Universes in Toposes

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Abstract. We discuss a notion of *universe* in toposes which from a logical point of view gives rise to an extension of Higher Order Intuitionistic Arithmetic (HAH) that allows one to construct families of types in such a universe by structural recursion and to quantify over such families. Further, we show that (hierarchies of) such universes do exist in all sheaf and realizability toposes but neither in the free topos nor in the $V_{\omega+\omega}$ model of Zermelo set theory.

Though universes in **Set** are necessarily of strongly inaccessible cardinality it remains an open question whether toposes with a universe allow one to construct internal models of Intuitionistic Zermelo Fraenkel set theory (IZF).

The background information about toposes and fibred categories as needed for our discussion in this paper can be found e.g. in the fairly accessible sources [MM, Jac, Str2].

1 Background and Motivation

It is commonly agreed on that elementary toposes with a natural numbers object (NNO) provide a concise and flexible notion of model for constructive Higher Order Arithmetic (HAH). Certainly, a lot of mathematics can be expressed within HAH. So what is the need then for set theory (ZFC) which is generally accepted as *the* foundation for mainstream mathematics? Well, ZFC is much stronger than HAH in the following respects:

(1) ZFC is based on classical logic whereas HAH is based on the weaker intuitionistic logic.

(2) ZFC postulates the axiom of choice whereas HAH does not.

(3) ZFC postulates the axiom of replacement which cannot even be formulated in HAH.[1]

[1] Notice, however, that the axiom of replacement obtains its full power only in presence of

The logic of toposes (with NNO) is inherently intuitionistic and in HAH the axiom of choice implies classical logic. Therefore, we have to give up (1) and (2) above when considering Grothendieck and realizability toposes as models of some kind of set theory. But what about (3), the axiom of replacement?

First of all notice that there are models of set theory without replacement but satisfying classical logic and choice, namely $V_{\omega+\omega}$.[2] On the other hand a lot of toposes, in particular Grothendieck toposes and realizability toposes, do model the axiom of replacement whereas in most cases they refute classical logic and the axiom of choice. More precisely, the above mentioned toposes model IZF, i.e. ZF with intuitionistic logic and axiom of regularity reformulated as $\in$-induction.[3]

Though a large class of toposes validates IZF one still may complain that the formulation of IZF suffers from "epsilonitis", i.e. that it "implements" informal mathematics via the $\in$-relation rather than axiomatizing mathematical practice in terms of its basic notions. So one may ask *what is the mathematical relevance of the set-theoretic replacement axiom?* Maybe a set-theorist would answer "for constructing ordinals greater than $\omega+\omega$" which, however, may seem a bit disappointing because most mathematics can be formulated without reference to transfinite ordinals.[4] Actually, what axiom of replacement is mainly needed for in mathematical practice is to define families of sets indexed by some set I carrying some inductive structure as, typically, the set $\mathbb{N}$ of natural numbers. For example, most mathematicians would not hesitate to construct the sequence $(\mathcal{P}^n(\mathbb{N}))_{n\in\mathbb{N}}$ by (primitive) recursion over $\mathbb{N}$. Already in ZC, however, this is impossible because $\{\langle n, \mathcal{P}^n(\mathbb{N})\rangle \mid n\in\mathbb{N}\} \notin V_{\omega+\omega}$. Usually, in ZFC the sequence $(\mathcal{P}^n(\mathbb{N}))_{n\in\mathbb{N}}$ is constructed by applying the axiom of replacement to an appropriately defined class function from the set of natural numbers to the class of all sets. However, in a sense that does not properly reflect the mathematician's intuition who thinks of $(\mathcal{P}^n(\mathbb{N}))_{n\in\mathbb{N}}$ as a function f from $\mathbb{N}$ to sets defined recursively as $f(0) = \mathbb{N}$ and $f(n{+}1) = \mathcal{P}(f(n))$. There is, however, a "little" problem, namely that the collection of all sets does not form a set but a proper class. Notice, however, that *a posteriori* the image of f does form a set

the full separation scheme. In recent, yet unpublished work by S. Awodey, C. Butz, A. Simpson and T. Streicher [ABSS] it has been shown that set theory with *bounded separation*, i.e. separation restricted to bounded formulas, but with replacement (and even strong collection) is equiconsistent to HAH as long as the underlying logic is intuitionistic. Otherwise the classical Principle of Excluded Middle allows one to derive full separation from replacement.

In the context of this paper when we say replacement we mean the power of replacement together with full separation (although the latter does not make sense from a type-theoretic point of view!).

[2]But notice that $V_{\omega+\omega}$ validates ZC, i.e. ZFC *without* replacement, which, however, is still stronger than HAH as already Z proves the consistency of HAH.

[3]Axiom of regularity and $\in$-induction are equivalent only classically as in IZF the principle of excluded middle follows from the axiom of regularity just as in HAH the principle of excluded middle follows from the least number principle.

[4]There are notable exceptions typically in the area of descriptive set theory as e.g. *Borel determinacy* which is provable in ZF (as shown by D. A. Martin) but not in Z (as shown by H. Friedman). Even IZF does not decide Borel determinacy as it holds in **Set** but not in Hyland's effective topos **Eff**.

as ensured by the replacement axiom. Notice, moreover, that the image of f is contained as a subset in the set $V_{\omega+\omega} = \bigcup_{n\in\mathbb{N}} \mathcal{P}^n(V_\omega)$ whose existence can be ensured again by the axiom of replacement in a similar way as above. Thus, if we had $V_{\omega+\omega}$ available as a set beforehand we could define f as a function from the set $\mathbb{N}$ to the set $V_{\omega+\omega}$ simply by primitive recursion. The distinguishing feature of $V_{\omega+\omega}$ is that it is closed under subsets, cartesian products, powersets (and, thus, also under exponentiation) and contains V_ω, the set of hereditarily finite sets, and, therefore, also $\mathbb{N}$ as elements. Actually, the set $V_{\omega+\omega}$ has even stronger closure properties, namely, that

(1) every element of $V_{\omega+\omega}$ is also a subset of $V_{\omega+\omega}$, i.e. $V_{\omega+\omega}$ is a so-called transitive set

(2) $V_{\omega+\omega}$ with $\in$ restricted to it provides a model of Zermelo set theory (with choice), i.e. ZF(C), without the axiom of replacement.

Sets satisfying these two properties are called *Zermelo universes* and they are abundant because V_λ is a Zermelo universe for all limit ordinals $\lambda > \omega$. Thus, there are as many Zermelo universes as there are ordinals.

A Zermelo universe U satisfying the additional requirement that

(3) if f is a function with $\mathsf{dom}(f) \in U$ and $\mathsf{rng}(f) \subseteq U$ then $\mathsf{rng}(f) \in U$

is called a *Grothendieck universe* because it was A. Grothendieck who introduced this notion for the purpose of a convenient and flexible set-theoretic foundation of category theory. More precisely, he suggested to use ZFC together with the requirement that every set A be contained in some Grothendieck universe guaranteeing at least[5] an infinite sequence

$$U_0 \in U_1 \in \ldots U_{n-1} \in U_n \in U_{n+1} \in \ldots$$

of Grothendieck universes. As Grothendieck universes are transitive such a sequence is also *cumulative* in the sense that

$$U_0 \subseteq U_1 \subseteq \ldots U_{n-1} \subseteq U_n \subseteq U_{n+1} \subseteq \ldots$$

holds as well. One can show that V_λ is a Grothendieck universe if and only if λ is a *strongly inaccessible* cardinal, i.e. λ is an infinite regular cardinal with $2^\kappa < \lambda$ for all $\kappa < \lambda$.

Obviously, ZFC does not prove the existence of Grothendieck universes (or of strongly inaccessible cardinals) as otherwise ZFC could prove its own consistency (as a Grothendieck universe provides a small inner model of ZFC) which is impossible by Gödel's 2$^{\text{nd}}$ Incompleteness Theorem.[6]

[5] Actually, postulating choice for classes gives rise to a class function Un that assigns to every set a a Grothendieck universe $Un(a)$ with $a \in Un(a)$. Then by transfinitely iterating the function Un one obtains incredibly big hierarchies of Grothendieck universes.

[6] Notice, however, that the notion of Grothendieck universe is stronger than the notion of *small inner model* which is a Zermelo universe required to satisfy condition (3) above only for those f which are first order definable in the language of set theory!

2 Universes in Toposes

We now define a notion of *universe in an (elementary) topos* that is stronger than the set-theoretic axiom of replacement but adapted to the "spirit" of type theory and thus freed from "epsilonitis".

We do not claim any originality for the subsequent notion as it is inspired by the categorical semantics (see e.g. [Str1]) for an impredicative version of Martin-Löf's universes as can be found in the Extended Calculus of Constructions (see [Luo]). Categorical semantics of universes was anticipated by Jean Bénabou's influential paper [Bén] (from 1971 !) introducing (among other important things) a notion of *topos internal to a topos*.[7]

Definition 2.1 *A* universe *in a topos $\mathcal{E}$ is given by a class $\mathcal{S}$ of morphisms in $\mathcal{E}$ satisfying the following conditions*

(1) *$\mathcal{S}$ is stable under pullbacks along morphisms in $\mathcal{E}$, i.e. for every pullback*

$$\begin{tikzcd} B \arrow[r] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ J \arrow[r, "f"'] & I \end{tikzcd}$$

in $\mathcal{E}$ it holds that $b \in \mathcal{S}$ whenever $a \in \mathcal{S}$.

(2) *$\mathcal{S}$ contains all monos of $\mathcal{E}$.*

(3) *$\mathcal{S}$ is closed under composition, i.e. if $f : A \to I$ and $g : B \to A$ are in $\mathcal{S}$ then $\Sigma_f g = f \circ g \in \mathcal{S}$.*

(4) *$\mathcal{S}$ is closed under dependent products, i.e. if $f : A \to I$ and $g : B \to A$ are in $\mathcal{S}$ then $\Pi_f g \in \mathcal{S}$ (where Π_f is right adjoint to $f^* : \mathbb{E}/I \to \mathbb{E}/A$).*

(5) *In $\mathcal{S}$ there is a* generic[8] *morphism, i.e. a morphism $El : E \to U$ in $\mathcal{S}$ such that for every $a : A \to I$ in $\mathcal{S}$ we have*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "El"] \\ I \arrow[r, "f"'] & U \end{tikzcd}$$

for some morphism $f : I \to U$ in $\mathcal{E}$, i.e. $a \cong f^* El$.

A universe $\mathcal{S}$ is called impredicative *iff $\Omega \to 1$ is in $\mathcal{S}$. If $\mathcal{E}$ has a natural numbers object N then we say that $\mathcal{S}$* contains *N iff $N \to 1$ is in $\mathcal{S}$.* $\Diamond$

[7] Alas, later work on categorical semantics of type theories usually does not refer to [Bén] but rather implicitly to some "folklore" dating back to the early 70ies when [Bén] was written.

[8] Notice that we do not require uniqueness of f, i.e. *El* is not a "classifying" but only a "generic" family for $\mathcal{S}$. Some authors use also the word "weakly classifying" instead of "generic".

As already mentioned this notion of universe is inspired by a similar notion introduced by J. Bénabou in [Bén]. His main motivation was to provide a notion of *internal topos* capturing finite cardinals, i.e. Kuratowski finite sets with decidable equality, inside a topos $\mathcal{E}$ with narural numbers object N for which a generic morphism is provided by

$$k = \pi_2 : K = \{(i,n) \in N^2 \mid i < n\} \longrightarrow N$$

The class $\mathbb{F}$ of all morphisms which can be obtained as pullback of the generic family k coincides with the class of families of finite cardinals. It satisfies conditions (1) and (3)–(5) of Definition 2.1. The monos in $\mathbb{F}$ are classified by $\mathsf{inl} : 1 \to 1{+}1$ and, accordingly, $\mathbb{F}$ contains all monos if and only if $\mathcal{E}$ is boolean. Thus, the logic of the internal topos of finite cardinals as given by $\mathbb{F}$ coincides with the logic of the ambient topos $\mathcal{E}$ if and only if $\mathcal{E}$ is boolean. On the other hand the morphism k is not only generic but also classifying[9] for $\mathbb{F}$ because in the internal logic of $\mathcal{E}$ from $K(n) \cong K(m)$ it follows that $n = m$.

Obviously, there is a lot of possibilities for varying the notion of universe according to one's needs. The minimal notion is given by a class $\mathcal{S}$ of morphism satisfying condition (1) (which already entails that $\mathcal{S}$ is closed under isomorphism). Such classes $\mathcal{S}$ satisfying (1) coincide with full (and replete) subfibrations of the fundamental fibration $P_{\mathcal{E}} = \partial_1 : \mathcal{E}^{\mathbf{2}} \to \mathcal{E}$ of $\mathcal{E}$ and, therefore, are abundant in category theory and categorical logic, in particular, as they provide the correct fibrational generalisation of the notion of full (and replete) subcategory. In semantics of dependent type theories such pullback stable classes $\mathcal{S}$ are called "classes of display maps" for some internal collection of types (see e.g. [Str1, Jac] for a more detailed treatment). In the *Algebraic Set Theory* of Joyal and Moerdijk [JM] they are called classes of "small" maps and are thought of as "families of sets indexed by a class". In a sense the word "small" is somewhat misleading here because for $a : A \to I$ in $\mathbb{F}$ and $m : A' \rightarrowtail A$ the composite $a \circ m$ need not be in $\mathbb{F}$ even for $I = 1$ because, constructively, finite cardinals are not closed under subsets. Already subterminals need not be finite cardinals and, accordingly, in general $\mathbb{F}$ will not satisfy condition (2) claiming that families of subterminals are "small" families. But this phenomenon has nothing to do with finiteness *per se* as in some (nonstandard) set theories subclasses of sets need not be sets themselves.[10]

[9] In [Bén] the more general notion of generic family was not considered. Probably because unique existence is "more categorical" than mere existence and certainly because of the above example of finite cardinals. One might wonder whether one always can construct a classifying family from a generic one. However, this seems to be unlikely because it amounts to choosing representatives from isomorphism classes which is not only unnatural but also impossible constructively.

[10] In Gödel-Bernays-vonNeumann class theory GBN every subclass of a set is guaranteed to be a set. But GBN guarantees only the existence of classes which are *first order definable* in the language of set theory. However, there is no reason why the intersection of a set with an *arbitrary nonstandard* class should be a set in general. Consider e.g. the class of *standard* elements of the set $\mathbb{N}$ of all natural numbers. Such phenomena lie at the heart of nonstandard set/class theories like E. Nelson's Inner Set Theory, P. Vopenka's Alternative Set Theory or E. Gordon's Nonstandard Class Theory. Nonstandard Class Theory was investigated and developed by J. Bénabou in the early 1970ies to quite some detail but, alas, never published.

Condition (3) says that (in each fibre) for a family of "small" sets indexed over a "small" set its sum (disjoint union) is small, too. From (1) and (3) it follows that for $a : A \to I$ and $b : B \to I$ the fibrewise product $a \times_I b : A \times_I B \to I$ is in $\mathcal{S}$, too. If, moreover, condition (2) is assumed then for $a : A \to I$ in $\mathcal{S}$ and arbitrary subobjects $m : A' \rightarrowtail A$ the composite $a \circ m$ is in $\mathcal{S}$, too. Thus, under assumption of (1), (2) and (3) for every object I in $\mathcal{E}$ the full subcategory $\mathcal{S}/I$ of $\mathcal{E}/I$ (on maps of $\mathcal{S}$ with codomain I) is finitely complete and inherits its finite limits from $\mathcal{E}/I$.[11]

Condition (4) says that the full subcategory as given by $\mathcal{S}$ is closed under dependent products and, therefore, under exponentiation. Under assumption of (1), (2) and (3) condition (4) is equivalent to the requirement that every $\mathcal{S}/I$ is closed under exponentiation in $\mathcal{E}/I$, i.e. that $\mathcal{S}/I$ is a full sub-cartesian-closed-category of $\mathcal{E}/I$.

Under assumption of conditions (1)–(4) condition (5) is equivalent to the requirement that the subfibration of $P_{\mathcal{E}}$ as given by $\mathcal{S} \hookrightarrow \mathcal{E}^{\mathbf{2}}$ is *equivalent*[12] to a *small* fibration. This still holds even if condition (2) is weakened to the requirement that for all $I \in \mathcal{E}$ the subcategory $\mathcal{S}/I$ of $\mathcal{E}/I$ is closed under equalisers. In general, a generic family need not be classifying as it can well happen that for distinct $f_1, f_2 : I \to U$ the families $f_1^* El$ and $f_2^* El$ are isomorphic as families over I. The family El is classifying iff in the internal logic of $\mathcal{E}$ it holds that $\forall a, b \in U \big(El(a) \cong El(b) \Rightarrow a =_U b\big)$. Notice that this requirement fails already when $\mathcal{E}$ is **Set** and $\mathcal{S}$ is the family $(a)_{a \in U}$ for some Grothendieck universe U in **Set** because U will contain an awful lot of distinct but equipollent sets. This, of course, can be overcome by restricting U to the cardinal numbers in U which, however, is only possible in presence of the axiom of choice and in any case does not seem very natural.

One of the useful consequences of condition (5) is that the maps of $\mathcal{S}$ are closed under $+$ which can be seen as follows. Suppose $a : A \to I$ and $b : B \to J$ are maps in $\mathcal{S}$. By condition (5) there exist maps $f : I \to U$ and $g : J \to U$ with $a \cong f^* El$ and $b \cong g^* El$. Due to the extensivity properties of toposes we then have $a + b \cong f^* El + g^* El \cong [f, g]^* El \in \mathcal{S}$ as desired.

Notice that $\mathcal{S} = \mathrm{Mono}(\mathcal{E})$ is a class of maps satisfying conditions (1)–(5). Thus, a universe need not contain the terminal projection $\Omega_{\mathcal{E}} \to 1_{\mathcal{E}}$. However, if it does, i.e. if $\mathcal{S}$ is "impredicative", then every $\mathcal{S}/I$ contains the object $\pi : I \times \Omega \to I$ and thus $\mathcal{S}/I$ is a subtopos of $\mathcal{E}/I$ in the sense that $\mathcal{S}/I \hookrightarrow \mathcal{E}/I$ is a logical functor. Moreover, for $\mathcal{E} = \mathbf{Set}$ the class $\mathbb{F}$ of families of finite cardinals gives rise to an impredicative universe which, however, does not contain N.

One of the useful consequences of impredicativity is that $\mathcal{S}$ is closed under

[11] Actually, a weaker condition than (2) suffices for this purpose, namely that every *regular monomorphism* is in $\mathcal{S}$. Under assumption of (1) and (3) this weakening of (2) is equivalent to the requirement that $\mathcal{S}$ contains all isos and $fg \in \mathcal{S}$ implies $g \in \mathcal{S}$. Thus, it follows in particular that morphisms between small maps are small themselves.

[12] This fibration need not itself be small as there need not be a classifying family, only a generic one. But it is equivalent to the small fibration arising from the internal category in $\mathcal{E}$ whose set of objects is given by U and whose family of morphisms is given by the exponential El^{El} in $\mathcal{E}/U$.

quotients in the sense that if $a \circ e \in \mathcal{S}$ and e is epic then $a \in \mathcal{S}$. This follows immediately from the facts that for all objects I in $\mathcal{E}$ the inclusion $\mathcal{S}/I \hookrightarrow \mathcal{E}/I$ is logical and that in a topos for every epimorphism $e : A \twoheadrightarrow B$ the object B appears as subobject of $\mathcal{P}(A)$ via $e^{-1} : B \rightarrowtail \mathcal{P}(A)$.

Finally, we explain how condition (5) allows one (in presence of the other conditions of Def. 2.1) to define families of sets in U via recursion over some index set (as e.g. the natural numbers object N) and to quantify over families of small sets (over a fixed index set).

Let $\pi_1, \pi_2 : U \times U \to U$ be first and second projection, respectively. Then by condition (1) the maps $\pi_1^* El$ and $\pi_2^* El$ are in $\mathcal{S}$. From conditions (3) and (4) it follows that then the exponential $\pi_2^* El^{\pi_1^* El}$ is also in $\mathcal{S}$. By condition (5) there exists a map $fun : U \times U \to U$ such that $\pi_2^* El^{\pi_1^* El} \cong fun^* El$. Obviously, the map $fun : U \times U \to U$ *internalizes* the exponentiation of sets in U. We often write b^a as an abbreviation for $fun(a, b)$. If $\mathcal{S}$ is impredicative then there exists an $\omega : 1 \to U$ such that $\omega^* El$ is isomorphic to the terminal projection $!_\Omega : \Omega \to 1$. Then the map $pow = fun \circ \langle \omega \circ !_U, id_U \rangle : U \to U$ *internalizes* the powerset operation on U since $pow(a) = \omega^a$. If $\mathcal{S}$ contains N then there exists an $n : 1 \to U$ such that $n^* El$ is isomorphic to $!_N : N \to 1$. Due to the universal property of the n.n.o. N there exists a unique map $p : N \to U$ with $p(0) = n$ and $p(k{+}1) = pow(p(k))$ for all $k \in N$. Obviously, the family $p^* El$ provides the desired generalisation of $(\mathcal{P}^k(\mathbb{N}))_{k \in \mathbb{N}}$ to toposes with a universe.

For every object A of $\mathcal{E}$ the exponential U^A exists. Thus, we can quantify over U^A, i.e. A-indexed families of smalls sets. As the family $\left(U^{El(a)}\right)_{a \in U}$ is given by the exponential $(!_U^* U)^{El}$ in $\mathcal{E}/U$ and every topos is in particular locally cartesian closed we have available in toposes with a universe also quantifications such as $\left(\forall a : U\right)\left(\forall b : U^{El(a)}\right)\left(\forall f : (\Pi x{:}El(a)) El(b(x))\right) \ldots$, i.e. quantification over all sections of families of small sets indexed by small sets.

In particular, a topos with a universe $\mathcal{S}$ provides *internal quantification* over all sorts of small structures (where the notion of smallness is given by $\mathcal{S}$).

Hierarchies of Universes

If one accepts one universe then there is no good reason why one shouldn't accept a further universe containing the previous one. The next definition makes precise what "containing" actually means. For a similar definition in the context of (semantics of) dependent type theory see the Appendix of [Str1].

Definition 2.2 *Let $\mathcal{E}$ be a topos and $\mathcal{S}_1$ and $\mathcal{S}_2$ universes in $\mathcal{E}$. We say that* $\mathcal{S}_1$ is included in $\mathcal{S}_2$ *iff there is a generic family $El_1 : E_1 \to U_1$ for $\mathcal{S}_1$ such that both El_1 and $!_{U_1} : U_1 \to 1$ are maps in $\mathcal{S}_2$.* $\Diamond$

Obviously $El_1 \in \mathcal{S}_2$ is equivalent to $\mathcal{S}_1 \subseteq \mathcal{S}_2$. But we also want that U_1 appears as element of U_2, i.e. $!_{U_1} \cong u_1^* El_2$ for some global element $u_1 : 1 \to U_2$, which, obviously, is equivalent to the requirement that $!_{U_1} : U_1 \to 1$ is in $\mathcal{S}_2$.

We suggest that a reasonable notion of model for impredicative constructive mathematics is provided by a topos $\mathcal{E}$ with a natural numbers object N together with a sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of impredicative universes containing N such that every $\mathcal{S}_n$ is contained in $\mathcal{S}_{n+1}$ (in the sense of Def. 2.2 above). An appropriate internal language for such a structure is given by Z. Luo's *Extended Calculus of Constructions* (ECC), see [Luo], together with the Axiom of Unique Choice (AUC), extensionality for functions, the *propositional extensionality principle* $\forall p, q{\in}Prop\big((p{\Leftrightarrow}q) \Rightarrow p{=}q\big)$ and the principle of *proof-irrelevance* stating that propositional types, i.e. types in *Prop*, contain at most one element. We call this formal system ECCT as an acronym for *Extended Calculus of Constructions within a Topos*.

Determining the proof-theoretic strength of ECCT is an open problem which, however, seems to be fairly difficult for the following reasons. On the one hand in **Set** an impredicative universe containing N has at least strongly inaccessible cardinality (because it is infinite, regular and closed under $\mathcal{P}(-)$, i.e. $2^{(-)}$). Thus, in **Set** an infinite cumulative sequence of such universes requires the existence of infinitely many strongly inaccessible cardinals. For this reason one might expect that ECCT is as strong as IZF with an external[13] cumulative sequence of Grothendieck universes. Postulating the Axiom of Choice (AC) this has been achieved by B. Werner in [Wer]. However, in toposes AC does not hold unless their logic is boolean. But in [JM] A. Joyal and I. Moerdijk have constructed so-called initial ZF-algebras, i.e. internal models of IZF, from universes $\mathcal{S}$ which on the one hand are a little weaker than our notion (see discussion in section 4) but on the other hand are stronger in the sense that they validate the so-called type-theoretic *Collection Axiom*

$$\text{(CA)} \qquad (\forall X)(\forall A{:}U)(\forall e{:}X{\to}A)\ \mathrm{Epic}(e) \Rightarrow (\exists C{:}U)(\exists f{:}C{\to}X)\ \mathrm{Epic}(e{\circ}f)$$

which is needed for verifying that the initial ZF-algebra validates the set-theoretic replacement axiom and they have verified the existence of such universes for all Grothendieck and realizability toposes. Thus, alas, these comparatively well-known models cannot serve the purpose of disproving the claim that ECCT proves consistency of IZF. On the other hand one has got the impression that something like (CA) is needed for verifying that the initial ZF-algebra does actually validate the replacement axiom.

3 Existence of Universes in Toposes

After having introduced the notion of universe in a topos we now discuss the question of their existence. We are primarily interested in the existence of impredicative universes containing N and from now on refer to them simply as universes. Accordingly, we assume all toposes to have a n.n.o. denoted as N.

[13] Apparently ECCT does not prove the consistency of IZF together with the axiom that every set is an element of a Grothendieck universe. For this purpose one would have to extend ECCT with a further universe U_ω containing all U_n for $n < \omega$.

First of all notice that generally in toposes universes need *not* exist. Consider for example the free topos $\mathcal{T}$ with a n.n.o N. If in $\mathcal{T}$ there existed a(n impredicative) universe (containing N) then HAH could prove its own consistency as the universe would allow one to construct a model for HAH inside $\mathcal{T}$ which is impossible by Gödel's 2^{nd} Incompleteness Theorem.

Quite expectedly, the situation is different for toposes whose construction depends on **Set** like realizability and Grothendieck toposes.

Let us first consider the somewhat simpler case of realizability toposes. Let $\mathcal{A}$ be a *partial combinatory algebra* (pca). Then the $\mathcal{P}(\mathcal{A})$-valued cumulative hierarchy $V^{(\mathcal{A})}$ is defined as

$$V^{(\mathcal{A})} = \bigcup_{\alpha \in \mathsf{Ord}} V_\alpha^{(\mathcal{A})}$$

where

$$V_\alpha^{(\mathcal{A})} = \bigcup_{\beta < \alpha} \mathcal{P}(\mathcal{A} \times V_\beta^{(\mathcal{A})})$$

is defined by transfinite recursion over α. Again by transfinite recursion one may define $\in$ and $=$ as $\mathcal{P}(\mathcal{A})$-valued binary predicates on $V^{(\mathcal{A})}$ as follows[14]

$$\begin{array}{lll} e \Vdash x \subseteq y & \text{iff} & \forall \langle a, z \rangle \in x.\ e{\cdot}a \Vdash z \in y \\ e \Vdash x = y & \text{iff} & \mathrm{pr}_1(e) \Vdash x \subseteq y \quad \text{and} \quad \mathrm{pr}_2(e) \Vdash y \subseteq x \\ e \Vdash x \in y & \text{iff} & \exists z \in V^{(\mathcal{A})}.\ \mathrm{pr}_1(e) \Vdash x = z \wedge \langle \mathrm{pr}_2(e), z \rangle \in y \end{array}$$

where $e \Vdash x \in y$ and $e \Vdash x = y$ stand for $e \in [\![x \in y]\!]$ and $e \in [\![x = y]\!]$, respectively. One may show that this gives rise to a model for IZF as was done for the first Kleene algebra (of Gödel numbers for partial recursive functions) in McCarty's Thesis [McC] but his proof extends to arbitrary pca's without any further effort. It has recently been shown by A. Simpson and the author (see [ABSS]) that the topos derived from this model of IZF is actually equivalent to $\mathrm{RT}(\mathcal{A})$, the realizability topos over $\mathcal{A}$. Now for strongly inaccessible cardinals κ one may easily show that $V_\kappa^{(\mathcal{A})}$ is a Grothendieck universe within $V^{(\mathcal{A})}$ and thus gives rise to a universe inside $V^{(\mathcal{A})}$.

For Grothendieck toposes the situation is somewhat more delicate. Suppose $\mathbf{U}$ is a Grothendieck universe in **Set**. Now if $\mathbb{C}$ is a category in $\mathbf{U}$ then this gives rise to a universe $\mathcal{U}$ inside the presheaf topos $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ which is defined as follows (see [HS]). First recall that for every $A \in \widehat{\mathbb{C}}$ the slice category $\widehat{\mathbb{C}}/A$ is equivalent to $\widehat{\mathsf{Elts}(A)}$ (where $\mathsf{Elts}(A)$ is the category of elements of A obtained via the Grothendieck construction). We define a morphism $b : B \to A$ to be contained in $\mathcal{U}$ iff the corresponding presheaf (via $\widehat{\mathbb{C}}/A \simeq \widehat{\mathsf{Elts}(A)}$) is isomorphic to one factoring through $\mathbf{U}$, i.e., more explicitly, iff $b_I^{-1}(x)$ is isomorphic to a set in $\mathbf{U}$ for all $I \in \mathbb{C}$ and $x \in A(I)$. It is more or less straightforward to verify that $\mathcal{U}$ satisfies the conditions required for a universe. The only slightly delicate

[14] We write $a{\cdot}b$ for "a applied to b" and pr_1 and pr_2 for first and second projection w.r.t. the coding of pairs available in any pca.

point is the existence of a morphism $El : E \to U$ generic for $\mathcal{U}$. We define U as the presheaf over $\mathbb{C}$ with $U(I) = \mathbf{U}^{(\mathbb{C}/I)^{\mathsf{op}}}$ and for $\alpha : J \to I$ in $\mathbb{C}$ we put $U(f) = f^* = \mathbf{U}^{(\mathbb{C}/f)^{\mathsf{op}}}$. We define the generic family $El : E \to U$ as the object of $\widehat{\mathbb{C}}/U$ corresponding (via $\widehat{\mathbb{C}}/U \simeq \widehat{\mathsf{Elts}(U)}$) to the presheaf $\mathrm{E} : \mathsf{Elts}(U)^{\mathsf{op}} \to \mathbf{U}$ which is defined as follows: with every object (I, A) in $\mathsf{Elts}(U)$ we associate the set $\mathrm{E}(I, A) = A(id_I)$ in $\mathbf{U}$ and with every morphism $f : (J, f^*A) \to (I, A)$ in $\mathsf{Elts}(U)$ we associate the map $\mathrm{E}(f) = A(f) : \mathrm{E}(I, A) \to \mathrm{E}(J, f^*A)$.

This construction for $\widehat{\mathbb{C}}$ extends to sheaf toposes $\mathcal{E} = \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ for Grothendieck topologies $\mathcal{J}$ on $\mathbb{C}$ in the following way. Let $\mathcal{U}$ be the universe in $\widehat{\mathbb{C}}$ as constructed above. We define a universe $\mathcal{U}_\mathcal{E}$ in $\mathcal{E}$ as the intersection of $\mathcal{U}$ and $\mathcal{E} \subseteq \widehat{\mathbb{C}}$. The class $\mathcal{U}_\mathcal{E}$ consists of all maps of $\mathcal{E}$ that are isomorphic to some arrow $\mathsf{a}(f)$ where $f \in \mathcal{U}$ and $\mathsf{a} : \widehat{\mathbb{C}} \to \mathsf{Sh}(\mathbb{C}, \mathcal{J})$ is the sheafification functor left adjoint to the inclusion $\mathcal{E} = \mathsf{Sh}(\mathbb{C}, \mathcal{J}) \hookrightarrow \widehat{\mathbb{C}}$. (One readily checks that the image of $\mathcal{U}$ under a is contained in $\mathcal{U}$ (using the assumption that $\mathbb{C}$ is internal to $\mathbf{U}$) and therefore $\mathcal{U}_\mathcal{E}$ coincides with the image of $\mathcal{U}$ under a.) From this observation it follows that $\mathcal{U}_\mathcal{E}$ is closed under composition and as the sheafification functor a preserves finite limits it is immediate that $\mathcal{U}_\mathcal{E}$ is stable under pullbacks along arbitrary arrows in $\mathcal{E}$. Moreover, the map $\mathsf{a}(El)$ is generic for $\mathcal{U}_\mathcal{E}$ because El is generic for $\mathcal{U}$. As a preserves monos and $\mathcal{U}$ contains all monos of $\widehat{\mathbb{C}}$ it follows that $\mathcal{U}_\mathcal{E}$ contains all monos of $\mathcal{E}$. That $\mathcal{U}_\mathcal{E}$ satisfies condition (4) of Definition 2.1 can be seen as follows. Under the conditions (1), (2) and (3) (already established for $\mathcal{U}_\mathcal{E}$) condition (4) is equivalent to the requirement that for all $A \in \mathcal{E}$ the slice $\mathcal{U}_\mathcal{E}/A$ has exponentials inherited from $\mathcal{E}/A$. Suppose $b_1 : B_1 \to A$ and $b_2 : B_2 \to A$ are objects in $\mathcal{U}_\mathcal{E}/A$. Then their exponential $b_2^{b_1}$ taken in $\widehat{\mathbb{C}}/A$ stays within $\mathcal{U}$ as $\mathcal{U}$ is a universe in $\widehat{\mathbb{C}}$ and it stays within $\mathcal{E}/A$ as $\mathcal{E}/A$ is a subtopos[15] of $\widehat{\mathbb{C}}/A$ and subtoposes are closed under exponentiation. Thus $b_2^{b_1}$ is in $\mathcal{U}_\mathcal{E}/A$ concluding the argument that $\mathcal{U}_\mathcal{E}/A$ is closed under exponentiation taken in $\mathcal{E}/A$.

Next we show that the universe $\mathcal{U}_\mathcal{E}$ is impredicative. First notice that $\mathsf{a}(\top_{\widehat{\mathbb{C}}})$ is a generic mono for $\mathcal{E}$.[16] Thus, there exists a map $s : \Omega_\mathcal{E} \to \mathsf{a}(\Omega_{\widehat{\mathbb{C}}})$ with $\top_\mathcal{E} \cong s^*\mathsf{a}(\top_{\widehat{\mathbb{C}}})$. As there is also a map $p : \mathsf{a}(\Omega_{\widehat{\mathbb{C}}}) \to \Omega_\mathcal{E}$ with $\mathsf{a}(\top_{\widehat{\mathbb{C}}}) \cong p^*\top_\mathcal{E}$ it follows that $(p \circ s)^*\top_\mathcal{E} \cong \top_\mathcal{E}$. Thus, we have $p \circ s = id_{\Omega_\mathcal{E}}$ and, therefore, the map $s : \Omega_\mathcal{E} \to \mathsf{a}(\Omega_{\widehat{\mathbb{C}}})$ is a split mono. As $\mathcal{U}$ is impredicative it contains the terminal projection of $\Omega_{\widehat{\mathbb{C}}}$ and, accordingly, $\mathcal{U}_\mathcal{E}$ contains the terminal projection of $\mathsf{a}(\Omega_{\widehat{\mathbb{C}}})$. Thus, as $\mathcal{U}_\mathcal{E}$ is closed under subobjects in $\mathcal{E}$ it follows that $\mathcal{U}_\mathcal{E}$ contains also the terminal projection of $\Omega_\mathcal{E}$, i.e. that $\mathcal{U}_\mathcal{E}$ is impredicative.

As $\mathcal{U}_\mathcal{E}$ contains the terminal projection of $N_\mathcal{E} = \mathsf{a}(N)$, the n.n.o. of $\mathcal{E}$, the universe $\mathcal{U}_\mathcal{E}$ contains $N_\mathcal{E}$.

[15] in the geometric sense because $\mathsf{a}_{/A} : \widehat{\mathbb{C}}/A \to \mathcal{E}/A$ is a finite limit preserving left adjoint to the inclusion $\mathcal{E}/A \hookrightarrow \widehat{\mathbb{C}}/A$

[16] In general, sheafification does not preserve subobject classifiers. Actually, it does if and only if the corresponding Lawvere-Tierney topology j preserves implication in the sense that $j \circ \to \; = \; \to \circ (j \times j)$.

4 Further Properties and Generalisations

It is a desirable property of a universe in a topos that for arbitrary families of types $a : A \to I$ the collection of those $i \in I$ with A_i small constitutes a subobject of I. The following mathematical precision of this informal idea is due to J. Bénabou[17] (see e.g. [Str2]).

Definition 4.1 *A pullback stable class $\mathcal{S}$ of morphisms in a topos $\mathcal{E}$ is called* definable *if for every morphism $a : A \to I$ in $\mathcal{E}$ there is a subobject $m : I_0 \rightarrowtail I$ such that $m^*a \in \mathcal{S}$ and every $f : J \to I$ with $f^*a \in \mathcal{S}$ factors through m.* $\diamondsuit$

At first sight in presence of a generic family $El : E \to U$ for $\mathcal{S}$ definability of $\mathcal{S}$ seems to be evident by considering the subobject

$$I_0 = \{i \in I \mid \exists u \in U.\, A_i \cong El(u)\}$$

of I expressible in the internal language of $\mathcal{E}$. However, when unfolding the definition of I_0 following precisely the rules of Kripke-Joyal semantics (see [MM]) one observes that I_0 is the *greatest* subobject m of I with $e^*m^*a \in \mathcal{S}$ for some epi e, i.e. $m^*a \in \mathcal{S}_\ell$ where $\mathcal{S}_\ell$, the so-called *stack completion of* $\mathcal{S}$, is the collection of all $b : B \to J$ with $e^*b \in \mathcal{S}$ for some epi $e : K \twoheadrightarrow J$. Again using Kripke-Joyal semantics one sees that $b : B \to J$ is in $\mathcal{S}_\ell$ iff $\forall j \in J. \exists u \in U.\, B_j \cong El(u)$ holds in $\mathcal{E}$. Notice that $b : B \to J$ might well be in $\mathcal{S}_\ell$ even if there is no $g : J \to U$ with $b \cong g^*El$ because every B_j may be isomorphic to some $El(u)$ though one might not be able to *choose* such a $u \in U$ uniformly in $j \in J$. This discussion shows that $\mathcal{S}$ is definable in the sense of Bénabou already if $\mathcal{S}$ satisfies the following *descent property* : $a \in \mathcal{S}$ whenever $e^*A \in \mathcal{S}$ for some epi e. One easily shows that definability of $\mathcal{S}$ implies[18] the descent property for $\mathcal{S}$. Thus, we have

Theorem 4.1 *A universe $\mathcal{S}$ in a topos $\mathcal{E}$ is definable in the sense of Bénabou if and only if $\mathcal{S}$ satifies the descent property.*

A further characterisation of definability of $\mathcal{S}$ can be found in [Sim1]. By definability for every object A of $\mathcal{E}$ there exists a subobject $m : \mathcal{P}_\mathcal{S}(A) \rightarrowtail \mathcal{P}(A)$ with $m^*(\in_A; \pi') \in \mathcal{S}$, i.e.

$$\begin{tikzcd}
\in_A^{\mathcal{S}} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \in_A \arrow[d, tail] \\
A \times \mathcal{P}_\mathcal{S}(A) \arrow[r, tail, "A \times m"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times \mathcal{P}(A) \arrow[d, "\pi'"] \\
\mathcal{P}_\mathcal{S}(A) \arrow[r, tail, "m"'] & \mathcal{P}(A)
\end{tikzcd}$$

[17] He introduced the notion of definability for (full) subfibrations of *arbitrary* fibrations and not just for the particular case of (full) subfibrations of the fundamental fibration $\partial_1 : \mathcal{E}^{\mathbf{2}} \to \mathcal{E}$ of a topos $\mathcal{E}$.

[18] If $e^*a \in \mathcal{S}$ for some epi $e : J \twoheadrightarrow I$ then e factors through some $m : I_0 \rightarrowtail I$ with $m^*a \in \mathcal{S}$. But then m is an iso and, therefore, already $a \in \mathcal{S}$.

with $\in_A^{\mathcal{S}};\pi' \in \mathcal{S}$, such that for every $r : R \rightarrowtail A \times B$ with $r;\pi' \in \mathcal{S}$ there exists a unique $\rho : B \to \mathcal{P}_{\mathcal{S}}(A)$ with

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \in_A^{\mathcal{S}} \arrow[d, tail] \\ A \times B \arrow[r, "A\times\rho"'] & A \times \mathcal{P}_{\mathcal{S}}(A) \end{tikzcd}$$

Obviously, the subobject $\mathcal{P}_{\mathcal{S}}(A)$ consists of those subsets of A which are small in the sense of $\mathcal{S}$. In [Sim1] it has been shown that existence of such *small power objects* entails the descent property for $\mathcal{S}$.[19]

Currently it is not (yet) clear (to us) whether the universes introduced in section 3 are definable. Though this seems to be very likely the case for Grothendieck toposes the question for realizability toposes seems to be much harder[20].

However, instead of a universe $\mathcal{S}$ one might instead consider its stack completion $\mathcal{S}_\ell$ still satisfying conditions (1)–(4) of Definition 2.1 but instead of condition (5) only

(5.1) (descent)
for every $a : A \to I$ if $e^*a \in \mathcal{S}_\ell$ for some epi $e : J \twoheadrightarrow I$ then already $a \in \mathcal{S}_\ell$

(5.2) (weakly generic family)
there exists a map $El : E \to U$ in $\mathcal{S}_\ell$ such that for every $a : A \to I$ in $\mathcal{S}_\ell$ there is an epi $e : J \twoheadrightarrow I$ and $f : J \to U$ with $e^*a \cong f^*El$.

We leave it for future investigations to find out whether the stack completions $\mathcal{S}_\ell$ of the universes $\mathcal{S}$ constructed in section 3 do validate the type-theoretic Comprehension Axiom (CA) discussed at end of section 2.

[19] If $b \in \mathcal{S}$ and

$$\begin{tikzcd} B \arrow[r, two heads, "e'"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ J \arrow[r, two heads, "e"'] & I \end{tikzcd}$$

then $a \cong \phi^*(\in_A^{\mathcal{S}};\pi')$ where the classifying map $\phi : I \to \mathcal{P}_{\mathcal{S}}(A)$ is given by

$$x = \phi(i) \iff \exists j \in J.\, i = e(j) \wedge x = e'_!(b^{-1}[j])$$

where $e'_!$ is the direct image map for e' and b^{-1} is the inverse image map for b, i.e.

$$\phi(i) = \{x \in A \mid \exists y \in B.\, e(b(y)) = i \wedge e'(y) = x\} .$$

[20] One would have to show for example that for the class $\mathcal{S}$ of small maps in the realizability topos **Eff** as considered in [JM] there exists a generic family and not only a weakly generic one. But this is difficult as one doesn't even know the size of $\Gamma^{-1}(1)$, i.e. how many (up to isomorphism) objects $X \in$ **Eff** exist such that X has precisely one global element.

5 Conclusions and Open Questions

We have introduced a (not too surprising) notion of (hierarchy of) universe(s) in toposes which we consider as an alternative to IZF. We think that the corresponding language ECCT being based on type theory is closer to mathematical practice than the first order language of IZF where everything is coded up in terms of $\in$. Nevertheless ECCT provides the possibility of defining families of types (in a universe) by recursion over the index set which is usually achieved by appeal to the set-theoretic replacement axiom in a much less direct way.

It is not clear, though very likely, that ECCT does not allow one to construct models for IZF without postulating further axioms like the type-theoretic Collection Axiom of [JM]. Moreover, as far as we know one has not yet found "mathematical", i.e. not meta-mathematical, statements expressible in the language of HAH which are not provable in HAH but are derivable in ECCT or IZF.[21] However, in [Sim2] it has been shown that HAH does not allow one to prove the existence of solutions of domain equations (for quite general functors on domains) although IZF does and similarly so does ECCT. The reason is that solutions of domain equations arise as inverse limits of recursively defined families of domains, i.e. particular sets in the setting of [Sim2]. It would be a pity if this quite convincing example from applied mathematics would remain the only one demonstrating the need for universes!

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[21] However, recently (in June 2005) Harvey Friedman has informed me that he has found mathematically natural Π^0_1 sentences A derivable in ZF but not in classical HAH. Thus, by Friedman's $\neg\neg$-translation of ZF to IZF it follows that such Π^0_1 sentences are provable in IZF but not in HAH.

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