Isomorphic Types are Equal!

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Motivation

Modern 20^{th} century mathematics leads us to think that

> isomorphic structures are (sort of) equal

which, however, is in conflict with set-theoretic foundations.
Because: if $A \cong B$ and $x \in A$ then in general $x \notin B$.
However: if $i : A \stackrel{\cong}{\to} B$ and $x \in A$ then $i(x) \in B$.

This opens up the possibility to use **intensional** Martin-Löf type theory (ITT) where from $e \in \mathrm{Id}_U(A, B)$ and $t \in A$ on cannot conclude that $t \in B$ but only $\mathrm{repl}(e, t) \in B$ where repl is constructed via the eliminator J for identity types.

Identity Types (1)

are the most intriguing concept of ITT. They are given by the rules

$$\frac{\Gamma \vdash A}{\Gamma, x, y{:}A \vdash \mathsf{Id}_A(x,y)}\ (\text{Id-F}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash r_A(x) : \mathsf{Id}_A(x,x)}\ (\text{Id-I})$$

$$\frac{\Gamma, x, y{:}A, z : \mathsf{Id}_A(x,y) \vdash C(x,y,z) \qquad \Gamma, x{:}A \vdash d : C(x,x,r_A(x))}{\Gamma, x, y{:}A, z : \mathsf{Id}_A(x,y) \vdash J((x)d)(z) : C(x,y,z)}\ (\text{Id-E})$$

together with the conversion rule

$$J((x)d)(r_A(t)) = d[t/x]$$

NB Id_A is an *inductively defined family of types*.

Identity Types (2)

Using J one can define operations

$cmp_A \in (\Pi x, y, z{:}A)\,\mathsf{Id}_A(x,y)\rightarrow\mathsf{Id}_A(y,z)\rightarrow\mathsf{Id}_A(x,z)$

$inv_A \in (\Pi x, y{:}A)\,\mathsf{Id}_A(x,y) \rightarrow \mathsf{Id}_A(y,x)$

validating (where we write id_x for $r_A(x)$)

(a) $(\Pi x, y, z, u{:}A)(\Pi f{:}\mathsf{Id}_A(x,y))(\Pi g{:}\mathsf{Id}_A(y,z))(\Pi f{:}\mathsf{Id}_A(z,u))$
$\mathsf{Id}_{\mathsf{Id}_A(x,u)}(cmp(f, cmp(g,h)), cmp(cmp(f,g),h))$

(b) $(\Pi x, y{:}A)\ \mathsf{Id}(cmp(id_x, f), f) \wedge \mathsf{Id}(cmp(g, id_y), g)$

(c) $(\Pi x, y{:}A)(\Pi f{:}\mathsf{Id}_A(x,y))$
$\mathsf{Id}(cmp(f, inv(f)), id_x) \wedge \mathsf{Id}(cmp(inv(f), f), id_y)$

Identity Types (3)

If U is a universe then we have

$$A : U \vdash \lambda x{:}A.x : A \to A$$

from which we get via the eliminator J that

$$A, B : U, e : \mathsf{Id}_U(A, B) \vdash \underbrace{J((A)\lambda x{:}A.x)(e)}_{\mathsf{repl}(e)} : A \to B$$

Again using J one can show that

$$A, B : U, e : \mathsf{Id}_U(A, B), x : A \vdash \mathsf{Id}_A(x, \mathsf{repl}(inv(e))(\mathsf{repl}(e)(x)))$$
$$A, B : U, e : \mathsf{Id}_U(A, B), y : B \vdash \mathsf{Id}_B(y, \mathsf{repl}(e)(\mathsf{repl}(inv(e))(y)))$$

exhibiting $\mathsf{repl}(e)$ as a (weak) iso.

Identity Types (4)

Every type A is an **internal groupoid** where **the groupoid equations hold only in the sense of propositional equality**.

For instance (a) means that there is a term

$$assoc_A(f,g,h) \in \mathrm{Id}_{\mathrm{Id}_A(x,u)}(cmp(f,cmp(g,h)),cmp(cmp(f,g),h))$$

which may be thought of as a **2-cell** in the sense of bicategories.

Since we may iterate Id-types we arrive at

n-cells in the sense of **weak higher dimensional categories**.

The Groupoid Model (1)

In early 1990ies I observed that one can prove

$$(\Pi A{:}\mathsf{Set})(\Pi x,y{:}A)(\Pi f,g{:}\mathsf{Id}_A(x,y))\ \mathsf{Id}_{\mathsf{Id}_A(x,y)}(f,g)$$

i.e. *Uniqueness of Equality Proofs* (UEP)
using the following natural extension of MLTT

$$\frac{\Gamma, x{:}A, z : \mathsf{Id}_A(x,x) \vdash C(x,z) \qquad \Gamma, x{:}A \vdash d : C(x, r_A(x))}{\Gamma, x{:}A, z : \mathsf{Id}_A(x,x) \vdash K((x)d)(z) : C(x,z)}\ (\mathsf{Id}\text{-}E')$$

together with the conversion rule

$$K((x)d)(r_A(t)) = d[t/x]$$

The Groupoid Model (2)

In 1994 [HS95] M. Hofmann and I constructed a groupoid model for ITT where K does not exist and (a)-(c) hold in the sense of judgemental equality.

The **key idea** was to interpret **types as groupoids** and **families of types as fibrations of groupoids** and

$$\mathrm{Id}_A(x,y) \quad \text{as} \quad A(x,y)$$

which may contain more than one element if the groupoid is not posetal. Thus

UEP fails in the groupoid model!

Towards Weak ω-Groupoids (1)

Already in [HS95] it was observed that

(1) ∞-groupoids might be more appropriate since in ITT the types $\mathrm{Id}_A(x, y)$ are groupoids and not just sets

(2) strict ω-groupoids are not sufficient either because in ITT the conditions (a), (b) and (c) do **not hold in the sense of judgemental equality** but **only in the sense of propositional equality**, i.e. that **weak ω-groupoids are more appropriate**.

But what is a **weak ∞-groupoid** *?*

Towards Weak ω-Groupoids (2)

In a talk in Uppsala (Nov. 2006) I suggested to consider the simplest notion of weak higher dimensional groupoid, namely **Kan complexes** in the category (topos) $\mathcal{SS} = \widehat{\mathbf{\Delta}}$ of *simplicial sets*. Accordingly, families of types will be modeled as **Kan fibrations**.

The latter form part of the classical Quillen model structure on $\mathcal{SS}$. Following a suggestion of I. Moerdijk, Awodey and Warren explained how to interpret Id-types in Quillen model structures.

Independently, V. Voevodsky (Oct. 2006) suggested to interpret type theory in simplicial sets (see `www.math.ias.edu/~vladimir`).
In particular, he came up with a construction of universes and, more recently, suggested his **Univalence Axiom** roughly saying that types are **equal** iff they are **isomorphic**.

A Recap of $\mathcal{SS}$

Let $\mathbf{\Delta}$ be the category of finite nonempty ordinals and monotone maps between them. We write $[n]$ for $\{0, 1, \ldots, n\}$. The maps of $\mathbf{\Delta}$ are generated by the morphisms

$$d_i^n : [n{-}1] \to [n] \qquad s_i^n : [n] \to [n{-}1]$$

where the first one is monic and omits i and the second one is epic and "repeats" i.

We write $\mathcal{SS}$ for $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ and $\Delta[n]$ for Yoneda of $[n]$.

For $0 \leq i \leq n$ let $\partial_i \Delta[n]$ be the subobject of $\Delta[n]$ of all maps $u : [m] \to [n]$ with $i \notin \mathrm{im}[u]$. We call $\partial\Delta[n] = \bigcup_{i=0}^{n} \partial_i \Delta[n]$ the *boundary* of $\Delta[n]$. For $0 \leq k \leq n$ let $\Lambda_k^n = \bigcup_{i \neq k} \partial_i \Delta[n]$, i.e. the union of all $(n{-}1)$-faces of $\Delta[n]$ containing the vertex k. Such objects are called **horns**.

Pictures of Horns (1)

The horn Λ^2_1 can be depicted as

$$\begin{tikzcd} & 2 \\ 0 \arrow[ur, dashed] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

where the omitted faces are indicated by broken lines.

Pictures of Horns (2)

Λ_1^2 is an inner horn as opposed to the horns Λ_0^2 and Λ_2^2 depicted as

$$\begin{tikzcd} & 2 \\ 0 \arrow[ur] \arrow[r] & 1 \arrow[u, dashed] \end{tikzcd} \qquad \begin{tikzcd} & 2 \\ 0 \arrow[ur] \arrow[r, dashed] & 1 \arrow[u] \end{tikzcd}$$

respectively.

Kan Complexes

A **horn** in a simplicial set X is a morphism $h : \Lambda^n_k \to X$.
A **Kan complex** is a simplicial set X such that every horn $h : \Lambda^n_k \to X$ in X can be extended to some $\bar{h} : \Delta[n] \to X$ making

$$\begin{tikzcd} & X \\ \Lambda^n_k \arrow[ur, "h"] \arrow[r, hook] & \Delta[n] \arrow[u, "\bar{h}"'] \end{tikzcd}$$

commute (this extension need not be unique!).

Remark Requiring this only for inner horns gves rise to Joyal's notion of **quasi-category**.

Kan Fibrations

A **Kan fibration** is a morphism $p : E \to B$ in $\mathcal{SS}$ such that every commuting square

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d, hook] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "k"'] \arrow[ur, dashed, "\overline{h}"] & B \end{tikzcd}$$

has some (not necessarily unique) filler $\overline{h}$.

Classical Quillen structure on $\mathcal{SS}$

There is an obvious functor from Δ to $\mathbf{Sp}$ whose left Kan extension

$$|\cdot| : \mathcal{SS} \to \mathbf{Sp}$$

is called **geometric realization**. We call a map w in $\mathcal{SS}$ a **weak equivalence** iff $|w|$ is a homotopy equivalence in $\mathbf{Sp}$.

The **classical Quillen model structure** on $\mathcal{SS}$ is given by $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ where

$\mathcal{C}$ = class of monomorphisms

$\mathcal{W}$ = class of weak equivalences

$\mathcal{F}$ = class of Kan fibrations.

Closure Properties of $\mathcal{F}$

Since $\mathcal{SS}$ is a topos it is in particular locally cartesian closed. As $\mathcal{F}$ is defined by a weak orthogonality condition it is obvious that $\mathcal{F}$ is closed under Σ. It is also closed under Π since the class $\mathcal{C} \cap \mathcal{W}$ is stable under pullbacks along maps in $\mathcal{F}$.

Thus $(\mathcal{SS}, \mathcal{F})$ gives a model of type theory without Id-types.

Let $\Delta \dashv \Gamma : \mathcal{SS} \to \mathsf{Set}$. Then all discrete simplicial sets $\Delta(S)$ are Kan complexes and all $\Delta(f)$ are Kan fibrations.

Thus $(\mathcal{SS}, \mathcal{F})$ contains Set as a submodel.
Ordinary Martin-Löf type theory stays within this fragment!

Interpreting Id-Types (1)

Awodey and Warren have suggested to interpret Id-types in Quillen model structures as follows.

For a fibration $a : A \to I$ the map δ_a

$$\begin{tikzcd}
A \arrow[dr, "\delta_a"] \arrow[drr, equal] \arrow[ddr, equal] & & \\
& A\times_I A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\
& A \arrow[r, "a"'] & I
\end{tikzcd}$$

gives the extensional identity type but will not be a fibration in general.

Interpreting Id-Types (2)

We may consider

$$\begin{tikzcd}
A \arrow[r, tail, "r_A"] \arrow[dr, "\delta_A"'] & \mathsf{Id}_A \arrow[d, "p_A"] \\
& A \times_I A
\end{tikzcd}$$

with $p_A \in \mathcal{F}$ and $r_A \in \mathcal{C} \cap \mathcal{W}$.

If I is terminal one may choose Id_A as $A^{\Delta[1]}$, r_A as $A^{!_{\Delta[1]}}$ and the components of p_A as $A^{d^1_1}$ and $A^{d^1_0}$, respectively.
This can be adapted easily to the slice over I.

Interpreting Id-Types (3)

Given a fibration $p_C : C \to \mathrm{Id}_A$ and $d : A \to C$ with $p_C \circ d = r_A$ then

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "r_A"'] & C \arrow[d, "p_C"] \\ \mathrm{Id}_A \arrow[ur, "J(d)"] \arrow[r, equal] & \mathrm{Id}_A \end{tikzcd}$$

for some $J(d)$.

But there is the **problem** that $J(d)$ is not unique and thus one does not know how to make a choice which is stable under pullbacks along substitutions $u : J \to I$.

Interpreting Id-Types (4)

This problem, however, can be overcome when instantiating I by the **generic** context

$$A : \mathsf{Set}, C : (x, y{:}A)\mathsf{Set}^{\mathsf{Id}_A(x,y)}, d : (x{:}A)C(x, x, r_A(x))$$

where Set is some appropriate universe since then one has to split just **once and for all** !

Lifting Universes (1)

If $\mathcal{U}$ is a (Grothendieck) universe in $\mathbf{Set}$ and $\mathcal{C}$ is a small category then this gives rise to a type-theoretic universe $p_U : \widetilde{U} \to U$ in $\mathbf{Set}^{\mathcal{C}^{\mathsf{op}}}$.
The object U is defined as

$$U(I) = \mathcal{U}^{(\mathcal{C}/I)^{\mathsf{op}}} \qquad U(\alpha) = \mathcal{U}^{\Sigma_\alpha^{\mathsf{op}}}$$

where for $\alpha : J \to I$ the functor $\Sigma_\alpha : \mathcal{C}/J \to \mathcal{C}/I$ is $\alpha \circ (-)$.
The presheaf $\widetilde{U}$ is defined as

$$\widetilde{U}(I) = \{\langle A, a\rangle \mid A \in U(I) \text{ and } a \in A(id_I)\}$$

$$\widetilde{U}(\alpha)(\langle A, a\rangle) = \langle U(\alpha)(A), A(\alpha \stackrel{\alpha}{\to} id_I)(a)\rangle$$

for $\alpha : J \to I$ in $\mathcal{C}$.
The map $p_U : \widetilde{U} \to U$ sends $\langle A, a\rangle$ to A.

Lifting Universes (2)

One easily checks that p_U is **generic** for maps with fibres small in the sense of $\mathcal{U}$: these maps are up to iso precisely those which can be obtained as pullback of p_U along some map in $\widehat{\mathcal{C}}$.

Lifting Universes to $\mathcal{SS}$ (1)

Now in case $\mathcal{C} = \Delta$ we adapt this idea in such a way that p_U is generic for Kan fibrations with fibres small in the sense of $\mathcal{U}$.

For this purpose we **redefine** U as

$$U([n]) = \{A \in \mathcal{U}^{(\Delta/[n])^{\mathsf{op}}} \mid P_A \text{ is a Kan fibration}\}$$

where $P_A : \mathsf{Elts}(A) \to \Delta[n]$ is obtained from A by the Grothendieck construction. For maps α in Δ we can define $U(\alpha)$ as above since Kan fibrations are stable under pullbacks.
We define $\widetilde{U}$ and p_U using the same formulas as above but understood as restricted to U in its present form.

Lifting Universes to $\mathcal{SS}$ (2)

Families of simplicial sets with $\mathcal{U}$-small fibres are closed under Σ, Π.

It has been shown that U is a Kan complex and p_U is a Kan fibration.

Thus p_U gives rise to a universe Set appropriate for interpreting Id-types.

Prop **in** $\mathcal{SS}$ **(1)**

From $\mathcal{P} = \{0,1\} = \{\emptyset, \{\emptyset\}\}$ one gets a universe Prf $\to$ Prop.

Notice that Prop($[n]$) consists of all monos $m : P \rightarrowtail [n]$ which are Kan fibrations. These are known to be trivial, i.e. either minimal or maximal. Thus, in $\mathcal{SS}$ we have Prop $\cong 2 = 1 + 1$ and

$$\begin{tikzcd} \mathsf{Prf} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ \mathsf{Prop} \arrow[r, tail, "{[\top, \bot]}"] & \Omega \end{tikzcd}$$

i.e. this way we obtain an interpretation of Prop which is 2-valued, boolean and proof-irrelevant.

Prop **in $\mathcal{SS}$ (2)**

The universe Prop is closed under arbitrary Π's along maps classified by p_U.

Thus, we get a model of the *Calculus of Constructions* underlying Coq.

Although the interpretation of logic is quite as in $\mathbf{Set}$ equality on Set is fairly noncanonical because it validates Voevodsky's

Univalence Axiom (1)

We first introduce a few abbreviations

$\text{iscontr}(X : \text{Set}) = (\Sigma x : X)(\Pi y : X)\ \text{Id}_X(x, y)$

$\text{hfiber}(X, Y : \text{Set})(f : X \to Y)(y : Y) =$
$\qquad = (\Sigma x : X)\ \text{Id}_Y(f(x), y)$

$\text{isweq}(X, Y : \text{Set})(f : X \to Y) =$
$\qquad = (\Pi y : Y)\ \text{iscontr}(\text{hfiber}(X, Y, f, y))$

$\text{Weq}(X, Y : \text{Set}) = (\Sigma f : X \to Y)\ \text{isweq}(X, Y, f)$

One can show that $\text{isweq}(X, Y : \text{Set})(f : X \to Y)$ is equivalent to

$$(\Sigma g : Y \to X)\Big((\Pi x{:}X)\text{Id}_X(g(fx), x)\Big) \times \Big((\Pi y{:}Y)\text{Id}_Y(f(gy), y)\Big)$$

i.e. that f is an isomorphism.

Univalence Axiom (2)

Using the eliminator J for identity types one easily constructs a map

$$\mathsf{eqweq}(X, Y : \mathsf{Set}) : \mathsf{Id}_{\mathsf{Set}}(X, Y) \to \mathsf{Weq}(X, Y)$$

Then the **Univalence Axiom**

$$\mathsf{EquAx} : (\Pi X, Y : \mathsf{Set})\ \mathsf{isweq}(\mathsf{eqweq}(X, Y))$$

postulates that all maps $\mathsf{eqweq}(X, Y)$ are weak equivalences.

Thus, for $X, Y \in \mathsf{Set}$ the type $\mathsf{Id}_{\mathsf{Set}}(X, Y)$ is isomorphic $\mathsf{Iso}(X, Y)$.

Voevodsky has shown that the Univalence Axiom holds in the model in simplicial sets.

Univalence Axiom (3)

The exponential $\mathsf{Hom}_{U \times U}(p_u, p_u)$ looks as follows: its fibre over $[n]$ consists of functors $P_A \to P_B$ over $\Delta[n]$ with $A, B \in U([n])$ and reindexing along $\alpha : J \to I$ is given by by pullback along Yoneda of u.
The subobject $\mathsf{Weq}(p_u, p_u)$ consists of those functors $P_A \to P_B$ that are weak equivalences.
For proving that p_U validates the univalence axiom one has to show that the map δ_U sending A to the identity on P_A is a weak equivalence.

$$\begin{tikzcd} U \arrow[r, "\delta_U"] \arrow[dr, "\Delta"'] & \mathsf{Weq}(p_U, p_U) \arrow[d] \\ & U \times U \end{tikzcd}$$

Conclusion and Problems

- Simplicial sets provide a classical model of impredicative type theory extending the naive model in $\mathbf{Set}$.
  Types are interpreted as Kan complexes, i.e. weak higher dimensional groupoids. Families of types are Kan fibrations.

- Types in the universe Set validate the Univalence Axiom saying that types in Set are propositionally equal iff they are isomorphic iff they are weakly equivalent. Since weakly equivalent types are equal the type theory sees Kan complexes as homotopy types.

- Is there a computational meaning of the Univalence Axiom?

- Since isweq(f) holds in $\mathcal{SS}$ iff f is a weak equivalence iff f is a homotopy equivalence one may develop **Synthetic Homotopy Theory** in type theory (Coq), see Voevodsky's files.