \mathcal{O}(X) \mid \exists p \in U.\, \pi_0(p) < i\}$ and thus $\exists = \bigcup_{i\in\mathbb{N}} G_i$. Thus, as the sequence (G_i) is increasing it follows that $\exists \in \overline{\{\eta_X(x) \mid x \in X\}}$. □

Theorem 3.2. The set $E(X) = \{\mathcal{U} \in \mathcal{O}^2(X) \mid \forall \Phi \in \mathcal{O}^3(X).\ \Phi(\mathcal{U}) = \mathcal{U}(\Phi \circ \eta_X)\}$ contains $\exists$ as an element. Thus η_X is not the equalizer of $\eta_{\mathcal{O}^2(X)}$ and $\mathcal{O}^2(\eta_X)$, i.e. the equalizer construction of repletion fails for X.

Proof. For showing that $\exists \in E(X)$ we have to show that $\Phi(\exists) = \exists(\Phi \circ \eta_X)$ for all $\Phi \in \mathcal{O}^3(X)$. Let $\Phi \in \mathcal{O}^3(X)$. Suppose $\exists(\Phi \circ \eta_X) = \top$. Then $\eta_X(x) \in \Phi$ for some $x \in X$. Thus $\Phi(\exists) = \top$ since $\eta_X(x) \subseteq \exists$ and Φ preserves the specialization order. For the reverse direction suppose $\Phi(\exists) = \top$, i.e. $\exists \in \Phi$. Then by Lemma 3.1 we have $\eta_X(x) \in \Phi$ for some $x \in X$. Thus $\Phi(\eta_X(x)) = \top$ from which it follows that $\exists(\Phi \circ \eta_X) = \top$ as desired.

One easily checks that η_X equalizes $\eta_{\mathcal{O}^2(X)}$ and $\mathcal{O}^2(\eta_X)$. But since $\exists \in E(X)$ and $\exists \notin \{\eta_X(x) \mid x \in X\}$ the map η_X is not the equalizer of $\eta_{\mathcal{O}^2(X)}$ and $\mathcal{O}^2(\eta_X)$. □

Thus we have shown that all known attempts to simplify the construction of repletion do not work for **QCB** and topological predomains, i.e. for realizability models over $\mathcal{P}\omega$. The same holds for function realizability since $\mathbf{ExPer}(\mathbb{N}^{\mathbb{N}}) \simeq \mathbf{ExPer}(\mathcal{P}\omega)$ as shown in (Bauer 2002). We leave it as a task for future investigations whether our counterexample can be adapted to number realizability.

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