Krivine's Classical Realizability from a Categorical Perspective

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The Scenario

Krivine's **Classical Realizability** will turn out as a **generalization of forcing** as known from set theory.

Following Hyland with every **partial combinatory algebra** (pca) $\mathbb{A}$ one associates a **realizability topos** $\mathrm{RT}(\mathbb{A})$. However,

$$\mathrm{RT}(\mathbb{A}) \text{ Groth. topos or boolean } \Rightarrow \ \mathbb{A} \text{ trivial pca}$$

thus classical realizability is not given by a pca.

However, the **order pca**'s of J. van Oosten and P. Hofstra provide a common generalization of realizability and Heyting valued models. We will use them for capturing classical realizability categorically.

Classical Realizability (1)

The collection of (possibly open) **terms** is given by the grammar

$$t ::= x \mid \lambda x.t \mid ts \mid \mathsf{cc}\,t \mid \mathsf{k}_\pi$$

where π ranges over **stacks** (i.e. lists) of closed terms. A **quasi-proof** is a term without occurrences of k. We write Λ for the set of closed terms, QP for the set of closed quasi-proofs and Π for the set of stacks of closed terms. A **process** is a pair $t * \pi$ with $t \in \Lambda$ and $\pi \in \Pi$.

The operational semantics of Λ is given by the relation $\succeq$ (*head reduction*) on processes defined inductively by the clauses

(pop)	$\lambda x.t * s.\pi$	$\succeq$	$t[s/x] * \pi$
(push)	$ts * \pi$	$\succeq$	$t * s.\pi$
(store)	$\mathsf{cc}\,t * \pi$	$\succeq$	$t * \mathsf{k}_\pi.\pi$
(restore)	$\mathsf{k}_\pi * t.\pi'$	$\succeq$	$t * \pi$

Classical Realizability (2)

This language has a natural interpretation within the recursive domain

$$D \cong \Sigma^{\mathsf{List}(D)} \cong \prod_{n \in \omega} \Sigma^{D^n}$$

We have $D \cong \Sigma \times D^D$. Thus D^D is a retract of D and, accordingly, D is a model for λ_β-calculus. The interpretation of Λ is given by

$$\begin{aligned}
[\![x]\!]_\varrho &= \varrho(x) & [\![ts]\!]_\varrho\, k &= [\![t]\!]\, \varrho\langle [\![s]\!]_\varrho, k\rangle \\
[\![\lambda x.t]\!]_\varrho \langle\rangle &= \top & [\![\lambda x.t]\!]_\varrho \langle d, k\rangle &= [\![t]\!]_{\varrho[d/x]}\, k \\
[\![\mathsf{cc}\, t]\!]_\varrho\, k &= [\![t]\!]_\varrho \langle \mathsf{ret}(k), k\rangle & [\![\mathsf{k}_\pi]\!]_\varrho &= \mathsf{ret}([\![\pi]\!]_\varrho)
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{ret}(k)\langle\rangle &= \top & \mathsf{ret}(k)\langle d, k'\rangle &= d(k) \\
[\![\langle\rangle]\!]\, \varrho &= \langle\rangle & [\![t.\pi]\!]_\varrho &= \langle [\![t]\!]_\varrho, [\![\pi]\!]_\varrho\rangle
\end{aligned}$$

Classical Realizability (3)

A **pole** is a set $\perp\!\!\!\perp$ of processes s.t. $q \in \perp\!\!\!\perp$ whenever $q \succeq p \in \perp\!\!\!\perp$.
We write $t \perp \pi$ for $t * \pi \in \perp\!\!\!\perp$. For $X \subseteq \Pi$ and $Y \subseteq \Lambda$ we put

$$X^\perp = \{t \in \Lambda \mid \forall \pi \in X.\, t \perp \pi\} \qquad Y^\perp = \{\pi \in \Pi \mid \forall t \in Y.\, t \perp \pi\}$$

Obviously $(-)^\perp$ is antitonic and $Z \subseteq Z^{\perp\perp}$ and thus $Z^\perp = Z^{\perp\perp\perp}$.
For a pole $\perp\!\!\!\perp$ second order logic over a set M of individuals is interpreted as follows: n-ary predicate variables range over functions $M^n \to \mathcal{P}(\Pi)$ and formulas A are interpreted as $||A|| \subseteq \Pi$

$$\begin{aligned}
&||X(t_1,\dots,t_n)||_\varrho = \varrho(X)(\llbracket t_1\rrbracket_\varrho,\dots,\llbracket t_1\rrbracket_\varrho)\\
&||A \to B||_\varrho = |A|_\varrho.||B||_\varrho\\
&||\forall x A(x)||_\varrho = \textstyle\bigcup_{a \in M} ||A||_{\varrho[a/x]}\\
&||\forall X A[X]||_\varrho = \textstyle\bigcup_{R \in \mathcal{P}(\Pi)^{M^n}} ||A||_{\varrho[R/X]}
\end{aligned}$$

where $|A|_\varrho = ||A||_\varrho^\perp$. The proposition A is **valid** iff $|A|_\varrho \cap \mathrm{QP} \neq \emptyset$.

Classical Realizability (4)

We have $|\forall X A| = \bigcap_{R \in \mathcal{P}(\Pi)^{M^n}} |A[R/X]|$.

In general $|A \to B|$ is a **proper** subset of

$$|A| \to |B| = \{t \in \Lambda \mid \forall s \in |A|\ ts \in |B|\}$$

unless $ts * \pi \in \perp\!\!\!\perp \Rightarrow t * s.\pi \in \perp\!\!\!\perp$

But for every $t \in |A| \to |B|$ its η-expansion $\lambda x.tx \in |A \to B|$ and, of course, we have $|A \to B| = |A| \to |B|$ whenever $\perp\!\!\!\perp$ is also *closed under head reduction*, i.e. $\perp\!\!\!\perp \ni p \succeq q$ implies $q \in \perp\!\!\!\perp$.

One may even assume that $\perp\!\!\!\perp$ is stable w.r.t. the semantic equality $=_D$ induced by the model D. However, there are interesting situations where one has to go beyond such a framework.

Classical Realizability (5)

For realizing the Countable Axiom of Choice CAC Krivine introduced a new language construct χ^* with the reduction rule

$$\chi^* * t.\pi \succeq t * n_t.\pi$$

where n_t is the Church numeral representation of a Gödel number for t, *c.f.* quote(t) of LISP.

NB quote is in conflict with β-reduction!

NB The term χ^* realizes *Krivine's Axiom*

$$\exists S \forall x \Big(\forall n^{\mathsf{Int}} Z(x, S_{x,n}) \to \forall X Z(x, X) \Big)$$

which entails CAC.

Axiomatic Classical Realizability (1)

Instead of pca's we consider **Abstract Krivine Structures** (aks's) :

- a set Λ of "terms" together with a binary application operation (written as juxtaposition) and distinguished elements K, S, $\mathsf{cc} \in \Lambda$
- a subset $\mathsf{QP} \subseteq \Lambda$ of "quasi-proofs" closed under application and containing K, S and cc as elements
- a set Π of "stacks" together with a push operation (push) from $\Lambda \times \Pi$ to Π (written $t.\pi$) and a unary operation $\mathsf{k} : \Pi \to \Lambda$
- a "pole" $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ satisfying

  (S1) $ts \star \pi \in \perp\!\!\!\perp$ whenever $t \star s.\pi \in \perp\!\!\!\perp$

  (S2) $\mathsf{K} \star t.s.\pi \in \perp\!\!\!\perp$ whenever $t \star \pi \in \perp\!\!\!\perp$

  (S3) $\mathsf{S} \star t.s.u.\pi \in \perp\!\!\!\perp$ whenever $tu(su) \star \pi \in \perp\!\!\!\perp$

  (S4) $\mathsf{cc} \star t.\pi \in \perp\!\!\!\perp$ whenever $t \star \mathsf{k}_\pi.\pi$ in $\perp\!\!\!\perp$

  (S5) $\mathsf{k}_\pi \star t.\pi' \in \perp\!\!\!\perp$ whenever $t \star \pi \in \perp\!\!\!\perp$.

Axiomatic Classical Realizability (2)

A **proposition** A is given by a subset $||A|| \subseteq \Pi$. Its set of **realizers** is

$$|A| = ||A||^{\perp} = \{t \in \Lambda \mid \forall \pi \in ||A|| \;\; t \star \pi \in \perp\!\!\!\perp\}$$

and A is **valid** iff $|A| \cap \mathsf{QP} \neq \emptyset$. Logic is interpreted as follows

$$||R(\vec{t})|| = R\Big([\![\vec{t}]\!]\Big)$$

$$||A \to B|| = |A|.||B|| = \{t.\pi \mid t \in |A|, \pi \in ||B||\}$$

$$||\forall x A(x)|| = \bigcup_{a \in M} ||A(a)||$$

$$||\forall X A(X)|| = \bigcup_{R \in \mathcal{P}(\Pi)^{M^n}} ||A(R)||$$

where M is the underlying set of the model.

Axiomatic Classical Realizability (3)

One could define propositions more restrictively as

$$\mathcal{P}_{\perp\!\!\perp}(\Pi) = \{X \in \mathcal{P}(\Pi) \mid X = X^{\perp\perp}\}$$

without changing the meaning of $|A|$ for closed formulas.

Notice that $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ is in 1-1-correspondence with

$$\mathcal{P}_{\perp\!\!\perp}(\Lambda) = \{X \in \mathcal{P}(\Lambda) \mid X = X^{\perp\perp}\}$$

via $(-)^{\perp}$.

In case (S1) holds as an equivalence, i.e. we have

(SS1) $\quad ts \star \pi$ in $\perp\!\!\perp$ $\quad$ iff $\quad t \star s.\pi$ in $\perp\!\!\perp$

one may define $|\cdot|$ directly as

Axiomatic Class Realiz. (4)

$$|R(\vec{t})| = R\Big(\llbracket\vec{t}\rrbracket\Big)$$

$$|A{\to}B| = |A|{\to}|B| = \{t \in L \mid \forall s \in |A|\ ts \in |B|\}$$

$$|\forall x A(x)| = \bigcap_{a \in M} |A(a)|$$

$$|\forall X A(X)| = \bigcap_{R \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)^{M^n}} |A(R)|$$

and it coincides with the previous definition for closed formulas.

Abstract Krivine structures validating the reasonable assumption (SS1) are called **strong abstract Krivine structures** (saks's).

Axiomatic Class Realiz. (5)

Obviously, for $A, B \in \mathcal{P}_{\perp\perp}(\Lambda)$ we have

$$|A{\to}B| \subseteq |A|{\to}|B| = \{t \in \Lambda \mid \forall s \in |A|\ ts \in |B|\}$$

But for any $t \in |A| \to |B|$ we have

$$\mathsf{E}t \in |A{\to}B|$$

where $\mathsf{E} = \mathsf{S}(\mathsf{KI})$ with $\mathsf{I} = \mathsf{SKK}$.

Axiomatic Class Realiz. (5a)

Proof. One easily checks that

$$\mathsf{I} * t.\pi \in \perp\!\!\!\perp \;\Leftarrow\; t * \pi \in \perp\!\!\!\perp$$

and thus we have

$$\mathsf{E}t * s.\pi \in \perp\!\!\!\perp \;\Leftarrow\; ts * \pi \in \perp\!\!\!\perp$$

because

$$\mathsf{E}t * s.\pi \in \perp\!\!\!\perp \;\Leftarrow\; \mathsf{KI}s(ts).\pi \in \perp\!\!\!\perp \;\Leftarrow\; \mathsf{I} * ts.\pi \in \perp\!\!\!\perp \;\Leftarrow\; ts * \pi \in \perp\!\!\!\perp$$

Then for $s \in |A|$, $\pi \in ||B||$ we have $\mathsf{E}t * s.\pi \in \perp\!\!\!\perp$
because $ts * \pi \in \perp\!\!\!\perp$ since $t \in |A| \to |B|$.

Thus $\mathsf{E}t \in |A{\to}B|$ as desired.

Forcing as an Instance (1)

Let $\mathbb{P}$ a $\wedge$-semilattice (with top element 1) and $\mathcal{D}$ a *downward closed* subset of $\mathbb{P}$. We write pq for $p \wedge q$.
Such a situation gives rise to a saks where

- $\Lambda = \Pi = \mathbb{P}$
- $\mathsf{QP} = \{1\}$
- application and the push operation are interpreted as $\wedge$ in $\mathbb{P}$
- k is the identity on $\mathbb{P}$ and constants K, S and cc are interpreted as 1
- $\perp\!\!\!\perp = \{(p,q) \in \mathbb{P}^2 \mid pq \in \mathcal{D}\}$.

We write $p \perp q$ for $p * q \in \perp\!\!\!\perp$, i.e. $pq \in \mathcal{D}$.

NB This is **not** a pca since application is commutative and associative and thus $a = kab = kba = b$.

Forcing as an Instance (2)

For $X \subseteq \mathbb{P}$ we have

$$X^{\perp} = \{p \in \mathbb{P} \mid \forall q \in X\ pq \in \mathcal{D}\}$$

which is downward closed and contains $\mathcal{D}$ as a subset.

For such X we have

$$X^{\perp} = \{p \in \mathbb{P} \mid \forall q \leq p\ (q \in X \Rightarrow q \in \mathcal{D})\}$$

Thus, for arbitrary $X \subseteq \mathbb{P}$ we have

$$\begin{aligned} X^{\perp\perp} &= \{p \in \mathbb{P} \mid \forall q \leq p\ (q \in X^{\perp} \Rightarrow q \in \mathcal{D})\} \\ &= \{p \in \mathbb{P} \mid \forall q \leq p\ (q \notin \mathcal{D} \Rightarrow q \notin X^{\perp})\} \\ &= \{p \in \mathbb{P} \mid \forall q \leq p\ (q \notin \mathcal{D} \Rightarrow\ \exists r \leq q\ (r \notin \mathcal{D} \wedge r \in X))\} \end{aligned}$$

as familiar from **Cohen forcing**.

Forcing as an Instance (3)

Accordingly, we define **propositions** as $A \subseteq \mathbb{P}$ with $A = A^{\perp\perp}$.

In case $\mathcal{D} = \{0\}$ then $\mathbb{P}^{\uparrow} = \mathbb{P} \setminus \{0\}$ is a conditional $\wedge$-semilattice and propositions are in 1-1-correspondence with *regular* subsets A of $\mathbb{P}^{\uparrow}$, i.e. $p \in A$ whenever $\forall q{\leq}p\, \exists r{\leq}q\; r \in A$, as in **Cohen forcing** over $\mathbb{P}^{\uparrow}$.

For propositions A, B, C we have

$$A \to B :\, = \{p \in \mathbb{P} \mid \forall q \in A\; pq \in B\} = \{p \in \mathbb{P} \mid \forall q \leq p\; (q \in A \Rightarrow q \in B)\}$$

and thus $\qquad C \subseteq A \to B \quad$ iff $\quad C \cap A \subseteq B$

The least proposition $\perp$ is given by $\mathbb{P}^{\perp} = \mathcal{D}$ and thus we have

$$\neg A \equiv A \to \perp = \{p \in \mathbb{P} \mid \forall q \in A\; pq \in \mathcal{D}\} = A^{\perp}$$

Characterization of Forcing

One can show that a saks arises (up to iso) from a downward closed subset of a $\wedge$-semilattice iff

(1) $\mathsf{k} : \Pi \to \Lambda$ is a bijection

(2) application is associative, commutative and idempotent and has a neutral element 1

(3) application coincides with the push operation (when identifying Λ and Π via k).

Remark The downset $\mathcal{D} = \{t \in \Lambda \mid (t, 1) \in \perp\!\!\!\perp\}$ (where 1 in Π via k).

In this sense **forcing = commutative realizability**

AKS's as total OPCAs (1)

Hofstra and van Oosten's notion of **order partial combinatory algebra** (opca) generalizes both pca's and complete Heyting algebras (cHa's).

We will show how every aks can be organised into a total opca.

A **total opca** is a triple $(\mathbb{A}, \leq, \bullet)$ where $\leq$ is a partial order on $\mathbb{A}$ and $\bullet$ is a binary monotone operation on $\mathbb{A}$ such that for some $k, s \in \mathbb{A}$

$$k \bullet a \bullet b \leq a \qquad s \bullet a \bullet b \bullet c \leq a \bullet c \bullet (b \bullet c)$$

for all $a, b, c \in \mathbb{A}$.

AKS's as total OPCAs (2)

With every aks we may associate the total opca whose underlying set is $\mathcal{P}_{\perp\!\!\perp}(\Pi)$, where $a \leq b$ iff $a \supseteq b$ and application is defined as

$$a \bullet b = \{\pi \in \Pi \mid \forall t \in |a|, s \in |b| \;\; t * s.\pi \in \perp\!\!\perp\}^{\perp\perp}$$

where $|a| = a^{\perp}$. Obviously $a \leq b$ iff $|a| \subseteq |b|$.

NB In case of a saks we have

$$|a \bullet b| = \{ts \mid t \in |a|, s \in |b|\}^{\perp\perp}$$

Lemma 1

From $a \leq b \to c$ it follows that $a \bullet b \leq c$.

Lemma 2

If $t \in |a|$ and $s \in |b|$ then $ts \in |a \bullet b|$.

$(\mathcal{P}_{\perp\!\!\perp}(\Pi), \supset, \bullet)$ is a total OPCA

One easily shows that $\{\mathsf{K}\}^{\perp}ab \leq a$.

For showing that $\{\mathsf{S}\}^{\perp} \bullet a \bullet b \bullet c \leq a \bullet c \bullet (b \bullet c)$ it suffices by (multiple applications of) Lemma 1 to show that $s \leq a \to b \to c \to (a \bullet c \bullet (b \bullet c))$. It suffices to show that

$$\mathsf{S} \in |a \to b \to c \to (a \bullet c \bullet (b \bullet c))|$$

For this purpose suppose $t \in |a|$, $s \in |b|$, $u \in |c|$ and $\pi \in a \bullet c \bullet (b \bullet c)$. Applying Lemma 2 iteratively we have $tu(su) \in |a \bullet c \bullet (b \bullet c)|$ and thus $tu(su) * \pi \in \perp\!\!\perp$. Since $\perp\!\!\perp$ is closed under expansion it follows that $\mathsf{S} * t.s.u.\pi \in \perp\!\!\perp$ as desired.

AKS's as total OPCAs (3)

A **filter** in a total opca $(\mathbb{A}, \leq, \bullet)$ is a subset Φ of $\mathbb{A}$ closed under $\bullet$ and containing (some choice of) k and s (for $\mathbb{A}$).

(1) In case of a saks induced by a downclosed set $\mathcal{D}$ in a $\wedge$-semilattice $\mathbb{P}$ a natural choice of a filter is $\{\mathbb{P}\}$.

(2) $\Phi = \{a \in \mathcal{P}_{\perp\perp}(\Pi) \mid |a| \cap \mathsf{QP} \neq \emptyset\}$ is a filter on $\mathcal{P}_{\perp\perp}(\Pi)$ by Lemma 2.

With a filtered opca one may associate a $\mathbf{Set}$-indexed preorder $[-, \mathbb{A}]_\Phi$

- $[I, \mathbb{A}]_\Phi = \mathbb{A}^I$ is the set of all functions from set I to $\mathbb{A}$
- endowed with the preorder $\varphi \vdash_I \psi$ iff $\exists a \in \Phi \forall i \in I \; a \bullet \varphi_i \leq \psi_i$
- for $u : J \to I$ the **reindexing map** $[u, \mathbb{A}]_\Phi = u^* : \mathbb{A}^I \to \mathbb{A}^J$ sends φ to $u^*\varphi = (\varphi_{u(j)})_{j \in J}$.

Krivine Tripos (1)

In case $\mathbb{A}$ arises from an AKS and $\Phi = \{a \in \mathcal{P}_{\perp\!\!\perp}(\Pi) \mid |a| \cap \mathsf{QP} \neq \emptyset\}$ the indexed preorder $[-,\mathbb{A}]_\Phi$ is a **tripos**, i.e.

- all $[I,\mathbb{A}]_\Phi$ are pre-Heyting-algebras whose structure is preserved by reindexing
- for every $u : J \to I$ in $\mathbf{Set}$ the reindexing map u^* has a left adjoint $\exists_u$ and a right adjoint $\forall_u$ satisfying (Beck-)Chevalley condition
- there is a *generic predicate* $T \in [\Sigma,\mathbb{A}]_\Phi$, namely $\Sigma = \mathbb{A}$ and $T = \mathrm{id}_\mathbb{A}$, of which all predicates arise by reindexing since $\varphi = \varphi^* \,\mathrm{id}_\mathbb{A}$

It coincides with Krivine's Classical Realizability since for $\varphi, \psi \in [M,\mathbb{A}]_\Phi$

$$\varphi \vdash_M \psi \quad \text{iff} \quad \exists t \in \mathsf{QP}\, \forall i \in M\ t \in |\varphi_i \to \psi_i|$$

Krivine Tripos (2)

Proof :

Suppose $\varphi \vdash_M \psi$. Then there exists $a \in \Phi$ such that $\forall i \in M \;\; a {\bullet} \varphi_i \leq \psi_i$. For all $i \in M$, $u \in |a|$ and $v \in |\varphi_i|$ by Lemma 2 we have $uv \in |a \bullet \varphi_i| \subseteq |\psi_i|$. Let $u \in |a| \cap \mathsf{QP}$. Then for all $i \in M$ we have $u \in |\varphi_i| \to |\psi_i|$ and thus $\mathsf{E}u \in |\varphi_i \to \psi_i|$. Thus $t = \mathsf{E}u \in \mathsf{QP}$ does the job.

Suppose there exists a $t \in \mathsf{QP}$ such that $\forall i \in M \; t \in |\varphi_i \to \psi_i|$. Then we have $\forall i \in M \; \{t\}^{\perp\perp} \subseteq |\varphi_i \to \psi_i|$. Thus, for $a = \{t\}^{\perp} \in \Phi$ we have

$$\forall i \in M \forall u \in |a| \forall v \in |\varphi_i| \forall \pi \in \psi_i \;\; u * v.\pi \in \perp\!\!\!\perp$$

from which it follows that

$$\forall i \in M \; a \bullet \varphi_i \leq \psi_i$$

i.e. $\varphi \vdash_M \psi$ as desired.

Forcing in Classical Realizability (1)

Let P be a meet-semilattice and C an upward closed subset of P. With every $X \subseteq P$ one associates[*]

$$|X| = \{p \in P \mid \forall q\,(\mathsf{C}(pq) \to X(q))\}$$

Such subsets of P are called propositions. We say

$$p \text{ forces } X \quad \text{iff} \quad p \in |X|$$

and want to have that

$$\begin{array}{lll} p \text{ forces } X \to Y & \text{iff} & \forall q\,(|X|(q) \to |Y|(pq)) \\ p \text{ forces } \forall i \in I.X_i & \text{iff} & \forall i \in I.\, p \text{ forces } X_i \end{array}$$

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[*] Traditionally, one would associate with X the set $X^{\perp} = \{p \in P \mid \forall q \in X\ \neg\mathsf{C}(pq)\}$. But, classically, we have $|X| = (P \setminus X)^{\perp}$.

Forcing in Classical Realizability (2)

Apparently, we have

$$\begin{aligned}
&p \text{ forces } X \to Y \quad \text{iff} \\
&\forall q\,(|X|(q) \to \forall r(\mathsf{C}(pqr) \to Y(r))) \quad \text{iff} \\
&\forall q, r\;(\mathsf{C}(pqr) \to |X|(q) \to Y(r)) \quad \text{iff} \\
&p \in \Big|\{qr \mid |X|(q) \to Y(r)\}\Big|
\end{aligned}$$

$$p \text{ forces } \forall i \in I.X_i \quad \text{iff} \qquad p \in \Big|\textstyle\bigcap_{i\in I} X_i\Big|$$

telling us how to interpret → and ∀.

Forcing in Classical Realizability (3)

Actually, in most cases P is not a meet-semilattice **but** it is so "from point of view" of $\mathsf{C} \subseteq P$, i.e. we have a binary operation on P and an element $1 \in P$ such that

$$\mathsf{C}(p(qr)) \leftrightarrow \mathsf{C}((pq)r)$$

$$\mathsf{C}(pq) \leftrightarrow \mathsf{C}(qp)$$

$$\mathsf{C}(p) \leftrightarrow \mathsf{C}(pp)$$

$$\mathsf{C}(1p) \leftrightarrow \mathsf{C}(p)$$

$$\Big(\mathsf{C}(p) \leftrightarrow \mathsf{C}(q)\Big) \to \Big(\mathsf{C}(pr) \leftrightarrow \mathsf{C}(qr)\Big)$$

together with

$$\mathsf{C}(pq) \to \mathsf{C}(p)$$

expressing that C is upward closed.

Forcing in Classical Realizability (3a)

On P we may define a congruence

$$p \simeq q \equiv \forall r.\, (\mathsf{C}(rp) \leftrightarrow \mathsf{C}(rq))$$

w.r.t. which P is a commutative idempotent monoid, i.e. a meet-semilattice, of which C is an upward closed subset.

Thus, on P we may consider the partial order relation $p \preceq q$ defined as $p \simeq pq$ which is equivalent to $\forall r.\, (\mathsf{C}(rp) \to \mathsf{C}(rq))$.

Forcing in Classical Realizability (4)

We have seen that p forces $X \to Y$ iff $\forall q, r\,(C(pqr) \to |X|(q) \to Y(r))$
Thus a term t realizes p forces $X \to Y$ iff

$$(\dagger) \quad \forall q, r \forall u {\in} \mathsf{C}(pqr) \forall s {\in} |X|(q) \forall \pi {\in} Y(r)\ t * u.s.\pi \in \perp\!\!\!\perp$$

Thus, one might want to define when a pair (t, p) realizes $X \to Y$.

For this purpose one has to find an aks structure whose term part is $\Lambda \times P$. We procrustinate the definition of application. The set of quasi-proofs is given by $\mathsf{QP} \times \{1\}$. The set of stacks is given by $\Pi \times P$ with $(t, p).(\pi, q) = (t.\pi, pq)$. From $\perp\!\!\!\perp$ one defines a new pole $\perp\!\!\!\perp\!\!\!\perp$ as

$$(t, p) * (\pi, q) \in \perp\!\!\!\perp\!\!\!\perp \quad \text{iff} \quad \forall u \in \mathsf{C}(pq)\ t * \pi^u \in \perp\!\!\!\perp$$

where π^u is obtained from π by inserting u at its bottom.

Forcing in Classical Realizability (4a)

Thus, we have

$$
\begin{aligned}
&(t,p) \in |X \to Y| \\
&\qquad \text{iff} \\
&\forall (s,q) \in |X| \forall (r,\pi) \in Y \ (t,p) * (s,q).(\pi,r) \in \perp\!\!\!\perp\!\!\!\perp \\
&\qquad \text{iff} \\
&\forall (s,q) \in |X| \forall (r,\pi) \in Y \forall u \in \mathsf{C}(pqr) \ t * s.\pi^u \in \perp\!\!\!\perp
\end{aligned}
$$

in accordance with explication (†) of t realizes p forces $X \to Y$ as

$$\forall q,r \forall u \in \mathsf{C}(pqr) \forall s \in |X|(q) \forall \pi \in Y(r) \ t * u.s.\pi \in \perp\!\!\!\perp$$

Forcing in Classical Realizability (5)

In order to jump back and forth between

t realizes p forces A and $(t', p) \in |A|$

one needs "read" and "write" constructs in the original AKS, i.e. command χ and χ' s.t.

$$
\begin{array}{lll}
\text{(read)} & \chi * t.\pi^s & \succeq & t * s.\pi \\
\text{(write)} & \chi' * t.s.\pi & \succeq & t * \pi^s
\end{array}
$$

Using these one can transform t into t' and *vice versa*.

Krivine concludes from this that for **realizing forcing one needs global memory**.

Forcing in Classical Realizability (6)

Now we can define application for the new aks. Let α be a uniform realizer of $\mathsf{C}((pq)r) \to \mathsf{C}(p(qr))$ and $\underline{\alpha}$ a term with

$$\underline{\alpha} * t.\pi^{u} \in \perp\!\!\!\perp \text{ whenever } t * \pi^{\alpha u} \in \perp\!\!\!\perp$$

which may be taken as $\lambda^{*}x.\chi(\lambda^{*}y.\chi' x(\alpha y))$. Now application in the new aks is defined as

$$(t,p)(s,q) \equiv (\underline{\alpha}(ts), pq)$$

for which it holds that

Forcing in Classical Realizability (6a)

$$\begin{array}{ll}
(t,p)(s,q) * (\pi,r) \in \perp\!\!\perp\!\!\perp & \text{iff} \\
\forall u \in \mathsf{C}((pq)r)\, \underline{\alpha}(ts) * \pi^{u} \in \perp\!\!\!\perp & \text{if} \\
\forall u \in \mathsf{C}((pq)r)\, ts * \pi^{\alpha u} \in \perp\!\!\!\perp & \text{if} \\
\forall u \in \mathsf{C}((pq)r)\, t * s.\pi^{\alpha u} \in \perp\!\!\!\perp & \text{if} \\
\forall u \in \mathsf{C}((p(qr))\, t * s.\pi^{u} \in \perp\!\!\!\perp & \text{iff} \\
(t,p) * (s,q).(\pi,r) \in \perp\!\!\perp\!\!\perp &
\end{array}$$

as required by condition (S1).

Generic Set and Ideal (1)

In forcing one usually considers the **generic set** $\mathcal{G}$ which is the predicate on P with $\mathcal{G}(p) = \{p\}^{\perp\perp}$.

Equivalently one my consider its complement, the **generic ideal** $\mathcal{J}$ with $|\mathcal{J}(p)| = \{p\}^{\perp}$, i.e.

$$\mathcal{J}(p) = \{q \in P \mid p \neq q\}$$

since $q \in |\mathcal{J}(p)|$ iff $\forall r\, (\mathsf{C}(qr) \to p \neq r)$ iff $\neg\mathsf{C}(qp)$.

Generic Set and Ideal (2)

Obviously, we have

$$p \simeq q \quad \text{iff} \quad \forall r\,(|\mathcal{J}(p)|(r) \leftrightarrow |\mathcal{J}(q)|(r))$$

and also

$$p \preceq q \quad \text{iff} \quad \forall r\,(|\mathcal{J}(q)|(r) \to |\mathcal{J}(p)|(r))$$

since its right hand side is equivalent to $\forall r\,(\mathsf{C}(rp) \to \mathsf{C}(rq))$.

Equivalently, we may define

$$||\mathcal{J}(p)|| = \Pi \times \{p\}$$

since $(t,q) \in |\mathcal{J}(p)|$ iff $\forall \pi\,(t,q) * (\pi,p) \in \perp\!\!\!\perp\!\!\!\perp$ iff $\forall u \in \mathsf{C}(qp) \forall \pi\, t * \pi^u \in \perp\!\!\!\perp$.

$\mathcal{P}(P)$ as a cBa

For $X \in \mathcal{P}(P)$ define $\mathcal{J}(X)$ such that

$$|\mathcal{J}(X)|(q) \quad \text{iff} \quad \forall p \in X\, \neg \mathsf{C}(qp)$$

i.e. $|\mathcal{J}|(X) = X^{\perp}$. We may extend $\preceq$ to $\mathcal{P}(P)$ as follows

$$X \preceq Y \equiv \forall r \left(|\mathcal{J}(Y)|(r) \to |\mathcal{J}(X)|(r) \right)$$

Thus $X \preceq Y$ iff $Y^{\perp} \subseteq X^{\perp}$ iff $X^{\perp\perp} \subseteq Y^{\perp\perp}$.
This endows $\mathcal{P}(P)$ with the structure of a complete boolean preorder denoted by B. Writing $\mathcal{E}$ for the classical realizability topos arising from the original AKS the classical topos arising from the new AKS is (equivalent to) the topos $\mathsf{Sh}_{\mathcal{E}}(B)$.

Warning B is not an assembly in $\mathcal{E}$ as it is uniform. Thus the construction of $\mathsf{Sh}_{\mathcal{E}}(B)$ from $\mathcal{E}$ is **not** induced by an opca morphism.