Choice Sequences vs. Formal Topology

T. Streicher

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It is a fact of classical mathematics that the adjunction $\mathcal{O} \dashv \mathsf{Pt} : \mathbf{Loc} \to \mathbf{Sp}$ is not an equivalence which, however, boils down to an equivalence between sober spaces and spatial locales.[1] However, classically, most of the usual spaces are sober as e.g. all Hausdorff spaces and all algebraic domains. But there are notable examples of non-spatial locales which do not have any points as e.g. the measure algebra (Borel subsets of $\mathbb{R}$ identified when their symmetric difference has measure 0) or the regular open subsets of $\mathbb{R}$. Locales, i.e. point free topology, have their origin in Grothendieck's topos theory. The (2-)category $\mathfrak{Top}$ of Grothendieck toposes and geometric morphisms hosts the full reflective sub-(2-)category of *localic* Grothendieck toposes equivalent to **Loc**.

In a classical framework ordinary mathematics usually does not require the consideration of pointless spaces, i.e. locales which haven't got enough points[2]. But, of course, they are important for constructing Boolean (or Heyting) valued models of, say, set theory and related systems. E.g. for refuting the Axiom of Choice one may consider the boolean valued model $V^{(B)}$ where B is the measure algebra, i.e. Borel subsets of $\mathbb{R}$ modulo the equivalence relation identifying Borel sets A and B iff their symmetric difference $A \nabla B$ has Lebesgue measure 0. In any case many non-atomic complete boolean algebras haven't got enough points (e.g. regular open subsets of $\mathbb{R}$ give rise to a complete Boolean algebra without any points).

Constructively, however, many locales haven't go enough points because their construction requires non-constructive means. For example in the effective topos $\mathcal{E}\!f\!f$ formal Cantor space hasn't got enough points because the Kleene tree provides a family of basic open sets which do not cover formally but cover all recursive binary sequences.
This applies even more so to formal Baire space which observation by Kleene was his motivation to consider *function realizability* which provides a model for Brouwerian intuitionism.

Function realizability is based on the partial combinatory algebra $\mathcal{K}_2$ (2nd Kleene algebra) whose underlying set is the Baire space $\mathbb{N}^\mathbb{N}$, i.e. arbitrary sequences of natural numbers including the non-effective ones). The applica-

[1] A space X is sober iff $\eta_X : X \to \mathsf{Pt}(\mathcal{O}(X))$ is an isomorphism and locale A is spatial iff $\varepsilon_A : \mathcal{O}(\mathsf{Pt}(A)) \to A$ is an isomorphism.

[2] A locale A has enough points iff $a, b \in A$ are equal whenever $p(a) = p(b)$ for all points $p : A \to \Sigma$.

tion operation of $\mathcal{K}_2$ is based on Brouwer's insight that one can use elements of $\mathbb{N}^\mathbb{N}$ for representing continuous functionals $\mathbb{N}^\mathbb{N} \to \mathbb{N}$. Formally, we have $\alpha\beta = \gamma$ iff $\gamma(n) + 1 = \alpha(\langle n \rangle * \bar{\beta}(k))$ where k is the least natural number with $\alpha(\langle n \rangle * \bar{\beta}(k)) > 0$.

In their 1965 book *Foundations of Intuitionistic Mathematics* Kleene and Vesley showed that function realizability validates the axioms of their system **FIM** which is based on **EL** (basic constructive theory of numbers and sequences of numbers as used in Troelstra's writings) to which one adds

(1) induction over natural numbers for arbitrary formulas

(2) countable choice

(3) decidable or monotone **bar induction** stating the principle of transfinite recursion for well-founded trees (given as subsets of $\mathbb{N}^*$)[3]

(4) the **continuity principle** $\forall\alpha\exists\beta\, A(\alpha,\beta) \to \exists\gamma\forall\alpha\, A(\alpha,\gamma\alpha)$.

Principles (1–3) are compatible with classical logic as opposed to (4) which is genuinely non-classical but is function realized by (essentially) the identity on Baire space.

Notice that classically bar induction is equivalent (by contraposition) to **dependent choice** formulated as follows: every tree $T \subseteq \mathbb{N}^*$ without leaves contains an infinite path α, i.e. $\forall n\, \bar{\alpha}(n) \in T$. Instantiating T by the Kleene tree this guarantees the existence of a non-effective object, namely an infinite path in the Kleene tree. But bar induction is weaker than this since as shown by J. R. Moschovakis (1971) **FIM** is consistent with the axiom that there are no non-recursive functions.[4]

Nevertheless, as shown in Fourman & Grayson's seminal paper *Formal Spaces* (1982) the principle of **monotone bar induction** suffices for showing that formal Baire has enough points, i.e. any cover covering all points already contains $\langle\rangle$. From this it follows that all formal spaces covered by formal Baire space have enough points.

A different model for **FIM** goes back to Fourman, Grayson, van der Hoeven and Moerdijk from the early 1980s. Let $\mathcal{M}$ be the monoid of continuous endomaps of Baire space $\mathcal{B} = \mathbb{N}^\mathbb{N}$. A sieve S is a cover iff there is a family of homeomorphisms $h_i : \mathcal{B} \xrightarrow{\sim} U_i$ such that U_i is an open subset of $\mathcal{B}$, $\bigcup U_i = \mathcal{B}$ and all $h_i \in S$. Let $\mathcal{J}$ be the Grothendieck topology on $\mathcal{M}$ generated by all covering sieves. Then the sheaf topos $\mathsf{Sh}(\mathcal{M},\mathcal{J})$ is a model of **FIM**. The nno N of $\mathsf{Sh}(\mathcal{M},\mathcal{J})$ is given by the set of continuous maps from $\mathcal{B}$ to the discrete space $\mathbb{N}$ acted on by $\mathcal{M}$ by composition from the right. One can show that N^N is isomorphic to $\mathcal{M}$ acted on by $\mathcal{M}$ via composition from the right. The **lawlike** objects of $\mathsf{Sh}(\mathcal{M},\mathcal{J})$ is the subsheaf of N^N generated by the **locally constant**

[3]For staying constructive it is essential that the bar is decidable or at least monotone since as shown in [KV65] un restricted bar induction entails PEM for Π^0_1-sentences.

[4]Her paper is not so easy to read but in modern terminology she considers modified realizability over $\mathsf{ICF}_{\mathsf{eff}}$, i.e. intensional continuous functionals restricted to their effective part.

endomaps of $\mathcal{B}$. In $\mathsf{RT}(\mathcal{K}_2)$ the lawlike objects are given by the subobject of N^N consisting of recursive functions.[5]

The advantage of the functional realizability model is that it provides a natural interpretation of "lawlike" as "computable". There is also the full-on-objects subcategory $\mathcal{KV}$ of $\mathsf{RT}(\mathcal{K}_2)$ consisting of those morphisms which have got an effective realizer. Though $\mathcal{KV}$ is not a subtopos of $\mathsf{RT}(\mathcal{K}_2)$ (in the technical sense) it is a topos, called *Kleene–Vesley topos*. It is the *relative realizability* topos arising from the pca $\mathcal{K}_2$ and its subpca $\mathcal{K}_2^{\mathsf{eff}}$.

The topos $\mathsf{RT}(\mathcal{K}_2)$ is particularly useful for computable analysis since it hosts a vast generalization of the *Kleene-Kreisel continuous functionals*. The category $\mathbf{Mod}(\mathcal{K}_2)$ may be thought of as the category of represented sets in the sense of Weihrauch's TTE (Type Two Effectivity). Sierpiński space Σ can be represented by the map $\rho_\Sigma : \mathcal{B} \rightharpoonup \Sigma$ with $\rho_\Sigma(\alpha) = \bot$ iff $\alpha = \lambda n.0$. An object A of $\mathbf{Mod}(\mathcal{K}_2)$ is called Σ-extensional iff $\eta_A : A \to \Sigma^{\Sigma^A}$ is a regular, i.e. $\neg\neg$-closed, mono. By a theorem of M. Schröder the Σ-extensional objects correspond to so-called **admissible representations**. This full subcategory of $\mathbf{Mod}(\mathcal{K}_2)$ has an obvious functor to the category $\mathbf{Sp}$ of topological spaces which is full and faithful. Up to equivalence these are those T_0 spaces which arise as quotients of subspaces of Baire space. The corresponding category is referred to as $\mathbf{QCB}_0$ and its rich theory has been developed by A. Simpson and M. Schröder. It hosts ω-continuous domains and complete separable metric spaces as full subcategories. The full on objects subcategory $\mathbf{QCB}_0^{\mathsf{eff}}$ of $\mathbf{QCB}_0$ is obtained as the pullback

$$\begin{array}{ccc} \mathbf{QCB}_0^{\mathsf{eff}} & \longrightarrow & \mathcal{KV} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{QCB}_0 & \longrightarrow & \mathsf{RT}(\mathcal{K}_2) \end{array}$$

in **Cat** and it povides a notion of computability for maps between spaces with admissible representations.

Which of the alternatives is better suited for constructive mathematics based on type theory?

Escardó and Xu (2015) have shown that intensional type theory is inconsistent with Brouwer's continuity principle. But bar induction is consistent with type theory (see work of Coquand and Spiwack). In this sense constructive mathematics based on "choice sequences" is compatible with a development in type theory. Thus, one is not forced to use formal topology methods when working in type theory. But one has to use bar induction whereas when using formal topology one can perform induction over elements of a basis which is presented as an inductive type which is more in the spirit of type theory than bar induction.

[5] There is a variation of this sheaf model giving a nice model for the theory of lawlike sequences.

But even if one does not postulate bar induction one may keep the illusion of working with elements. This can be achieved (following M. Fourman's *Continuous Truth* (1984)) by working in a gros topos over a small category of "separable[6] locales" with the open cover topology which can be shown to validate **FIM**. Thus one may work in higher order **FIM** and translate all results obtained in there via Kripke-Joyal semantics back into a statement of formal topology holding in the base topos. But what one obtains this way may not be too telling!

Final questions and remarks

To which extent does Fan Theorem replace arguments using formal topology? It suffices to show that formal Cantor space and thus all its subquotients have enough points.

If one thinks that **Loc** or $\mathfrak{Top}$ provide suitable generalizations of spaces there arises the question where one want's to reason about them. Their internal logic is not even regular it's just cartesian (if in case of $\mathfrak{Top}$ one restricts to bounded Grothendieck toposes).

[6]This means that it is given by a countable base for which $\leq$ is decidable and the covering relation is generated by countably many inhabited sieves.